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Quasiautomorphism groups of type F_∞

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The groups QF, QT, $\bar{Q}T$, $\bar{Q}V$ and QV are groups of quasiautomorphisms of the infinite binary tree. Their names indicate a similarity with Thompson's well-known groups F, T and V.

We will use the theory of diagram groups over semigroup presentations to prove that all of the above groups (and several generalizations) have type F_∞. Our proof uses certain types of hybrid diagrams, which have properties in common with both planar diagrams and braided diagrams. The diagram groups defined by hybrid diagrams also act properly and isometrically on CAT(0) cubical complexes.

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1 Introduction

Let Γ be a graph. A *quasiautomorphism* of Γ is a bijection of the vertices that preserves adjacency and nonadjacency, with at most finitely many exceptions. Following the notation of Nucinkis and St John-Green [16], we will let QV denote the group of quasiautomorphisms h of the infinite binary tree that also take the left and right children of a given vertex v to the left and right children of $h(v)$, again with at most finitely many exceptions. The notation "QV" indicates that QV is a collection of quasiautomorphisms that bears a family resemblance to Thompson's group V. (A standard reference for Thompson's groups is Cannon, Floyd and Parry [7]. We will assume a basic familiarity with that source or its equivalent throughout this article.)

Groups of quasiautomorphisms have been the subject of several recent studies. Lehnert conjectured in his thesis that the group QV is a universal group with context-free coword problem, ie a universal coCF group. Bleak, Matucci and Neunhöffer [3] have produced an embedding of QV into Thompson's group V, and thus proved that Lehnert's conjecture is equivalent to the conjecture that V is itself a universal coCF group. More recently, Nucinkis and St John-Green [16] have studied the finiteness properties of QV and related groups. They introduced additional groups QF, QT,

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$\bar{Q}T$ and $\bar{Q}V$. The groups QF and QT are natural subgroups of QV that preserve (respectively) the linear and cyclic orderings of the ends of the infinite binary tree (and therefore bear a family resemblance to Thompson's groups F and T, respectively). The groups $\bar{Q}T$ and $\bar{Q}V$ are analogous groups that act as quasiautomorphisms on the union of an infinite binary tree with an isolated point. Nucinkis and St John-Green show that the groups QF, $\bar{Q}T$ and $\bar{Q}V$ have type F_∞, and also compute explicit finite presentations for these groups. Whether QT and QV are finitely presented (and thus, more particularly, of type F_∞) is left as an open problem in [16].

The third author showed (in [12]) that QV is a braided diagram group over a semigroup presentation. This description suggests an approach to proving the F_∞ property for QV. Since Farley [11] shows that a class of braided diagram groups (including Thompson's group V) have type F_∞, and in fact other classes of diagram groups were shown to have type F_∞ in Farley [9; 11], it is at least plausible that some approach inspired by the theory of diagram groups could establish the F_∞ property for QV and QT. (We note that the original proofs that Thompson's groups F, T and V have type F_∞ were given by Ken Brown [5] and by Brown and Geoghegan [6] in the 1980s.)

Nucinkis and St John-Green show, however, that the hypotheses of the main theorem in [11] are satisfied by neither QT nor QV. In fact, as also noted in [16], even the much more general main theorem of Thumann [17] does not apply to either of QT or QV.

The goal of the present article is to extend the diagram-group methods of [9; 11] to the groups QF, QT, QV, $\bar{Q}T$ and $\bar{Q}V$. We will show that all of these groups can be described using the theory of diagram groups over semigroup presentations. Indeed, all of these groups are diagram groups over the same semigroup presentation, namely $\mathcal{P} = \langle x, a \mid x = xax \rangle$, although the specific types of diagram vary from group to group. Three types of diagram groups have been considered in the literature: planar, annular and braided diagram groups. All were introduced by Guba and Sapir [13], which devotes by far the greatest attention to planar diagram groups (which are usually simply called diagram groups). Farley [10; 11] considers the annular and braided diagram groups in more detail. Here we will introduce hybrid diagram groups that combine properties of multiple diagram group types. For instance, the group QF is a special type of diagram group over $\mathcal{P}$, in which the diagrams exhibit both planar and braided behavior at the same time. We will use such hybrid diagrams to prove that the groups QF, QT, QV, $\bar{Q}T$ and $\bar{Q}V$ all act properly by isometries on CAT(0) cubical complexes, and that all have type F_∞. In fact, our methods extend with equal ease to the case of an arbitrary finite number of binary trees and isolated vertices (see Section 5), and the case of n–ary trees (for fixed $n \geq 2$) is different only in the details. It even seems likely that our argument generalizes to other, nonregular, trees, although this is more speculative, and we attempt no general statement about such cases here.

We note that it is probably possible to extend the main theorem of Thumann [17] to prove the F_∞ property in the cases under consideration here. This is only a guess, however, since the authors claim little familiarity with the methods of [17].

We will now offer an outline of the argument. In Section 2, we will give a rapid introduction to the basic theory of diagram groups (including all three types: planar, annular and braided), and describe the natural cubical complexes on which such groups act, including a description of the links of vertices. In Section 3, we will give careful definitions of the groups QF, QT and QV, and describe how to represent elements of each group as "hybrid" diagrams. Section 3 also includes a description of natural complexes on which QF and QT act; these arise as convex (and thus CAT(0) by Crisp and Wiest [8]) subcomplexes of QV. (We will in fact confine our attention to QF, QT and QV alone, without sketching a general theory of "hybrid" diagrams. Nevertheless, we hope that the ideas indicated in Section 3 may be of some independent interest.) Section 4 shows that QF, QT and QV have type F_∞. Our argument follows the long-established method given by Brown [5]. Section 5 sketches some possible further developments, including sketches of the proofs that $\bar{Q}T$ and $\bar{Q}V$ have type F_∞ (as already proved by [16]), and the other generalizations briefly described above.

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2 Braided diagram groups and actions on associated complexes

2.1 Definition of braided diagram groups

To define braided diagram groups, we must first define braided diagrams over semigroup presentations. The reader might like to refer to Example 2.7 and the accompanying figure while reading the definition.

Definition 2.1 (braided diagrams over a semigroup presentation) Let $\mathcal{P}$ be a *semigroup presentation*; thus, $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$, where Σ is a set (to be regarded as an alphabet) and $\mathcal{R} \subseteq \Sigma^+ \times \Sigma^+$, where Σ^+ is the set of all nonempty positive words in the symbols Σ. We view the elements of $\mathcal{R}$ as equalities between elements of Σ^+ (ie as *relations*). For technical reasons, we impose the additional restriction that $(u, u) \notin \mathcal{R}$ for any $u \in \Sigma^+$.

A *braided diagram* Δ *over* $\mathcal{P}$ is a labeled ordered topological space formed by making identifications among three types of components: wires, transistors and the frame.

- A *wire* is a homeomorphic copy of $[0, 1]$. The "0" end is the bottom of the wire, and the "1" end is the top.
- A *transistor* is a homeomorphic copy of $[0, 1]^2$. Each transistor has well-defined top, bottom, left and right sides (in the obvious senses: the top is $[0, 1] \times \{1\}$, the bottom is $[0, 1] \times \{0\}$, etc). These sides are part of the transistor's definition. The top and bottom sides have equally obvious left-to-right orderings. (We make no use of any ordering of the sides.)
- The *frame* is a homeomorphic copy of $\partial[0, 1]^2$. It has well-defined top, bottom, left and right sides, just as a transistor does. The top and bottom sides have obvious left-to-right orderings.

To form a braided diagram Δ over $\mathcal{P}$, we begin with a finite nonempty collection $W(\Delta)$ of wires, a labeling function $\ell\colon W(\Delta) \to \Sigma$, a finite (possibly empty) collection $T(\Delta)$ of transistors and a frame. Each endpoint of each wire is then attached either to a transistor or to the frame. The bottom of a wire is attached either to the top of a transistor or to the bottom of the frame; the top of a wire is attached either to the bottom of a transistor or to the top of the frame. Moreover, the images of any two wires in the quotient must be disjoint.

The resulting labeled oriented quotient space is called a *braided diagram over* $\mathcal{P}$ if the following two conditions are also satisfied:

(1) Note that each transistor T inherits a top and bottom label from the labels of the wires it touches. (The points at which the wires meet transistors are called *contacts*.) These labels are words in Σ^+, obtained by reading the labels of connecting wires from left to right.

Let $\ell_{\text{top}}(T)$ and $\ell_{\text{bot}}(T)$ denote the top and bottom labels, respectively. We require that $(\ell_{\text{top}}(T), \ell_{\text{bot}}(T)) \in \mathcal{R}$ or that $(\ell_{\text{bot}}(T), \ell_{\text{top}}(T)) \in \mathcal{R}$, for each transistor T. If u and v are the top and bottom labels of T, then we say that T is a (u, v)*–transistor*.

(2) For transistors T_1 and T_2 of Δ, write $T_1 \preccurlyeq T_2$ if there is a wire whose bottom contact is a point on the top of T_1 and whose top contact is a point on the bottom of T_2. Let $<$ be the transitive closure of $\preccurlyeq$. We require that $<$ be a strict partial order on the transistors of Δ.

(Equivalently, suppose that the braided diagram Δ is drawn in the plane, such that each transistor is enclosed by the frame and the sides of the transistors and frame are parallel to the coordinate axes. We require that it be possible to arrange the transistors and the frame in this fashion in such a way that each wire can be embedded monotonically, ie so that the y–coordinate in the embedded image increases as we move from the bottom of the wire to the top.)

Definition 2.2 (planar and annular diagrams) Let Δ be a braided diagram over the semigroup presentation $\mathcal{P}$. If Δ admits an embedding $h\colon \Delta \to \mathbb{R}^2$ into the plane that preserves the left–right and top–bottom orientations on the transistors and frame, then we say that Δ is *planar*.

We say that Δ is *annular* if it can be similarly embedded in an annulus. Or, more precisely, suppose that we replace the frame $\partial([0,1]^2)$ with a pair of disjoint circles, each of which is given the standard counterclockwise orientation, in place of the usual left–right orientations on the top and bottom of $\partial([0,1])^2$. We further give both circles basepoints, which are to be disjoint from all contacts. Transistors and wires are defined as before, and their attaching maps are subject to the same restrictions as before. We say that the resulting diagram is *annular* if the result may be embedded in the plane, again preserving the left–right orientations of the transistors. We think of the "top" circle as the inner ring of the annulus and the "bottom" circle as the outer ring.

(Here it may be helpful to view the transistors as having the counterclockwise orientation on their "top" and "bottom" faces, where the "top" faces the interior boundary circle of the annulus and the "bottom" faces the external boundary of the annulus.)

Definition 2.3 (equivalence of braided diagrams) Two braided diagrams Δ_1 and Δ_2 are *equivalent* if there is a homeomorphism h between them such that h preserves the labelings of wires and all orientations (left–right and top–bottom) on all transistors and the frame.

In the sequel, we will consider diagrams up to equivalence without explicitly mentioning it.

Definition 2.4 ((u,v)–diagrams) Let Δ be a braided semigroup diagram; let u and v be in Σ^+. We can define the *top* and *bottom labels* of Δ by reading the labels of the wires that connect to the frame, from left to right, just as we defined the top and bottom labels of an individual transistor above. We say that Δ is a *braided* (u,v)*–diagram* if the top label of Δ is u and the bottom label is v.

In some cases, it is not important to specify the bottom label. We say that Δ is a *braided* $(u,*)$*–diagram* if the top label of Δ is u and the bottom label is arbitrary.

Definition 2.5 (concatenation) If Δ' is a braided (u,v)–diagram and Δ'' is a braided (v,w)–diagram, then the *concatenation* $\Delta' \circ \Delta''$ is defined by stacking the diagrams, Δ' on top of Δ''.

Remark 2.6 We note (for the sake of clarity) that the basepoints on the inner and outer circles of an annular diagram Δ are needed in Definitions 2.4 and 2.5. Here, the "top" label of Δ is to be read counterclockwise from the top (inner) basepoint, while the bottom label of Δ is similarly read counterclockwise from the outer circle's basepoint.

If Δ_1 and Δ_2 are annular (u,v)– and (v,w)–diagrams, respectively, for u, v, $w \in \Sigma^+$, then the concatenation $\Delta_1 \circ \Delta_2$ is the result of identifying the outer circle of Δ_1 with the inner circle of Δ_2 at the chosen basepoints (while also, of course, matching the other contacts in counterclockwise order).

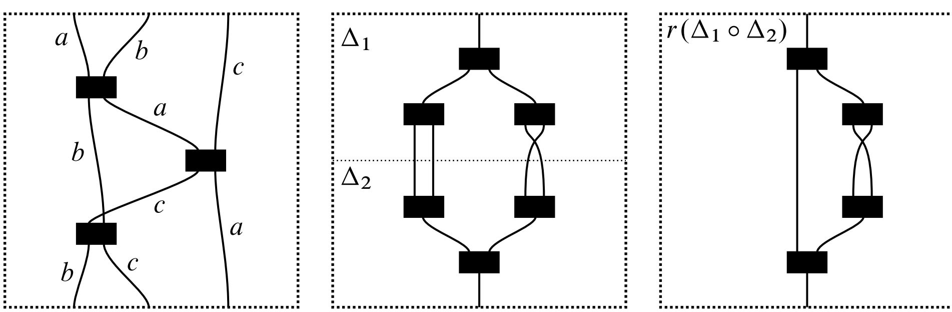

Figure 1: Three examples of braided diagrams over semigroup presentations

Example 2.7 Figure 1 gives three examples of braided diagrams over semigroup presentations. On the left, we have a braided (abc, bca)–diagram Δ over the semigroup presentation $\mathcal{P} = \langle a, b, c, \mid ab = ba,\ bc = cb,\ ac = ca \rangle$. More properly speaking, this is the immersed image of such a diagram under a mapping into the plane. All of the defining features of Δ are illustrated. The frame appears as a dotted box, and the left-to-right orientations of the transistors are the obvious ones. Note that the apparent crossing of wires above the bottom-left transistor represents a double point of the projection, since the wires are necessarily disjoint (by Definition 2.1) in the original diagram Δ. Note also that it is unnecessary to specify whether the "c" wire crosses over the "b" wire or vice versa, since such overcrossing data is not part of the definition of Δ. (In effect, we allow any two wires of a braided diagram to pass through each other.) This means that the descriptor "braided" is a misnomer; there is no true braiding.

In the center is the concatenation $\Delta_1 \circ \Delta_2$ of Δ_1 with Δ_2, where both are diagrams over the semigroup presentation $\mathcal{P} = \langle x \mid x = x^2 \rangle$. (We omit the labels of the wires, since each such label is "x".) Here Δ_1 is a braided (x, x^4)–diagram and Δ_2 is a braided (x^4, x)–diagram.

On the right is the result of removing all dipoles from the concatenation $\Delta_1 \circ \Delta_2$. (See Definition 2.8.)

It is reasonably clear that, for a fixed word $w \in \Sigma^+$, the braided (w, w)–diagrams over $\mathcal{P}$ form a semigroup under concatenation. We can define inverses using the idea of a dipole.

Definition 2.8 (dipoles) Suppose that T_1 and T_2 with $T_1 \preccurlyeq T_2$ are transistors in a braided semigroup diagram Δ over $\mathcal{P}$. Let $w_1, \dots, w_n$ be a complete list of wires attached at the top of T_1, listed in the left-to-right order of their attaching contacts. We say that T_1 and T_2 form a *dipole* if

(1) the tops of the wires $w_1, \dots, w_n$ are glued in left-to-right order (w_1 leftmost, etc) to the bottom of T_2, and no other wires are attached to the bottom of T_2, and

(2) the top label of T_2 is the same as the bottom label of T_1.

In this case, the result of removing the transistors T_1 and T_2 and the wires $w_1, \dots, w_n$, and then attaching the top contacts of T_2 to the bottom contacts of T_1 (in left-to-right order-preserving fashion) is called *removing a dipole*. The inverse operation is called *inserting a dipole*.

A diagram that contains no dipoles is called *reduced*.

Two diagrams are *equivalent modulo dipoles* if one diagram can be obtained from the other by repeatedly inserting or removing dipoles. The relation "equivalent modulo dipoles" is indeed an equivalence relation.

Remark 2.9 The inverse of a (w, w)–diagram Δ is simply the result of reflecting Δ across an axis parallel to the top of the frame. The proof that this is indeed Δ^{-1} is straightforward. Note that the unique planar (w, w)–diagram with no transistors functions as an identity.

Remark 2.10 A standard argument using Newman's lemma [15] shows that each equivalence class modulo dipoles contains a unique reduced diagram. If Δ is a diagram, then we let $r(\Delta)$ denote the unique reduced representative in its equivalence class. For the proof, see Lemma 2.2 from [10].

Definition 2.11 (braided diagram groups) Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation; let $w \in \Sigma^+$. The set of all equivalence classes of (w, w)–diagrams over $\mathcal{P}$ (under the dipole equivalence relation) is a group under concatenation. It is denoted by $D_b(\mathcal{P}, w)$ and called the *braided diagram group over* $\mathcal{P}$ *based at* w.

The set of all equivalence classes of planar (w, w)–diagrams over $\mathcal{P}$ is a group under concatenation, denoted by $D(\mathcal{P}, w)$. This is the (*planar*) *diagram group over* $\mathcal{P}$ *based at* w.

Similarly, the group of all equivalence classes of annular (w, w)–diagrams over $\mathcal{P}$ is denoted by $D_a(\mathcal{P}, w)$ and called the *annular diagram group over* $\mathcal{P}$ *based at* w.

2.2 The diagram complex

In this section, we will describe a CAT(0) cubical complex $\widetilde{K}_b(\mathcal{P}, w)$, the *diagram complex*, on which $D_b(\mathcal{P}, w)$ acts properly and isometrically. The groups $D(\mathcal{P}, w)$ and $D_a(\mathcal{P}, w)$ admit actions on similar complexes, $\widetilde{K}(\mathcal{P}, w)$ and $\widetilde{K}_a(\mathcal{P}, w)$, respectively. The definitions of the latter complexes may be obtained from the definition of $\widetilde{K}_b(\mathcal{P}, w)$ simply by replacing all mentions of braided diagrams with planar or annular diagrams, respectively. We will therefore concentrate on the case of $\widetilde{K}_b(\mathcal{P}, w)$.

We continue (in this subsection and the next) to let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be an arbitrary semigroup presentation such that $(u, u) \notin \mathcal{R}$ (for all $u \in \Sigma^+$), and let $w \in \Sigma^+$.

Definition 2.12 (vertices of $\widetilde{K}_b(\mathcal{P}, w)$) A vertex of $\widetilde{K}_b(\mathcal{P}, w)$ is an equivalence class of reduced braided $(w, *)$–diagrams over $\mathcal{P}$; the equivalence relation in question is

$$\Delta_1 \sim \Delta_2 \iff \Delta_1 \circ \Pi = \Delta_2,$$

where Π is a diagram with no transistors (ie a *permutation diagram*).

We can think of a vertex as simply a braided $(w, *)$–diagram such that the bottommost wires do not attach to the bottom of the frame. We may thus sometimes refer to a reduced braided $(w, *)$–diagram Δ over $\mathcal{P}$ as a vertex itself, even though (as above) a vertex is technically an equivalence class of diagrams. We hope that this causes no confusion.

Definition 2.13 (cubes of $\widetilde{K}_b(\mathcal{P}, w)$) A marked cube in $\widetilde{K}_b(\mathcal{P}, w)$ is determined by a pair (Δ, Ψ), where Δ is a reduced braided (w, v)–diagram (for some $v \in \Sigma^+$) and Ψ is a thin braided (v, u)–diagram (for some $u \in \Sigma^+$). Here, a *thin diagram* Ψ is such that no two transistors of Ψ are comparable in the partial order $<$ on transistors (as defined in Definition 2.1).

Definition 2.14 (realization of a marked cube) Given a marked cube (Δ, Ψ), we define its *realization* $|\Delta, \Psi|$ as follows. If Ψ contains n transistors, then choose a numbering of the transistors $1, \dots, n$. A corner of the cube $[0, 1]^n$ is (obviously) identified with a binary string of length n. We specify a labeling of each corner $w = (a_1, \dots, a_n)$ of $[0, 1]^n$ by a vertex of $\widetilde{K}_b(\mathcal{P}, w)$ as follows:

- If $a_i = 0$, then we remove the i^{th} transistor of Ψ by clipping the wires above it.
- If $a_i = 1$, then we leave the i^{th} transistor alone.

If we let Ψ_w denote the result of performing the above operations, then $r(\Delta \circ \Psi_w)$ is the label of the corner w.

We note, in particular, that two vertices Δ_1 and Δ_2 are adjacent exactly when one of the vertices may be obtained from the other by clipping the wires above a bottommost transistor. (Or, conversely, when one of the vertices may be obtained from the other by attaching a new transistor to the bottommost wires in a label-preserving way.) This characterization of adjacency in $\widetilde{K}_b(\mathcal{P}, w)$ plays a part in the description of links (see Proposition 2.19).

This labeling of the vertices is uniquely determined by the pair (Δ, Ψ) and the numbering of the transistors.

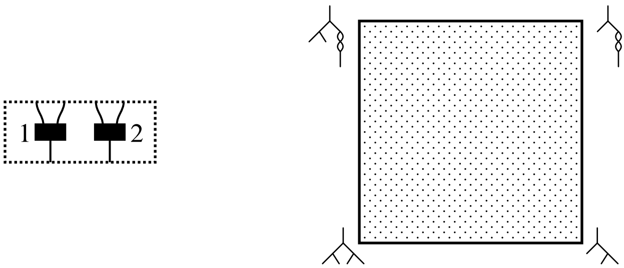

Figure 2: A thin braided (x^4, x^2)–diagram Ψ over $\mathcal{P} = \langle x \mid x = x^2 \rangle$ with a numbering of its transistors (left) and the associated marked square (Δ_1, Ψ) (right). Note that Δ_1 is as in Figure 1.

Example 2.15 Consider the pair (Δ_1, Ψ), where Δ_1 is as in Figure 1 and Ψ appears in the left side of Figure 2. Both are diagrams over $\mathcal{P} = \langle x \mid x = x^2 \rangle$. The associated marked cube appears in the right side of Figure 2, where Δ_1 appears as the label of the bottom-left corner. Note that we have used a modified notation to label the corners of the square. In particular, we have shrunk each transistor to a single vertex; this is harmless, since the defining data of the braided diagrams can still be read from the picture.

The frames have been omitted entirely. By Definition 2.12, the left–right ordering of the wires attached to the bottom of each frame is irrelevant, while the left–right ordering of the wires at the top of each frame is obvious, so the omission of the frames is also harmless.

We note finally that the figure-eights occurring in the top-left and top-right corners depict pairs of transistors with crossing wires in between (compare to the right-hand picture in Figure 1). In particular, the middle "vertices" in these figure-eights are wire crossings, not transistors.

Definition 2.16 (the diagram complex $\widetilde{K}_b(\mathcal{P}, w)$) Pick a realization $|\Delta, \Psi|$ for each cube (Δ, Ψ) as above. We glue two such realizations of cubes together along faces whose vertices have the same labeling.

Proposition 2.17 *The result of the above gluing is a* $\mathrm{CAT}(0)$ *cubical complex* [9; 10], *and thus contractible. There is a natural group action of $D_b(\mathcal{P}, w)$ on $\widetilde{K}_b(\mathcal{P}, w)$ by the rule*

$$\widetilde{\Delta} \cdot (\Delta, \Psi) = (r(\widetilde{\Delta} \circ \Delta), \Psi).$$

(Here $r(\Delta)$ is as defined in Remark 2.10.) *This action is proper and by isometries. It is not necessarily cocompact. In fact, if Δ_1 and Δ_2 are vertices, then Δ_1 and Δ_2 lie in the same orbit if and only if the bottom labels of the diagrams Δ_1 and Δ_2 are permutations of each other.* □

The proof that the complexes in question are CAT(0) uses Gromov's famous link condition. We refer the reader to [4] for one standard account.

2.3 The link of a vertex in the diagram complex

Definition 2.18 (disjoint applications of relations) Let Δ_1 and Δ_2 be thin braided $(w,*)$–diagrams over $\mathcal{P}$. Let $w_1, \dots, w_n$ be the wires of Δ_1 that meet the top of the frame, listed in the left-to-right order in which they meet the top of the frame. Let $\widehat{w}_1, \dots, \widehat{w}_n$ be the wires of Δ_2 that meet the top of its frame, ordered similarly. (Note that the number of such wires is the same in both cases, since both numbers are equal to the length of w.)

Let

$$S_1 = \{j \in \{1,\dots,n\} \mid \text{the bottom of } w_j \text{ is attached to a transistor of } \Delta_1\}$$

and

$$S_2 = \{j \in \{1,\dots,n\} \mid \text{the bottom of } \widehat{w}_j \text{ is attached to a transistor of } \Delta_2\}.$$

We say that Δ_1 and Δ_2 represent *disjoint applications of relations to* w, or simply that Δ_1 and Δ_2 are *disjoint*, if $S_1 \cap S_2 = \varnothing$.

Proposition 2.19 [9; 10] (description of the link) *Let v be a vertex of $\widetilde{K}_b(\mathcal{P}, w)$; let Δ represent v. Assume Δ is a braided $(w,\widehat{w})$–diagram. Define an abstract simplicial complex $L(v)$ as follows: The vertices are the braided $(\widehat{w},*)$–diagrams over $\mathcal{P}$ that contain exactly one transistor. A finite collection of such vertices spans a simplex if and only if the vertices are pairwise disjoint* (in the sense of Definition 2.18).

The link of v in $\widetilde{K}_b(\mathcal{P}, w)$ is isomorphic to $L(v)$. □

2.4 The links associated to $\mathcal{P} = \langle x, a \mid xax = x \rangle$

The links in the cubical complex $\widetilde{K}_b(\mathcal{P}, w)$, where $\mathcal{P} = \langle x, a \mid xax = x\rangle$ and $w \in \{x,a\}^+$, will be especially important in our main argument. We offer a direct description here. The reader should refer to Remark 2.23, which describes the intuition behind Definitions 2.20 and 2.21.

Definition 2.20 (abstract links in $\widetilde{K}_b(\mathcal{P},w)$) Let $v = x^k a^l \in \{x,a\}^+$ for $k,l \geq 0$. We let

$$D_v = \big\{(m,n,p) \mid m,n \in \{1,\dots,k\},\ p \in \{1,\dots,l\},\ m \neq n\big\},$$

and

$$A_v = \big\{(m) \mid m \in \{1,\dots,k\}\big\}.$$

We define an abstract simplicial complex $\mathrm{lk}(v)$ as follows. The vertex set of $\mathrm{lk}(v)$ is $D_v \cup A_v$. A nonempty collection

$$S = \{(m_1,n_1,p_1),\dots,(m_\alpha,n_\alpha,p_\alpha)\} \cup \{(m'_1),\dots,(m'_\beta)\}$$

is a simplex in $\mathrm{lk}(v)$ if and only if there are no repetitions in the lists

$$m_1,n_1,\dots,m_\alpha,n_\alpha,m'_1,\dots,m'_\beta$$

and

$$p_1,\dots,p_\alpha.$$

(We note that one of α or β may be 0, but not both.)

We say that $\mathrm{lk}(v)$ is the (*abstract*) *link of the word* v in $\widetilde{K}_b(\mathcal{P},w)$.

Definition 2.21 (abstract descending links in $\widetilde{K}_b(\mathcal{P},w)$) Let v and D_v be as above. The vertex set of $\mathrm{lk}_\downarrow(v)$ is D_v. A collection of vertices S determines a simplex exactly under the conditions specified in Definition 2.20.

We say that $\mathrm{lk}_\downarrow(v)$ is the (*abstract*) *descending link of the word* v in $\widetilde{K}_b(\mathcal{P},w)$.

Proposition 2.22 *Let $v = x^k a^l$. The simplicial complex $\mathrm{lk}(v)$ is isomorphic to the link of any vertex whose bottom label is a permutation of $x^k a^l$.*

The simplicial complex $\mathrm{lk}_\downarrow(v)$ is the full subcomplex of $\mathrm{lk}(v)$ determined by the (xax,x)–transistors.

Proof The first statement is a special case of Proposition 2.19. The second statement follows easily from the identification of the link of v with $L(v)$ from Proposition 2.19. See also Remark 2.23 for more details. □

Remark 2.23 The complexes $\mathrm{lk}(v)$ and $\mathrm{lk}_\downarrow(v)$, as defined in Definitions 2.21 and 2.20, are both flag complexes, as may be easily checked. The latter fact is used in the proof that the larger cubical complexes are CAT(0) (see Proposition 2.17).

The members of D_v represent "descending" applications of relations (ie descending attachments of transistors, in which three contacts are on top of the transistor). The numbers m and n in the triple (m,n,p) describe how the left and right top "x"–contacts of the transistor are connected to the top of the frame, where the number m (for instance) indicates that the top left "x"–contact of the transistor is connected by a wire to the m^{th} "x"–contact from the left at the top of the frame. The third coordinate, p, describes how the top "a"–contact of the transistor is connected to the top of the frame (namely, at the p^{th} "a"–contact from the left).

In a similar way, the members of A_v represent "ascending" applications of relations, in which a single "x"–contact appears at the top of a transistor.

3 The groups QF, QT and QV

In this section, we describe the groups QF, QT and QV, first as groups of quasiautomorphisms, and second as braided diagram groups over semigroup presentations (or as subgroups of such groups). We will first briefly review the description of F, T and V as diagram groups, since such an understanding of these groups will be necessary in what follows.

3.1 The groups F, T and V as diagram groups

According to the introduction of [13], Victor Guba was the first to observe the isomorphism $D(\mathcal{P}',x) \cong F$, where $\mathcal{P}' = \langle x \mid x = x^2 \rangle$ and F is Thompson's group. Guba and Sapir [13] subsequently developed the theory of annular and braided diagram groups in order to describe Thompson's group T and V (respectively) as diagram groups. They proved that $D_a(\mathcal{P}',x) \cong T$ and $D_b(\mathcal{P}',x) \cong V$ (see Chapter 16 of [13]).

The first published proof of the isomorphism $D(\mathcal{P}',x) \cong F$, from [13], used a general procedure for producing presentations of diagram groups. The resulting presentation for $D(\mathcal{P}',x)$ is identical to a standard one for F.

The third author gave a direct description of all three isomorphisms in [10, Section 6]. We review this description here. Let Δ be a reduced (x,x)–semigroup diagram over $\mathcal{P}'$. Such a diagram contains two types of transistors: (x,x^2)–transistors and (x^2,x)–transistors. Call the first type *positive* and the second type *negative*. Since Δ is reduced, it follows that no positive transistor is less than a negative transistor in the partial order $<$ from Definition 2.1. Thus, there is an arc c connecting the left and right sides of Δ that separates all of the positive transistors from the negative ones. This arc exhibits Δ as a product $\Delta_1 \circ \Delta_2^{-1}$, where both Δ_1 and Δ_2 contain only positive transistors. (Recall the description of inverses from Remark 2.9.) The diagrams Δ_1 and Δ_2 each directly determine finite binary trees T_1 and T_2, respectively. The original diagram Δ now corresponds to the triple (T_2, σ, T_1), where T_2 is the domain tree, T_1 is the range tree and σ is a bijection between the leaves of T_2 and T_1 that is determined by wires in Δ. The triple (T_2, σ, T_1) is a standard tree pair for an element of F, T or V (as from [7]), depending upon whether Δ is planar, annular or braided.

For instance, let Δ be the diagram in the top-right of Figure 1. If we perform the above procedure, we arrive at a tree pair (T', σ, T''), where T' and T'' are both full binary trees of depth 2, and σ is the bijection which sends the first leaf of T' (from the left) to the first leaf of T'', the second leaf to the second leaf, the third leaf to the fourth leaf, and the fourth leaf to the third leaf.

3.2 The groups QF, QT and QV as quasiautomorphism groups

The results in this subsection are taken from [16] and included for the reader's convenience. Most of the definitions are from Section 2 of that source.

Definition 3.1 (QV) We let $\mathcal{T}$ denote the infinite rooted binary tree. The vertices of $\mathcal{T}$ may be identified with the members of the monoid $\{0,1\}^*$, which consists of all finite binary sequences, including the empty sequence, which we denote by ϵ.

For a given $v \in \mathcal{T}^0$, we regard $v0$ as the left child of v and $v1$ as the right child.

A bijection $h\colon \mathcal{T}^0 \to \mathcal{T}^0$ is a member of QV if, for almost all $v \in \mathcal{T}^0$, h sends the left (resp. right) child of v to the left (resp. right) child of $h(v)$. (Here, "almost all" means "with at most finitely many exceptions".) Note, in particular, that h need not preserve adjacency.

The set QV is a group under composition of functions.

Proposition 3.2 (tree pair representatives) *Any $h \in QV$ can be specified by a pair $((T_1, \sigma, T_2), f)$ where:*

(1) *(T_1, σ, T_2) is the usual tree pair representative of an element of V. Thus, T_1 and T_2 are finite rooted ordered binary trees with the same number of leaves, and σ is a bijection of the leaves.*

(2) *f is a bijection between the set of interior nodes of T_1 and the set of interior nodes of T_2.* (Here, an *interior node* of T_i is a vertex of T_i that is not a leaf.) ◻

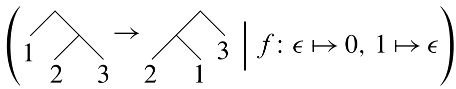

Figure 3: An element h of QV, described by a tree pair (left) and a bijection f of interior nodes (right).

Example 3.3 Consider the pair in Figure 3. Here we have indicated the bijection σ of leaves by direct enumeration. Thus, the leaf 0 goes to the leaf 01, 10 to 00, and 11 to 1. The resulting assignment determines a bijection h between the vertices of $\mathcal{T}$ that are not interior nodes of T_1 and the vertices of $\mathcal{T}$ that are not interior nodes of T_2 once we specify that h takes the left and right children of any noninterior vertex of T_1 to the left and right children (respectively) of its image. Thus,

$$h(010) = \sigma(0)10 = 0110.$$

The bijection f specifies the values of h on the remaining vertices.

Proposition 3.4 *The function $\pi\colon QV \to V$ determined by projection onto the first coordinate is a surjective homomorphism; the kernel is isomorphic to the group S_∞ of self-bijections of $\mathbb{N}$ having finite support.* ◻

We easily define QF and QT in terms of π:

Definition 3.5 (definitions of QF and QT)

(1) $QF = \pi^{-1}(F)$.

(2) $QT = \pi^{-1}(T)$.

We recall that elements of Thompson's group F may be represented by tree pairs (as described in Proposition 3.2(1)) in which the bijection of the leaves is order-preserving, and elements of T may be similarly represented by tree pairs in which the bijection permutes the order of the leaves cyclically. We refer the reader to [7] for a more extended discussion.

Thus, simply put, QF consists of the pairs, as in Proposition 3.2, such that the bijection between leaves in the first coordinate preserves the left-to-right order, and QT consists of the pairs in which the same bijection preserves the cyclic order.

3.3 The groups QF, QT and QV as diagram groups

The observation that QV is a braided diagram group first appeared in [12]; see Example 4.4 from that source. Here we will develop the idea in somewhat more depth.

Proposition 3.6 *QV is isomorphic to $D_b(\mathcal{P}, x)$, where $\mathcal{P} = \langle x, a \mid x = xax \rangle$.*

Proof We sketch the isomorphism. Let Δ be a braided (x,x)–diagram over $\mathcal{P}$. We can classify each $T \in \Delta$ as either positive or negative, as follows: a transistor is *positive* if the top label is x and the bottom label is xax; it is *negative* in the opposite case. If a positive transistor of Δ is less than a negative transistor in the partial order on transistors, then Δ is necessarily not reduced. We can then remove dipoles until no positive transistor is less than a negative transistor. The diagram Δ can now be expressed in the form $\Delta = \Delta_1 \circ \Delta_2^{-1}$, where all transistors in Δ_1 and Δ_2 are positive. We call such braided $(x,*)$–diagrams Δ_1 and Δ_2 *positive*.

A given positive braided $(x, x^{n+1}a^n)$–diagram $(n \geq 0)$, read from bottom to top, can be interpreted as a set of instructions for assembling an infinite binary tree (represented by "x") out of $n+1$ infinite binary trees and n vertices (represented by "a"). Specifically, a positive transistor selects two infinite binary trees (two "x"s) and a single vertex (an "a") and combines them into a single binary tree (an "x"). (Or: the transistor encodes the action of assembling these two trees and a vertex into a single binary tree.) The bottom-left "x" contact becomes the rooted subtree with root 0 in the new tree, the bottom-right "x" contact becomes the rooted subtree with root 1 and the vertex represented by "a" becomes the root. Conversely, a negative transistor (read from bottom to top) represents a dissection of a binary tree into three pieces.

Assume, without loss of generality, that Δ_1 and Δ_2 are braided $(x, x^{n+1}a^n)$–diagrams. It follows from the above discussion that Δ_2^{-1} represents a dissection of the standard infinite binary tree into $n+1$ binary trees and n vertices; Δ_1 represents the subsequent reassembly of these pieces into a single binary tree. (Thus, Δ_2^{-1} is a dissection of the domain tree, and Δ_1 shows how to reassemble the pieces into the range tree.) We let $f_\Delta \in QV$ denote the function so determined by the diagram Δ.

The proof is completed by noting that

(1) the function f_Δ indicated above is always in QV, and any element of QV is f_Δ for appropriate Δ;

(2) if Δ' and Δ'' are braided (x,x)–diagrams over $\mathcal{P}$ and Δ' and Δ'' are equivalent modulo dipoles, then $f_{\Delta'} = f_{\Delta''}$; and

(3) the indicated correspondence is a homomorphism, ie $f_{\Delta' \circ \Delta''} = f_{\Delta'} \circ f_{\Delta''}$, for all braided (x,x)–diagrams Δ' and Δ'' over $\mathcal{P}$. □

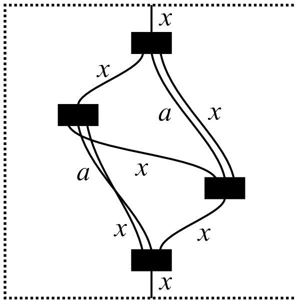

Figure 4: The element $h \in QV$ from Figure 3 described in diagram form

Example 3.7 Figure 4 shows how the element h of QV from Figure 3 may be specified by a braided (x,x)–diagram over the presentation $\mathcal{P} = \langle a, x \mid xax = x\rangle$. Here, the negative transistors represent a dissection of the domain tree and the positive transistors represent a dissection of the range tree.

We read the diagram from the bottom up. The bottommost transistor represents the dissection of the basic binary tree $\mathcal{T}$ into three pieces: the binary tree rooted at "0" (represented by the leftmost wire at the top of the transistor; hereafter, we simply call this the 0–tree), the root (represented by the "a" wire) and the 1–tree (represented by the rightmost wire). We note that the rightmost wire leads to another negative transistor; this transistor represents the dissection of the subtree rooted at 1 into three more pieces: the 10–tree (represented by the left wire), the root 1 (represented by the "a" wire) and the 11–tree (represented by the right wire). There are no more negative transistors, so the domain tree is dissected into five pieces (in left-to-right order): the 0–tree, the root ϵ, the 10–tree, the vertex 1 and the 11–tree.

In a similar way, the dissection of the range tree is encoded by the positive transistors, which are read from the top down. Thus, the range tree is dissected into (respectively): the 00–tree, the vertex 0, the 01–tree, the root ϵ and the 1–tree.

The mapping between these pieces is determined by wire connections; in particular, we have that h sends the 0–tree to the 01–tree, the root ϵ to 0, the 10–tree to the 00–tree, the vertex 1 to ϵ, and the 11–tree to the 1–tree.

3.4 Actions of QF, QT and QV on CAT(0) cubical complexes

In this subsection, we will describe actions of QF, QT and QV on CAT(0) cubical complexes. All of the actions under consideration will be proper and by isometries. We have already seen an action of QV with the desired properties: indeed, $QV \cong D_b(\mathcal{P}, x)$ (where $\mathcal{P} = \langle a, x \mid x = xax \rangle$), so QV acts properly and isometrically on the CAT(0) cubical complex $\widetilde{K}_b(\mathcal{P}, x)$. As subgroups of QV, both QF and QT also act properly and isometrically on $\widetilde{K}_b(\mathcal{P}, x)$. We will, however, want QF and QT to act on more economical complexes when we attempt to establish the F_∞ property for these groups. Here we will find such complexes as convex subcomplexes of $\widetilde{K}_b(\mathcal{P}, x)$.

Proposition 3.8 (diagram description of elements in QF and QT) *Let Δ be a braided (x, x)–diagram over $\mathcal{P} = \langle a, x \mid x = xax \rangle$.*

1. *The diagram Δ represents an element of QF if and only if the result of deleting each edge labeled “a” results in a planar (x, x)–diagram over the presentation $\mathcal{P}' = \langle x \mid x = x^2 \rangle$.*
2. *The diagram Δ represents an element of QT if and only if the result of deleting each edge labeled “a” results in an annular (x, x)–diagram over the presentation $\mathcal{P}'$.*

Indeed, the above operation of deleting “a” edges induces the homomorphism π from Proposition 3.4.

Proof We sketch the proof. A braided (x, x)–diagram Δ over $\mathcal{P}$ describes a quasiautomorphism of $\mathcal{T}$ via the isomorphism from Proposition 3.6. Under this correspondence, the wires labeled by a describe the action of Δ on a finite number of individual vertices, while the wires labeled by x describe the action of Δ on infinite binary subtrees. If we forget the action on the individual vertices represented by “a” wires (by deleting them), then the remaining “x” wires determine the action of Δ on the ends of $\mathcal{T}$. The resulting transformation of the ends of $\mathcal{T}$ is an element of F or T under the given assumptions, essentially by the discussion in Section 3.1.

Thus, in the notation of Figure 3, the “a” wires determine the right half of the ordered pair, while the left half of the ordered pair corresponds to the diagram Δ with “a” wires removed. □

The above descriptions of QF and QT suggest the following definition:

Definition 3.9 (the complexes K_{QF} and K_{QT}) Let $\mathcal{P} = \langle a, x \mid x = xax \rangle$. We denote the complex $\widetilde{K}_b(\mathcal{P}, x)$ by K_{QV}.

For a braided $(x,*)$–diagram Δ over $\mathcal{P}$, let $\pi(\Delta)$ denote the diagram over $\mathcal{P}' = \langle x \mid x = x^2 \rangle$ obtained by deleting each wire labeled by "a".

We define subcomplexes K_{QF} and K_{QT} of K_{QV} to be the full subcomplexes of K_{QV} determined by vertex sets K^0_{QF} and K^0_{QT}, which are defined as follows: a vertex v of K_{QV} is in K^0_{QF} (resp. in K^0_{QT}) if and only if it has a representative Δ such that $\pi(\Delta)$ is planar (resp. annular). (Recall that a vertex is an equivalence class of braided diagrams (see Definition 2.12), so a representative of v is a choice of diagram from the equivalence class.)

Proposition 3.10 *The complexes K_{QF} and K_{QT} are path-connected subcomplexes of K_{QV} such that, for each $v \in K^0_{QF}$ (resp. K^0_{QT}), the link of v in K_{QF} (resp. K_{QT}) embeds in the link of v in K_{QV} as a full subcomplex.*

In particular, K_{QF} and K_{QT} are CAT(0) *cubical complexes. Moreover, QF and QT act properly and isometrically on K_{QF} and K_{QT}.*

Proof We first show that K_{QF} and K_{QT} are path-connected subcomplexes of K_{QV}. Since the proofs in both cases are similar, it will be sufficient to consider K_{QF}.

Let v be a vertex in QF. We will show that v can be connected to the natural basepoint v^* of K_{QF}, which is the equivalence class of the permutation (x,x)–diagram Δ'. (Recall that a permutation diagram has no transistors (see Definition 2.12). Thus, Δ' consists simply of a frame and a single wire, which runs from the top of the frame to the bottom.) We prove that v can be connected to v^* by an edge-path using induction on the number n of transistors in a diagram representative Δ for v. This is trivial if $n = 0$. If $n \geq 1$, then we pick a transistor T of Δ that is minimal in the partial order on transistors. We let Δ_1 denote the diagram that is obtained from deleting T (and all depending wires) from Δ, and we let v_1 denote the vertex represented by Δ_1. Clearly, v_1 and v are adjacent in K_{QF}, and, by induction, v_1 can be connected to the basepoint v^* by a path. It follows that v can also be so connected to v^*, and it then follows that K_{QF} is path-connected.

To show that K_{QF} embeds as a convex subspace of K_{QV}, it now suffices to prove that the link of an arbitrary vertex $v \in K_{QF}$ includes into the link of $v \in K_{QV}$ as a full subcomplex. (We are appealing to Theorem 1(2) from [8].) Assume that Δ

represents v, and the bottom label of Δ is $x^k a^l$ for some integers k and l, with $k > 0$. Let Ψ_1, Ψ_2, ..., Ψ_m be $(x^k a^l, *)$–diagrams over the semigroup presentation $\mathcal{P}$ be such that (i) each Ψ_i has exactly one transistor; (ii) the product $\Delta \circ \Psi_i$ determines a vertex of K_{QF}, possibly after reducing a dipole; (iii) any two Ψ_i and Ψ_j represent disjoint applications of relations to $x^k a^l$ (in the sense of Definition 2.18). We note that (i)–(iii) are equivalent to the condition that the Ψ_i are pairwise joined by edges in the link $\mathrm{lk}^{QF}(v)$. We are required to show that the collection of all of the Ψ_i spans a single $(m-1)$–simplex in $\mathrm{lk}^{QF}(v)$. But the condition that the Ψ_i are pairwise disjoint shows that

$$\{\Psi_1, \ldots, \Psi_m\}$$

is a simplex in $\mathrm{lk}^{QV}(v)$, and it is easy to see that the "union" of the Ψ_i (see Remark 2.23) is planar after the application of π. This means that the link $\mathrm{lk}^{QF}(v)$ is a full subcomplex of $\mathrm{lk}^{QV}(v)$. It now follows that K_{QF} is a convex subset of K_{QV}, and therefore $\mathrm{CAT}(0)$.

Since the action of QF on K_{QF} is determined by stacking diagrams, and such stacking preserves planarity under the projection π, QF acts on K_{QF} by cell-permuting automorphisms, and thus by isometries. The properness of the action follows from the properness of the action of QF on K_{QV}. □

4 The F_∞ property

In this section, we will prove that each of the groups QF, QT and QV has type F_∞. Our proof uses Brown's well-known finiteness criterion, which we recall below in Section 4.1 (Theorem 4.1). In the remaining subsections we establish the various hypotheses of Theorem 4.1. The section ends in Section 4.4 with a proof that each of the groups QF, QT and QV has the F_∞ property.

4.1 Brown's finiteness criterion

Our proof that QF, QT and QV have type F_∞ uses a well-known theorem due to Brown.

Theorem 4.1 (Brown's finiteness criterion [5]) *Let X be a CW–complex. Let G be a group acting on X. If*

(1) *X is contractible,*

(2) *G acts cellularly on X, and*

(3) *there is a sequence of subcomplexes $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$ such that*
   (a) $X = \bigcup_{n=1}^{\infty} X_n$,
   (b) *G acts cocompactly on X_i and leaves each X_i invariant,*
   (c) *G acts with finite cell stabilizers, and*
   (d) *for every $k \geq 0$, there exists an N such that X_n is k–connected for every $n \geq N$,*

then G is of type F_∞. □

We note that QF, QT and QV each act cellularly on contractible CW–complexes by the results of Section 3.4. It therefore remains to introduce suitable filtrations $\{X_i \mid i \in \mathbb{N}\}$ that satisfy (3).

4.2 Filtrations by subcomplexes

Definition 4.2 If Σ is an alphabet, $p \in \Sigma$ and $w \in \Sigma^+$, then we let $|w|_p$ denote the number of occurrences of the letter p in w.

Definition 4.3 Let $X = K_{QF}$, K_{QT} or K_{QV}. For $n \geq 1$, let L_n denote the collection of words w in the alphabet $\{a, x\}$ such that $|w|_a = j - 1$ and $|w|_x = j$ for some $j \in \{1, \ldots, n\}$.

We let X_n denote the subcomplex generated by the collection of all vertices of X having bottom labels in L_n. Thus, X_n^0 consists of all vertices having bottom labels in L_n, and a higher-dimensional cube C of X lies in X_n if and only if the bottom label of each corner of C lies in L_n.

Proposition 4.4 *If $X = K_{QF}$, K_{QT} or K_{QV}, then $\{X_n \mid n \in \mathbb{N}\}$ is a filtration of X by subcomplexes that satisfies Theorem 4.1(3)(a)–(c), where G is QF, QT or QV (respectively).*

Proof We prove the proposition for $X = K_{QV}$. The proofs in the other two cases are similar.

We first prove that property (3)(a) holds. Let C be any cube in K_{QV}. Each corner v of the cube may be represented by a braided $(x, *)$–diagram Δ. It is easy to see (by induction on the number of transistors in Δ) that the bottom label of Δ is $x^j a^{j-1}$, up to permutation of the letters, for some $j \geq 1$. Since C has a finite number of corners, there is some particular $k \geq 1$ such that the bottom label of each corner lies in L_k.

It follows that all corners of C lie in X_k, so that $C \subseteq X_k$. Thus, any cube of X is contained in some X_k, proving (3)(a).

Next we establish property (3)(b). Recall that two vertices of K_{QV} are in the same orbit if and only if they may be represented by diagrams with the same bottom labels (Proposition 2.17). It follows that each X_k is invariant under the action of QV, and that there are only finitely many vertices of X_k modulo the action of QV. Since the link of each vertex in QV is finite (Proposition 2.19 and Definition 2.20), the action of QV on each X_k is cocompact. This proves (3)(b).

The group QV acts on each X_k with finite cell stabilizers since QV acts on K_{QV} itself with finite cell stabilizers. This proves (3)(c). □

Remark 4.5 The groups QF and QT both act on K_{QV}. Indeed, these actions satisfy all of the conditions of Theorem 4.1, with the sole exception of the cocompactness condition 3(b). This defect necessitates that we work with the complexes K_{QF} and K_{QT}.

4.3 Analysis of descending links

In this subsection, we consider the descending links in the complexes K_{QF}, K_{QT} and K_{QV}. We first demonstrate (in Section 4.3.1) that the connectivity of the complexes X_n is determined by the connectivity of the descending links $\mathrm{lk}_{\downarrow}(w)$, where $w \in L_n$. We then compute the connectivity of $\mathrm{lk}_{\downarrow}(w)$ as a function of n in Section 4.3.2.

4.3.1 The descending link and connectivity of the complexes X_n Fix $n \geq 2$. Let X denote any of the complexes K_{QF}, K_{QT} or K_{QV}. (The current discussion applies equally to all three complexes, with inessential differences, so we will assume that $X = K_{QV}$ for simplicity.) For a small $\epsilon > 0$, consider the union of all ϵ–neighborhoods around the vertices $v \in X_n$ having bottom labels $x^n a^{n-1}$. Let A denote the complement of this union and let B denote its closure. Clearly, $X_n = A \cup B$.

Since a vertex $v \in X_n$ with bottom label $x^n a^{n-1}$ is adjacent in X_n only to vertices with the bottom label $x^{n-1} a^{n-2}$, the link of v in X_n is isomorphic to the descending link $\mathrm{lk}_{\downarrow}(v)$, as described in Definition 2.21. It follows that $A \cap B$ may be identified with a countable disjoint union of the links $\mathrm{lk}_{\downarrow}(v)$, as v runs over all vertices in X_n with bottom label $x^n a^{n-1}$. Moreover, the subspace A strong deformation retracts onto the complex X_{n-1}. It follows that, up to homotopy, we can describe X_n as

$$X_n = X_{n-1} \cup_{\bigsqcup_v \mathrm{lk}_{\downarrow}(v)} \left(\bigsqcup_v C_v \right),$$

where the disjoint unions are over all vertices v having bottom label $x^n a^{n-1}$ and the spaces C_v are cones on the descending links, and therefore contractible.

We note that the above setup is a special case of a more general situation considered in [1, Lemma 2.5]. As in [1], standard arguments using van Kampen's theorem, the Mayer–Vietoris sequence and the Hurewicz theorem yield the following result:

Proposition 4.6 *If the descending link $\mathrm{lk}_\downarrow(x^k a^{k-1})$ is n–connected ($n \geq 0$), then the inclusion map $X_{k-1} \hookrightarrow X_k$ induces isomorphisms $\pi_j(X_{k-1}) \to \pi_j(X_k)$ for $0 \leq j \leq n$.*

In particular, if $\mathrm{lk}_\downarrow(x^k a^{k-1})$ is n–connected for all $k \geq N$ (for some $N \in \mathbb{N}$), then X_m is n–connected for all $m \geq N$. □

For future reference, we now give a combinatorial description of the descending links.

Definition 4.7 Let $v = x^k a^l$. We let $\mathrm{lk}_\downarrow^{QV}(v) = \mathrm{lk}_\downarrow(v)$ (as defined in Definition 2.21). We also let $\mathrm{lk}_\downarrow^{QF}(v)$ and $\mathrm{lk}_\downarrow^{QT}(v)$ denote the subcomplexes of $\mathrm{lk}_\downarrow^{QV}(v)$ spanned by the vertex sets

$$\big\{(m, m+1, p) \mid m \in \{1, \dots, k-1\},\ p \in \{1, \dots, l\}\big\}$$

and

$$\big\{(m, m+1, p) \mid m \in \{1, \dots, k-1\},\ p \in \{1, \dots, l\}\big\} \cup \big\{(k, 1, p) \mid p \in \{1, \dots, l\}\big\},$$

respectively.

Proposition 4.8 *Let $v = x^k a^l$. The complexes $\mathrm{lk}_\downarrow^{QF}(v)$ and $\mathrm{lk}_\downarrow^{QT}(v)$ are isomorphic to the descending links of the word v in the subcomplexes $D(\mathcal{P}, v)$ and $D_a(\mathcal{P}, v)$ of $D_b(\mathcal{P}, v)$, respectively.*

Proof This follows from the description of $\mathrm{lk}(v)$ from Proposition 2.22, and from the description of the embedding of the complexes for QF and QT into the complex for QV (Proposition 3.10). □

Remark 4.9 As defined in Definition 4.7, the vertices of $\mathrm{lk}^{QF}(v)$ and $\mathrm{lk}^{QT}(v)$ are 3–tuples of integers. We will sometimes think of the final coordinate as a "color" coordinate, since it has greater freedom of movement than the first two coordinates (which are the "x"–coordinates, and therefore constrained by the requirements that the associated diagrams be planar or annular). We will especially use the color language in the proof of Proposition 4.11.

4.3.2 The connectivity of the descending links in K_{QF} and K_{QT} We will determine the connectivity of the links $\mathrm{lk}_{\downarrow}^{QF}(v)$ and $\mathrm{lk}_{\downarrow}^{QT}(v)$ (for $v = x^k a^l$) with the aid of covers by subcomplexes. The nerve theorem will be an important tool for us.

Theorem 4.10 [2] (the nerve theorem) *Let Δ be a simplicial complex and $(\Delta_i)_{i\in I}$ a family of subcomplexes such that $\Delta = \bigcup_{i\in I} \Delta_i$. If every nonempty finite intersection $\Delta_{i_1} \cap \cdots \cap \Delta_{i_t}$ is $(k-t+1)$–connected, then Δ is k–connected if and only if the nerve $\mathcal{N}(\Delta_i)$ is k–connected.* □

Proposition 4.11 *The complexes $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$ and $\mathrm{lk}_{\downarrow}^{QT}(x^k a^l)$ are n–connected ($n \geq 0$) if $k \geq 3n+5$ and $l \geq 2n+3$.*

Proof The proofs are nearly identical, whether we consider QF or QT; we will give a detailed proof in the former case.

We prove the statement by induction on n. Note that the given link is nonempty when $k \geq 2$ and $l \geq 1$.

Consider the base case $n = 0$. The sequence

$$(1,2,\alpha),\ (3,4,\beta),\ (1,2,\gamma)$$

determines an edge-path in $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$, where α, β, $\gamma \in \{1,\ldots,l\}$ are all distinct. It follows that any two vertices of the form $(1,2,\alpha)$ can be connected by an edge-path. Clearly, any vertex of the form $(m, m+1, \beta)$ $(m > 2)$ is adjacent to some $(1,2,\alpha)$. Finally, note that any vertex $(2,3,\beta)$ can be connected to a vertex of the form $(1,2,\alpha)$ by the edge-path

$$(2,3,\beta),\ (4,5,\gamma),\ (1,2,\alpha).$$

It now follows easily that $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$ is connected if $k \geq 5$ and $l \geq 3$.

Now assume that the statement of the proposition is true for $0 \leq j \leq n$. We consider $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$, where $k \geq 3(n+1)+5$ and $l \geq 2(n+1)+3$. Following the basic strategy of [9] (in the proof of Proposition 4.11 there), we would like to cover $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$ by the simplicial neighborhoods of the vertices

$$\mathcal{C} = \big\{(1,2,\alpha), (2,3,\alpha) \mid \alpha \in \{1,\ldots,l\}\big\}.$$

Unfortunately, these simplicial neighborhoods do not cover $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$ in general. (If $k \gg l$, then it is possible to find a simplex S that "uses" all of the colors $1,\ldots,l$, but none of the vertices in the above collection; such a simplex clearly cannot be part of the simplicial neighborhood of any of the above vertices.) We will instead consider the $(n+2)$–skeleton of $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$. We claim that the $(n+2)$–skeleton is indeed covered by the $(n+2)$–skeletons of the simplicial neighborhoods considered above, as we now prove.

Let $S = \{(m_1, m_1+1, \alpha_1), \dots, (m_p, m_p+1, \alpha_p)\}$ be a simplex in the $(n+2)$–skeleton. If one of the m_i is 1 or 2, then it is clear that the given simplex is in the simplicial neighborhood of some vertex in $\mathcal{C}$, and the desired conclusion follows. If all of the m_i are greater than 2, then, since $p \le n+3 < 2n+5 \le l$, there is a color $\beta \in (\{1, \dots, l\} - \{\alpha_1, \dots, \alpha_p\})$. The simplex S is therefore in the simplicial neighborhood of $(1, 2, \beta)$. Thus, clearly S is in the $(n+2)$–skeleton of the simplicial neighborhood of $(1, 2, \beta)$. This proves the claim.

We let $C(m, m+1, \beta)$ denote the $(n+2)$–skeleton of the simplicial neighborhood of $(m, m+1, \beta)$ in $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$. We consider the cover

$$\widehat{\mathcal{C}} = \bigl\{C(m, m+1, \beta) \mid m \in \{1, 2\},\ \beta \in \{1, \dots, l\}\bigr\}.$$

We first claim that the nerve $\mathcal{N}(\widehat{\mathcal{C}})$ is $(n+1)$–connected. We will prove this by showing that every set of $n+3$ vertices of the nerve span an $(n+2)$–simplex of the nerve. It will then follow that the nerve contains the $(n+2)$–skeleton of a high-dimensional simplex, and is therefore $(n+1)$–connected. So, take any $n+3$ members of $\widehat{\mathcal{C}}$. These elements

$$C(m_1, m_1+1, \beta_1), \dots, C(m_{n+3}, m_{n+3}+1, \beta_{n+3})$$

$(m_i \in \{1, 2\})$ collectively "use" at most $n+3$ colors, out of a total of at least $2n+5$. Thus, there is a $\beta \in (\{1, \dots, l\} - \{\beta_1, \dots, \beta_{n+3}\})$, and it follows that the intersection

$$\bigcap_{i=1}^{n+3} C(m_i, m_{i+1}, \beta_i)$$

is nonempty. (It contains the vertex $(4, 5, \beta)$, for instance.) This proves that the nerve $\mathcal{N}(\widehat{\mathcal{C}})$ is $(n+1)$–connected.

Now we want to prove that any nonempty t–fold intersection of members of $\widehat{\mathcal{C}}$ is $(n-t+2)$–connected. Thus, we consider the intersection

$$\bigcap_{i=1}^{t} C(m_i, m_i+1, \beta_i),$$

where $m_i \in \{1,2\}$. One easily checks that this intersection consists of all q–simplices $(q \leq n+2)$ on the set

$$\big\{(m, m+1, \gamma) \mid m \in \{3, \dots, k-1\},\ \gamma \in (\{1, \dots, l\} - \{\beta_1, \dots, \beta_i\})\big\}$$

or on the set

$$\big\{(m, m+1, \gamma) \mid m \in \{4, \dots, k-1\},\ \gamma \in (\{1, \dots, l\} - \{\beta_1, \dots, \beta_i\})\big\}.$$

It follows directly that the intersection may be identified with the $(n+2)$–skeleton of $\mathrm{lk}_{\downarrow}^{QF}(x^{k'} a^{l'})$, where $k' \geq k-3$ and $l' \geq l-t$. We note first that

$$k' \geq k-3 \geq 3(n+1)+5-3 = 3n+5 \geq 3(n-t+2)+5$$

and

$$l' \geq l-t \geq 2n+5-t.$$

We want to prove that $2n+5-t \geq 2(n-t+2)+3$, but this is easily seen to be equivalent to the inequality $t \geq 2$.

We can now conclude that the intersection in question is the same as the $(n+2)$–skeleton of $\mathrm{lk}_{\downarrow}^{QF}(x^{k'} a^{l'})$, where $k' \geq 3(n-t+2)+5$ and $l' \geq 2(n-t+2)+3$. It follows directly that the intersection is $(n-t+2)$–connected, as required.

It follows from the nerve theorem that $\mathrm{lk}_{\downarrow}^{QF}(x^k a^l)$ is $(n+1)$–connected for $k \geq 3(n+1)+5$ and $l \geq 2(n+1)+3$, as required. This completes the induction, and proves the proposition in the case of QF.

Note that the required connectivity of the complex $\mathrm{lk}_{\downarrow}^{QT}(x^k a^l)$ is proved almost exactly as above. In fact, the intersection of the cones $C(m, m+1, \beta)$ from the last part of the proof is isomorphic to a descending link $\mathrm{lk}_{\downarrow}^{QF}(x^{k'} a^{l'})$ (rather than a descending link of the form $\mathrm{lk}_{\downarrow}^{QT}(x^{k'} a^{l'})$, as one might expect). □

4.3.3 The connectivity of the descending links in K_{QV} Here we will determine the connectivity of the descending links $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$ by a method analogous to the one from Section 4.3.2.

The argument makes use of the following lemma [11, Lemma 6].

Lemma 4.12 *Let K be a finite flag complex and let $n \geq 0$. The complex K is n–connected if, for any collection of vertices $S \subseteq K^{(0)}$ with $|S| \geq 2$, the intersection $\bigcap_{v \in S} \mathrm{lk}(v)$ is $(n+1-|S|)$–connected.* □

Proposition 4.13 *For $n \geq 0$, the descending link $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$ is n–connected when $k \geq 4n+5$ and $l \geq 2n+3$.*

Proof We will use induction to show that this is true for any $n \geq 0$.

Consider $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$; let $k \geq 5$ and $l \geq 3$. We will show that $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$ is 0–connected. Let (m,n,α) and (m',n',α') be arbitrary vertices of the link $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$. Note that, if $\{m,n\} \cup \{m',n'\}$ has three or fewer elements, then the vertices in question are clearly connected by a path, since (m'',n'',α'') is adjacent to both vertices if $m'',n'' \notin \{m,n\} \cup \{m',n'\}$ and $\alpha'' \notin \{\alpha',\alpha''\}$. Thus, we may assume that m, n, m' and n' are all different. In this case, there is nothing to prove unless $\alpha = \alpha'$ (otherwise the vertices are adjacent by definition). Let $M \notin \{m,n,m',n'\}$. We note that (m', M, β) $(\beta \neq \alpha)$ is adjacent to (m,n,α). Now, since (m', M, β) may be connected to (m',n',α') by the previous argument, it follows that (m,n,α) and (m',n',α') may be connected by a path. This proves that $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$ is 0–connected.

Now let $k \geq 4(n+1)+5$ and $l \geq 2(n+1)+3$, where $n \geq 0$. We assume that the statement of the proposition is true for $0 \leq j \leq n$. Let S be an arbitrary collection of vertices in $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$. We may assume that $|S| \leq n+2$ (otherwise there is nothing to prove). We note that

$$\bigcap_{v \in S} \mathrm{lk}(v)$$

is simply the full subcomplex of $\mathrm{lk}_{\downarrow}^{QV}(x^k a^l)$ on the vertices $\hat{v}$ that have no overlap with any of the $v \in S$. (That is, $\hat{v}$ and v represent disjoint applications of relations, as in Definition 2.18, to $x^k a^l$ for all $v \in S$.) It follows that

$$\bigcap_{v \in S} \mathrm{lk}(v) \cong \mathrm{lk}_{\downarrow}^{QV}(x^{k'} a^{l'}),$$

where $k' \geq 4n+9-2|S|$ and $l' \geq 2n+5-|S|$. By Lemma 4.12 and induction, it now suffices to show that

$$4n+9-2|S| \geq 4(n+1-|S|)+5$$

and

$$2n+5-|S| \geq 2(n+1-|S|)+3.$$

Both of these inequalities are obvious. This completes the induction. □

4.4 Conclusion: proof of the F_∞ property

It is now straightforward to complete the proof of the F_∞ property.

Proposition 4.14 *Let $X = K_{QF}, K_{QT}$ or K_{QV}.*

(1) *If $X = K_{QF}$ or K_{QT}, then X_k is n–connected if $k \geq 3n + 5$.*

(2) *If $X = K_{QV}$, then X_k is n–connected if $k \geq 4n + 5$.*

Proof If $X = K_{QF}$ or K_{QT}, then, by Propositions 4.6 and 4.11, the descending link $\mathrm{lk}_{\downarrow}(x^k a^{k-1})$ is n–connected provided that $k \geq 3n + 5$.

Similarly, if $X = K_{QV}$, then, by Propositions 4.6 and 4.13, the descending link $\mathrm{lk}_{\downarrow}(x^k a^{k-1})$ is n–connected provided that $k \geq 4n + 5$. □

Theorem 4.15 *The groups QF, QT and QV have type F_∞.*

Proof The complexes $X = K_{QF}$, K_{QT} and K_{QV} admit actions by QF, QT and QV (respectively) satisfying (1) and (2) from Theorem 4.1. This was established by the end of Section 4.1. In Section 4.2, we established that the complexes X admit filtrations by subcomplexes $\{X_k\}$ such that (3)(a)–(c) from Theorem 4.1 are satisfied. By Proposition 4.14, these filtrations satisfy (3)(d) from Theorem 4.1 as well. It now follows from Theorem 4.1 that QF, QT and QV have type F_∞. □

5 Some generalizations

Nucinkis and St John-Green introduced the groups $\bar{Q}T$ and $\bar{Q}V$, and proved that both groups have type F_∞. The group $\bar{Q}V$ is defined just as QV was (see Definition 3.1), except that $\bar{Q}V$ is a set of automorphisms of $\mathcal{T} \cup \{*\}$, where $*$ is an isolated point, rather than a set of automorphisms of $\mathcal{T}$, as in the case of QV. There is still a natural action of $\bar{Q}V$ on the set of ends of $\mathcal{T}$, which yields a homomorphism $\bar{Q}V \to V$. The inverse image of Thompson's group T under this homomorphism is $\bar{Q}T$.

In the vocabulary of the main body of the paper, $\bar{Q}V$ is the braided diagram group $D_b(\mathcal{P}, xa)$, where $\mathcal{P} = \langle x, a \mid x = xax \rangle$. The group $\bar{Q}T$ is the subgroup of $\bar{Q}V$ consisting of diagrams that become (or remain) annular when all wires labeled "a" are erased. The groups $\bar{Q}V$ and $\bar{Q}T$ then act on cubical complexes analogous to the ones described in Sections 2.2 and 3.3: $\bar{Q}V$ acts on $\widetilde{K}_b(\mathcal{P}, xa)$ and $\bar{Q}T$ acts on the convex subcomplex spanned by diagrams that are annular (in the above extended sense). The proofs of the F_∞ property from the main body of the paper apply to these groups with no essential changes.

More generally, we can define a class of groups as follows. Let $\mathcal{F}_{k,l}$ denote an ordered forest consisting of k infinite binary trees and l isolated vertices. We let $QV_{k,l}$ denote the group of all bijections of the vertices of $\mathcal{F}_{k,l}$ that preserve left- and right-children, with at most finitely many exceptions (as in Definition 3.1). We let $QF_{k,l}$ and $QT_{k,l}$ be the subgroups of $QV_{k,l}$ that preserve the linear and the cyclic ordering, respectively, of the ends. The group $QV_{k,l}$ is simply the braided diagram group $D_b(\mathcal{P}, x^k a^l)$ (with $\mathcal{P}$ as above), and the groups $QF_{k,l}$ and $QT_{k,l}$ have obvious definitions, analogous to the ones given above. A minor variant of the main argument now proves:

Theorem 5.1 *The groups $QF_{k,l}$, $QT_{k,l}$ and $QV_{k,l}$ have type F_∞ for $k \geq 1$ and $l \geq 0$.* □

It is not clear whether the above theorem specifies infinitely many different groups of type F_∞. The groups $QV_{k,l}$ and $QV_{k',l'}$ are isomorphic if $k-l = k'-l'$. This can be seen in at least two ways:

(i) If $k-l = k'-l'$, then there is a quasi-isomorphism between $\mathcal{F}_{k,l}$ and $\mathcal{F}_{k',l'}$ that conjugates $QV_{k,l}$ to $QV_{k',l'}$.

(ii) Under the same hypothesis, $x^k a^l$ is equivalent to $x^{k'} a^{l'}$ modulo the presentation $\mathcal{P} = \langle x, a \mid xax = x \rangle$ and up to permutation of the letters; thus, the groups in question are isomorphic by what amounts to a change-of-basepoint isomorphism (see [13] for a discussion of this principle in connection with diagram groups).

On the other hand, if $k-l \neq k'-l'$, then the authors do not know whether the groups can be isomorphic. The above discussion carries over to the cases of $QF_{k,l}$ and $QT_{k,l}$ without any changes.

Further extensions are possible. For instance, we could consider the case of rooted ordered n–ary trees. Let $\mathcal{T}^n_{k,l}$ denote the ordered forest of k infinite n–ary trees and l isolated vertices. We can let $QV^n_{k,l}$ denote the group of bijections h of $(\mathcal{T}^n_{k,l})^0$ that send the j^{th} child of a vertex v to the j^{th} child of the vertex $h(v)$, with at most finitely many exceptions. (These groups might equally well be denoted by $QG^n_{k,l}$, since they extend Higman's groups $G_{n,k}$ [14] in the same way that QV extends V. We caution the reader, however, that the groups $QV_{k,l}$ are not obviously extensions of $V_{k,l}$ or $G_{k,l}$, since the subscripts k and l in the latter groups refer to the (downward) degree of the trees and the number of trees, respectively, in contrast to the meanings of the same subscripts in $QV_{k,l}$.) This group is isomorphic to the braided diagram group $D_b(\mathcal{P}_n, x^k a^l)$, where $\mathcal{P}_n = \langle x, a \mid x = x^n a \rangle$. This construction also yields its own "QF" and "QT" versions, in a natural way. All such groups will have type F_∞ provided that $n \geq 2$. The argument differs from the main argument of this paper in only minor ways (for instance, the bounds in the analogs of Propositions 4.11 and 4.13 will be different).

Even more generally, we could define a tree using a tree-like semigroup presentation $\mathcal{P}' = \langle \Sigma \mid \mathcal{R} \rangle$ (see Definition 4.1 from [12]). In such a presentation, each relation has the form $x = x_1 x_2 \dots x_i$, and a given $x \in \Sigma$ is the left-hand side of at most one relation in $\mathcal{R}$. For any fixed $x \in \Sigma$, there is a natural simplicial tree $T_{(\mathcal{P}',x)}$ (as defined in the proof of Theorem 4.12 from [12]). We can construct $T_{(\mathcal{P}',x)}$ inductively as follows. We begin with a root labeled x. This vertex is adjacent to children labeled (from left to right) $x_1, x_2, \dots, x_i$ (respectively) if $(x = x_1 x_2 \dots x_i) \in \mathcal{R}$. One similarly introduces and labels the children of $x_1, \dots, x_i$ using the relations of the form $(x_\alpha = w_\alpha) \in \mathcal{R}$ (for $1 \le \alpha \le i$, w_α is a word in the generators Σ). The result is easily seen to be a rooted, ordered, labeled simplicial tree. The reader can easily verify that $T_{(\mathcal{P},x)}$ is the usual infinite binary tree if $\mathcal{P} = \langle x \mid x = x^2 \rangle$, for instance. Now, assuming that $a \notin \Sigma$, we define $\mathcal{P}'' = \langle \Sigma \cup \{a\} \mid \mathcal{R}' \rangle$, where $\mathcal{R}'$ is the result of replacing each relation $(x = x_1 \dots x_i) \in \mathcal{R}$ with the relation $x = x_1 \dots x_i a$. (Here, the letter "a" is again being used to represent an isolated vertex.) The braided diagram group $D_b(\mathcal{P}'', x)$ should be isomorphic to the group of quasiautomorphisms of $T_{(\mathcal{P}',x)}$ under appropriate hypotheses on the original tree-like semigroup presentation $\mathcal{P}'$. (For instance, one would need to ensure that the trees $T_{(\mathcal{P}',x)}$ and $T_{(\mathcal{P}',y)}$ are not isomorphic for different x and y; failure to do this would make the braided diagram group strictly smaller than the corresponding quasiautomorphism group.) It is reasonable to expect that such groups will often be of type F_∞, but we will not try to guess at the appropriate hypotheses here.

It is worth adding a final cautionary note. We cannot expect all groups in the family sketched above to have type F_∞. The Houghton group H_n can be described as the "QF" group associated to the forest $\mathcal{T}^1_{n,0}$. This group was shown to have type F_{n-1} but not type F_n by Brown [5]. In fact, it seems that adding even a single rooted 1–ary tree (ie a cellulated ray) to a forest of higher-valence rooted trees will destroy the F_∞ property.

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