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Representations of the Kauffman bracket skein algebra II: Punctured surfaces

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In part I, we constructed invariants of irreducible finite-dimensional representations of the Kauffman bracket skein algebra of a surface. We introduce here an inverse construction, which to a set of possible invariants associates an irreducible representation that realizes these invariants. The current article is restricted to surfaces with at least one puncture, a condition that is lifted in subsequent work relying on this one. A step in the proof is of independent interest, and describes the arithmetic structure of the Thurston intersection form on the space of integer weight systems for a train track.

57M27, 57R56; 57M27

This article is a continuation of [9] and is part of the program described in Bonahon and Wong [6], devoted to the analysis and construction of finite-dimensional representations of the Kauffman bracket skein algebra of a surface.

Let S be an oriented surface of finite topological type without boundary. The *Kauffman bracket skein algebra* $\mathcal{S}^A(S)$ depends on a parameter $A = \mathrm{e}^{\pi i \hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: One first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0,1]$, and then one takes the quotient of this space by two relations. The first and main relation is the *skein relation*, which states that

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

whenever the three links K_1, K_0 and $K_\infty \subset S \times [0,1]$ differ only in a little ball where they are as represented in Figure 1. The second relation is the *trivial knot relation*, which asserts that

$$[K \cup O] = -(A^2 + A^{-2})[K]$$

whenever O is the boundary of a disk $D \subset K \times [0,1]$ disjoint from K, and is endowed with a framing transverse to D. The algebra multiplication is provided by the operation of superposition, for which the product $[K] \cdot [L]$ is represented by the union $[K' \cup L']$ where $K' \subset S \times \left[0, \frac{1}{2}\right]$ and $L' \subset S \times \left[\frac{1}{2}, 1\right]$ are respectively obtained by rescaling the framed links $K \subset S \times [0,1]$ and $L' \subset S \times [0,1]$ in the $[0,1]$ direction.

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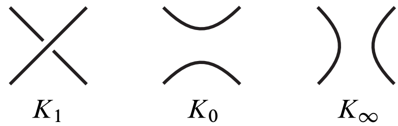

Figure 1: A Kauffman triple

Turaev [33], Bullock, Frohman and Kania-Bartoszyńska [14; 15] and Przytycki and Sikora [29] showed that the skein algebra $\mathcal{S}^A(S)$ provides a quantization of the *character variety*

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\} /\!\!/ \mathrm{SL}_2(\mathbb{C}),$$

where $\mathrm{SL}_2(\mathbb{C})$ acts on homomorphisms by conjugation, and where the double bar indicates that the quotient is to be taken in the sense of geometric invariant theory; see Mumford, Fogarty and Kirwan [27]. In fact, if one follows the physical tradition that a quantization of a space X replaces the commutative algebra of functions on X by a noncommutative algebra of operators on a Hilbert space, an element of the quantization should be a *representation* of the skein algebra.

When A is a root of unity, a classical example of a finite-dimensional representation of the skein algebra $\mathcal{S}^A(S)$ arises from the Witten–Reshetikhin–Turaev topological quantum field theory associated to the fundamental representation of the quantum group $U_q(\mathfrak{sl}_2)$; see Blanchet, Habegger, Masbaum and Vogel [3], Bonahon and Wong [8], Reshetikhin and Turaev [30], Turaev [34] and Witten [35]. The main purpose of the current article is to provide a wider family of such representations when the surface S has at least one puncture. The case of closed surfaces is considered in our subsequent article [10], which builds on this one.

In part I [9], we identified invariants for irreducible finite-dimensional representations $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ in the case where A^2 is a primitive N^{th} root of unity with N odd. These invariants are a little easier to describe when $A^N = -1$, and most of the current article will be devoted to this case. We indicate in Section 6 how the other possible case when $A^N = +1$ can be deduced from this one. Because N is odd, the property that A^2 is a primitive N^{th} root of unity with $A^N = -1$ is equivalent to the property that A is a primitive N^{th} root of -1.

When $A^N = -1$, our main invariant is a point of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. Its definition involves the n^{th} normalized Chebyshev polynomial $T_n(x)$ of the first kind, determined by the trigonometric identity that $2\cos n\theta = T_n(2\cos\theta)$. Equivalently, $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$ for every matrix $M \in \mathrm{SL}_2(\mathbb{C})$.

A character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ associates a trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$ to each closed curve K on the surface S. This trace is independent of the homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$

used to represent r, and of the representative chosen in the conjugacy class of $\pi_1(S)$ representing K. In fact, the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is defined in such a way that two homomorphisms $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ correspond to the same character if and only if they induce the same trace function $K \mapsto \operatorname{Tr} r(K)$.

Theorem 1 (Bonahon and Wong [9]) *Suppose that A is a primitive N^{th} root of -1 with N odd, and let $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ be an irreducible finite-dimensional representation of the Kauffman bracket skein algebra. Let $T_N(x)$ be the N^{th} normalized Chebyshev polynomial of the first kind.*

(1) *There exists a unique character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that*

$$T_N(\rho([K])) = -(\operatorname{Tr} r_\rho(K)) \operatorname{Id}_E$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

(2) *Let P_k be a small simple loop going around the k^{th} puncture of S, and consider it as a knot in $S \times [0,1]$ with vertical framing. Then there exists a number $p_k \in \mathbb{C}$ such that $\rho([P_k]) = p_k \operatorname{Id}_E$.*

(3) *The number p_k of (2) is related to the character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ of (1) by the property that $T_N(p_k) = -\operatorname{Tr} r_\rho(P_k)$.*

The character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ associated to the irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ by part (1) of Theorem 1 is the *classical shadow* of ρ. The numbers p_k defined by part (2) are the *puncture invariants* of the representation ρ. Part (3) shows that, once the classical shadow r_ρ is known, there are at most N possible values for each of the puncture invariants p_k.

The classical shadow provides one more example of a situation where a quantum object determines one of the classical objects that are being quantized. See also Lê [25] for another approach to the key results underlying Theorem 1.

The main result of this article is the following converse statement.

Theorem 2 *Assume that the surface S has at least one puncture, that its Euler characteristic is negative, that A is a primitive N^{th} root of -1 with N odd, and that we have*

(1) *a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ which realizes some ideal triangulation of S in the sense discussed in Section 3;*

(2) *for each puncture of S, a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = -\operatorname{Tr} r(P_k)$, where as in Theorem 1, P_k is a small loop going around the puncture.*

Then there exists an irreducible finite-dimensional representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ whose classical shadow is equal to r and whose puncture invariants are the p_k.

The requirement that r realizes some ideal representation is fairly mild. It can be shown to be satisfied by all points outside of an algebraic subset of complex codimension $2|\chi(S)|-1$ in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

The sequel [10; 11] to this paper greatly improves Theorem 2. In particular, it removes the requirements that r realizes an ideal triangulation, and that S has at least one puncture. It also shows that the representation provided by our construction is independent of the many choices made during the argument, so that its output is natural, in particular with respect to the action of the mapping class group $\pi_0\mathrm{Diff}(S)$ of the surface. The constructions and results of the current article are a key ingredient in the proofs of [10; 11].

The proof of Theorem 2 uses as a fundamental tool the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S)\to\mathcal{T}^\omega(\lambda)$, constructed in Bonahon and Wong [7], which embeds the skein algebra in the quantum Teichmüller space. The *quantum Teichmüller space* is here incarnated as the Chekhov–Fock algebra $\mathcal{T}^\omega(\lambda)$ of an ideal triangulation λ of the surface, and is a quantization of an object that is closely related to the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. It is not as natural as the Kauffman bracket skein algebra, but its algebraic structure is very simple. In particular, its representation theory is relatively easy to analyze; see Bonahon and Liu [5]. The same holds for a smaller algebra $\mathcal{Z}^\omega(\lambda)\subset\mathcal{T}^\omega(\lambda)$ containing the image of the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda$. Composing representations of $\mathcal{Z}^\omega(\lambda)$ with the homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S)\to\mathcal{Z}^\omega(\lambda)$ provides an extensive family of representations of the skein algebra $\mathcal{S}^A(S)$, which can then be used to prove Theorem 2.

The main technical challenge in this strategy is to compute the classical shadow of the representations of $\mathcal{S}^A(S)$ so obtained, in terms of the parameters controlling the original representations of $\mathcal{Z}^\omega(\lambda)$. This is provided by the miraculous cancellations discovered in [9]. These properties show that the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S)\to\mathcal{Z}^\omega(\lambda)$ is well behaved with respect to the Chebyshev homomorphism $\mathcal{S}^{-1}(S)\to\mathcal{S}^A(S)$ used to define the classical shadow of a representation of $\mathcal{S}^A(S)$, and with respect to the Frobenius homomorphism $\mathcal{Z}^\iota(\lambda)\to\mathcal{Z}^\omega(\lambda)$ which computes the invariants of representations of $\mathcal{Z}^\omega(\lambda)$.

One of the steps in the proof, used to determine the algebraic structure of the algebra $\mathcal{Z}^\omega(\lambda)$, may be of interest by itself. This statement describes the structure of the Thurston intersection form on the set $\mathcal{W}(\tau;\mathbb{Z})$ of integer-valued edge weight systems for a train track τ. The result is well known for real-valued weights. However, the integer-valued case has subtler number-theoretic properties, resulting in the unexpected simultaneous occurrence of blocks $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&2\\-2&0\end{smallmatrix}\right)$ in the block diagonalization of the Thurston form. See Theorem 26 in the appendix. Because of the ubiquity of the

Thurston intersection form in many geometric problems (for instance, the relationship between complex lengths and the shear-bend cocycle $\beta \in \mathcal{W}(\tau; \mathbb{C}/2\pi i\mathbb{Z})$ of a pleated surface; see Bonahon [4]), this statement is probably of interest beyond the quantum topology scope of the current article.

Recent works of Abdiel and Frohman [1; 20], Frohman and Kania-Bartoszyńska [21], and Frohman, Kania-Bartoszyńska and Lê [22] develop another construction of representations of $\mathcal{S}^A(S)$ with a given classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, valid for r in a Zariski dense open subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. In particular, the recent preprint [22] abstractly shows that these representations are isomorphic to ours. It would be interesting to compare the two approaches, as the construction pioneered by Abdiel and Frohman in [1; 20] is simple and elegant while ours is more explicit. See also Takenov [31] for an earlier viewpoint on the case of small surfaces.

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1 The Chekhov–Fock algebra and the quantum trace homomorphism

1.1 The Chekhov–Fock algebra

The Chekhov–Fock algebra (introduced in [5] as a reinterpretation of key insights from [19; 17; 18]) is the avatar of the quantum Teichmüller space associated to an ideal triangulation of the surface S. See also [24] for a related construction, and [5; 26] for more discussion.

If S is obtained from a compact surface $\overline{S}$ by removing finitely many points $v_1, \dots, v_s$, an *ideal triangulation* of S is a triangulation λ of $\overline{S}$ whose vertex set is exactly $\{v_1, v_2, \dots, v_s\}$. The surface S admits an ideal triangulation if and only if it is noncompact and if its Euler characteristic is negative; we will consequently assume that these properties are satisfied throughout the article. If the surface has genus g and s punctures, an ideal triangulation then has $n = 6g+3s-6$ edges and $4g+2s-4$ faces.

Let $e_1, e_2, \dots, e_n$ denote the edges of λ. Let $a_i \in \{0,1,2\}$ be the number of times an end of the edge e_j immediately succeeds an end of e_i when going counterclockwise around a puncture of S, and set $\sigma_{ij} = a_{ij} - a_{ji} \in \{-2,-1,0,1,2\}$. The *Chekhov–Fock algebra* $\mathcal{T}^\omega(\lambda)$ of λ is the algebra defined by generators $Z_1^{\pm 1}, Z_2^{\pm 1}, \dots, Z_n^{\pm 1}$ associated to the edges $e_1, e_2, \dots, e_n$ of λ, and by the relations

$$Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i.$$

Remark 3 The actual Chekhov–Fock algebra $\mathcal{T}^q(\lambda)$ that is at the basis of the quantum Teichmüller space uses the constant $q = \omega^4$ instead of ω. The generators Z_i of $\mathcal{T}^\omega(\lambda)$ appearing here are designed to model square roots of the original generators of $\mathcal{T}^q(\lambda)$.

An element of the Chekhov–Fock algebra $\mathcal{T}^\omega(\lambda)$ is a linear combination of monomials $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \cdots Z_{i_l}^{n_l}$ in the generators Z_i, with $n_1, n_2, \dots, n_l \in \mathbb{Z}$. Because of the skew-commutativity relation $Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i$, the order of the variables in such a monomial does matter. It is convenient to use the following symmetrization trick. The *Weyl quantum ordering* for $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \cdots Z_{i_l}^{n_l}$ is the monomial

$$[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \cdots Z_{i_l}^{n_l}] = \omega^{-\sum_{u<v} n_u n_v \sigma_{i_u i_v}} Z_{i_1}^{n_1} Z_{i_2}^{n_2} \cdots Z_{i_l}^{n_l}.$$

The formula is specially designed so that $[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \cdots Z_{i_l}^{n_l}]$ is invariant under any permutation of the $Z_{i_u}^{n_u}$. Note that the algebraic structure of the Chekhov–Fock algebra $\mathcal{T}^\omega(\lambda)$ depends only on the square ω^2, but that the Weyl quantum ordering depends on the choice of ω.

1.2 The quantum trace homomorphism

Theorem 4 [7] *For $A = \omega^{-2}$, there exists an injective algebra homomorphism*

$$\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S) \to \mathcal{T}^\omega(\lambda).$$

The specific homomorphism $\mathrm{Tr}_\lambda^\omega$ constructed in [7] is the *quantum trace homomorphism.* It is uniquely determined by certain properties stated in that article, but for now we need only use that it exists and satisfies the properties given in Section 1.3 below.

1.3 The Chebyshev and Frobenius homomorphisms

We now assume that A is a primitive N^{th} root of -1 with N odd. Recall that T_N denotes the N^{th} normalized Chebyshev polynomial, defined by the property that $\cos N\theta = \frac{1}{2} T_N(2\cos\theta)$ for every θ.

Theorem 5 [9] *When A is a primitive N^{th} root of -1 with N odd, there is a unique algebra homomorphism $\boldsymbol{T}^A\colon \mathcal{S}^{-1}(S) \to \mathcal{S}^A(S)$ such that*

$$\boldsymbol{T}^A([K]) = T_N([K])$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical. In addition, the image of $\boldsymbol{T}^A$ is central in $\mathcal{S}^A(S)$. □

For a framed link $K \subset S \times [0,1]$ whose projection to S is allowed to have crossings, the image $\boldsymbol{T}^A([K])$ is equal to the element $[K^{T_N}] \in \mathcal{S}^A(S)$ defined by threading the Chebyshev polynomial T_N along all components of K; see [9] for a precise definition.

The homomorphism $\boldsymbol{T}^A$ provided by Theorem 5 is the *Chebyshev homomorphism*. It is a key ingredient in the definition of the invariants of Theorem 1.

There is an analogous and much simpler homomorphism at the level of the Chekhov–Fock algebra, namely the following *Frobenius homomorphism*.

Proposition 6 *If $\iota = \omega^{N^2}$, there is an algebra homomorphism*

$$\boldsymbol{F}^\omega\colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$$

which maps each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ to $Z_i^N \in \mathcal{T}^\omega(\lambda)$, where in the first instance $Z_i \in \mathcal{T}^\iota(\lambda)$ denotes the generator associated to the i^{th} edge e_i of λ, whereas the second time $Z_i \in \mathcal{T}^\omega(\lambda)$ denotes the generator of $\mathcal{T}^\omega(\lambda)$ associated to the same edge e_i. □

Note that $\iota^2 = \omega^{2N^2} = A^{-N^2} = (-1)^N = -1$, so $\iota = \pm \mathrm{i}$.

The following compatibility statement, which connects the Chebyshev homomorphism to the Frobenius homomorphism through appropriate quantum trace homomorphisms, is fundamental for our arguments. This result encapsulates the miraculous cancellations of [9].

Theorem 7 [9] *The diagram*

$$\begin{array}{ccc} \mathcal{S}^A(S) & \xrightarrow{\mathrm{Tr}_\lambda^\omega} & \mathcal{T}^\omega(\lambda) \\ \uparrow \boldsymbol{T}^A & & \uparrow \boldsymbol{F}^\omega \\ \mathcal{S}^{-1}(S) & \xrightarrow{\mathrm{Tr}_\lambda^\iota} & \mathcal{T}^\iota(\lambda) \end{array}$$

is commutative. Namely, for every skein $[K] \in \mathcal{S}^{-1}(S)$, the quantum trace $\mathrm{Tr}_\lambda^\omega(\boldsymbol{T}^A([K]))$ of $\boldsymbol{T}^A([K])$ is obtained from the classical trace polynomial $\mathrm{Tr}_\lambda^\iota([K])$ by replacing each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ by $Z_i^N \in \mathcal{T}^\omega(\lambda)$. □

2 The balanced Chekhov–Fock algebra

2.1 Definition of the balanced Chekhov–Fock algebra

The quantum trace homomorphism $\mathrm{Tr}_\lambda^\omega$ of Theorem 4 (and [7]) is far from being surjective. Indeed, for a skein $[K] \in \mathcal{S}^A(S)$ represented by a framed link $K \subset S\times[0,1]$, the exponents of the monomials $Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}$ appearing in the expression of $\mathrm{Tr}_\lambda^\omega([K])$ are *balanced*, in the sense that they satisfy the following parity condition: for every triangle T_j of the ideal triangulation λ, the sum $k_{i_1} + k_{i_2} + k_{i_3}$ of the exponents of the generators $Z_{i_1}, Z_{i_2}, Z_{i_3}$ associated to the sides of T_j is even.

Let $\mathcal{Z}^\omega(\lambda)$ denote the subalgebra of $\mathcal{T}^\omega(\lambda)$ generated by all monomials satisfying this exponent parity condition. By definition, $\mathcal{Z}^\omega(\lambda)$ is the *balanced Chekhov–Fock algebra* of the ideal triangulation λ. It is designed so that the quantum trace homomorphism restricts to a homomorphism $\mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$.

To keep track of the exponent parity condition defining the monomials of $\mathcal{Z}^\omega(\lambda)$, it is convenient to consider a train track τ_λ which, on each triangle T_j of the ideal triangulation λ, looks as in Figure 2. In particular, τ_λ has one switch for each edge of λ, and three edges for each triangle of λ. Let $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ be the set of integer edge weight systems α for τ_λ, assigning a number $\alpha(e)\in\mathbb{Z}$ to each edge e of τ_λ in such a way that, at each switch, the weights of the edges incoming on one side add up to the sum of the weights of the edges outgoing on the other side.

There is a natural map $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z}^n$ which, given an edge weight system, associates to each of the n switches of τ_λ the sum of the weights of the edges incoming on any side of the switch. Then an element $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ is in the image of this map if and only if it satisfies the parity condition defining the monomials of $\mathcal{Z}^\omega(\lambda)$, namely if and only if the sum of the coordinates associated to the sides of each triangle of λ is even. Also, the map $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z}^n$ is easily seen to be injective. Since the image of this map has finite index, it follows that $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^n$ as an abelian group.

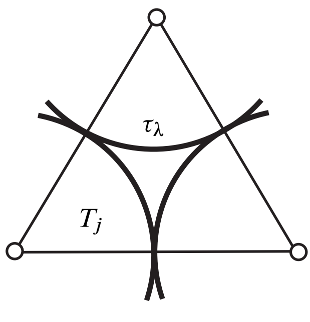

Figure 2

This enables us to give a different description of $\mathcal{Z}^\omega(\lambda)$. For a weight system $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, which assigns a weight $\alpha_i \in \mathbb{Z}$ to the i^{th} edge e_i of λ (= the i^{th} switch of τ_λ), define

$$Z_\alpha = [Z_1^{\alpha_1} Z_2^{\alpha_2} \cdots Z_n^{\alpha_n}] \in \mathcal{Z}^\omega(\lambda),$$

where the bracket [ ] denotes the Weyl quantum ordering defined in Section 1.1.

The above discussion proves the following fact.

Lemma 8 *As $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ ranges over all weight systems for the train track τ_λ, the associated monomials Z_α form a basis for the vector space $\mathcal{Z}^\omega(\lambda)$.* □

We can elaborate a little on the structure of the group $\mathcal{W}(\tau_\lambda;\mathbb{Z})$. By definition of the parity condition, $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \subset \mathbb{Z}^n$ contains the subset $(2\mathbb{Z})^n$ consisting of all switch weight systems $(\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{Z}^n$ where the α_i are even. Also, given $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, we can define a chain with coefficients in $\mathbb{Z}_2$ by endowing each edge e of the train track τ_λ with the modulo 2 reduction of the weight $\alpha(e) \in \mathbb{Z}$. The switch relations guarantee that this chain is closed, and this defines a natural homomorphism $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to H_1(S;\mathbb{Z}_2)$.

Lemma 9 *The inclusion map and homomorphism above define an exact sequence*

$$0 \to (2\mathbb{Z})^n \to \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to H_1(S;\mathbb{Z}_2) \to 0.$$

Proof The homomorphism $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to H_1(S;\mathbb{Z}_2)$ can also be expressed in terms of the dual graph Γ_λ of the triangulation λ. Indeed, the class $[\alpha] \in H_1(S;\mathbb{Z}_2)$ induced by $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ is also realized by endowing each edge f_i of Γ_λ with the modulo 2 reduction of the switch weight α_i associated by α to the edge e_i of λ that is dual to f_i; the parity condition guarantees that this chain is really closed. The result then immediately follows from the definitions, and from the isomorphism $H_1(\Gamma_\lambda;\mathbb{Z}_2) \cong H_1(S;\mathbb{Z}_2)$ coming from the fact that the surface S deformation retracts to the dual graph Γ_λ. □

Note that the exact sequence of Lemma 9 admits no partial splitting.

2.2 The algebraic structure of the balanced Chekhov–Fock algebra

We first describe the multiplicative structure of the balanced Chekhov–Fock algebra $\mathcal{Z}^\omega(\lambda)$ in the context of Lemma 8.

The weight system space $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ of the train track τ_λ carries a very natural antisymmetric bilinear form

$$\Omega\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \times \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z},$$

the *Thurston intersection form* defined by the property that, for $\alpha, \beta \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$,

$$\Omega(\alpha, \beta) = \frac{1}{2} \sum_{e \text{ right of } e'} (\alpha(e)\beta(e') - \alpha(e')\beta(e)),$$

where the sum is over all pairs (e, e') of edges of τ_λ such that e and e' come out of the same side of some switch of τ_λ, with e to the right of e'. See Lemma 28 in the appendix for a more conceptual interpretation of Ω, and for a proof that $\Omega(\alpha, \beta)$ is really an integer.

Lemma 10 *For every $\alpha, \beta \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$,*

$$Z_\alpha Z_\beta = \omega^{2\Omega(\alpha,\beta)} Z_{\alpha+\beta}.$$

In particular, $Z_\alpha Z_\beta = \omega^{4\Omega(\alpha,\beta)} Z_\beta Z_\alpha$.

Proof The second statement, that $Z_\alpha Z_\beta = \omega^{4\Omega(\alpha,\beta)} Z_\beta Z_\alpha$, is a simple computation. After observing that this property holds for any ω (not just roots of unity), the first statement, that $Z_\alpha Z_\beta = \omega^{2\Omega(\alpha,\beta)} Z_{\alpha+\beta}$, then follows by definition of the Weyl quantum ordering. □

This is particularly simple if we replace ω by $\iota = \omega^{N^2}$, with the assumption that $A^{2N} = 1$ so that $\iota^4 = \omega^{4N^2} = A^{-2N^2} = 1$.

Corollary 11 *If $\iota^4 = 1$, the algebra $\mathcal{Z}^\iota(\lambda)$ is commutative.* □

In general, the key to understanding the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ is Lemma 12.

For $k = 1, \dots, s$, the k^{th} puncture of S specifies an element $\eta_k \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$ defined as follows: for every edge e of the train track τ_λ, the edge weight $\eta_k(e)$ is equal to 1 if e is adjacent to the annulus component of $S - \tau_\lambda$ that surrounds this puncture, and is equal to 0 otherwise.

Recall that the surface S has genus g and s punctures.

Lemma 12 *The lattice $\mathcal{W}(\tau_\lambda; \mathbb{Z}) \cong \mathbb{Z}^n$ admits a basis in which the matrix of the Thurston intersection form Ω is block diagonal with g blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $2g + s - 3$ blocks $\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$ and s blocks (0). In addition, the kernel of Ω is freely generated by the elements $\eta_1, \eta_2, \dots, \eta_s \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$ associated to the punctures of S as above.*

Proof This is a special case of a result given by Theorem 26 in the appendix, which determines the algebraic structure of the Thurston intersection form for a general train track τ. When applying this result to the train track τ_λ, the numbers h, n_{even} and n_{odd} of Theorem 26 are respectively equal to the genus g of the surface S, to the number s of punctures of S, and to the number $4g+2s-4$ of triangles of the ideal triangulation λ. □

The combination of Lemmas 8, 10 and 12 now provides the complete algebraic structure of the balanced Chekhov–Fock algebra $\mathcal{Z}^\omega(\lambda)$. Let $\mathcal{W}^q$ denote the algebra, known as the quantum torus, defined by generators $X^{\pm 1}$, $Y^{\pm 1}$ and by the relation $XY = qYX$.

Corollary 13 *For $q=\omega^4$, the balanced Chekhov–Fock algebra $\mathcal{Z}^\omega(\lambda)$ is isomorphic to*

$$\mathcal{W}_1^q \otimes \mathcal{W}_2^q \otimes \cdots \otimes \mathcal{W}_g^q \otimes \mathcal{W}_{g+1}^{q^2} \otimes \mathcal{W}_{g+2}^{q^2} \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^{q^2} \otimes \mathbb{C}[H_1] \otimes \mathbb{C}[H_2] \otimes \cdots \otimes \mathbb{C}[H_s],$$

where each $\mathcal{W}_i^q$ is a copy of the quantum torus $\mathcal{W}^q$, each $\mathcal{W}_j^{q^2}$ is a copy of $\mathcal{W}^{q^2}$, and each $\mathbb{C}[H_k]$ is a polynomial algebra in the variable H_k.

In addition, the s central generators $H_k = Z_{\eta_k} \in \mathcal{Z}^\omega(\lambda)$ are naturally associated to the punctures of S, and are defined by the edge weight systems $\eta_k \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ generating the kernel of the Thurston intersection form Ω as in Lemma 12. □

2.3 Representations of the balanced Chekhov–Fock algebra

The algebraic structure of the balanced Chekhov–Fock algebra $\mathcal{Z}^\omega(\lambda)$ determined in Corollary 13 is relatively simple. This makes it easy to classify its irreducible finite-dimensional representations.

As usual, we assume that $A=\omega^{-2}$ is a primitive N^{th} root of -1, with N odd.

Proposition 14 *Let $\mu\colon \mathcal{Z}^\omega(\lambda) \to \operatorname{End}(E)$ be an irreducible finite-dimensional representation of $\mathcal{Z}^\omega(\lambda)$. There exists a map $\zeta_\mu\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ and numbers $h_k \in \mathbb{C}^*$, with $k=1,\dots,s$, associated to the punctures of the surface S such that*

(1) $\mu(Z_\alpha^N) = \zeta_\mu(\alpha)\,\mathrm{Id}_E$ *for every edge weight system $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ with associated monomial $Z_\alpha \in \mathcal{Z}^\omega(\lambda)$;*

(2) $\zeta_\mu(\alpha+\beta) = (-1)^{\Omega(\alpha,\beta)}\zeta_\mu(\alpha)\zeta_\mu(\beta)$ *for every $\alpha,\beta \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, where Ω is the Thurston intersection form;*

(3) $\mu(H_k) = h_k\,\mathrm{Id}_E$ *for the central element $H_k = Z_{\eta_k} \in \mathcal{Z}^\omega(\lambda)$ associated to the k^{th} puncture of S as in Corollary 13;*

(4) $\zeta_\mu(\eta_k) = h_k^N$ *for the weight system $\eta_k \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ associated to the k^{th} puncture of S as in Lemma 12.*

Proof For every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, Lemma 10 shows that the element $Z_\alpha^N = Z_{N\alpha}$ is central in $\mathcal{Z}^\omega(\lambda)$. In particular, if $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ is an irreducible finite-dimensional representation of $\mathcal{Z}^\omega(\lambda)$, there is a number $\zeta_\mu(\alpha) \in \mathbb{C}^*$ such that $\mu(Z_\alpha^N) = \zeta_\mu(\alpha)\,\mathrm{Id}_E$. In addition, Lemma 10 shows that

$$Z_\alpha^N Z_\beta^N = \omega^{2N^2\Omega(\alpha,\beta)} Z_{\alpha+\beta}^N = (-1)^{\Omega(\alpha,\beta)} Z_{\alpha+\beta}^N,$$

so the map $\zeta_\mu\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ satisfies property (2).

Similarly, Corollary 13 shows that each H_k is central in $\mathcal{Z}^\omega(\lambda)$, so $\mu(H_k) = h_k\,\mathrm{Id}$ for some $h_k \in \mathbb{C}^*$. Then $h_k^N\,\mathrm{Id}_E = \mu(H_k^N) = \mu(Z_{\eta_k}^N) = \zeta_\mu(\eta_k)\,\mathrm{Id}_E$ since $H_k = Z_{\eta_k}$, so $\zeta_\mu(\eta_k) = h_k^N$. □

A map $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ that satisfies condition (2) of Proposition 14 is a *twisted homomorphism* twisted by the Thurston form Ω, or more precisely twisted by the symmetric map $(\alpha,\beta) \mapsto (-1)^{\Omega(\alpha,\beta)}$. This notion will probably look less intimidating once one realizes that a twisted homomorphism is completely determined by the assignment of a nonzero complex number to each of the n generators of the group $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \cong \mathbb{Z}^n$.

Proposition 15 *Suppose that we are given a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ twisted by the Thurston form Ω and, for each of the punctures of S, a number $h_k \in \mathbb{C}^*$ such that $h_k^N = \zeta(\eta_k)$. Then, up to isomorphism, there exists a unique irreducible finite-dimensional representation $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ such that*

1. *$\zeta_\mu = \zeta$, namely $\mu(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_E$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$;*
2. *$\mu(H_k) = h_k\,\mathrm{Id}_E$ for $k = 1, \dots, s$.*

In addition, for such a representation, the vector space E has dimension N^{3g+s-3}.

Proof Using elementary linear algebra, this follows immediately from Corollary 13. More precisely, consider the isomorphism

$$\mathcal{Z}^\omega(\lambda) \cong \mathcal{W}_1^q \otimes \cdots \otimes \mathcal{W}_g^q \otimes \mathcal{W}_{g+1}^{q^2} \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^{q^2} \otimes \mathbb{C}[H_1] \otimes \cdots \otimes \mathbb{C}[H_s]$$

provided by Corollary 13.

For $1 \leqslant i \leqslant 3g+s-3$, let $X_i^{\pm 1}$ and $Y_i^{\pm 1}$ denote the generators of $\mathcal{W}_i^q$ or $\mathcal{W}_i^{q^2}$ (satisfying the relation $X_iY_i = qY_iX_i$ if $1 \leqslant i \leqslant g$ and $X_iY_i = q^2Y_iX_i$ if $g < i \leqslant 3g+s-3$). The proof of Corollary 13 shows that these generators are of the form $X_i = Z_{\alpha_i}$, $Y_i = Z_{\beta_i}$ and $H_k = Z_{\eta_k}$ for some edge weight systems $\alpha_i, \beta_i, \eta_k \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. In addition, the α_i, β_i and η_k form a basis for $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \cong \mathbb{Z}^n$.

Because N is odd, $q=\omega^4$ and q^2 are both primitive N^{th} roots of unity. Arbitrarily pick N^{th} roots $\zeta(\alpha_i)^{1/N}$ and $\zeta(\beta_i)^{1/N}$, and define $\mu_i\colon \mathcal{W}_i^q \to \operatorname{End}(E_i)$ by the property that, if $v_1, v_2, \dots, v_N$ form a basis for $E_i \cong \mathbb{C}^N$,

$$\mu_i(X_i)(v_j) = \begin{cases} -\zeta(\alpha_i)^{1/N} q^j v_j & \text{if } 1 \le i \le g, \\ \zeta(\alpha_i)^{1/N} q^{2j} v_j & \text{if } g < i \leqslant 3g+s-3, \end{cases}$$

$$\mu_i(Y_i)(v_j) = \zeta(\beta_i)^{1/N} v_{j+1}.$$

Then for $E = E_1 \otimes E_2 \otimes \cdots \otimes E_{3g+s-3}$, define $\mu\colon \mathcal{Z}^\omega(\lambda) \to \operatorname{End}(E)$ by the property that μ coincides with $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{3g+s-3}$ on $\mathcal{W}_1^q \otimes \mathcal{W}_2^q \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^q$, and $\mu(H_k) = h_k \operatorname{Id}_E$ for every $k = 1, \dots, s$.

It is immediate that μ satisfies the required properties. The fact that μ is irreducible, and that every irreducible representation is isomorphic to μ, is easily proved by elementary linear algebra; see for instance [5, Section 4] for details. □

3 Pleated surfaces and homomorphisms to $\mathrm{SL}_2(\mathbb{C})$

Let us consider the special case of Proposition 15 when $N = 1$. In particular, $A = -1$ and $\iota = \omega = \pm \mathrm{i}$. Since the Chebyshev polynomial $T_1(x)$ is equal to x, the choice of puncture invariants h_k is irrelevant and Proposition 15 associates to any twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ a representation $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \operatorname{End}(\mathbb{C})$. By composition with the quantum trace homomorphism $\operatorname{Tr}_\lambda^\iota\colon \mathcal{S}^{-1}(S) \to \mathcal{Z}^\iota(\lambda)$ of Theorem 4, we now have a homomorphism

$$\rho_\zeta = \mu_\zeta \circ \operatorname{Tr}_\lambda^\iota\colon \mathcal{S}^{-1}(S) \to \operatorname{End}(\mathbb{C}) = \mathbb{C}.$$

We can then apply the case $N = 1$ of Theorem 1 (which actually is an observation of Doug Bullock, Charlie Frohman, Joanna Kania-Bartoszyńska, Jozef Przytycki and Adam Sikora [12; 13; 14; 15; 29] and plays a crucial rôle in the proof of Theorem 1 in its full generality). It provides a character $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that

$$\rho_\zeta([K]) = -\operatorname{Tr} r_\zeta(K)$$

for every framed knot $K \subset S \times [0,1]$. The property is valid for all knots, not just those whose projection to S has no double point [12; 13; 14; 15; 29].

It is natural to ask which elements of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ are obtained in this way. The answer involves the following geometric definition.

Let $\widetilde{S}$ be the universal cover of S, and let $\widetilde{\lambda}$ be the ideal triangulation of $\widetilde{S}$ obtained by lifting the edges and faces of λ. Identify $\mathrm{PSL}_2(\mathbb{C})$ to the isometry group of the hyperbolic 3–space $\mathbb{H}^3$. A *pleated surface* with *pleating locus* λ is the data $(\widetilde{f}, \overline{r})$ of a map $\widetilde{f}\colon \widetilde{S} \to \mathbb{H}^3$ and a group homomorphism $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ such that

(1) $\widetilde{f}$ homeomorphically sends each edge of $\widetilde{\lambda}$ to a complete geodesic of the hyperbolic space $\mathbb{H}^3$, and every face of $\widetilde{\lambda}$ to a totally geodesic ideal triangle of $\mathbb{H}^3$, with vertices on the sphere at infinity $\partial_\infty \mathbb{H}^3$;

(2) $\widetilde{f}$ is $\overline{r}$–equivariant, in the sense that $\widetilde{f}(\gamma\widetilde{x}) = \overline{r}(\gamma)(\widetilde{f}(\widetilde{x}))$ for every $\gamma \in \pi_1(S)$ and every $\widetilde{x} \in \widetilde{S}$.

Following the terminology introduced in [32], we say that the group homomorphism $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ *realizes* the ideal triangulation λ if there exists a pleated surface $(\widetilde{f}, \overline{r})$ with pleating locus λ. By extension, a point in the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ *realizes* λ if it can be represented by a homomorphism $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ realizing λ. Finally, a character in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *realizes* λ if it is sent to a point of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ realizing λ by the canonical projection $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \to \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

We are now ready to state the result promised. At the beginning of this section, we associated a character $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ to each twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$.

Proposition 16 *A character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is associated to a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ as above if and only it realizes the ideal triangulation λ.*

Proof Suppose that $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ realizes the ideal triangulation λ. By definition, there exists a pleated surface $(\widetilde{f}, \overline{r})$ with pleating locus λ, where the homomorphism $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ represents the image of r under the projection $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \to \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

The pleated surface $(\widetilde{f}, \overline{r})$ determines, for each edge $\widetilde{e}_i$ of the ideal triangulation $\widetilde{\lambda}$ of $\widetilde{S}$, a complex weight $\widetilde{x}_i \in \mathbb{C}^*$ defined as follows: If $\widetilde{Q}_i \subset \widetilde{S}$ is the quadrilateral formed by the two faces of $\widetilde{\lambda}$ meeting along the edge $\widetilde{e}_i$, then $-\widetilde{x}_i$ is the cross-ratio of the four vertices of $\widetilde{f}(\widetilde{Q}_i)$ in the sphere at infinity $\mathbb{C} \cup \{\infty\}$ of $\mathbb{H}^3$. These edge weights $\widetilde{x}_i$ are equivariant under the action of $\pi_1(S)$, and therefore descend to a system of weights x_i for the edges e_i of λ. The edge weights $x_i \in \mathbb{C}^*$ are the *shear-bend parameters* of the pleated surface $(\widetilde{f}, \overline{r})$.

Choose square roots $z_i = \sqrt{x_i}$. Then for every closed curve K in S, there is an explicit formula that expresses the trace $\operatorname{Tr} \overline{r}(K)$ as a Laurent polynomial in the z_i; see for instance [7, Sections 1.3–1.4]. Note that there necessarily is a sign ambiguity in this formula, as the trace of an element of $\mathrm{PSL}_2(\mathbb{C})$ is only defined up to sign. Another sign ambiguity occurs in the choice of the square roots $z_i = \sqrt{x_i}$.

We will use these edge weights $z_i \in \mathbb{C}^*$ to construct representations of $\mathcal{Z}^\iota(\lambda)$ and $\mathcal{S}^{-1}(S)$. Recall that a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ is equivalent to the data of its value on a set of generators of $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \cong Z^n$. We can therefore find such a twisted homomorphism such that

$$\zeta(\alpha) = \pm z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$$

for every edge weight system $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ assigning weight $\alpha_i \in \mathbb{Z}$ to the edge e_i of λ. The $\pm$ signs are here required by the twisting. In addition, a simple manipulation of the formula for the Thurston intersection form (or a use of Lemma 10) show that $\Omega(\alpha,\beta)$ is even whenever $\alpha \in (2\mathbb{Z})^n \subset \mathcal{W}(\tau_\lambda;\mathbb{Z})$ assigns even weights $\alpha_i \in \mathbb{Z}$ to all edges of λ; in particular, there is no twisting on $(2\mathbb{Z})^n \subset \mathcal{W}(\tau_\lambda;\mathbb{Z})$. Using Lemma 9, we can therefore arrange that, for every $\alpha \in (2\mathbb{Z})^n$,

$$\zeta(\alpha) = +z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Note that there are several possible choices for ζ, coming from the signs $\pm$. In fact, Lemma 9 shows that there are exactly 2^d possibilities for ζ, where d is the dimension of $H_1(S;\mathbb{Z}_2)$. We will later adjust the choice of ζ so that it fits our purposes.

Let $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$ be the representation of $\mathcal{Z}^\iota(\lambda)$ associated to the twisted homomorphism ζ by Proposition 15. Namely, $\mu(Z_\alpha) = \zeta(\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$.

The definition of the quantum trace homomorphism $\mathrm{Tr}_\lambda^\iota\colon \mathcal{S}^{-1}(S) \to \mathcal{Z}^\iota(\lambda)$ in [7] was specially designed to copy the formula expressing the trace $\mathrm{Tr}\,\bar r(K)$ as a Laurent polynomial in the square roots $z_i = \sqrt{x_i}$ of the shear-bend parameters of the pleated surface $(\tilde f, \bar r)$. In particular, because of the key property that $\mu_\zeta(Z_i^2) = +z_i^2$,

$$\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = \pm \mathrm{Tr}\, r(K)$$

for every framed knot $K \subset S \times [0,1]$, where the sign $\pm$ depends on K and on the choice of the square roots $z_i = \sqrt{x_i}$; see the discussion in [7, Sections 1.3–1.4].

As discussed at the beginning of this section, the homomorphism $\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota\colon \mathcal{S}^{-1}(S) \to \mathbb{C}$ also defines a character $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that

$$\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = -\mathrm{Tr}\, r_\zeta(K)$$

for every framed knot $K \subset S \times [0,1]$. As a consequence, $\mathrm{Tr}\, r_\zeta(K) = \pm \mathrm{Tr}\, r(K)$ for every knot K.

At this point, there is no reason for the two characters r and $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ to coincide. However, by construction, they project to the same $\mathrm{PSL}_2(\mathbb{C})$–valued character in $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. Their difference can therefore be encoded by a cohomology class $\varepsilon \in H^1(S;\mathbb{Z}_2)$ such that, for every knot $K \subset S \times [0,1]$,

$$\mathrm{Tr}\, r(K) = (-1)^{\varepsilon(K)}\, \mathrm{Tr}\, r_\zeta(K).$$

Each edge e_i of the ideal triangulation λ is Poincaré dual to a cohomology class $\varepsilon_i \in H^1(S;\mathbb{Z}_2)$. Replacing the square root $z_i = \sqrt{x_i}$ by the other square root $-z_i$ has the effect of replacing r_ζ with $\varepsilon_i r_\zeta$. Since the ε_i generate $H^1(S;\mathbb{Z}_2)$, we can therefore adjust the choice of the square roots $z_i = \sqrt{x_i}$ so that the characters r and $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ are now equal.

This proves that, if the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ realizes the ideal triangulation λ, there exists a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ whose associated character $r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is equal to r.

Conversely, suppose that $r = r_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is associated to a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ as above. More precisely, consider the corresponding representation $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$, defined by the property that $\mu_\zeta(Z_\alpha) = \zeta(\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. Then for every framed knot $K \subset S \times [0,1]$,

$$\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = -\operatorname{Tr} r(K).$$

The generator $Z_i \in \mathcal{T}^\iota(\lambda)$ associated to the edge e_i of λ does not satisfy the exponent parity condition defining the balanced Chekhov–Fock algebra $\mathcal{Z}^\iota(\lambda)$, but its square does. We can therefore consider $x_i = \mu_\zeta(Z_i^2) \in \mathbb{C}$, which is different from 0 since Z_i^2 is invertible.

We can then construct a pleated surface $(\widetilde{f}, \overline{r})$ whose pleating locus is equal to λ and whose shear-bend parameters are equal to the edge weights x_i. In particular, this pleated surface is equivariant with respect to a homomorphism $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, which defines a character $\overline{r} \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

By our discussion of the geometric interpretation of the trace homomorphism $\mathrm{Tr}_\lambda^\iota$, the character $\overline{r} \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is the projection of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. In particular, r realizes the ideal triangulation λ. This concludes the proof of Proposition 16. □

4 Representations of the skein algebra

We are now ready to prove Theorem 2. We begin with an elementary lemma about the Chebyshev polynomials T_n. Remember that the polynomial T_n is defined by the property that $\operatorname{Tr} M^n = T_n(\operatorname{Tr} M)$ for every $M \in \mathrm{SL}_2(\mathbb{C})$. Applying this to a rotation matrix gives the trigonometric interpretation that $\cos n\theta = \frac{1}{2} T_n(2\cos\theta)$.

Lemma 17 (1) *If $x = a + a^{-1}$, then $T_n(x) = a^n + a^{-n}$.*

(2) *If $y = b + b^{-1}$, the set of solutions to the equation $T_n(x) = y$ consists of the numbers $x = a + a^{-1}$ as a ranges over all n^{th} roots of b.*

Proof For a matrix $M \in \mathrm{SL}_2(\mathbb{C})$, the data of its trace x is equivalent to the data of its spectrum $\{a, a^{-1}\}$. The first property is then a straightforward consequence of the fact that $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$. The second property immediately follows. □

We will also need the following quantum trace computation, which connects the skein $[P_k] \in \mathcal{S}^A(S)$ and the central element $H_k \in \mathcal{Z}^\omega(\lambda)$ that are associated to the same k^{th} puncture of S.

Lemma 18 *For the quantum trace homomorphism* $\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$,

$$\mathrm{Tr}_\lambda^\omega([P_k]) = H_k + H_k^{-1}.$$

Proof Let $e_{i_1}, e_{i_2}, \dots, e_{i_u}$ be the edges of λ that lead to the k^{th} puncture, indexed in counterclockwise order around the puncture; in particular, the e_{i_j} are not necessarily distinct.

The construction of $\mathrm{Tr}_\lambda^\omega([P_k])$ in [7] requires a careful control of elevations (namely $[0,1]$–coordinates) along the knot $P_k \subset S \times [0,1]$. Choose this knot so that it steadily goes up from e_{i_1} to e_{i_u}, and then sharply goes down to return to its starting point in e_{i_1}. In this setup, the formula of [7] yields

$$\mathrm{Tr}_\lambda^\omega([P_k]) = \omega^{-u+2} Z_{i_1} Z_{i_2} \cdots Z_{i_u} + \omega^{-u+2} Z_{i_1}^{-1} Z_{i_2}^{-1} \cdots Z_{i_u}^{-1}.$$

This is relatively straightforward when only one end of the edge e_{i_1} leads to the k^{th} puncture, namely when the projection of P_k to S crosses e_{i_1} only once, but otherwise requires the consideration of correction terms in a bigon neighborhood of e_{i_1}, of the type given by [7, Lemma 22]. Fortunately, these correction terms turn out to be trivial in this case.

We need to connect this formula to $H_k = [Z_{i_1} Z_{i_2} \cdots Z_{i_u}]$. Computing the Weyl quantum ordering is again straightforward when each edge e_{i_j} has only one end leading to the k^{th} puncture. For the general case, we could use a brute force computation as in [10, Lemma 12]. We prefer to give here a more indirect argument, based on the invariance of $\mathrm{Tr}_\lambda^\omega([P_k])$ under isotopy of P_k.

For this, choose the elevation of P_k so that it now goes *down* from e_{i_1} to e_{i_u}, and then goes up near e_{i_1} to return to its starting point. In this setup, the formulas of [7] give

$$\mathrm{Tr}_\lambda^\omega([P_k]) = \omega^{u-2} Z_{i_u} Z_{i_{u-1}} \cdots Z_{i_1} + \omega^{u-2} Z_{i_u}^{-1} Z_{i_{u-1}}^{-1} \cdots Z_{i_1}^{-1}.$$

Comparing the two expressions for $\mathrm{Tr}_\lambda^\omega([P_k]) \in \mathcal{Z}^\omega(\lambda)$ shows in particular that

$$\omega^{-u+2} Z_{i_1} Z_{i_2} \cdots Z_{i_u} = \omega^{u-2} Z_{i_u} Z_{i_{u-1}} \cdots Z_{i_1}.$$

By definition of the Weyl quantum ordering, there exists an integer $a \in \mathbb{Z}$ such that

$$H_k = \omega^a Z_{i_1} Z_{i_2} \cdots Z_{i_u} = \omega^{-a} Z_{i_u} Z_{i_{u-1}} \cdots Z_{i_1}.$$

We can then rephrase the above equality as $\omega^{-a-u+2}H_k = \omega^{a+u-2}H_k$. Although the current article usually focuses on the case where ω is a root of unity, these computations are valid for all ω. It follows that $a = -u+2$. This proves that $H_k = \omega^{-u+2}Z_{i_1}Z_{i_2}\cdots Z_{i_u}$, and our first computation then shows that $\mathrm{Tr}_\lambda^\omega([P_k]) = H_k + H_k^{-1}$. □

We are now ready to prove Theorem 2, which we repeat here for convenience. Recall that a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ associates a number $\mathrm{Tr}\, r(P_k)$ to the k^{th} puncture of S, where P_k is a small loop going around the puncture.

Theorem 19 *Assume that the surface S has at least one puncture, that its Euler characteristic is negative, that A is a primitive N^{th} root of -1 with N odd, and that we are given*

1. *a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ realizing some ideal triangulation λ of S;*
2. *a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = -\mathrm{Tr}\, r(P_k)$ for each puncture of S.*

Then there exists an irreducible finite-dimensional representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ whose classical shadow is equal to r and whose puncture invariants are the p_k.

Proof Since $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ realizes the ideal triangulation λ, Proposition 16 provides a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ and an associated representation $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$, such that

$$\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = -\mathrm{Tr}\, r(K)$$

for every framed knot $K \subset S \times [0,1]$

By Lemma 18, the image of $[P_k] \in \mathcal{S}^A(S)$ under the quantum trace homomorphism $\mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ is equal to $\mathrm{Tr}_\lambda^\omega([P_k]) = H_k + H_k^{-1}$ in $\mathcal{Z}^\omega(\lambda)$. Similarly, $\mathrm{Tr}_\lambda^\iota([P_k]) = H_k + H_k^{-1}$ in $\mathcal{Z}^\iota(\lambda)$. (Beware that we are using the same symbols to denote the skeins $[P_k] \in \mathcal{S}^A(S)$ and $[P_k] \in \mathcal{S}^{-1}(S)$, and the central elements $H_k \in \mathcal{Z}^\omega(\lambda)$ and $H_k \in \mathcal{Z}^\iota(\lambda)$.) Then for $[P_k] \in \mathcal{S}^{-1}(S)$,

$$\mathrm{Tr}\, r(P_k) = -\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([P_k]) = -\mu_\zeta(H_k + H_k^{-1}) = -g_k - g_k^{-1}$$

if we set $g_k = \mu_\zeta(H_k) \in \mathrm{End}(\mathbb{C}) = \mathbb{C}$.

For each k, we are given a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = -\mathrm{Tr}\, r(P_k) = g_k + g_k^{-1}$. Lemma 17 then provides an N^{th} root $h_k = \sqrt[N]{g_k}$ of such that $p_k = h_k + h_k^{-1}$.

Proposition 15 associates to the homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ and to the N^{th} roots $h_k = \mu_\zeta(H_k)^{1/N}$ an irreducible representation $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ such that

(1) $\mu(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_E$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$;
(2) $\mu(H_k) = h_k\,\mathrm{Id}_E$ for every $k = 1, \dots, s$.

Composing with the quantum trace map $\mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$, we now define a representation

$$\rho = \mu \circ \mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathrm{End}(E).$$

To determine the classical shadow of ρ, let K be a framed knot whose projection to S has no crossing and whose framing is vertical. Then for the associated skein $[K] \in \mathcal{S}^A(S)$,

$$T_N(\rho([K])) = \rho(T_N([K])) = \mu\circ\mathrm{Tr}_\lambda^\omega(T_N([K])) = \mu\circ\mathrm{Tr}_\lambda^\omega\circ\boldsymbol{T}^A([K]) = \mu\circ\boldsymbol{F}^\omega\circ\mathrm{Tr}_\lambda^\iota([K])$$

by using the fact that ρ is an algebra homomorphism for the first equality, by definition of the Chebyshev homomorphism $\boldsymbol{T}^A\colon \mathcal{S}^{-1}(S) \to \mathcal{S}^A(S)$ in Section 1.3 for the third equality, and by the miraculous cancellations of Theorem 7 for the last relation. In terms of the Frobenius homomorphism $\boldsymbol{F}^\omega\colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$ introduced in Section 1.3 and of the representation $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$, the property that $\mu(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_E = \mu_\zeta(Z_\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ can be rephrased as $\mu \circ \boldsymbol{F}^\omega = \mu_\zeta$. Therefore,

$$T_N(\rho([K])) = \mu \circ \boldsymbol{F}^\omega \circ \mathrm{Tr}_\lambda^\iota([K]) = \mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K])\,\mathrm{Id}_E = -\mathrm{Tr}\, r(K)\,\mathrm{Id}_E\,.$$

Also, for the k^{th} puncture of S, the corresponding puncture invariant is determined by the property that

$$\rho([P_k]) = \mu \circ \mathrm{Tr}_\lambda^\omega([P_k]) = \mu(H_k + H_k^{-1}) = (h_k + h_k^{-1})\,\mathrm{Id}_E = p_k\,\mathrm{Id}_E\,.$$

If we knew that ρ was irreducible, we would be done with the proof of Theorem 19. At this point, there is no reason for this property to hold. However, if ρ is not irreducible, it suffices to consider an irreducible component $\rho'\colon \mathcal{S}^A(S) \to \mathrm{End}(F)$ with $F \subset E$. Restricting the above computations to F shows that the classical shadow of the representation ρ' is equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, and that its puncture invariants are equal to the numbers p_k. □

Remark 20 We conjecture that, when r is sufficiently generic in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, the representation $\rho = \mu \circ \mathrm{Tr}_\lambda^\omega$ used in the proof of Theorem 19 is already irreducible, and that there is no need to restrict to an irreducible factor. In earlier versions of this article we also conjectured that, for generic $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, there is a unique representation ρ satisfying the conclusions of Theorem 19 up to isomorphism. This second conjecture was recently proved by Frohman, Kania-Bartoszyńska and Lê [22]. See also Takenov [31] for an earlier proof of this second conjecture in the cases of the one-puncture torus and the four-puncture sphere (building on earlier work of Bullock and Przytycki [16] and Havlíček and Pošta [23] for the one-puncture torus).

Remark 21 In the very nongeneric case where $r(P_k)$ is the identity and where $p_k = -\omega^4 - \omega^{-4}$ for some punctures, the representation $\rho = \mu \circ \mathrm{Tr}_\lambda^\omega$ is definitely reducible. This is a key ingredient of the "puncture filling" process developed in [10].

5 A uniqueness property

We made choices in the proof of Theorem 19, and more precisely in its intermediate step the proof of Proposition 16. Indeed, when proving Proposition 16, we first took arbitrary square roots $z_i = \sqrt{x_i}$ for the shear-bend parameters $x_i \in \mathbb{C}^*$ of a pleated surface, and then adjusted these square roots in order to get the desired classical shadow for the representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$.

The goal of this section is to show that the output of the construction does not depend on these choices, provided we carefully specify our data and our conclusions. The resulting uniqueness statement will be used in the subsequent article [10]. Indeed, [10] heavily relies on Theorem 19 to construct representations of the skein algebra of a closed surface, by applying this statement to suitably chosen punctured surfaces.

5.1 Pleated surfaces and representations of $\mathcal{Z}^\iota(\lambda)$

The proof of Theorem 19 hinges on Proposition 16 which, given a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, provides a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ and its associated representation $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ such that

$$\mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = -\operatorname{Tr} r(K)$$

for every framed knot $K \subset S \times [0,1]$. Recall that μ_ζ and ζ are related by the property that $\mu_\zeta(Z_\alpha) = \zeta(\alpha) \in \mathbb{C}^*$ for every basis element $Z_\alpha \in \mathcal{Z}^\iota(\lambda)$ associated to an edge weight system $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$.

For most characters $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, the homomorphism $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ is uniquely determined by r and by the pleated surface $(\widetilde{f}, \overline{r})$. However this uniqueness fails, in a very specific way, when the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ admits a very special type of internal symmetry which we now describe.

The cohomology group $H^1(S;\mathbb{Z}_2)$ acts on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ by the property that, for every homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and cohomology class $\varepsilon \in H^1(S;\mathbb{Z}_2)$, the homomorphism εr is defined by

$$\varepsilon r(\gamma) = (-1)^{\varepsilon(\gamma)} r(\gamma) \in \mathrm{SL}_2(\mathbb{C})$$

for every $\gamma \in \pi_1(S)$. We say that $\varepsilon \in H^1(S;\mathbb{Z}_2)$ is a *sign-reversal symmetry* for the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if the action of ε on $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ fixes r. This is equivalent to the property that the trace $\operatorname{Tr} r(\gamma)$ is equal to 0 for every $\gamma \in \pi_1(S)$ with $\varepsilon(\gamma) \neq 0$.

The group $H^1(S;\mathbb{Z}_2)$ also acts on the balanced Chekhov–Fock algebra $\mathcal{Z}^\omega(\lambda)$ by the property that $\varepsilon Z_\alpha = (-1)^{\varepsilon([\alpha])} Z_\alpha$ for every $\varepsilon \in H^1(S;\mathbb{Z}_2)$ and every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, where $[\alpha] \in H_1(S;\mathbb{Z}_2)$ is the homology class associated to the edge weight system α as in Lemma 9.

Proposition 22 *Suppose the pleated surface $(\widetilde{f},\overline{r})$ has pleating locus the ideal triangulation λ, and let $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ be represented by a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ lifting the monodromy $\overline{r}\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ of $(\widetilde{f},\overline{r})$. Then there exists an algebra homomorphism $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$, associated to a twisted homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$, such that*

(1) *for each edge e_i of λ, we have that $\mu_\zeta(Z_i^2)$ is equal to the shear-bend parameter $x_i \in \mathbb{C}^*$ of e_i in the pleated surface $(\widetilde{f},\overline{r})$;*

(2) $\mu_\zeta \circ \operatorname{Tr}_\lambda^\iota([K]) = -\operatorname{Tr} r(K)$ *for every framed knot $K \subset S \times [0,1]$.*

In addition, μ_ζ is unique up to the action on $\mathcal{Z}^\iota(\lambda)$ of a sign-reversal symmetry $\varepsilon \in H^1(S;\mathbb{Z}_2)$ of the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

We say that a homomorphism $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ satisfying the above conclusions is *compatible* with the pleated surface $(\widetilde{f},\overline{r})$ and the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

Proof of Proposition 22 The existence is provided by Proposition 16, or more precisely by its proof to guarantee that $\mu_\zeta(Z_i^2) = x_i$ for every edge e_i of λ.

To prove the uniqueness, suppose that we are given another algebra homomorphism $\mu_{\zeta'}\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ satisfying the same conclusions, and that this homomorphism is associated to a twisted homomorphism $\zeta'\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$. From the property that $\mu_\zeta(Z_i^2) = \mu_{\zeta'}(Z_i^2) = x_i$, we conclude that $\mu_\zeta(Z_\alpha)^2 = \mu_{\zeta'}(Z_\alpha)^2$ and therefore $\mu_\zeta(Z_\alpha) = \pm\mu_{\zeta'}(Z_\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. Since μ_ζ and $\mu_{\zeta'}$ are both algebra homomorphisms, there consequently exists a group homomorphism $\varepsilon\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z}_2$ such that $\mu_\zeta(Z_\alpha) = (-1)^{\varepsilon(\alpha)}\mu_{\zeta'}(Z_\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. Another application of the property that $\mu_\zeta(Z_i^2) = \mu_{\zeta'}(Z_i^2)$ shows that ε is trivial on the subgroup $(2\mathbb{Z})^n \subset \mathcal{W}(\tau_\lambda;\mathbb{Z})$ of Lemma 9. This statement then shows that ε comes from a homomorphism $H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$, and can therefore be interpreted as a cohomology class $\varepsilon \in H^1(S;\mathbb{Z}_2)$.

In this cohomological interpretation of $\varepsilon \in H^1(S;\mathbb{Z}_2)$, we have that $\mu_\zeta(Z_\alpha) = (-1)^{\varepsilon([\alpha])}\mu_{\zeta'}(Z_\alpha)$ for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. Namely, the homomorphisms μ_ζ and $\mu_{\zeta'}\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ differ by the action of $\varepsilon \in H^1(S;\mathbb{Z}_2)$ on $\mathcal{Z}^\iota(\lambda)$.

Given a framed link $K \subset S \times [0,1]$, the construction of the quantum trace $\mathrm{Tr}_\lambda^\iota$ in [7] shows that $\mathrm{Tr}_\lambda^\iota([K]) \in \mathcal{Z}^\iota(\lambda)$ is a linear combination of monomials Z_α whose associated homology class $[\alpha] \in H_1(S;\mathbb{Z}_2)$, in the sense of Lemma 9, is the same as the class $[K] \in H_1(S;\mathbb{Z}_2)$ defined by K. As a consequence,

$$\mathrm{Tr}\, r(K) = -\mu_{\zeta'} \circ \mathrm{Tr}_\lambda^\iota([K]) = -(-1)^{\varepsilon(K)} \mu_\zeta \circ \mathrm{Tr}_\lambda^\iota([K]) = (-1)^{\varepsilon(K)}\, \mathrm{Tr}\, r(K)$$

for every framed link $K \subset S \times [0,1]$. This proves that $\varepsilon \in H^1(S;\mathbb{Z}_2)$ is a sign-reversal symmetry for the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

As a consequence, the homomorphisms $\mu_\zeta, \mu_{\zeta'} \colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ differ by the action on $\mathcal{Z}^\iota(\lambda)$ of a sign-reversal symmetry $\varepsilon \in H^1(S;\mathbb{Z}_2)$ of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. □

Characters with nontrivial sign-reversal symmetries exist, but are rare. In fact, the characters that have no (nontrivial) sign-reversal symmetries form a Zariski dense closed subset in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. (Hint: Choose a family of simple closed curves $\gamma_1, \gamma_2, \dots, \gamma_k$ in S that generate $H_1(S;\mathbb{Z}_2)$, and consider the set of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that $\mathrm{Tr}\, r(\gamma_i) \neq 0$ for some i.) This subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ includes all injective homomorphisms $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, since their images contain no matrix with trace 0. In particular all "geometric" characters, corresponding to fuchsian or quasifuchsian groups, admit no sign-reversal symmetries.

More precisely, a simple algebraic manipulation shows that every character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ with a nontrivial sign-reversal symmetry $\varepsilon \in H^1(S;\mathbb{Z}_2)$ is represented by a homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ of the following type: Considering ε as a group homomorphism $\varepsilon \colon \pi_1(S) \to \mathbb{Z}_2$ and for an arbitrary $\gamma_0 \in \pi_1(S)$ with $\varepsilon(\gamma_0) \neq 0$, there exists a group homomorphism $\theta \colon \ker \varepsilon \to \mathbb{C}/2\pi \mathrm{i}\mathbb{Z}$ such that

$$r(\gamma_0) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \quad \text{and} \quad r(\gamma) = \begin{pmatrix} \cosh\theta(\gamma) & \sinh\theta(\gamma) \\ \sinh\theta(\gamma) & \cosh\theta(\gamma) \end{pmatrix} \quad \text{for every } \gamma \in \ker\varepsilon.$$

In particular, noting the constraints that $\theta(\gamma_0^2) = \pi\mathrm{i}$ and $\theta(\gamma_0\gamma\gamma_0^{-1}) = -\theta(\gamma)$ for every $\gamma \in \ker\varepsilon$, the space of such characters has complex dimension $2g+s-2$ in the $(6g+3s-6)$–dimensional character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ (where g is the genus of the surface S and s is its number of punctures).

5.2 A strengthening of Theorem 19

Recall that, if the k^{th} puncture of S is adjacent to the edges $e_{i_1}, e_{i_2}, \dots, e_{i_u}$ of the ideal triangulation λ, it determines an element $H_k = [Z_{i_1} Z_{i_2} \cdots Z_{i_u}] \in \mathcal{Z}^\iota(\lambda)$.

Proposition 23 *Assume that the surface S has at least one puncture, that its Euler characteristic is negative, that A is a primitive N^{th} root of -1 with N odd, and that we are given*

(i) *a pleated surface* $(\widetilde{f}, \overline{r})$ *with pleating locus* λ, *a character* $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *lifting* $\overline{r} \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, *and an algebra homomorphism* $\mu_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ *compatible with* $(\widetilde{f}, \overline{r})$ *and* r *as in Proposition 22;*

(ii) *for each puncture of* S, *an* N^{th} *root* h_k *of* $\mu_\zeta(H_k) \in \mathbb{C}^*$.

Then, up to isomorphism, there exists a unique representation $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ *of the balanced Chekhov–Fock algebra* $\mathcal{Z}^\omega(\lambda)$ *with the following properties:*

(1) *the dimension of the vector space* E *is equal to* N^{3g+s-3}, *where* g *is the genus of the surface* S *and* s *its number of punctures;*

(2) $\mu(Z_\alpha^N) = \mu_\zeta(Z_\alpha)$ *for every edge weight system* $\alpha \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$, *where we use the same symbol to represent the associated base elements* $Z_\alpha \in \mathcal{Z}^\omega(\lambda)$ *and* $Z_\alpha \in \mathcal{Z}^\iota(\lambda)$;

(3) $\mu(H_k) = h_k \,\mathrm{Id}_E$ *for the central element* $H_k \in \mathcal{Z}^\omega(\lambda)$ *associated to the* k^{th} *puncture of* S.

Also, the representation μ *is irreducible and the representation* $\rho = \mu \circ \mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ *has classical shadow* $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, *in the sense that*

$$T_N(\rho([K])) = -\operatorname{Tr} r(K) \,\mathrm{Id}_E$$

for every knot $K \subset S \times [0,1]$ *whose projection to* S *has no crossing and whose framing is vertical (where* $T_N(x)$ *is the* N^{th} *Chebyshev polynomial of the first type).*

Proof The existence and uniqueness part is essentially a restatement of the classification of irreducible representations of $\mathcal{Z}^\omega(\lambda)$ in Proposition 15. The fact that ρ has classical shadow r follows from the proof of Theorem 19. □

Although the representation $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ of Proposition 23 is irreducible, the representation $\rho = \mu \circ \mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ is not necessarily irreducible; see Remark 21.

6 The case where $A^N = +1$

The case where $A^N = +1$ can be deduced from the case where $A^N = -1$ by the *Barrett isomorphism* $B_\sigma\colon \mathcal{S}^A(S) \to \mathcal{S}^{-A}(S)$ associated to a spin structure σ on the surface S. This isomorphism is defined by the property that, for every framed link $K \subset S \times [0,1]$ with k components,

$$B_\sigma([K]) = (-1)^{k+\sigma(K)}[K] \in \mathcal{S}^{-A},$$

where $\sigma(K) \in \mathbb{Z}_2$ is the monodromy of the framing of K with respect to σ. See [2] and [29, Section 2] for a proof that $B_\sigma\colon \mathcal{S}^A(S) \to \mathcal{S}^{-A}(S)$ is an algebra isomorphism.

If $A^N = +1$, an irreducible finite-dimensional representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ defines an irreducible representation $\rho' = \rho \circ B_\sigma\colon \mathcal{S}^{-A}(S) \to \mathrm{End}(E)$, to which we can apply Theorems 1 and 2 since $(-A)^N = -1$ as N is assumed to be odd. This process depends on the choice of a spin structure σ, but we can make it more canonical by the following construction.

Let $\mathrm{Spin}(S)$ denote the set of isotopy classes of spin structures on S. Any two spin structures differ by an obstruction in $H^1(S;\mathbb{Z}_2)$, which defines an action of $H^1(S;\mathbb{Z}_2)$ on $\mathrm{Spin}(S)$. The cohomology group $H^1(S;\mathbb{Z}_2)$ also acts on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ by the property that, if $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\varepsilon \in H^1(S;\mathbb{Z}_2)$, then $\varepsilon r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is defined by

$$\varepsilon r(\gamma) = (-1)^{\varepsilon(\gamma)} r(\gamma) \in \mathrm{SL}_2(\mathbb{C})$$

for every $\gamma \in \pi_1(S)$.

The *twisted character variety* $\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ is then defined as the quotient

$$\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})} = (\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S))/H^1(S;\mathbb{Z}_2).$$

If the twisted character $\hat{r} \in \mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ is represented by $(r,\sigma) \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)$ and if K is a framed knot in $S \times [0,1]$, the definition is designed so that the *trace*

$$\mathrm{Tr}\, \hat{r}(K) = -(-1)^{\sigma(K)}\, \mathrm{Tr}\, r(K)$$

depends only on the twisted character $\hat{r} \in \mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$, and not on its representative $(r,\sigma) \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)$.

The correspondence $\rho \leftrightarrow \rho \circ B_\sigma$ is used in [9] to establish the following result.

Theorem 24 [9] *Suppose that A is a primitive N^{th} root of $+1$ with N odd, and let $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ be an irreducible finite-dimensional representation of the Kauffman bracket skein algebra. Let $T_N(x)$ be the N^{th} normalized Chebyshev polynomial of the first kind.*

(1) *There exists a unique twisted character $\hat{r}_\rho \in \mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ such that*

$$T_N(\rho([K])) = -(\mathrm{Tr}\, \hat{r}_\rho(K))\, \mathrm{Id}_E$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

(2) *Let P_k be a small simple loop going around the k^{th} puncture of S, and consider it as a knot in $S\times[0,1]$ with vertical framing. Then there exists a number $p_k\in\mathbb{C}$ such that $\rho([P_k]) = p_k \operatorname{Id}_E$.*

(3) *The number p_k of (2) is related to the twisted character $\widehat{r}_\rho \in \mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ of (1) by the property that $T_N(p_k) = -\operatorname{Tr}\widehat{r}_\rho(P_k)$.* □

The same correspondence $\rho \leftrightarrow \rho\circ B_\sigma$ can be used to prove the following analogue of Theorem 2. We say that the twisted character $\widehat{r}\in\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ *realizes* the ideal triangulation λ of S if the image $\overline{r}\in\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of $\widehat{r}$ under the natural projection $\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}\to\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ realizes λ in the sense of Section 3.

Theorem 25 *Assume that the surface S has at least one puncture, that its Euler characteristic is negative, that A is a primitive N^{th} root of $+1$ with N odd, and that we are given*

(1) *a twisted character $\widehat{r}\in\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ which realizes some ideal triangulation λ of S;*

(2) *a number $p_k\in\mathbb{C}$ such that $T_N(p_k) = -\operatorname{Tr}\widehat{r}(P_k)$ for each of the punctures of S.*

Then there exists an irreducible finite-dimensional representation $\rho\colon \mathcal{S}^A(S)\to\operatorname{End}(E)$ whose classical shadow is equal to $\widehat{r}$ and whose puncture invariants are the p_k.

Proof Represent $\widehat{r}\in\mathcal{R}^{\mathrm{Spin}(S)}_{\mathrm{PSL}_2(\mathbb{C})}$ by a pair $(r,\sigma)\in\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)\times\mathrm{Spin}(S)$. Theorem 2 provides an irreducible representation $\rho'\colon \mathcal{S}^{-A}(S)\to\operatorname{End}(E)$ with classical shadow $r\in\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and puncture invariants equal to the p_k. Then $\rho=\rho'\circ B_\sigma\colon \mathcal{S}^A(S)\to\operatorname{End}(E)$ satisfies the required properties. □

Appendix: The Thurston intersection form of a train track

Let τ be a train track in an oriented surface S, and let $\mathcal{W}(\tau;\mathbb{Z})$ be the space of integer-valued edge weights for τ. Namely, an element $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$ assigns a weight $\alpha(e)\in\mathbb{Z}$ to each edge e of τ in such a way that, at each switch s of τ, the sum of the weights of the edges of τ coming in on one side of s is equal to the sum of the weights of the edges going out on the other side. This abelian group comes with an additional structure provided by the Thurston intersection form

$$\Omega\colon \mathcal{W}(\tau;\mathbb{Z})\times\mathcal{W}(\tau;\mathbb{Z})\to\mathbb{Z}$$

defined as in Section 2.2. Namely,

$$\Omega(\alpha,\beta)=\frac{1}{2}\sum_{e \text{ right of } e'}(\alpha(e)\beta(e')-\alpha(e')\beta(e)),$$

where the sum is over all pairs (e, e') such that e and e' are two "germs of edges" emerging on the same side of a switch of τ with e to the right of e' (note e and e' are not necessarily adjacent at that switch) for the orientation of S. At this point, $\Omega(\alpha, \beta)$ is only a half-integer, but Theorem 26 below will prove that it is indeed an integer.

We want to determine the algebraic structure of $\mathcal{W}(\tau;\mathbb{Z})$ endowed with Ω. This is a classical property in the case of real-valued edge weights (see for instance [28, Section 3.2] or [4, Section 3]), but the subtleties of the integer-valued case seem less well known. The result is of independent interest because, beyond the scope of this article, integer-valued edge weight do occur in geometric situations where the Thurston intersection form is also relevant. One such instance arises for general pleated surfaces where the pleating locus is allowed to have uncountably many leaves, as opposed to the simpler pleated surfaces considered in Section 3. The bending of such a pleated surface is measured by an edge weight system valued in $\mathbb{R}/2\pi\mathbb{Z}$ for a train track carrying the pleating locus, and this edge weight system is related to rotation numbers by the Thurston intersection form [4].

The complement $S - \tau$ of the train track τ admits a certain number of "spikes", each locally delimited by two edges of τ that approach the same side of a switch of τ. Thicken τ to a subsurface $U \subset S$ that deformation retracts to τ. Each component of $U - \tau$ is then an annulus that contains one component of ∂U and a certain number of spikes of $S - \tau$. We can then consider the genus h of U, and the number n_{even} (resp. n_{odd}) of components of $U - \tau$ that contain an even (resp. odd) number of spikes.

A component U_1 of $U - \tau$ that contains an even number $n_1 > 0$ of spikes of $S - \tau$ determines, up to sign, an element of $\mathcal{W}(\tau;\mathbb{Z})$ as follows. The core of U_1 is homotopic to a closed curve γ_1 in τ that is made up of arcs $k_1, k_2, \dots, k_{n_1}, k_{n_1+1} = k_1$, in this order, such that each arc k_1 is immersed in τ and such that two consecutive arcs k_i and k_{i+1} locally bound a spike of U_1 at their common end point. For each edge e of τ, we can then consider

$$\alpha(e) = \sum_{i=1}^{n_1} (-1)^i \alpha_i(e) \in \mathbb{Z},$$

where $\alpha_i(e) \in \{0, 1, 2\}$ is the number of times the arc k_i passes over the edge e. Because the signs $(-1)^i$ alternate at the spikes of U_1 (using the fact that n_1 is even for $i = n_1$), one easily sees that these edge weights $\alpha(e)$ satisfy the switch conditions, and therefore define an edge weight system $\alpha \in \mathcal{W}(\tau;\mathbb{Z})$.

A component U_1 of $U - \tau$ that contains no spike similarly determines an edge weight system $\alpha \in \mathcal{W}(\tau;\mathbb{Z})$. The core of U_1 is now homotopic to a closed curve γ_1 immersed in τ, and α associates to each edge e the number $\alpha(e)$ of times γ_1 passes over e.

Also, recall that the train track τ is *orientable* if its edges can be oriented in such a way that the orientations match at the switches of τ.

Theorem 26 *For a connected train track τ in the surface S, let the numbers h, n_{even} and n_{odd} be defined as above. Then the lattice $\mathcal{W}(\tau;\mathbb{Z})$ of integer-valued edge weight systems for τ admits a basis in which the Thurston intersection form Ω is block diagonal with*

- *h blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\frac{1}{2}n_{\mathrm{odd}}-1$ blocks $\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$, and n_{even} blocks (0) if $n_{\mathrm{odd}}>0$;*
- *h blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and n_{even} blocks (0) if $n_{\mathrm{odd}}=0$ and τ is nonorientable;*
- *h blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $n_{\mathrm{even}}-1$ blocks (0) if $n_{\mathrm{odd}}=0$ and τ is orientable.*

In addition, in all cases, we can choose the base elements corresponding to the blocks (0) to be the edge weight systems associated as above to the components of $U-\tau$ that contain an even number of spikes.

In particular, n_{odd} is always even.

Proof We will subdivide the proof into several lemmas. The reader may recognize many analogies with the arguments used in the proof of [5, Proposition 5].

We first discuss a classical homological interpretation of the elements of $\mathcal{W}(\tau;\mathbb{Z})$ and of the Thurston intersection form Ω.

Because the edges of τ are not oriented, an edge weight system does not directly define a homology class in $H_1(\tau;\mathbb{Z})$. Instead consider the 2–fold orientation covering $\hat\tau$ of τ, consisting of all pairs (x,o) where $x\in\tau$ and o is a local orientation of the train track τ at x. Note that $\hat\tau$ is a canonically oriented train track, and that the covering involution $\sigma\colon \hat\tau\to\hat\tau$ that exchanges the two sheets of the covering reverses the orientation of $\hat\tau$.

An edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$ lifts to a weight system $\hat\alpha\in\mathcal{W}(\hat\tau;\mathbb{Z})$. Endowing each (oriented) edge of $\hat\tau$ with the weight assigned by $\hat\alpha$ defines a chain, which is closed because of the switch condition and therefore defines a homology class $[\hat\alpha]\in H_1(\hat\tau;\mathbb{Z})$. Note that $\sigma_*([\hat\alpha])=-[\hat\alpha]$ since the covering involution σ reverses the canonical orientation of $\hat\tau$.

Conversely, each homology class $[\hat\alpha]\in H_1(\hat\tau;\mathbb{Z})$ is represented by a unique linear combination of the edges of $\hat\tau$, and therefore determines an edge weight system $\hat\alpha\in\mathcal{W}(\hat\tau;\mathbb{Z})$. Assuming in addition that $\sigma_*([\hat\alpha])=-[\hat\alpha]$, this edge weight system is invariant under the action of σ, and therefore comes from an edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$. This proves:

Lemma 27 *The above correspondence identifies the space $\mathcal{W}(\tau;\mathbb{Z})$ of edge weight systems to the eigenspace*

$$H_1(\widehat{\tau};\mathbb{Z})^- = \big\{[\widehat{\alpha}] \in H_1(\widehat{\tau};\mathbb{Z}) : \sigma_*([\widehat{\alpha}]) = -[\widehat{\alpha}]\big\} \subset H_1(\widehat{\tau};\mathbb{Z})$$

of the homomorphism $\sigma_*\colon H_1(\widehat{\tau};\mathbb{Z}) \to H_1(\widehat{\tau};\mathbb{Z})$ that is induced by the covering involution σ. □

To describe the Thurston intersection form in this homological framework, consider the subsurface U deformation retracting to τ. The covering $\widehat{\tau} \to \tau$ uniquely extends to a 2–fold covering $\widehat{U} \to U$, whose covering involution $\sigma\colon \widehat{U} \to \widehat{U}$ extends our previous involution σ. The orientation of $U \subset S$ lifts to an orientation of $\widehat{U}$.

Lemma 28 *If $[\widehat{\alpha}], [\widehat{\beta}] \in H_1(\widehat{\tau})^-$ are associated to the edge weight systems $\alpha, \beta \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$,*

$$\Omega(\alpha,\beta) = \tfrac{1}{2}\,[\widehat{\alpha}]\cdot[\widehat{\beta}],$$

where $\cdot$ denotes the algebraic intersection number of classes of $H_1(\widehat{U};\mathbb{Z}) \cong H_1(\widehat{\tau};\mathbb{Z})$. In addition, $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ is even, and $\Omega(\alpha,\beta)$ is an integer.

Proof To prove the first statement push the oriented train track $\widehat{\tau}$ to its left to obtain a train track $\widehat{\tau}' \subset \widehat{U}$ that is transverse to $\widehat{\tau}$, realize the homology class $[\widehat{\alpha}]$ by $\widehat{\tau}$ endowed with the edge multiplicities coming from α, realize $[\widehat{\beta}]$ by $\widehat{\tau}'$ endowed with the edge multiplicities coming from β, and use this setup to compute the algebraic intersection number $[\widehat{\alpha}]\cdot[\widehat{\beta}]$. Evaluating the contribution to $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ of each point of $\widehat{\tau}\cap\widehat{\tau}'$ then shows that this algebraic intersection number is equal to $2\Omega(\alpha,\beta)$.

The second statement is obtained by a similar but different computation of $[\widehat{\alpha}]\cdot[\widehat{\beta}]$. Perturb τ to a train track τ'' that is transverse to τ, and let $\widehat{\tau}''$ be the preimage of τ'' in $\widehat{U}$. Now compute $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ by realizing the homology class $[\widehat{\beta}]$ by $\widehat{\tau}''$ endowed with the edge multiplicities coming from β, while still realizing $[\widehat{\alpha}]$ by $\widehat{\tau}$ endowed with the edge multiplicities coming from α. The intersection $\widehat{\tau}\cap\widehat{\tau}''$ splits into pairs of points interchanged by the covering involution σ, and the two points in each pair have the same contribution to $[\widehat{\alpha}]\cdot[\widehat{\beta}]$. It follows that $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ is even. □

We now need to better understand the action of σ_* on the homology group $H_1(\widehat{U};\mathbb{Z})$.

It will be convenient to systematically use a notation which already appeared in Lemma 27. If V is a space where some restriction of the covering involution σ induces a homomorphism σ_*, then

$$V^- = \{\alpha \in V : \sigma_*(\alpha) = -\alpha\}.$$

For instance, Lemma 27 provides a natural isomorphism $\mathcal{W}(\tau;\mathbb{Z}) \cong H_1(\widehat{U};\mathbb{Z})^-$.

Let $\partial_{\text{even}} U$ be the union of the n_{even} components of ∂U that are adjacent to an even number of spikes of $S - \tau$, and set $\partial_{\text{odd}} U = \partial U - \partial_{\text{even}} U$.

Lemma 29 *Let γ_1 be a component of $\partial_{\text{even}} U$, and let $\widehat{\gamma}_1$ be its preimage in $\widehat{U}$. Then $H_1(\widehat{\gamma}_1; \mathbb{Z})^-$ is isomorphic to $\mathbb{Z}$. In addition, the image in $H_1(\widehat{U}; \mathbb{Z})^- \cong \mathcal{W}(\tau; \mathbb{Z})$ of one of the generators of $H_1(\widehat{\gamma}_1; \mathbb{Z})^-$ coincides up to sign with the edge weight system that, right before Theorem 26, we associated to the component U_1 of $U - \tau$ that contains γ_1.*

Proof As right above Theorem 26, the curve γ_1 is homotopic to a closed curve γ_1' in τ that is made up of n_1 arcs $k_1, k_2, \dots, k_{n_1}, k_{n_1+1} = k_1$, in this order, such that each arc k_1 is immersed in τ and such that two consecutive arcs k_i and k_{i+1} locally bound a spike of U_1 at their common end point. Because n_1 is even, there are two possible ways to orient these arcs in such a way that consecutive arcs have opposite orientations. This shows that γ_1' has two distinct lifts to $\widehat{\tau}$, and therefore that the preimage $\widehat{\gamma}_1$ of γ_1 in $\widehat{U}$ consists of two components of $\partial \widehat{U}$ that are exchanged by the covering involution. This provides an isomorphism $H_1(\widehat{\gamma}_1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ where σ_* exchanges the two factors. It immediately follows that $H_1(\widehat{\gamma}_1; \mathbb{Z})^- \cong \mathbb{Z}$.

If $\widehat{\gamma}_1' \subset \widehat{\tau}$ denotes one of the two lifts of γ_1' to $\widehat{\tau}$, the image of $H_1(\widehat{\gamma}_1; \mathbb{Z})^-$ in $H_1(\widehat{U}; \mathbb{Z})^- \cong H_1(\widehat{\tau}; \mathbb{Z})^- \cong \mathcal{W}(\tau; \mathbb{Z})$ is generated by $[\widehat{\gamma}_1'] - \sigma_*([\widehat{\gamma}_1'])$. The second statement easily follows. □

To prove Theorem 26, we will first restrict attention to the case where $n_{\text{odd}} > 0$. This is equivalent to the property that $\partial_{\text{odd}} U$ is nonempty.

We just saw that the restriction of the covering $\widehat{U} \to U$ above $\partial_{\text{even}} U$ is trivial; similarly, its restriction above each component of $\partial_{\text{odd}} U$ is nontrivial. Therefore, the covering $\widehat{U} \to U$ is classified by a cohomology class in $H^1(U; \mathbb{Z}_2)$ which evaluates to 0 on the elements of $\partial_{\text{even}} U$ and to 1 on the components of $\partial_{\text{odd}} U$.

Since the subset $\partial_{\text{odd}} U$ is nonempty, and can therefore realize the cohomology class classifying the covering $\widehat{U} \to U$ as the Poincaré dual of a family $K \subset U$ of disjoint arcs whose boundary $\partial K = K \cap \partial U$ consists of one point in each component of $\partial_{\text{odd}} U$.

Split U along a separating simple closed curve γ to isolate K inside of a planar surface $U_1 \subset S$ with boundary $\partial U_1 = \gamma \cup \partial_{\text{odd}} U$, while the closure U_2 of $U - U_1$ has genus h and boundary $\partial U_2 = \gamma \cup \partial_{\text{even}} U$. Let $\widehat{U}_1$ and $\widehat{U}_2$ be the respective preimages of U_1 and U_2 in $\widehat{U}$.

Since K is disjoint from U_2, the covering $\widehat{U}_2 \to U_2$ is trivial, and $\widehat{U}_2$ consists of two disjoint copies of the surface U_2 which are exchanged by σ.

The covering $\widehat{U}_1 \to U_1$ is nontrivial above each component of $\partial_{\mathrm{odd}}U$ and trivial above γ. Since the surface U_1 is planar, an Euler characteristic computation shows that $\widehat{U}_1$ has genus $\frac{1}{2}n_{\mathrm{odd}}-1$ and has $n_{\mathrm{odd}}+2$ boundary components.

Consider the Mayer–Vietoris exact sequence

$$0 \to H_1(\widehat{\gamma};\mathbb{Z}) \to H_1(\widehat{U}_1;\mathbb{Z}) \oplus H_1(\widehat{U}_2;\mathbb{Z}) \to H_1(\widehat{U};\mathbb{Z}) \to 0,$$

where $\widehat{\gamma}$ denotes the preimage of γ in $\widehat{U}$. (To explain the 0 on the right, note that the map $H_0(\widehat{\gamma};\mathbb{Z}) \to H_0(\widehat{U}_2;\mathbb{Z})$ is injective.)

Lemma 30 *Remembering that V^- denotes the (-1)–eigenspace of the action of σ_* over a space V, the above exact sequence induces another exact sequence*

$$0 \to H_1(\widehat{\gamma};\mathbb{Z})^- \to H_1(\widehat{U}_1;\mathbb{Z})^- \oplus H_1(\widehat{U}_2;\mathbb{Z})^- \to H_1(\widehat{U};\mathbb{Z})^- \to 0.$$

Proof The only point that requires some thought is the fact that the third homomorphism is surjective.

Given $u \in H_1(\widehat{U};\mathbb{Z})^-$, the first exact sequence provides $u_1 \in H_1(\widehat{U}_1;\mathbb{Z})$ and $u_2 \in H_1(\widehat{U}_2;\mathbb{Z})$ such that $u=u_1+u_2$ in $H_1(\widehat{U};\mathbb{Z})$. Since $\sigma_*(u)=-u$, we conclude that there exists $v \in H_1(\widehat{\gamma};\mathbb{Z})$ such that $\sigma_*(u_1)=-u_1+v$ in $H_1(\widehat{U}_1;\mathbb{Z})$ and $\sigma_*(u_2)=-u_2-v$ in $H_1(\widehat{U}_2;\mathbb{Z})$. Note that $v \in H_1(\widehat{\gamma};\mathbb{Z})$ is invariant under σ_*. Therefore, for the isomorphism $H_1(\widehat{\gamma};\mathbb{Z}) \cong H_1(\gamma;\mathbb{Z}) \oplus H_1(\gamma;\mathbb{Z})$ coming from the fact that each of the two components of $\widehat{\gamma}$ is naturally identified to γ, $v=(w,w)$ for some $w \in H_1(\gamma;\mathbb{Z})$. If we replace u_1 by $u_1'=u_1-(w,0)$ and u_2 by $u_2'=u_2+(w,0)$, we now have that $u=u_1'+u_2'$ with $\sigma_*(u_1')=-u_1'$ and $\sigma_*(u_2')=-u_2'$, as requested. □

We now analyze the terms of the exact sequence of Lemma 30.

The space $H_1(\widehat{U}_2;\mathbb{Z})^-$ is easy to understand, because $\widehat{U}_2$ is made up of two disjoint copies of U_2, which are exchanged by the covering involution σ. Therefore, $H_1(\widehat{U}_2;\mathbb{Z}) \cong H_1(U_2;\mathbb{Z}) \oplus H_1(U_2;\mathbb{Z})$ and, for this isomorphism, $H_1(\widehat{U}_2;\mathbb{Z})^-$ corresponds to $\{(\alpha,-\alpha) : \alpha \in H_1(U_2;\mathbb{Z})\}$. This defines an isomorphism $H_1(\widehat{U}_2;\mathbb{Z})^- \cong H_1(U_2;\mathbb{Z})$, for which the intersection form of $H_1(\widehat{U}_2;\mathbb{Z})^-$ corresponds to twice the intersection form of $H_1(U_2;\mathbb{Z})$.

Lemma 31 *There exists a basis for $H_1(\widehat{U}_2;\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$ and n_{even} blocks (0).*

In addition, we can arrange that the basis elements corresponding to the blocks (0) are the images of generators of $H(\widehat{\alpha};\mathbb{Z})^- \cong \mathbb{Z}$ as $\widehat{\alpha} \subset \widehat{U}_2$ ranges over all preimages of components α of $\partial_{\mathrm{even}}U$, and that a generator of $H_1(\widehat{\gamma};\mathbb{Z})^- \cong \mathbb{Z}$ is sent to the sum of these elements.

Proof The surface U_2 has genus h and has $n_{\text{even}}+1$ boundary components, and γ is one of these boundary components. We can therefore find a basis for $H_1(U_2;\mathbb{Z})$ in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and n_{even} blocks (0). In addition, since $\partial U_2=\gamma\cup\partial_{\text{even}}U$, we can arrange that the basis elements corresponding to the blocks (0) are the images of generators of $H_1(\alpha;\mathbb{Z})$ as α ranges over all components of $\partial_{\text{even}}U$, while the image of a generator of $H_1(\gamma;\mathbb{Z})$ is sent to the sum of these elements.

The result then follows by considering the isomorphism $H_1(\widehat{U}_2;\mathbb{Z})^-\cong H_1(U_2;\mathbb{Z})$ mentioned above. □

We now consider $H_1(\widehat{U}_1;\mathbb{Z})^-$.

Lemma 32 *There exists a basis for $H_1(\widehat{U}_1;\mathbb{Z})^-$ in which the intersection form is block diagonal with $\frac{1}{2}n_{\text{odd}}-1$ blocks $\left(\begin{smallmatrix}0 & 4\\ -4 & 0\end{smallmatrix}\right)$ and with one block (0). In addition, the block (0) corresponds to the image of the homomorphism $H_1(\widehat{\gamma};\mathbb{Z})^-\to H_1(\widehat{U}_1;\mathbb{Z})^-$ induced by the inclusion map.*

Proof We will use an explicit description of the covering $\widehat{U}_1\to U_1$, with a specific basis for $H_1(\widehat{U}_1;\mathbb{Z})$.

Recall that this covering is classified by a cohomology class in $H^1(U_1;\mathbb{Z}_2)$ that is dual to a family $K\subset U_1$ of $\frac{1}{2}n_{\text{odd}}$ disjoint arcs, with one boundary point in each component of $\partial_{\text{odd}}U$. Index the components of $\partial_{\text{odd}}U$ as $\alpha_1,\alpha_2,\dots,\alpha_{n_{\text{odd}}}$ and the components of K as $k_1,k_3,k_5,\dots,k_{n_{\text{odd}}-1}$ in such a way that k_{2i-1} joins α_{2i-1} to α_{2i}. Add to K a family of disjoint arcs $k_2,k_4,\dots,k_{n_{\text{odd}}-2}$, disjoint from the k_{2i-1}, such that each k_{2i} joins α_{2i} to α_{2i+1}. See Figure 3.

For $i=1,2,\dots,n_{\text{odd}}-1$, consider a small regular neighborhood of $k_i\cup\alpha_i\cup\alpha_{i+1}$ in U_1 and let β_i be the boundary component of this neighborhood which is neither α_i nor α_{i+1}; endow β_i by the corresponding boundary orientation. Orient each curve α_i by the boundary orientation of $\partial_{\text{odd}}U$.

The preimage of each curve α_i is a single curve $\widehat{\alpha}_i$, which we orient by the orientation of α_i. The preimage of β_j in $\widehat{U}_1$ consists of two disjoint curves. Arbitrarily choose one of these curves $\widehat{\beta}_j$ and orient it by the orientation of β_j. Then the $[\widehat{\alpha}_i]$ and $[\widehat{\beta}_j]$ form a basis for $H_1(\widehat{U}_1;\mathbb{Z})$. See Figure 3.

Consider an element $u\in H_1(\widehat{U}_1;\mathbb{Z})$, uniquely expressed in this basis as

$$u=\sum_{i=1}^{n_{\text{odd}}}a_i[\widehat{\alpha}_i]+\sum_{j=1}^{n_{\text{odd}}-1}b_j[\widehat{\beta}_j]$$

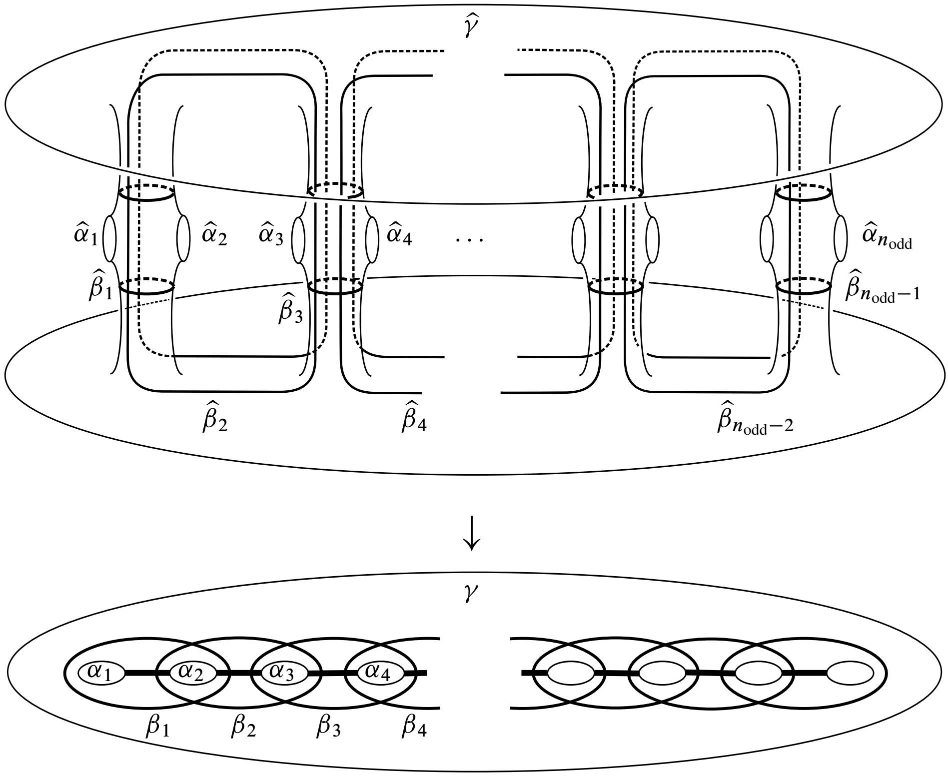

Figure 3

with all $a_i, b_j \in \mathbb{Z}$. By construction of the curves $\widehat{\alpha}_i$ and $\widehat{\beta}_j$,

$$\sigma_*([\widehat{\alpha}_i]) = [\widehat{\alpha}_i] \quad \text{and} \quad \sigma_*([\widehat{\beta}_j]) = -[\widehat{\beta}_j] - [\widehat{\alpha}_j] - [\widehat{\alpha}_{j+1}].$$

If u belongs to $H_1(\widehat{U}_1; \mathbb{Z})^-$, namely if $\sigma_*(u) = -u$, it follows from these observations and from the consideration of the coefficients of each $[\widehat{\alpha}_i]$ that we necessarily have

$$\begin{aligned} b_1 &= 2a_1, \\ b_i + b_{i-1} &= 2a_i \quad \text{for every } i \text{ with } 2 \leqslant i \leqslant n_{\mathrm{odd}} - 1, \\ b_{n_{\mathrm{odd}}-1} &= 2a_{n_{\mathrm{odd}}}. \end{aligned}$$

In particular, the coefficients b_j are all even, and

$$u = \tfrac{1}{2}(u - \sigma_*(u)) = \sum_{j=1}^{n_{\mathrm{odd}}-1} \tfrac{1}{2} b_j ([\widehat{\beta}_j] - \sigma_*([\widehat{\beta}_j])).$$

Therefore, the elements $[\widehat{\beta}_j]-\sigma_*([\widehat{\beta}_j])$ generate $H_1(\widehat{U}_1;\mathbb{Z})^-$. Since these elements $[\widehat{\beta}_j]-\sigma_*([\widehat{\beta}_j]) = 2[\widehat{\beta}_j]+[\widehat{\alpha}_j]+[\widehat{\alpha}_{j+1}]$ are linearly independent, they form a basis for $H_1(\widehat{U}_1;\mathbb{Z})^-$.

Note that $[\widehat{\beta}_j]\cdot[\widehat{\beta}_{j'}]=0$ if $|j-j'|>1$, and $[\widehat{\beta}_j]\cdot[\widehat{\beta}_{j+1}]=\varepsilon_j=\pm 1$, where the sign depends on which lift of β_j we chose for $\widehat{\beta}_j$. Also,

$$\sigma_*([\widehat{\beta}_j])\cdot[\widehat{\beta}_{j'}] = [\widehat{\beta}_j]\cdot\sigma_*([\widehat{\beta}_{j'}]) = -\sigma_*([\widehat{\beta}_j])\cdot\sigma_*([\widehat{\beta}_{j'}]) = -[\widehat{\beta}_j]\cdot[\widehat{\beta}_{j'}].$$

It follows that, in the basis of $H_1(\widehat{U}_1;\mathbb{Z})^-$ formed by the $[\widehat{\beta}_j]-\sigma_*([\widehat{\beta}_j])$, the matrix of the intersection form is

$$\begin{pmatrix} 0 & 4\varepsilon_1 & 0 & 0 & \cdots & 0 & 0 \\ -4\varepsilon_1 & 0 & 4\varepsilon_2 & 0 & \cdots & 0 & 0 \\ 0 & -4\varepsilon_2 & 0 & 4\varepsilon_3 & \cdots & 0 & 0 \\ 0 & 0 & -4\varepsilon_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 4\varepsilon_{n_{\mathrm{odd}}-2} \\ 0 & 0 & 0 & 0 & \cdots & -4\varepsilon_{n_{\mathrm{odd}}-2} & 0 \end{pmatrix}$$

By block diagonalizing this matrix, a final modification of the basis provides a new basis for $H_1(\widehat{U}_1;\mathbb{Z})^-$ in which the intersection form is block diagonal with $\frac{1}{2}n_{\mathrm{odd}}-1$ blocks $\left(\begin{smallmatrix} 0 & 4 \\ -4 & 0 \end{smallmatrix}\right)$ and with one block (0).

There remains to show that the block (0) corresponds to the image of $H_1(\widehat{\gamma};\mathbb{Z})^-$. This could be seen by explicitly analyzing the block diagonalization process of the above matrix. However, it is easier to note that $H_1(\widehat{\gamma};\mathbb{Z})^- \cong \mathbb{Z}$ is generated by $[\widehat{\gamma}_1]-[\widehat{\gamma}_2]$, where $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are the two components of the preimage $\widehat{\gamma}$ of γ and are oriented by the boundary orientation of $\partial\widehat{U}_1$. Then $[\widehat{\gamma}_1]-[\widehat{\gamma}_2]$ is in the kernel of the intersection form of $H_1(\widehat{U}_1;\mathbb{Z})^-$, since $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are in the boundary of $\widehat{U}_1$, and generate this kernel since it is isomorphic to $\mathbb{Z}$ and since $[\widehat{\gamma}_1]-[\widehat{\gamma}_2]$ is indivisible in $H_1(\widehat{U}_1;\mathbb{Z})$. □

We now only need to combine the computations of Lemmas 30, 31 and 32 to obtain a basis of $H_1(\widehat{U};\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, $\frac{1}{2}n_{\mathrm{odd}}-1$ blocks $\left(\begin{smallmatrix} 0 & -4 \\ 4 & 0 \end{smallmatrix}\right)$, and n_{even} blocks (0).

Applying Lemmas 27 and 28 to connect this to the Thurston intersection form on the edge weight space $\mathcal{W}(\tau;\mathbb{Z})$, we conclude that $\mathcal{W}(\tau;\mathbb{Z})$ admits a basis in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\frac{1}{2}n_{\mathrm{odd}}-1$ blocks $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, and n_{even} blocks (0). In addition, by the second half of Lemma 31 and using Lemma 29, the generators corresponding to the blocks (0) can be assumed to correspond to the elements of $\mathcal{W}(\tau;\mathbb{Z})$ associated to the components of $\partial_{\mathrm{even}}U$.

This proves Theorem 26, under our assumption that $n_{\mathrm{odd}}>0$.

We now consider the case where $n_{\mathrm{odd}} = 0$, namely where $\partial_{\mathrm{odd}} U = \varnothing$, and where the train track τ is nonorientable. This second property is equivalent to the property that the covering $\widehat{U} \to U$ is nontrivial. We can then realize the cohomology class of $H^1(U;\mathbb{Z}_2)$ classifying the covering $\widehat{U} \to U$ as the Poincaré dual of a nonseparating simple closed curve K. Let $U_1 \subset U$ be a surface of genus 1 containing K and bounded by a simple closed curve γ, and let U_2 be the closure of $U - U_1$. As before, let $\widehat{U}_1$, $\widehat{U}_2$ and $\widehat{\gamma}$ denote the respective preimages of U_1, U_2 and γ in $\widehat{U}$.

The computation of Lemma 31 applies to this case as well, and provides a basis for $H_1(\widehat{U}_2;\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$ and n_{even} blocks (0).

The surface $\widehat{U}_1$ is a twice-punctured torus. A simple analysis of the covering $\widehat{U}_1 \to U_1$ shows that $H_1(\widehat{U}_1;\mathbb{Z})^- \cong \mathbb{Z}$ is equal to the image of $H_1(\widehat{\gamma};\mathbb{Z})^-$. The intersection form of $H_1(\widehat{U}_1;\mathbb{Z})^-$ is then 0.

Again, combining these computations with the exact sequence

$$0 \to H_1(\widehat{\gamma};\mathbb{Z})^- \to H_1(\widehat{U}_1;\mathbb{Z})^- \oplus H_1(\widehat{U}_2;\mathbb{Z})^- \to H_1(\widehat{U};\mathbb{Z})^- \to 0$$

provides in this case a basis for $H_1(\widehat{U};\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$ and n_{even} blocks (0). Using Lemmas 27 and 28, this provides the result promised in Theorem 26 in this case as well. The fact that the generators corresponding to the blocks (0) can be chosen to be the elements associated to the components of $\partial_{\mathrm{even}} U$ is a byproduct of the proof as in the previous case.

Finally, we need to consider the case where $n_{\mathrm{odd}} = 0$ and the train track τ is orientable. Then the covering $\widehat{U} \to U$ is trivial, so $H_1(\widehat{U};\mathbb{Z})^- \cong H_1(U;\mathbb{Z})$ in such a way that the intersection form of $H_1(\widehat{U};\mathbb{Z})^-$ corresponds to twice the intersection form of $H_1(U;\mathbb{Z})$. By Lemma 28, the last case of Theorem 26 immediately follows. □

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