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Stable functorial decompositions of $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$

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We first construct a functorial homotopy retract of $\Omega^{n+1}\Sigma^{n+1}X$ for each natural coalgebra-split sub-Hopf algebra of the tensor algebra. Then, by computing their homology, we find a collection of stable functorial homotopy retracts of $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$.

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1 Introduction

In the 1970s, Snaith [12] proved iterated loop suspensions of a space can be split stably into simpler pieces. This is called the Snaith splitting. In detail, let X be a path-connected CW–complex, with $X^{(j)}$ the j–fold self smash product of X. Let $F(\mathbb{R}^{n+1}, j)$ be the j^{th} configuration space of $\mathbb{R}^{n+1}$ and Σ_j be the symmetric group on j letters. Let $D_j(X)$ denote the smash product $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$. There is a homotopy equivalence

$$\Sigma^\infty \Omega^{n+1}\Sigma^{n+1}X \simeq \bigvee_{j=0}^{\infty} \Sigma^\infty F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)} = \bigvee_{j=0}^{\infty} \Sigma^\infty D_j(X).$$

Subsequently, it was shown that similar splittings can be applied to a more general space CX; see Cohen, May and Taylor [4; 5] and May and Taylor [8].

A few years later, Bödigheimer [2] showed a unified form of all these splittings. Let K be a finite complex, K_0 a subcomplex and X a connected CW–complex. Let M be a smooth, parallelizable n–manifold with a submanifold M_0 such that $(M, M_0) \simeq (K, K_0)$. For the space $\mathrm{Map}(K, K_0; \Sigma^n X)$ of based maps from K/K_0 to $\Sigma^n X$, there is a stable splitting

$$\Sigma^\infty \mathrm{Map}(K, K_0; \Sigma^n X) \simeq \bigvee_{j=1}^{\infty} \Sigma^\infty D_k(M, M_0; X),$$

where $D_k(M, M_0; X)$ for $k \geqslant 1$ are simpler pieces constructed from the labeled configuration space $C(M, M_0; X)$.

Snaith splitting is one kind of stable splitting. Recently, the techniques of stable splittings have been applied to toric topology. For instance, Bahri, Bendersky, Cohen and

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Gitler [1] found various stable splittings of polyhedral product functors. Dobrinskaya [6] proved that the loop space of the polyhedral product shares similar decompositions as the Snaith splitting.

Here we study further functorial decompositions of the Snaith splitting. More precisely, we will focus on the functorial homotopy decompositions of $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$. When $n = 0$, we have $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)} = X^{(j)}$. Selick and the first author [11] showed that if $p = 2$ and $\overline{H}_*(X; \mathbb{Z}/p)$ has a nontrivial Steenrod operation then the irreducible functorial decomposition component of $X^{(j)}$ and the 2–row Young diagram with distinct row numbers are in one-to-one correspondence. In this paper, we will study the case when $n > 0$.

The main idea driving this paper comes from functorial homotopy decompositions of $\Omega\Sigma X$: For each natural coalgebra-split sub-Hopf algebra (see Definition 2.2), there is a functorial homotopy retract of $\Omega\Sigma X$ with the inclusion an Ω–map; see Li, Lei and Wu [7] and Selick and Wu [10]. Among all the natural coalgebra-split sub-Hopf algebras, we mainly focus on a special one. Let $L_m^{\max}$ be the maximal T_m–projective submodule functor of the free Lie algebra functor L_m (see Section 2.1). For a graded (alternatively ungraded) $\mathbb{Z}/p$–module V, the tensor algebra $T(L_m^{\max}(V))$ generated by $L_m^{\max}(V)$ is a natural coalgebra-split sub-Hopf algebra (Proposition 2.3). Following from Section 2.3, there is geometric realization of $L_m^{\max}(V)$, denoted by $L_m^{\max}(X)$, such that $\Omega\Sigma L_m^{\max}(X)$ is a functorial homotopy retract of $\Omega\Sigma X$. Furthermore, the inclusion is an Ω–map.

For a space $\Sigma^n X$, we have that $\Omega\Sigma L_m^{\max}(\Sigma^n X)$ is a functorial homotopy retract of $\Omega\Sigma^{n+1} X$ with the inclusion an Ω–map. Applying the loop functor n times, we can obtain a functorial homotopy retract of $\Omega^{n+1}\Sigma^{n+1} X$ with the functorial the homotopy inclusion an Ω^{n+1}–map. It can be shown that this retract is a $(n+1)$–iterated loop suspension (Lemma 3.1). Now a natural question is: what is the relation between the Snaith splitting of the retract and the Snaith splitting of the original $(n+1)$–iterated loop suspension? To answer this question, we have the following main result:

Theorem 1.1 *Let X be a 1–connected p–local suspension of finite type. For the natural coalgebra-split sub-Hopf algebra $T(L_m^{\max}(V))$, there is an n^{th} desuspension $\Sigma^{-n} L_m^{\max} \Sigma^n X$ of the topological space $L_m^{\max}(\Sigma^n X)$ and a sufficient large integer t such that $\Sigma^t D_j(\Sigma^{-n} L_m^{\max} \Sigma^n X)$ is a functorial homotopy retract of $\Sigma^t D_{jm}(X)$.*

This article is organized as follows. In Section 2, we give a brief introduction about natural coalgebra-split sub-Hopf algebras of the tensor algebra, functorial homotopy retracts of $\Omega\Sigma X$ and the homology of $\Omega^{n+1}\Sigma^{n+1} X$. Section 3 constructs natural homotopy retracts of $\Omega^{n+1}\Sigma^{n+1} X$ from natural coalgebra-split sub-Hopf algebras of the tensor algebra. In Section 4, we compute the homology image of $\Sigma^{-n} L_m^{\max} \Sigma^n X$ in the homology $\Omega^{n+1}\Sigma^{n+1}X$. In Section 5, a collection of the functorial stable homotopy retract of $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$ is constructed. Additionally, the proof of Theorem 1.1 is given in this section. An example is given in Section 6.

2 Preliminaries

Let $\mathbb{k} = \mathbb{Z}/p$ be the ground ring; p is a prime. All topological spaces are assumed to be p–local CW–complexes. All homology is taken with the coefficients $\mathbb{Z}/p$ unless otherwise stated.

2.1 T_n–projective module

Let V be a graded (ungraded) $\mathbb{k}$–module. Let $T(V)$ be the tensor algebra generated by V, namely
$$T(V) = \sum_{n=0}^{\infty} V^{\otimes n}.$$
A Hopf algebra structure can be given over $T(V)$ by setting V to be primitive. Let $T_n(V) = V^{\otimes n}$. Then T and T_n can be viewed as functors from the category of graded (ungraded) $\mathbb{k}$–modules to the category of graded (ungraded) $\mathbb{k}$–modules.

Let M and N be functors from the category of graded (ungraded) $\mathbb{k}$–modules to the category of graded (ungraded) $\mathbb{k}$–modules. M is a *submodule functor* of N if $M(V) \subseteq N(V)$ for each graded (ungraded) $\mathbb{k}$–module V, and M is a *retract* of N if there are natural transformations $i\colon M \to N$ and $r\colon N \to M$ of $\mathbb{k}$–modules such that $r \circ s = \mathrm{id}\colon M \to M$. A retract of T_n is related to a $\mathbb{k}(\Sigma_n)$–projective module (see [7, Proposition 2.10]). Hence, if M is a retract of T_n, we also call it T_n*–projective*.

Let $L(V)$ be the free Lie algebra generated by V. Then L is a submodule functor of T. Let $L_n(V) = L(V) \cap T_n(V)$. From Selick and the first author [10], there exists a submodule functor $L_n^{\max}$ of L_n with the following properties:

Proposition 2.1 [10, Section 6] (1) $L_n^{\max}$ *is* T_n*–projective.*

(2) *Each* T_n*–projective submodule functor of* L_n *is a retract of* $L_n^{\max}$*.*

Up to isomorphism, $L_n^{\max}$ is the *maximal* T_n*–projective submodule functor* of L_n.

2.2 Coalgebra-split sub-Hopf algebras

A coalgebra-split sub-Hopf algebra is a retract of $T(V)$ with additional Hopf algebra and coalgebra structures.

Definition 2.2 Let B be a submodule functor of T. We say $B(V)$ is *a natural coalgebra-split sub-Hopf algebra* of $T(V)$ if:

(1) $B(V)$ is a natural sub-Hopf algebra of $T(V)$ with natural inclusion of Hopf algebras $j_V\colon B(V) \to T(V)$.

(2) There is a natural coalgebra transformation $r_V\colon T(V) \to B(V)$ with $r_V \circ j_V = \mathrm{id}_{B(V)}$.

If $B(V)$ is a natural coalgebra-split sub-Hopf algebra defined as above, the natural maps j_V and r_V are called an *associated natural inclusion* and *associated natural retraction* of $B(V)$, respectively.

A natural coalgebra-split sub-Hopf algebra is a tensor algebra. Let $Q(V)$ be the set of indecomposable elements of $B(V)$; this is a $\Bbbk$–submodule of $B(V)$. We have a natural isomorphism of Hopf algebras

$$B(V) \cong T(Q(V)).$$

Define the maps k_V and ψ_V as the canonical inclusion and projection

$$k_V\colon Q(V) \to T(Q(V)) \cong B(V),$$
$$\psi_V\colon B(V) \cong T(Q(V)) \to Q(V).$$

These definitions imply the following commutative diagrams:

$$\begin{tikzcd} Q(V) \arrow[rr, "i_V"] \arrow[dr, "k_V"'] & & T(V) \\ & B(V) \arrow[ur, "j_V"'] & \end{tikzcd} \qquad \begin{tikzcd} Q(V) & & T(V) \arrow[ll, "\phi_V"'] \arrow[dl, "r_V"] \\ & B(V) \arrow[ul, "\psi_V"] & \end{tikzcd}$$

Here j_V is a Hopf algebra homomorphism, r_V is a coalgebra homomorphism, $r_V \circ j_V = \mathrm{id}_{B(V)}$, the maps k_V and ψ_V are homomorphisms of $\Bbbk$–modules, and i_V and ϕ_V are defined as the compositions of the other two maps in the triangle.

If $B(V)$ is a sub-Hopf algebra of $T(V)$ only, then properties of $Q(V)$ can imply a coalgebra-split structure of $B(V)$.

Proposition 2.3 [7, Theorem 5.2] *Let $B(V)$ be a natural sub-Hopf algebra of $T(V)$. Then the following statements are equivalent:*

(1) *$B(V)$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$.*

(2) *Each $Q_n(V) = Q(V) \cap T_n(V)$ is naturally equivalent to a T_n–projective subfunctor of L_n.*

(3) *Each Q_n is T_n–projective.*

Since $L_n^{\max}$ is a T_n–projective subfunctor of L_n, Proposition 2.3 implies $T(L_n^{\max}(V))$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$.

2.3 Functorial homotopy retracts of $\Omega\Sigma X$

Let A and B be functors from the (homotopy) category of path-connected p–local CW–complexes to the (homotopy) category of spaces. Let $\mathcal{C}$ be a subcategory of the (homotopy) category of path-connected p–local CW–complexes. A is a *functorial homotopy retract* of B over $\mathcal{C}$ if, for each object X in $\mathcal{C}$, there are natural maps $i_X\colon A(X) \to B(X)$ and $r_X\colon B(X) \to A(X)$ such that $r_X \circ i_X \simeq \mathrm{id}_{A(X)}$. The homotopy need not be natural. The maps i_X and r_X are called an *associated natural inclusion* and *associated natural retraction* of A, respectively.

The functorial homotopy retracts of $\Omega\Sigma X$ are related to natural coalgebra-split sub-Hopf algebras of $T(V)$. Let X be a CW–complex. X is a *p–local suspension of finite type* if X is homotopic equivalent to $\Sigma Y_{(p)}$, the suspension of the p–localization of a finite CW–complex Y. Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and $Q(V)$ be the set of indecomposable elements of $B(V)$. A functorial homotopy retract of $\Omega\Sigma X$ can be constructed from $B(V)$ and $Q(V)$.

Theorem 2.4 [10, Theorem 1.1; 13, Theorem 3.3] *Let X be a 1–connected p–local suspension of finite type. Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ with associated natural inclusion $j_V\colon B(V) \to T(V)$, and $Q(V)$ the set of indecomposable elements of $B(V)$. Then there is a functorial space $Q(X)$ with a natural map $i_X\colon Q(X) \to \Omega\Sigma X$ such that:*

(1) *$\Omega\Sigma Q(X)$ is a natural homotopy retract of $\Omega\Sigma X$ with associated natural inclusion $\Omega\tilde{\imath}_X$, where $\tilde{\imath}_X\colon \Sigma Q(X) \to \Sigma X$ is the adjoint of $i_X\colon Q(X) \to \Omega\Sigma X$:*

$$\begin{tikzcd} Q(X) \arrow[rr, "i_X"] \arrow[dr] & & \Omega\Sigma X \\ & \Omega\Sigma Q(X) \arrow[ur, "\Omega\tilde{\imath}_X"'] & \end{tikzcd} \tag{1}$$

Here the map $Q(X) \to \Omega\Sigma Q(X)$ is the canonical suspension map.

(2) *$Q(X)$ has a wedge decomposition. In detail, there are elements $\lambda_m \in \mathbb{Z}(\Sigma_m)$ for $m \geqslant 2$ such that $Q(X) = \bigvee_{m=2}^{\infty} Q_m(X)$, where $Q_m(X) = \mathrm{hocolim}_{\lambda_m} X^{(m)}$. Here Σ_m acts on $X^{(m)}$ by permuting factors.*

(3) *$\bar{H}_*(Q(X)) \cong Q(\bar{H}_*(X))$ and $H_*(\Omega\Sigma Q(X)) \cong B(\bar{H}_*(X))$. Furthermore, the induced diagram from diagram (1) satisfies $(\Omega\tilde{\imath}_X)_* = j_{\bar{H}_*(X)}$:*

$$\begin{tikzcd} Q(\overline{H}_*(X)) \arrow[rr, "i_{X*}"] \arrow[dr] & & T(\overline{H}_*(X)) \\ & B(\overline{H}_*(X)) \arrow[ur, "(\Omega\tilde{\imath}_X)_*"'] & \end{tikzcd}$$

In following discussions, we denote the map $\Omega\tilde{\imath}_X$ by j_X. It follows from the theorem that $\Omega\Sigma Q(X)$ is a functorial homotopy retract of $\Omega\Sigma X$ with an associated natural inclusion $j_X\colon \Omega\Sigma Q(X) \to \Omega\Sigma X$ which is a loop map.

2.4 Homology of $\Omega^{n+1}\Sigma^{n+1}X$

Let X be a connected CW–complex. All homology is taken with $\mathbb{Z}/p$–coefficients. The homology of $\Omega^{n+1}\Sigma^{n+1}X$ can be formulated by H_*X, Dyer–Lashof operations Q^i, Browder operations λ_n (we will also use $[-,-]_n$), a function ξ_n and a function ζ_n. The function ζ_n is defined for $p>2$ only.

To formulate the homology of $\Omega^{n+1}\Sigma^{n+1}X$, a set T_nX will be defined first. For convenience, we list the construction of T_nX for $p>2$ only in the following. The case for $p=2$ is similar.

Let $V=\overline{H}_*X$. An element $x\in V$ is a λ_n–product of *weight* 1 ($\omega(x)=1$); the weight of $[a,b]_n$ is defined by $\omega([a,b]_n)=\omega(a)+\omega(b)$. We say $x\in V$ is a *basic* λ_n*–product of weight* 1. Assume the basic λ_n–product of weight $j<k$ has been defined and totally ordered; the *basic* λ_n*–product of weight* k is of the form $[a,b]_n$ such that:

(1) $\omega([a,b]_n)=k$.

(2a) a and b are basic λ_n–products, with $a<b$. If $b=[c,d]_n$ for c and d basic then $a\geqslant c<d$.

(2b) If a is a basic λ_n–product of weight 1 and $n+\text{degree}(a)$ is odd, then $[a,a]_n$ is also a basic λ_n–product of weight 2.

Let $I=(\varepsilon_1,s_1,\dots,\varepsilon_k,s_k)$ be a $2k$–tuple of integers with $s_j\geqslant\varepsilon_j$ and $\varepsilon=0$ or 1. I is *admissible* if $ps_j-\varepsilon_j\geqslant s_{j-1}$ for $2\leqslant j\leqslant k$. Define functions e, d, l and b as follows:

(i) **Excess** $e(I)=2s_1-\varepsilon_1-\sum_{j=2}^{k}[2s_j(p-1)-\varepsilon_j]$.

(ii) **Degree** $d(I)=\sum_{j=1}^{k}[2s_j(p-1)-\varepsilon_j]$.

(iii) **Length** $l(I)=k$.

(iv) $b(I)=\varepsilon_1$.

If $I = \varnothing$, then let $e(I) = \infty$ and $d(I) = l(I) = b(I) = 0$.

For $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$, let $Q^I y = \beta^{\varepsilon_1} Q^{s_1} \cdots \beta^{\varepsilon_k} Q^{s_k} y$. Define the set $T_n X$ by

$$T_n X = \big\{ Q^I y \bigm| y \text{ a basic } \lambda_n\text{–product, } I \text{ admissible, } e(I) + b(I) > |y|, \\ \text{if } I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k), \text{ then } s_k \leqslant \tfrac{1}{2}(n+q) \big\}.$$

Here we denote $\xi_n x$ by $Q^{(n+q)/2} x$ and $\zeta_n x$ by $\beta Q^{(n+q)/2} x$ for $x \in H_q X$, and $|y|$ is the degree of y.

For a prime p, the homology $H_* \Omega^{n+1} \Sigma^{n+1} X$ is a functor of $H_* X$, denoted by $W_n H_* X$. On the other hand, let $AT_n X$ be the free commutative algebra generated by the set $T_n X$. We have the following theorem:

Theorem 2.5 [3, Theorem 3.1, Lemma 3.8] *For a connected* X*, there is an isomorphism of algebras*

$$W_n H_* X \cong A T_n X.$$

Remark Here we use $W_n H_* X$ as another notation for $H_* \Omega^{n+1} \Sigma^{n+1} X$. In fact, it can be defined independently as an $A R_n \Lambda_n$–Hopf algebra with conjugation (see [3, Section 2]).

There is a weight filtration defined on $W_n H_* X$. For an element $Q^I y$ in $T_n X$, let its *weight* $\omega(Q^I y)$ be defined by

$$\omega(Q^I y) = p^{l(I)} \omega(y),$$

where $l(I)$ is the length of the tuple I and $\omega(y)$ is the weight of the basic λ_n–product y. Since $H_* \Omega^{n+1} \Sigma^{n+1} X$ is the commutative algebra generated by $T_n X$, we can define the weight of the product $Q^I y \cdot Q^{I'} y'$ as

$$\omega(Q^I y \cdot Q^{I'} y') = \omega(Q^I y) + \omega(Q^{I'} y').$$

This makes $H_* \Omega^{n+1} \Sigma^{n+1} X$ a filtered algebra by defining the filtration as

$$F_k W_n H_* X = \{ x \in H_* \Omega^{n+1} \Sigma^{n+1} X \mid \omega(x) \leqslant k \}.$$

Let $E_k W_n H_* X = F_k W_n H_* X / F_{k-1} W_n H_* X$. There is a geometric realization of $E_k W_n H_* X$.

Proposition 2.6 [3, Section 4] $\overline{H}_*(F(\mathbb{R}^{n+1}, k)^+ \wedge_{\Sigma_k} X^{(k)}) \cong E_k W_n H_* X.$

2.5 Homology suspensions and transgressions

The *homology suspension* is defined as the homomorphism

$$\sigma_* = p \circ \partial^{-1} \colon \overline{H}_*(\Omega B) \xleftarrow[\cong]{\partial} H_{*+1}(PB, \Omega B) \xrightarrow{p_*} H_{*+1}(B),$$

where $p\colon PB \to B$ is the map $p(u) = u(1)$. The transgression is the differential map in the Serre spectral sequences. Fix a fibration $F \to E \to B$ with connected B and F; in the associated Serre spectral sequence, the transgression τ is the differential

$$d_n\colon E^n_{n,0} \to E^n_{0,n-1}.$$

Some general properties of σ_* and τ are listed as follows:

Proposition 2.7 [9, Propositions 6.10 and 6.11] (1) *Let $f\colon X \to \Omega Y$ be a pointed map and $\widetilde{f}\colon \Sigma X \to Y$ be its adjoint; then the homology suspension σ_* and the suspension $\Sigma_*\colon H_*X \to H_{*+1}\Sigma X$ form a commutative diagram:*

$$\begin{tikzcd} \overline{H}_{q-1}(X) \arrow[r, "f_*"] \arrow[d, "\Sigma_*"] & \overline{H}_{q-1}(\Omega Y) \arrow[d, "\sigma_*"] \\ \overline{H}_q(\Sigma X) \arrow[r, "\widetilde{f}_*"] & H_q(Y) \end{tikzcd}$$

(2) *If B is simply connected, then in the Serre spectral sequence of $\Omega B \to PB \to B$ there is a commutative diagram:*

$$\begin{tikzcd} E^q_{q,0} \arrow[r, "d^q", "\cong"'] \arrow[d, hook] & E^q_{0,q-1} \\ H_q(B) & H_{q-1}(F) \arrow[l, "\sigma_*"'] \arrow[u, two heads] \end{tikzcd}$$

In particular, the image of σ_* is transgressive.

Consider the relation between τ and the Browder operation $[-,-]_n$; we have:

Proposition 2.8 *If X is connected, then in the Serre spectral sequence of*

$$\Omega^{n+1}\Sigma^{n+1}X \to P\Omega^n\Sigma^{n+1}X \to \Omega^n\Sigma^{n+1}X$$

we have

$$\tau\big([sx_1,\ldots,[sx_{k-1},sx_k]_{n-1}]_{n-1}\big) = [x_1,\ldots,[x_{k-1},x_k]_n]_n,$$
$$\tau Q^I sx = (-1)^{d(I)} Q^I x,$$

where sx is the image of $x \in H_*X$ under the isomorphism $\Sigma_*\colon H_*X \to H_{*+1}\Sigma X$.

This proposition is implicit in the proof of [3, Theorem 3.2].

3 Functorial homotopy retracts of $\Omega^{n+1}\Sigma^{n+1}X$

Let $B(V)$ be a natural coalgebra-split sub-Hopf algebra of $T(V)$ and $Q(V)$ the set of indecomposable elements of $B(V)$. Let X be a 1–connected p–local suspension of finite type. It follows from Theorem 2.4 that $\Omega\Sigma Q(\Sigma^n X)$ is a functorial homotopy retract of $\Omega\Sigma(\Sigma^n X)$. By applying the loop functor n times, we can obtain that $\Omega^{n+1}\Sigma Q(\Sigma^n X)$ is a homotopy retract of $\Omega^{n+1}\Sigma^{n+1}X$ and the natural inclusion

$$\Omega^n j_{\Sigma^n X}\colon \Omega^{n+1}\Sigma Q(\Sigma^n X) \hookrightarrow \Omega^{n+1}\Sigma^{n+1}X$$

is an Ω^{n+1}–map. If X is a co-H–space, the space $Q(\Sigma^n X)$ can be desuspended n times:

Lemma 3.1 *If X is a co-H–space, then there is a space $\bar{Q}(X)$ such that $Q(\Sigma^n X)$ is naturally homotopic to $\Sigma^n \bar{Q}(X)$.*

Proof Since $Q(X) = \bigvee_{m=2}^{\infty} Q_m(X)$, it is sufficient to prove $Q_m(\Sigma^n X)$ can be desuspended n times. Let $X^{(m)}$ be the m–fold self smash product of X. The definition of $Q_m(\Sigma^n X)$ implies a homotopy commutative diagram:

$$\begin{tikzcd}
(\Sigma^n X)^{(m)} \arrow[r, "\phi"] & (\Sigma^n X)^{(m)} \\
\Sigma^{mn} X^{(m)} \arrow[u, "\text{shuffling isomorphism}"] \arrow[r, "\Sigma^{mn}\bar{\phi}"] & \Sigma^{mn} X^{(m)} \arrow[u, "\text{shuffling isomorphism}"']
\end{tikzcd} \tag{2}$$

Here

$$\begin{aligned} \phi = \lambda_m &= \sum_{\sigma\in\Sigma_m} k_\sigma \sigma\colon (\Sigma^n X)^{(m)} \to (\Sigma^n X)^{(m)}, \\ \bar{\phi} &= \sum_{\sigma\in\Sigma_m} k_\sigma \sigma (-1)^{n^2 \operatorname{Sign}\sigma}\colon X^{(m)} \to X^{(m)}, \end{aligned} \tag{3}$$

and the vertical maps are the natural shuffling homeomorphisms.

Let $\bar{Q}_m(X) = \operatorname{hocolim}_{\bar{\phi}} X^{(m)}$. It is obvious that

$$\Sigma^{mn}\bar{Q}_m(X) \simeq \operatorname{hocolim}_{\Sigma^{mn}\bar{\phi}} \Sigma^{mn} X^{(m)} \cong \operatorname{hocolim}_{\phi} (\Sigma^n X)^{(m)} = Q_m(\Sigma^n X).$$

Thus,

$$Q(\Sigma^n X) = \bigvee_{m=2}^{\infty} Q_m(\Sigma^n X) = \bigvee_{m=2}^{\infty} \Sigma^{mn}\bar{Q}_m(X) = \Sigma^n \bigvee_{m=2}^{\infty} \Sigma^{n(m-1)}\bar{Q}_m(X).$$

It is clear that all homotopy equivalences are natural. □

Remark This lemma shows that $\bigvee_{m=2}^{\infty}\Sigma^{n(m-1)}\bar{Q}_m(X)$ is the n^{th} desuspension of $Q(\Sigma^n X)$. For convenience, in later discussion, $\Sigma^{-n}Q(\Sigma^n X)$ is used to denote the space $\bigvee_{m=2}^{\infty}\Sigma^{n(m-1)}\bar{Q}_m(X)$. Similarly, we use $\Sigma^{-n}Q_m\Sigma^n X$ to denote $\Sigma^{n(m-1)}\bar{Q}_m(X)$.

For the space $\Sigma^{-n}Q(\Sigma^n X)$, there is a natural inclusion

$$\Sigma^{-n}Q\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}Q\Sigma^n X) \xrightarrow{\Omega^n j_{\Sigma^n X}} \Omega^{n+1}\Sigma^{n+1}X.$$

Up to homotopy, this map is the adjoint map of

$$Q\Sigma^n X \to \Omega\Sigma(Q\Sigma^n X) \xrightarrow{j_{\Sigma^n X}} \Omega\Sigma(\Sigma^n X).$$

This composition is exactly the functorial map $i_Y\colon Q(Y) \to \Omega\Sigma Y$, where $Y = \Sigma^n X$. In summary, we have the following theorem:

Theorem 3.2 *Let X be a 1–connected p–local suspension of finite type. If $B(V)$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$ and $Q(V)$ is the set of indecomposable elements of $B(V)$, then there exists a functorial homotopy retract $\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}Q\Sigma^n X)$ with a natural inclusion*

$$i\colon \Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}Q\Sigma^n X) \to \Omega^{n+1}\Sigma^{n+1}X,$$

which is an Ω^{n+1}–map. Furthermore,

$$\bar{H}_*(\Sigma^{-n}Q\Sigma^n X) \cong Q(\bar{H}_*(\Sigma^n X)).$$

4 $\Sigma^{-n}L^{\max}\Sigma^n X$ and its homology image in $\Omega^{n+1}\Sigma^{n+1}X$

Let $L_m^{\max}$ be the maximal T_m–projective submodule functor of L_m. The tensor algebra $T(L_m^{\max}(V))$ is a natural coalgebra-split sub-Hopf algebra with the set of indecomposable elements $L_m^{\max}(V)$. Then we have two spaces $L_m^{\max}(X)$ and $\Sigma^{-n}L_m^{\max}\Sigma^n X$. Furthermore, $\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X)$ is a functorial homotopy retract of $\Omega^{n+1}\Sigma^{n+1}X$. The inclusion map is

$$\tilde{\imath}_{n,X}\colon \Sigma^{-n}L_m^{\max}\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X) \xrightarrow{\Omega^n j_{\Sigma^n X}} \Omega^{n+1}\Sigma^{n+1}X, \tag{4}$$

which is the adjoint of the map

$$i_{n,X}\colon L_m^{\max}\Sigma^n X \to \Omega\Sigma(L_m^{\max}\Sigma^n X) \xrightarrow{j_{\Sigma^n X}} \Omega\Sigma(\Sigma^n X).$$

To analyze the homology image of $\Sigma^{-n}L_m^{\max}\Sigma^n X$ in $\Omega^{n+1}\Sigma^{n+1}X$, we need to compute

$$(\tilde{\imath}_{n,X})_*\colon H_*\Sigma^{-n}L_m^{\max}\Sigma^n X \to H_*\Omega^{n+1}\Sigma^{n+1}X.$$

From the properties of the homology suspension σ_* (see Proposition 2.7), we obtain a commutative diagram

$$\begin{tikzcd}
H_*\Sigma^{-n}L_m^{\max}\Sigma^n X \arrow[r] \arrow[d, "\Sigma_*^{(n)}"] & H_*\Omega^n\Sigma^n(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, "(\Omega^n i_{n,X})_*"] \arrow[d, "\sigma_*^{(n)}"] & H_*\Omega^{n+1}\Sigma^{n+1}X \arrow[d, "\sigma_*^{(n)}"] \\
H_{*+n}L_m^{\max}(\Sigma^n X) \arrow[r, equal] & H_{*+n}L_m^{\max}(\Sigma^n X) \arrow[r, "(i_{n,X})_*"] & H_{*+n}\Omega\Sigma(\Sigma^n X)
\end{tikzcd}$$

where $\Sigma_*^{(n)}$ and $\sigma_*^{(n)}$ mean n–fold compositions.

For $x \in H_*X$, denote the image of x under the isomorphism $\Sigma_*\colon H_*X \to H_{*+1}\Sigma X$ by sx. Consequently, $s^n x$ is used to denote $\Sigma_*^{(n)}(x)$. Let $[x_1, x_2, \dots, x_m]_n$ be an arbitrary λ_n–product of weight m formed by elements $x_1, \dots, x_m$. For an element $[s^n x_1, s^n x_2, \dots, s^n x_m]_0$ in $H_{*+n}L_m^{\max}(\Sigma^n X)$, with $x_i \in H_*X$, denote its inverse image under the isomorphism

$$\Sigma_*^{(n)}\colon H_*\Sigma^{-n}L_m^{\max}\Sigma^n X \to H_{*+n}L_m^{\max}(\Sigma^n X)$$

by $s^{-n}[s^n x_1, s^n x_2, \dots, s^n x_m]_0$.

For the map $\tilde{\imath}_{n,X}$, we have the following lemma:

Lemma 4.1 *Under the homomorphism*

$$(\tilde{\imath}_{n,X})_*\colon H_*(\Sigma^{-n}L_m^{\max}\Sigma^n X) \to H_*(\Omega^{n+1}\Sigma^{n+1}X),$$

$s^{-n}[s^n x_1, s^n x_2, \dots, s^n x_m]_0$ *is mapped to* $[x_1, x_2, \dots, x_m]_n$*, with* $x_i \in H_*X$*.*

Proof We prove this lemma by induction on n. For $n = 1$, there is a commutative diagram:

$$\begin{tikzcd}
H_*\Sigma^{-1}L_m^{\max}\Sigma X \arrow[r] \arrow[d, "\Sigma_*"] & H_*\Omega\Sigma(\Sigma^{-1}L_m^{\max}\Sigma X) \arrow[r, "(\Omega i_{1,X})_*"] \arrow[d, "\sigma_*"] & H_*\Omega^2\Sigma^2 X \arrow[d, "\sigma_*"] \\
H_{*+1}L_m^{\max}(\Sigma X) \arrow[r, equal] & H_{*+1}L_m^{\max}(\Sigma X) \arrow[r, "(i_{1,X})_*"] & H_{*+1}\Omega\Sigma^2 X
\end{tikzcd}$$

The bottom row is the natural inclusion

$$(i_{1,X})_*\colon L_m^{\max}(sH_*X) \hookrightarrow T(sH_*X).$$

The upper row is exactly $(\tilde{\imath}_{1,X})_*$. Since the first map of the upper row is a natural inclusion, we only need to prove

$$(\Omega i_{1,X})_*(s^{-1}[sx_1, sx_2, \dots, sx_m]_0) = [x_1, x_2, \dots, x_m]_1.$$

To prove this, we consider a natural commutative diagram of Serre path fibrations

$$\begin{tikzcd}
\Omega L_m^{\max}(\Sigma X) \arrow[r] \arrow[d] & PL_m^{\max}(\Sigma X) \arrow[r] \arrow[d] & L_m^{\max}(\Sigma X) \arrow[d] \\
\Omega^2\Sigma^2 X \arrow[r] & P\Omega\Sigma^2 X \arrow[r] & \Omega\Sigma^2 X
\end{tikzcd}$$

which implies a natural morphism of Serre spectral sequences. Therefore, for the transgression τ, there is an equality by naturality,

$$\tau \circ (i_{1,X})_* = (\Omega i_{1,X})_* \circ \tau.$$

In the Serre spectral sequence of the path fibration

$$\Omega^2\Sigma^2 X \to P\Omega\Sigma^2 X \to \Omega\Sigma^2 X,$$

we have the equality (see Proposition 2.8)

$$\tau[sx_1, \dots, sx_m]_0 = [x_1, \dots, x_m]_1.$$

Hence,

$$\begin{aligned}
(\Omega i_{1,X})_*(s^{-1}[sx_1, sx_2, \dots, sx_m]_0) &= (\Omega i_{1,X})_* \circ \tau([sx_1, sx_2, \dots, sx_m]_0)) \\
&= \tau \circ (i_{1,X})_*([sx_1, sx_2, \dots, sx_m]_0) \\
&= \tau([sx_1, sx_2, \dots, sx_m]_0) \\
&= [x_1, \dots, x_m]_1.
\end{aligned}$$

Now assume this lemma is true for $n < k$. For $n = k$, there is a commutative diagram:

$$\begin{tikzcd}
H_*\Sigma^{-k}L_m^{\max}\Sigma^k X \arrow[r] \arrow[d, "\Sigma_*"] & H_*\Omega^k\Sigma^k(\Sigma^{-k}L_m^{\max}\Sigma^k X) \arrow[r, "(\Omega^k i_{k,X})_*"] \arrow[d, "\sigma_*"] & H_*\Omega^{k+1}\Sigma^{k+1}X \arrow[d, "\sigma_*"] \\
H_{*+1}\Sigma^{1-k}L_m^{\max}(\Sigma^k X) \arrow[r] \arrow[d, "\Sigma_*^{(k-1)}"] & H_{*+1}\Omega^{k-1}L_m^{\max}(\Sigma^k X) \arrow[r, "(\Omega^{k-1} i_{k-1,\Sigma X})_*"] \arrow[d, "\sigma_*^{(k-1)}"] & H_{*+1}\Omega^k\Sigma^{k+1}X \arrow[d, "\sigma_*^{(k-1)}"] \\
H_{*+k}L_m^{\max}\Sigma^k X \arrow[r, equal] & H_{*+k}L_m^{\max}\Sigma^k X \arrow[r, "(i_{k,X})_*"] & H_{*+k}\Omega\Sigma^{k+1}X
\end{tikzcd}$$

The composition of the second row is $(\tilde{\imath}_{k-1,\Sigma X})_*$. By induction,

$$(\tilde{\imath}_{k-1,\Sigma X})_*(s^{1-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) = [sx_1, sx_2, \dots, sx_m]_{k-1}.$$

The horizontal rows of left commutative squares are natural inclusions. So, the above identity implies

$$(\Omega^{k-1} i_{k-1,\Sigma X})_*(s^{1-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) = [sx_1, sx_2, \dots, sx_m]_{k-1}.$$

Note that we need to prove

$$(\Omega^k i_{k,X})_*(s^{-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) = [x_1, x_2, \dots, x_m]_k.$$

It follows from the commutative diagram

$$\begin{tikzcd}
\Omega^k L_m^{\max}(\Sigma^k X) \arrow[r] \arrow[d, "\Omega^k i_{k,X}"] & P\Omega^{k-1} L_m^{\max}(\Sigma^k X) \arrow[r] \arrow[d] & \Omega^{k-1} L_m^{\max}(\Sigma^k X) \arrow[d, "\Omega^{k-1} i_{k-1,\Sigma X}"] \\
\Omega^{k+1}\Sigma^{k+1} X \arrow[r] & P\Omega^k \Sigma^{k+1} X \arrow[r] & \Omega^k \Sigma^{k+1} X
\end{tikzcd}$$

and the induced Serre spectral sequences that

$$(\Omega^k i_{k,X})_* \circ \tau = \tau \circ (\Omega^{k-1} i_{k-1,\Sigma X})_*.$$

Thus,

$$\begin{aligned}
&(\Omega^k i_{k,X})_*(s^{-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) \\
&\qquad = (\Omega^k i_{k,X})_* \circ \tau(s^{1-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) \\
&\qquad = \tau \circ (\Omega^{k-1} i_{k-1,\Sigma X})_*(s^{1-k}[s^k x_1, s^k x_2, \dots, s^k x_m]_0) \\
&\qquad = \tau([s x_1, s x_2, \dots, s x_m]_{k-1}) \\
&\qquad = [x_1, \dots, x_m]_k.
\end{aligned}$$

This completes the proof. □

5 Further decompositions of the Snaith splitting

Fix an integer $n \geqslant 0$. The space $\Omega^{n+1}\Sigma^{n+1} X$ has the Snaith splitting

$$\Sigma^\infty \Omega^{n+1}\Sigma^{n+1} X \simeq \bigvee_{j=0}^\infty \Sigma^\infty F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)} = \bigvee_{j=0}^\infty \Sigma^\infty D_j(X).$$

Here $F(\mathbb{R}^{n+1}, j)$ is the j^{th} configuration space of $\mathbb{R}^{n+1}$, and $D_j(X)$ is the smash product $F(\mathbb{R}^{n+1}, j)^+ \wedge_{\Sigma_j} X^{(j)}$. From the above splitting, $D_j(X)$ is a natural stable retract of $\Omega^{n+1}\Sigma^{n+1} X$. The homology of $D_j(X)$ (see Proposition 2.6) is

$$\overline{H}_*(D_j(X)) \cong F_j W_n H_* X / F_{j-1} W_n H_* X = E_j W_n H_* X.$$

In other words, $\overline{H}_* D_j(X)$ consists of the homology classes in $H_*\Omega^{n+1}\Sigma^{n+1} X$ with weight j.

It follows from Theorem 3.2 that $\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n} L_m^{\max} \Sigma^n X)$ is a functorial homotopy retract of $\Omega^{n+1}\Sigma^{n+1} X$. Hence we can apply the Snaith splitting to both spaces and compare $D_j(\Sigma^{-n} L_m^{\max} \Sigma^n X)$ with $D_q(X)$ for nonnegative integers j, m and q.

Proof of Theorem 1.1 In Lemma 3.1, we have proved that the n^{th} desuspension $\Sigma^{-n}L_m^{\max}\Sigma^n X$ of $L_m^{\max}(\Sigma^n X)$ exists. The left part of the main theorem will be proved in two steps. First, the stable case will be proved. We claim that there are stable maps

$$\begin{tikzcd} \Sigma^\infty D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, shift left, "\phi"] & \Sigma^\infty D_{jm}X \arrow[l, shift left, "\psi"] \end{tikzcd}$$

such that

$$\psi_*\circ\phi_* = \mathrm{id},$$

that is:

$$\begin{tikzcd} H_*(\Sigma^\infty D_j\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, "\phi_*"] \arrow[dr, equal, "\mathrm{id}"'] & H_*\Sigma^\infty D_{jm}X \arrow[d, "\psi_*"] \\ & H_*\Sigma^\infty D_j\Sigma^{-n}L_m^{\max}\Sigma^n X \end{tikzcd}$$

Recall that $\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X)$ is a natural homotopy retract of $\Omega^{n+1}\Sigma^{n+1}X$, ie there exist maps

$$\begin{tikzcd} \Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, shift left, "g"] & \Omega^n\Sigma^n X \arrow[l, shift left, "h"] \end{tikzcd}$$

such that

$$h\circ g\simeq \mathrm{id}.$$

Furthermore, g is an Ω^{n+1}–map. In fact, g can be chosen to be $\Omega^n j_{\Sigma^n X}$ (see (4)). Applying the Snaith splitting, we have a diagram as follows:

$$\begin{tikzcd} \Sigma^\infty\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, shift left, "\Sigma^\infty g"] \arrow[d, "\wr"] & \Sigma^\infty\Omega^{n+1}\Sigma^{n+1}X \arrow[l, shift left, "\Sigma^\infty h"] \arrow[d, "\wr"] \\ \bigvee_{j\geqslant 1}\Sigma^\infty D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[d, shift left, "p'_j"] & \bigvee_{q\geqslant 1}\Sigma^\infty D_qX \arrow[d, shift left, "p_q"] \\ \Sigma^\infty D_j\Sigma^{-n}L_m^{\max}\Sigma^n X \arrow[u, shift left, "s'_j"] & \Sigma^\infty D_qX \arrow[u, shift left, "s_q"] \end{tikzcd} \tag{5}$$

where p'_q and p_q are the canonical projections to the q^{th} components, and s'_q and s_q are the canonical inclusions from the q^{th} component to the whole spaces.

Next, consider their induced maps on homology. Recall that

$$H_*\Sigma^\infty\Omega^{n+1}\Sigma^{n+1}X\cong H_*\Omega^{n+1}\Sigma^{n+1}X\cong\bigoplus_{q=1}^{\infty}H_*D_qX\cong\bigoplus_{q=1}^{\infty}H_*\Sigma^\infty D_qX$$

and

$$\overline{H}_* D_q X = E_q W_n H_* X.$$

Hence $(p_q)_*$ is isomorphic to the canonical projection from the direct sum to the q^{th} summand, and $(s_q)_*$ is isomorphic to the canonical inclusion from the q^{th} summand to the whole direct sum. That is,

$$\bigoplus_{q=1}^{\infty} E_q W_n H_* X \underset{(s_q)_*}{\overset{(p_q)_*}{\rightleftarrows}} E_q W_n H_* X.$$

Thus, we obtain a diagram of homology:

$$\begin{tikzcd}
W_n H_*(\Sigma^{-n} L_m^{\max} \Sigma^n X) \arrow[r, shift left, "(\Sigma^\infty g)_*"] \arrow[d, "\cong"] & W_n H_* X \arrow[l, shift left, "(\Sigma^\infty h)_*"] \arrow[d, "\cong"] \\
\bigoplus_{j \geqslant 1} E_j W_n H_*(\Sigma^{-n} L_m^{\max} \Sigma^n X) \arrow[d, shift left, "(p'_j)_*"] & \bigoplus_{q \geqslant 1} E_q W_n H_* X \arrow[d, shift left, "(p_q)_*"] \\
E_j W_n H_* \Sigma^{-n} L_m^{\max} \Sigma^n X \arrow[u, shift left, "(s'_j)_*"] & E_q W_n H_* X \arrow[u, shift left, "(s_q)_*"]
\end{tikzcd}$$

Now the claim below is obvious:

$$(s_h)_* \circ (p_h)_*|_{\overline{H}_* D_q X} = \begin{cases} 0 & \text{if } h \neq q, \\ \mathrm{id}_{\overline{H}_* D_q X} & \text{if } h = q. \end{cases}$$

Let us consider the composition

$$E_j W_n H_* \Sigma^{-n} L_m^{\max} \Sigma^n X \xrightarrow{(s'_j)_*} W_n H_*(\Sigma^{-n} L_m^{\max} \Sigma^n X) \xrightarrow{(\Sigma^\infty g)_*} W_n H_* X.$$

An element of $E_j W_n H_* \Sigma^{-n} L_m^{\max} \Sigma^n X$ can be written as

$$Q^{I_1} y_1(z_1, \dots, z_m) \cdot Q^{I_2} y_2(z_1, \dots, z_m) \cdots Q^{I_k} y_k(z_1, \dots, z_m),$$

where $y_i(z_1, \dots, z_m)$ $(1 \leqslant i \leqslant k)$ are basic λ_n–products formed by $z_1, \dots, z_m$ for $z_i \in H_* \Sigma^{-n} L_m^{\max} \Sigma^n X$, and the product $Q^{I_1} y_1 \cdots Q^{I_k} y_k$ is a homology class of $H_* \Omega^{n+1} \Sigma^{n+1} X$ of weight j. That g is an Ω^{n+1}–map implies that $g_* Q^I = Q^I g_*$ and $g_*[x, y]_n = [g_* x, g_* y]_n$. Thus,

$$g_*(Q^{I_i} y_i(z_1, \dots, z_m)) = Q^{I_i} y_i(g_* z_1, \dots, g_* z_m).$$

By Lemma 4.1, for an element $s^{-n}[s^n x_1, \dots, s^n x_m]_0$ in $H_* \Sigma^{-n} L_m^{\max} \Sigma^n X$, with $x_i \in \overline{H}_* X$, we have

$$g_*(s^{-n}[s^n x_1, \dots, s^n x_m]_0) = [x_1, \dots, x_m]_n.$$

It follows that g_*z_i is of weight m for each element $z_i \in H_*\Sigma^{-n}L_m^{\max}\Sigma^n X$. Thus, the weight of $Q^{I_i}y_i(g_*z_1,\dots,g_*z_m)$ is equal to the weight of $Q^{I_i}y_i$ multiplied by m. Finally,

$$((\Sigma^\infty g)_*\circ(s'_j)_*)(E_jW_nH_*\Sigma^{-n}L_m^{\max}\Sigma^n X)\subseteq E_{jm}W_nH_*X.$$

Now let $\phi_{j,q}=p_q\circ\Sigma^\infty g\circ s'_j$ and $\psi_{j,q}=p'_j\circ\Sigma^\infty h\circ s_q$. We can obtain that

$$(\psi_{j,q})_*\circ(\phi_{j,q})_*=(p'_j)_*\circ(\Sigma^\infty h)_*\circ((s_q)_*\circ(p_q)_*)\circ(\Sigma^\infty g)_*\circ(s'_j)_*.$$

Since

$$\operatorname{Im}((\Sigma^\infty g)_*\circ(s'_j)_*)\subseteq E_{jm}W_nH_*X,$$

we have:

(1) If $q\neq jm$, then
$$(s_q)_*\circ(p_q)_*|_{E_{jm}H_*X}=0.$$
Thus $(\psi_{j,q})_*\circ(\phi_{j,q})_*=0$.

(2) If $q=jm$, then
$$(s_q)_*\circ(p_q)_*|_{E_{jm}H_*X}=\mathrm{id}.$$
Thus $(\psi_{j,q})_*\circ(\phi_{j,q})_*=(p'_j)_*\circ(\Sigma^\infty h)_*\circ(\Sigma^\infty g)_*\circ(s'_j)_*=\mathrm{id}$.

Let $\phi=\phi_{j,jm}$ and $\psi=\psi_{j,jm}$. The discussion above implies that

$$\psi_*\circ\phi_*=\mathrm{id}.$$

This completes the proof of step one.

In step two, it will be proved that the stable maps ϕ and ψ can be induced from unstable maps. Recall diagram (5).

There are an integer t_1 and a map

$$\overline{p}_q\colon \Sigma^{t_1}\Omega^{n+1}\Sigma^{n+1}X\to\Sigma^{t_1}D_qX$$

such that

$$\Sigma^\infty\overline{p}_q\colon \Sigma^\infty\Omega^{n+1}\Sigma^{n+1}X\to\Sigma^\infty D_qX$$

is homotopic to the map p_q [4, Theorem 7.1]. Similarly, we have a map

$$\overline{s}_q\colon \Sigma^{t_2}D_qX\to\Sigma^{t_2}\Omega^{n+1}\Sigma^{n+1}X$$

for some integer t_2. This map induces the stable map s_q. Similarly, we can obtain maps $\overline{p}'_j$ and $\overline{s}'_j$ inducing maps p'_j and s'_j for integers t_3 and t_4, respectively:

$$\overline{p}'_j\colon \Sigma^{t_3}\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^nX)\to\Sigma^{t_3}D_j(\Sigma^{-n}L_m^{\max}\Sigma^nX),$$
$$\overline{s}'_j\colon \Sigma^{t_4}D_j(\Sigma^{-n}L_m^{\max}\Sigma^nX)\to\Sigma^{t_4}\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^nX).$$

Let $t = \max\{t_1, t_2, t_3, t_4\}$. There are four maps $\overline{p}_j$, $\overline{s}_j$, $\overline{p}'_j$ and $\overline{s}'_j$ up to Σ^t. For simplicity, we still denote them by $\overline{p}_j$, $\overline{s}_j$, $\overline{p}'_j$ and $\overline{s}'_j$. Then there is a diagram:

$$\begin{tikzcd}
\Sigma^t\Omega^{n+1}\Sigma^{n+1}(\Sigma^{-n}L_m^{\max}\Sigma^n X) \arrow[r, shift left, "\Sigma^t g"] \arrow[d, shift left, "\overline{p}'_j"] & \Sigma^t\Omega^{n+1}\Sigma^{n+1}X \arrow[l, shift left, "\Sigma^t h"] \arrow[d, shift left, "\overline{p}_q"] \\
\Sigma^t D_j\Sigma^{-n}L_m^{\max}\Sigma^n X \arrow[u, shift left, "\overline{s}'_j"] & \Sigma^t D_q X \arrow[u, shift left, "\overline{s}_q"]
\end{tikzcd}$$

Define two maps $\overline{\phi}$ and $\overline{\psi}$ as follows:

$$\overline{\phi} = \overline{p}_{jm}\circ\Sigma^t g\circ\overline{s}'_j\colon \Sigma^t D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X)\to\Sigma^t D_{jm}X,$$
$$\overline{\psi} = \overline{p}'_j\circ\Sigma^t h\circ\overline{s}_{jm}\colon \Sigma^t D_{jm}X\to\Sigma^t D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X).$$

The map $\overline{\psi}\circ\overline{\phi}$ induces an identity on the homology:

$$(\overline{\psi})_*\circ(\overline{\phi})_* = (\Sigma^\infty\overline{\psi})_*\circ(\Sigma^\infty\overline{\phi})_* = \psi_*\circ\phi_* = \mathrm{id}.$$

By the Whitehead theorem, we have $\overline{\psi}\circ\overline{\phi}$ is a homotopy equivalence. It follows that

$$(\overline{\psi}\circ\overline{\phi})^{-1}\circ\overline{\psi}\circ\overline{\phi}\simeq\mathrm{id}.$$

The maps

$$\overline{\phi}\colon \Sigma^t D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X)\to\Sigma^t D_{jm}X,$$
$$(\overline{\psi}\circ\overline{\phi})^{-1}\circ\overline{\psi}\colon \Sigma^t D_{jm}X\to\Sigma^t D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X),$$

imply that $\Sigma^t D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X)$ is a homotopy retract of $\Sigma^t D_{jm}X$. Note that we assume all spaces are CW–complexes, thus all constructions are natural up to homotopy. This completes the proof of step two. □

From the proof, we can obtain a corollary for the stable case.

Corollary 5.1 *Let X be a 1–connected p–local suspension of finite type. For the natural coalgebra-split sub-Hopf algebra $T(L_m^{\max}(V))$, the spectrum $\Sigma^\infty D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X)$ is a functorial stable homotopy retract of $\Sigma^\infty D_{jm}(X)$. In other words, there are maps*

$$\Sigma^\infty D_j(\Sigma^{-n}L_m^{\max}\Sigma^n X) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \Sigma^\infty D_{jm}X \qquad \text{such that } \psi\circ\phi\simeq\mathrm{id}.$$

6 Example

Let X be a p–local 2–cell complex. Denote the Steenrod algebra by A. Let $V = \overline{H}_*(X;\mathbb{Z}/p)$. Assume that there are two generators u and v in V such that $P^1_* v = u$, where P^1_* is the dual operation of Steenrod operation P^1. Furthermore, assume that the degrees of u and v are both odd; denote them by $|u|$ and $|v|$, respectively.

Recall $\Sigma^{-1}L^{\max}_p\Sigma X$ is a stable functorial homotopy retract of D_pX. Thus, we have a stable functorial homotopy decomposition

$$D_pX \overset{s}{\simeq} (\Sigma^{-1}L^{\max}_p\Sigma X) \vee M_pX.$$

In the following, the homology of this decomposition and the A–module structure of each piece for $p=5$ will be computed.

6.1 Additive basis

In $H_*\Omega^2\Sigma^2X$, denote the 1–bracket (of Browder operation) $[x_1,\dots,[x_{m-1},x_m]_1,]_1$ by $[x_1,\dots,x_m]_1$. The basic 1–bracket (ie basic λ_1–product) with weight no greater than 5 are

$$\begin{aligned} u<v<[u,v]_1&<[u,u,v]_1<[v,u,v]_1<[u,u,u,v]_1<[v,u,u,v]_1<[v,v,u,v]_1\\ &<[u,u,u,u,v]_1<[v,u,u,u,v]_1<[v,v,u,u,v]_1<[v,v,v,u,v]_1\\ &<[[u,v]_1,[u,u,v]_1]_1<[[u,v]_1,[v,u,v]_1]_1.\end{aligned}$$

Since $|u|$ and $|v|$ are odd, $[u,u]_1$ and $[v,v]_1$ are trivial. All the basic 1–products above are of odd degrees.

Recalling Proposition 2.6, we have the following additive basis of $\overline{H}_*D_pX$:

$$\begin{aligned} (6)\qquad & u\cdot[u,u,u,v]_1,\ u\cdot[v,u,u,v]_1,\ u\cdot[v,v,u,v]_1,\ v\cdot[u,u,u,v]_1,\ v\cdot[v,u,u,v]_1,\\ & v\cdot[v,v,u,v]_1,\ [u,v]_1\cdot[u,u,v]_1,\ [u,v]_1\cdot[v,u,v]_1,\ u\cdot v\cdot[u,u,v]_1,\ u\cdot v\cdot[v,u,v]_1,\\ & [u,u,u,u,v]_1,\ [v,u,u,u,v]_1,\ [v,v,u,u,v]_1,\ [v,v,v,u,v]_1,\\ & [[u,v]_1,[u,u,v]_1]_1,\ [[u,v]_1,[v,u,v]_1]_1,\ \zeta_1u,\ \zeta_1v,\ \xi_1u,\ \xi_1v.\end{aligned}$$

In $H_*\Omega^2\Sigma^2X$, the first two rows of this basis are decomposable. The others are indecomposable.

6.2 Module structures over the Steenrod algebra

Let $P^r_*\colon H_*X\to H_{*-2r(p-1)}X$ be the dual operation of the Steenrod operation P^r. We have a right A–module structure on $\overline{H}_*D_5X$. For convenience, we still write the Steenrod operation P^r_* on the left.

There is a new additive basis of $\overline{H}_*D_5X$ which is invariant under Steenrod operations (see [13, Proposition 5.2]):

$$
\begin{gathered}
u\cdot[u,u,u,v]_1,\ \ u\cdot[v,u,u,v]_1,\ \ u\cdot[v,v,u,v]_1,\ \ 2v\cdot[u,u,u,v]_1-u\cdot[v,u,u,v]_1,\\
2v\cdot[v,u,u,v]_1+u\cdot[v,v,u,v]_1,\ \ v\cdot[v,v,u,v]_1,\ \ [u,v]_1\cdot[u,u,v]_1,\\
[u,v]_1\cdot[v,u,v]_1,\ u\cdot v\cdot[u,u,v]_1,\ \ u\cdot v\cdot[v,u,v]_1,\ \ -[u,u,u,u,v]_1,\\
-[v,u,u,u,v]_1+[[u,v]_1,[u,u,v]_1]_1,\ \ 2[v,v,u,u,v]_1-[[u,v]_1,[v,u,v]_1]_1,\\
-[v,v,v,u,v]_1,\ \ [[u,v]_1,[u,u,v]_1]_1,\ \ [[u,v]_1,[v,u,v]_1]_1,\\
\zeta_1 u,\ \ \zeta_1 v,\ \ \xi_1 u,\ \ \xi_1 v.
\end{gathered}
\tag{7}
$$

For $x \in \overline{H}_* D_p X$, let $A\langle x\rangle$ be the right A–module generated by x. Define A–modules M_i for $1 \leqslant i \leqslant 5$ as follows:

(1) $M_1 = A\langle [[u,v]_1,[v,u,v]_1]_1\rangle$, with

$$[[u,v]_1,[v,u,v]_1]_1 \xrightarrow{P^1_*} [[u,v]_1,[u,u,v]_1]_1.$$

(2) $M_2 = A\langle u\cdot v\cdot[v,u,v]_1\rangle$, with

$$u\cdot v\cdot[v,u,v]_1 \xrightarrow{P^1_*} u\cdot v\cdot[u,u,v]_1.$$

(3) $M_3 = A\langle [u,v]_1\cdot[v,u,v]_1\rangle$, with

$$[u,v]_1\cdot[v,u,v]_1 \xrightarrow{P^1_*} [u,v]_1\cdot[u,u,v]_1.$$

(4) $M_4 = A\langle \xi_1 v\rangle$. The diagram shows the additive basis of M_4:

$$
\begin{tikzcd}
\xi_1 v \arrow[r, "\beta"] \arrow[d, "P^1_*"] \arrow[ddddd, bend left=80, "P^5_*"] & \zeta_1 v \arrow[ddddd, "P^5_*"] \\
-[v,v,v,u,v]_1 \arrow[d, "P^1_*"] & \\
2[v,v,u,u,v]_1-[[u,v]_1,[v,u,v]_1]_1 \arrow[d, "P^1_*"] & \\
-[v,u,u,u,v]_1+[[u,v]_1,[u,u,v]_1]_1 \arrow[d, "P^1_*"] & \\
-[u,u,u,u,v]_1 & \\
\xi_1 u \arrow[r, "\beta"] & \zeta_1 u
\end{tikzcd}
$$

(5) $M_5 = A\langle v\cdot[v,v,u,v]_1, u\cdot[v,v,u,v]_1\rangle$, with:

$$\begin{tikzcd}
v\cdot[v,v,u,v]_1 \arrow[d, "P^1_*"] & \\
2v\cdot[v,u,u,v]_1 + u\cdot[v,v,u,v]_1 \arrow[d, "P^1_*"] & u\cdot[v,v,u,v]_1 \arrow[d, "P^1_*"] \\
2v\cdot[u,u,u,v]_1 - u\cdot[v,u,u,v]_1 \arrow[d, "P^1_*"] & u\cdot[v,u,u,v]_1 \arrow[d, "P^1_*"] \\
u\cdot[u,u,u,v]_1 \arrow[r, equal] & u\cdot[u,u,u,v]_1
\end{tikzcd}$$

It is obvious that there is an isomorphism of right A–modules

$$\overline{H}_* D_5 X \cong M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5.$$

6.3 $\Sigma^{-1} L_p^{\max} \Sigma X$ and $M_p X$

$L_5^{\max}(V)$ has a basis $[[u,v],[u,u,v]]$, $[[u,v],[v,u,v]]$ [10, Proposition 11.6]. It follows from Lemma 4.1 that this basis is mapped by the map

$$i_*\colon \overline{H}_*\Sigma^{-1}L_p^{\max}\Sigma X \to \overline{H}_*\Omega^2\Sigma^2 X \quad \text{to} \quad [[u,v]_1,[u,u,v]_1]_1,\ \ [[u,v]_1,[v,u,v]_1]_1.$$

Thus we can obtain the homology of $\Sigma^{-1}L_5^{\max}\Sigma X$ and $M_5 X$. The following equations are isomorphisms of right A–modules:

$$\overline{H}_*(\Sigma^{-1}L_5^{\max}\Sigma X) \cong M_1, \quad \overline{H}_* M_5 X \cong M_2 \oplus M_3 \oplus M_4 \oplus M_5.$$

Remark As a right A–module, $\overline{H}_* M_p X$ is splittable, so it is natural to ask whether $M_p X$ is splittable as a topological space, particularly whether the functorial homotopy decomposition exists or not.

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