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Affine Hirsch foliations on 3–manifolds

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This paper is devoted to discussing affine Hirsch foliations on 3–manifolds. First, we prove that up to isotopic leaf-conjugacy, every closed orientable 3–manifold M admits zero, one or two affine Hirsch foliations. Furthermore, every case is possible.

Then we analyze the 3–manifolds admitting two affine Hirsch foliations (we call these *Hirsch manifolds*). On the one hand, we construct Hirsch manifolds by using exchangeable braided links (we call such Hirsch manifolds *DEBL Hirsch manifolds*); on the other hand, we show that every Hirsch manifold virtually is a DEBL Hirsch manifold.

Finally, we show that for every $n \in \mathbb{N}$, there are only finitely many Hirsch manifolds with strand number n. Here the strand number of a Hirsch manifold M is a positive integer defined by using strand numbers of braids.

57M50, 57R32; 37E10, 57M25

1 Introduction

In 1975, Hirsch [8] constructed an analytic 2–foliation on a closed 3–manifold such that the foliation contains exactly one exceptional minimal set. Let's briefly recall his construction here.

The foliation is constructed by starting with a solid torus and removing from the interior another solid torus which wraps around the original solid torus twice. This gives us a manifold, foliated by 2–punctured disks, with two transverse tori as boundary components. We then glue the exterior boundary component to the interior boundary component to obtain a foliated manifold without boundary. Hirsch chose a gluing map carefully so that the 2–punctured fibration structure induces a foliation and the induced foliation is analytic and contains exactly one exceptional minimal set.

There are many variations of Hirsch's construction in the literature, for instance, Ghys [7], Biś, Hurder and Shive [4]:

- Ghys [7] considered a variant of Hirsch's construction: Hirsch's gluing map is changed to a map that is "affine" in some sense. In [4], the authors call these foliations *affine Hirsch foliations*.

- In [4], the authors generalize Hirsch's construction in many cases. In the three-dimensional case, they generalize Hirsch's construction by starting with a solid torus V and removing from V a small solid torus V_0 so that V_0 can be regarded as a tubular neighborhood of a closed twisted braid in V.

Actually, it is natural to further generalize these foliations by using braids:

- For every n–braid b whose closure is a knot, starting with a solid torus V and removing from its interior a small solid torus V_0 which is a small tubular neighborhood of the closure of b, we get a compact 3–manifold, foliated by n–punctured disks, with two boundary components transverse to the n–punctured disk fibration.
- Then we glue the exterior boundary component to the interior boundary component to obtain a foliated manifold induced by the n–punctured disk fibration.

For simplicity, we still call the new foliations *Hirsch foliations*, which are the main objects in this paper. Similarly, if the gluing map is "affine" in some sense, we call the Hirsch foliation *affine*. More precise definitions can be found in Section 2.

There are several kinds of discussions about Hirsch foliations in the literature:

- Biš, Hurder and Shive [4] generalized Hirsch's construction to construct analytic foliations of arbitrary codimension with exotic minimal sets.
- Alvarez and Lessa [1] considered the Teichmüller space of a Hirsch foliation.
- Shive in his thesis [12] considered the following conjugacy question: fixing two Hirsch foliations $(M_1, \mathcal{H}_1)$ and $(M_2, \mathcal{H}_2)$, a C_r leaf-conjugacy diffeomorphism $H\colon M_1 \to M_2$ and an integer $k \in \mathbb{N}$, how does one find conditions on the foliations and the map H which ensure that the map H is $C_{k+\lambda}$?

In this paper, we also would like to discuss a conjugacy question. In contrast to what Shive did, we hope to understand the leaf-conjugacy classes of Hirsch foliations. We say two foliations $\mathcal{H}_1$ and $\mathcal{H}_2$ on a closed 3–manifold M are *isotopically leaf-conjugate* if there exists a homeomorphism $h\colon M \to M$ which maps every leaf of $\mathcal{H}_1$ to a leaf of $\mathcal{H}_2$ and is isotopic to the identity map on M. We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the same up to *isotopic leaf-conjugacy* if $\mathcal{H}_1$ and $\mathcal{H}_2$ are isotopically leaf-conjugate. In this paper, we will restrict ourselves to affine Hirsch foliations. The reasons why we focus on affine Hirsch foliations are the following:

- A Hirsch foliation always can be easily rebuilt (see Remark 4.4) by modifying the gluing map of an affine Hirsch foliation.
- Affine Hirsch foliations are natural objects in dynamical systems: the projection of the stable manifolds of a Smale solenoid attractor on the orbit space of the wandering

set (by a Smale solenoid mapping on a solid torus) is an affine Hirsch foliation. Our forthcoming paper [16] will focus on this topic.

Now we can naturally ask the following question as the main motivation for this paper.

Question 1.1 For a closed 3–manifold M, can we classify all affine Hirsch foliations up to isotopic leaf-conjugacy?

Alvarez and Lessa [1, Section 1.3] have discussed this question on the 3–manifolds constructed by Hirsch. As a first step toward answering Question 1.1, we have:

Theorem 1.2 *Let M be a closed orientable 3–manifold. Then M admits 0, 1 or 2 affine Hirsch foliations up to isotopic leaf-conjugacy.*

Then one naturally would like to answer:

Question 1.3 (1) Which 3–manifolds admit a Hirsch foliation?

(2) Which 3–manifolds admit two nonisotopically leaf-conjugate affine Hirsch foliations and what are the relations between these two foliations?

Actually, to the first item of Question 1.3, on the one hand, which manifolds these are is very clear, ie each one is precisely determined by a braid and a gluing map; on the other hand, it is not easy to describe all of these manifolds in a straightforward way. Nevertheless, we would like to give some characterizations of these 3–manifolds.

Proposition 1.4 *Let M be a closed orientable 3–manifold which admits an (affine) Hirsch foliation. Then*

(1) *M is a toroidal 3–manifold whose JSJ diagram is cyclic;*

(2) *each JSJ piece is either hyperbolic or a $S(0,2;q/p)$–type Seifert manifold where p and q $(0<q<p)$ are coprime.*

This proposition is a consequence of Lemma 3.7 and Corollary 3.8.

We are more interested in the second item of Question 1.3. We call a 3–manifold M a *Hirsch manifold* if M admits two nonisotopically leaf-conjugate Hirsch foliations. Notice that the 3–manifold constructed by Hirsch in [8] actually is a Hirsch manifold.

Actually, there are many Hirsch manifolds; see Section 4.2 and Proposition 4.3. The following are the reasons why we are interested in Hirsch manifolds:

- A Hirsch manifold has some nice symmetric structures.

- Hirsch manifolds and their two affine Hirsch foliations will play a central role in a class of dynamical systems: in [16], the author will use Hirsch manifolds and affine

Hirsch foliations to discuss a kind of Ω–stable diffeomorphism on 3–manifolds whose nonwandering set is the union of a Smale solenoid attractor and a Smale solenoid repeller.

The exchangeably braided links introduced by Morton [10] will play a crucial role in describing Hirsch manifolds. An *exchangeably braided link* is a two-component link $L = K_1 \cup K_2$ in S^3 such that each component is braided relative to the other one. More details about exchangeably braided links can be found in Section 2.

Motivated by the second item of Question 1.3, we will give two observations to describe the relationships between exchangeably braided links and Hirsch manifolds. The first observation is that for every exchangeably braided link $L = K_1 \cup K_2$, one can build a (unique) Hirsch manifold following a series of standard combinatorial surgeries (see Section 4). Such a Hirsch manifold is called a *Hirsch manifold derived from an exchangeably braided link* (abbreviated as a *DEBL Hirsch manifold*). The second observation is that every Hirsch manifold virtually is a DEBL Hirsch manifold. More precisely:

Theorem 1.5 *Let M be a Hirsch manifold. Then there exists a q_2–covering space of M, denoted by $\widetilde{M}$, such that $\widetilde{M}$ is a Hirsch manifold derived from an exchangeably braided link (a DEBL Hirsch manifold). Moreover, q_2 can be divided by $n^2 - 1$ where n is the strand number of M.*

Here, the *strand number* of a Hirsch manifold M (see Definition 4.2) is defined to be the strand number of a braid which can be used to build the Hirsch manifold M.

Hirsch manifolds have the following finiteness property.

Proposition 1.6 *For every $n \in \mathbb{N}$, there are only finitely many Hirsch manifolds with strand number n.*

In the final section (Section 5), we will build an example to show:

Proposition 1.7 *There exists a 3–manifold which admits a Hirsch foliation but is not a Hirsch manifold.*

Proposition 1.4, Proposition 4.3, the examples in Section 4 and Proposition 1.7 imply that there exist closed oriented 3–manifolds M_0, M_1 and M_2 such that

- M_0 doesn't admit any affine Hirsch foliations;
- M_1 admits exactly one affine Hirsch foliation;
- M_2 is a Hirsch manifold, ie M_2 admits two nonisotopically leaf-conjugate affine Hirsch foliations.

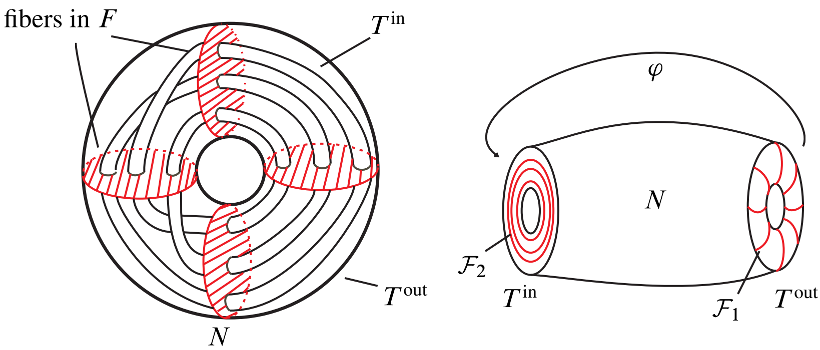

Figure 1: Constructing a Hirsch foliation

This means that every case in Theorem 1.2 can be realized (in some closed 3–manifold).

We can see that the results in this paper (in particular, Proposition 4.3 and Theorem 1.2) give a satisfying response to the problem of classifying all Hirsch foliations. They allow us to reduce classifying all Hirsch foliations to a classical problem in one-dimensional dynamical systems: classifying degree-n $(n \geq 2)$ endomorphisms[1] on S^1 up to conjugacy. More details can be found in Remark 4.4.

2 Preliminaries

Definition 2.1 Let $\mathcal{H}$ be a codimension-1 foliation on a closed oriented 3–manifold M. $\mathcal{H}$ is called a *Hirsch foliation* if there exists a torus T embedded into M such that

(1) the path closure of $M - T$, denoted by N, is a compact oriented 3–manifold with two tori T^{out} and T^{in} as its boundary;

(2) $\mathcal{H}|_N$ is an n–punctured disk fibration on N such that each fiber is transverse to ∂N;

(3) every leaf in $\mathcal{H}$ is orientable.

By Definition 2.1, a Hirsch foliation $\mathcal{H}$ on a closed oriented 3–manifold M can be constructed as follows (see Figure 1 for an illustration):

- Choose an n–braid b whose closure is a knot; b also can be used to represent a diffeomorphism on an n–punctured disk Σ.

- We denote the mapping torus of (Σ, b) by N. Notice that $F = \{\Sigma \times \{\star\}\}$ provides a natural n–punctured disk fibration on N, which provides T^{in} and T^{out} two S^1–fibration structures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

[1] An endomorphism on S^1 means a monotonic continuous map on S^1.

- We suppose that Σ is oriented. Then Σ naturally induces an orientation on each fiber of $\mathcal{F}_1$ and $\mathcal{F}_2$. We also give an orientation on N which naturally induces two orientations on T^{out} and T^{in}, respectively.

- Build an orientation-preserving homeomorphism $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ which maps every fiber of $\mathcal{F}_1$ to a fiber of $\mathcal{F}_2$ and preserves the corresponding orientations.[2] Let $M = N \setminus x \sim \varphi(x)$ $(x \in T^{\text{out}}N)$. Then the n–punctured disk fibration F on N naturally induces a Hirsch foliation $\mathcal{H}$ on M by φ.

There are some further comments about Hirsch foliations which will be useful:

- N also can be obtained by removing a small solid torus V_0 from a solid torus V where V_0 is a small tubular neighborhood of the closure of a braid b.

- There is a natural quotient map $P\colon N \to S^1$ where S^1 is the fiber quotient space of F.

- φ induces a map $\varphi_2\colon S^1 \to S^1$, which is called the *projective holonomy map* of $\mathcal{F}$ relative to the embedded torus T.

Definition 2.2 Let $\mathcal{H}$ be a Hirsch foliation on a closed 3–manifold M. $\mathcal{H}$ is called an *affine Hirsch foliation* if the projective holonomy map of $\mathcal{F}$ relative to an embedded torus T transverse to $\mathcal{H}$ is topologically conjugate to the map z^n on S^1 for some $n \in \mathbb{N}$ satisfying $n \geq 2$. Here we can parametrize S^1 by $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

In 1985, Morton [10] introduced exchangeably braided links. An *exchangeably braided link* is a two-component link $L = K_1 \cup K_2$ which admits a kind of very nice symmetry: each component is braided relative to the other one, ie K_1 is a closed braid $\widetilde{b}_1$ in the solid torus $S^3 - K_2$ and K_2 is a closed braid $\widetilde{b}_2$ in the solid torus $S^3 - K_1$. Such a braid b_1 is called an *exchangeable braid*. Automatically, every exchangeably braided link L can be regarded as the union of the closure of an exchangeable braid and an axis of the closed braid.

Morton [10] showed many nice properties of exchangeably braided links. For instance, he built some necessary and some sufficient conditions for exchangeability. For instance, he showed that the exchangeable braids belong to a family of braids introduced by Stallings [13].

Let's briefly introduce Stallings braids and the relationships between Stallings braids and exchangeable braids. Certainly, the closure of an exchangeable braid is a trivial

[2] $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ preserves the orientations since the glued manifold M should be orientable. φ preserves the corresponding orientations of the fibers of $\mathcal{F}_1$ and $\mathcal{F}_2$ since every leaf in the glued foliation $\mathcal{H}$ should be orientable.

knot. But the converse is not true, ie if the closure of a braid b is a trivial knot, b is not necessarily an exchangeable braid. Actually, Stallings [13] introduced a family of braids in which every braid b satisfies the following:

(1) $\widetilde{b}$ is a trivial knot.

(2) There is a disk D spanning $\widetilde{b}$ which intersects the axis at exactly n points.

Morton called these *Stallings braids*. The set of Stallings braids is a proper subset of the union of the braids whose closure is a trivial knot. In [10], Morton constructed a braid $\omega = \sigma_3\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1 \in B_4$ [3] which is a Stallings braid but not an exchangeable braid. Therefore, the union of the exchangeable braids is a proper subset of the union of Stallings braids.

Stallings braids have a very nice characterization.[4] Under this characterization, it is easy to obtain the following finiteness property.

Proposition 2.3 *For a given $n \in \mathbb{N}$, up to conjugacy, there are finitely many Stallings braids with strand number n.*

The following corollary is an immediate consequence of this proposition (this corollary exactly is [10, Corollary 1.2]).

Corollary 2.4 (1) *Up to conjugacy, there are finitely many exchangeable braids with strand number n.*

(2) *Up to isotopy, there are finitely many exchangeably braided links with linking number n.*

3 Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. We will prove an equivalent form of the theorem: if a closed 3–manifold M admits a Hirsch foliation $\mathcal{F}$, then up to isotopic leaf-conjugacy, M admits at most two affine Hirsch foliations.

First, we give more notation and parameters (see Figure 2 as an illustration):

- Assume that $i_1\colon T^{\text{out}} \to N$ and $i_2\colon T^{\text{in}} \to N$ are the associated embedding maps and $i_{1,\star}\colon H_1(T^{\text{out}}) \to H_1(N)$ and $i_{2,\star}\colon H_1(T^{\text{in}}) \to H_1(N)$ are the corresponding induced homomorphisms.

- We denote the oriented simple closed curve $\Sigma \cap T^{\text{out}}$ by m_1 and denote the oriented simple closed curves $\Sigma \cap T^{\text{in}}$ by $m_2^1, \dots, m_2^n$. Here, the orientations of the simple closed curves are induced by Σ. Sometimes we also use m_2 to represent m_2^1.

[3] Here and below, the notation for braids is standard in braid theory (see, for instance, Birman [3]).

[4] A careful reader can find the characterization in the beginning of [10, Section 2].

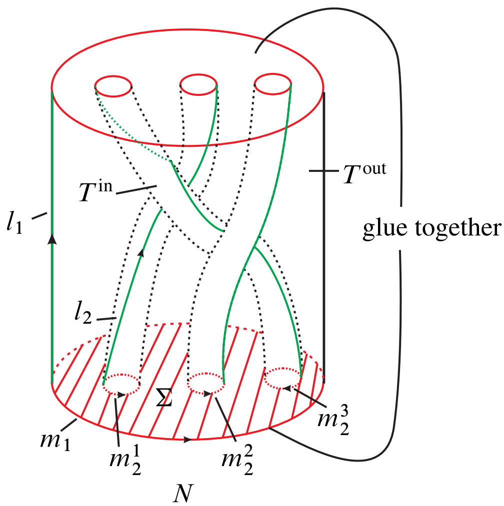

Figure 2: Notation used in the proof of Theorem 1.2

- l_1 is chosen to be an oriented simple closed curve in T^{out} which intersects m_1 at one point such that (m_1, l_1) endows T^{out} with an orientation which is consistent with the orientation of T^{out}.

- l_2 is chosen to be the unique (up to isotopy in T^{in}) oriented simple closed curve in T^{in} such that

  (1) l_2 intersects m_2 at one point;

  (2) (m_2, l_2) endows T^{in} with an orientation which is consistent with the orientation of T^{in};

  (3) $i_{2,\star}([l_2]) = n \cdot i_{1,\star}([l_1])$.

- If there are two oriented simple closed curves m and l on a torus T^2 which intersect at one point, we will use $pm + ql$ (p and q are coprime) to represent an oriented simple closed curve on T^2 which wraps p times around m and q times around l.

The existence and the uniqueness of l_2 can be shown by a short computation on homology, as follows. If we choose a simple closed curve $l_2' \in T^{\text{in}}$ so that l_2' intersects m_2 at one point and the orientation given by (m_2, l_2') is consistent with the orientation of T^{in} given by N, then $i_{2,\star}([l_2']) = n \cdot i_{1,\star}([l_1]) + x \cdot i_{2,\star}([m_2])$ for some $x \in \mathbb{Z}$. Since $i_{2,\star}([m_2])$ is nonzero in $H_1(N)$, there is, up to isotopy, a unique simple closed curve $l_2 = l_2' - x \cdot m_2$ in T^{in} such that $i_{2,\star}([l_2]) = n \cdot i_{1,\star}([l_1])$. For simplicity, we will use $[m_j]$ and $[l_j]$ ($j = 1, 2$) to represent the corresponding elements in $H_1(N)$.

We collect some information about $H_1(N)$ as follows, which can be obtained by Alexander duality.

Lemma 3.1 *$H_1(N) \cong \mathbb{Z} \oplus \mathbb{Z}$, and it is generated by $[l_1]$ and $[m_2]$. Moreover, $[m_1] = n[m_2]$ and $[l_2] = n[l_1]$.*

The following lemma shows that the set of all punctured disk fibrations on N is quite limited.

Lemma 3.2 *Let F be an s–punctured disk fibration on N. Assume that Σ is a fiber of F whose boundary is the union of a simple closed curve $c_1 \in T^{\mathrm{out}}$ and s pairwise parallel and pairwise disjoint simple closed curves $c_2^1, \dots, c_2^s$ in T^{in} (sometimes we also use c_2 to represent c_2^1). Assume that $c_i = p_i m_i + q_i l_i$ $(i = 1, 2)$ where p_i and q_i are coprime. Then there exists an orientation on Σ which induces an orientation on c_1 and an orientation on c_2 such that $s = n$, $p_1 = p_2 = 1$ and $q_1 = n^2 q_2$.*

Proof First let us prove that $p_2 = 1$. If we glue a solid torus V to N by a gluing map $\psi\colon \partial V \to T^{\mathrm{in}}$ so that c_2 bounds a disk in V, then the glued 3–manifold U is homeomorphic to a solid torus. On the one hand, it is obvious that $H_1(U) = \langle [l_1] \rangle \cong \mathbb{Z}$. On the other hand, $H_1(U) = \langle [m_2], [l_1] : p_2[m_2] + q_2[l_2] = p_2[m_2] + nq_2[l_1] = 0 \rangle$. Therefore, $p_2 = \pm 1$.

Furthermore, we can endow Σ with an orientation which induces two orientations on c_1 and c_2, respectively, so that $p_2 = 1$. These orientations satisfy the requirements in the lemma and will be used in the remainder of the proof.

To conclude, we will prove that $p_1 = 1$ and $s = n$. Since the union of $c_2^1, \dots, c_2^s$ and c_1 bound an s–punctured disk Σ, we have $[c_1] = s[c_2]$. Equivalently, $p_1[m_1] + q_1[l_1] = s([m_2] + nq_2[l_1])$, and so $q_1 = sq_2 n$ and $p_1 n = s$. We have $q_1 = p_1 q_2 n^2$. Recall that p_1 and q_1 are coprime, and therefore $p_1 = 1$, $s = n$ and $q_1 = n^2 q_2$. □

Remark 3.3 Actually, for every $q_2 \in \mathbb{Z}$, there always exists an associated punctured disk fibration F on N. One can construct it by a standard surgery in low-dimensional topology (for the surgery, see, for instance, Jaco [9, III.14]).

From now on, c_1, c_2 and Σ are oriented as Lemma 3.2.

Lemma 3.4 *Let $\varphi\colon T^{\mathrm{out}} \to T^{\mathrm{in}}$ be a diffeomorphism such that $\varphi(m_1) = m_2$ and $\varphi(l_1) = l_2 + ym_2$ $(y \in \mathbb{Z})$. If $\varphi(c_1)$ is isotopic to c_2 in T^{in}, then c_1 is isotopic to m_1 in T^{out} and c_2 is isotopic to m_2 in T^{in}.*

Proof On the one hand,

$$\begin{aligned}\varphi_\star([c_1]) &= \varphi_\star([m_1]+q_1[l_1]) = \varphi_\star([m_1]) + q_1\varphi_\star([l_1])\\ &= m_2 + q_1(l_2 + y m_2) = (1+q_1 y)m_2 + q_1 l_2;\end{aligned}$$

on the other hand,

$$\varphi_\star([c_1]) = [c_2] = [m_2] + q_2[l_2].$$

Therefore, $1+q_1 y = 1$ and $q_1 = q_2$. By Lemma 3.2, $q_1 = n^2 q_2$ and $|n| \geq 2$. Then $q_1 = q_2 = 0$. Also notice that $p_1 = p_2 = 1$ (Lemma 3.2). The conclusions of the lemma follow. □

Lemma 3.5 *If F_1 and F_2 are two n–punctured disk fibrations on N with two fibers Σ_1 and Σ_2, respectively, such that $\partial\Sigma_1 = \partial\Sigma_2$ is the union of m_1 and n simple closed curves $m_2^1, \dots, m_2^n$ which are pairwise isotopic, then Σ_1 is isotopic to Σ_2 relative to $\partial\Sigma_1 = \partial\Sigma_2$ in N.*

Proof Up to isotopy, we can assume that $\mathrm{int}(\Sigma_1) \cap \mathrm{int}(\Sigma_2)$ is the union of finitely many pairwise disjoint simple closed curves $\alpha_1, \dots, \alpha_m$. Here $\mathrm{int}(\Sigma_i)$ $(i = 1, 2)$ is defined to be the interior of Σ_i. Moreover, we assume that $m \geq 1$ and m is minimal up to isotopy relative to $\partial\Sigma_1 = \partial\Sigma_2$.

First, we will show that every α_i $(i \in \{1, \dots, m\})$ is essential in Σ_2. Otherwise, some α_i bounds a disk D_2 in Σ_2. Notice that Σ_1 is incompressible in N, and the union of D_1 and D_2, denoted by S, is a 2–sphere embedded in N. Since N is an irreducible 3–manifold, S bounds a 3–ball in N. This means that we can do a surgery on Σ_2 in a small neighborhood of the 3–ball to obtain Σ_2' so that Σ_2' is isotopic to Σ_2 and the number of connected components of $\Sigma_2' \cap \Sigma_1$ is smaller than m. This contradicts the assumption.

Then there is a nested k–punctured disk $D_1^k \subset \Sigma_1$ with boundary $\alpha_j \cup (m_2^{s_1} \cup \cdots \cup m_2^{s_k})$ for some $j \in \{1, \dots, m\}$, where α_j is an essential simple closed curve in the interior of Σ_1. Here the fact that D_1^k is a nested disk means that the interior of D_1^k is disjoint from Σ_2. We cut N along Σ_1 to obtain a 3–manifold N_0 which is homeomorphic to $\Sigma_1 \times [0, 1]$. Because $\partial\Sigma_1 = \partial\Sigma_2$ and D_1^k is a nested k–punctured disk, by a simple argument on N_0, one can obtain that ∂D_1^k also bounds a nested k–punctured disk D_2^k in N_0. We define Σ_3 to be $(\Sigma_1 - D_1^k) \cup D_2^k$, which is an incompressible k–punctured disk. Since N_0 is homeomorphic to $\Sigma_1 \times [0, 1]$, we have that Σ_3 is isotopic to Σ_1 relative to $\Sigma_1 - D_1^k$ in N_0. We can push Σ_3 a little into the interior of N_0 to Σ_3' so that the intersection number of Σ_3' and Σ_2 is strictly smaller than the intersection number of Σ_1 and Σ_2. This contradicts the minimality. □

Now we deal with the trouble that maybe there are many incompressible tori in a Hirsch manifold. For this purpose, we should observe more topological information about N.

First, we recall some classical facts about the geometry and topology of surface bundles. The Nielsen–Thurston theorem (see, for instance, Fathi, Laudenbach and Poenaru [6]) states that a homeomorphism f on a compact surface Σ is isotopic to one of three types according to its dynamics: periodic, reducible and pseudo-Anosov. The Thurston geometrization theorem for surface bundles (see Thurston [14]) implies that the Nielsen–Thurston theorem deeply involves the geometric structure of three-dimensional manifolds as follows: the mapping torus $M_f = \Sigma \times I/(s,1) \sim (f(s),0)$ is an irreducible 3–manifold, and moreover,

(1) M_f is hyperbolic if and only if f is pseudo-Anosov;

(2) M_f is Seifert-fibered if and only if f is periodic;

(3) M_f contains an essential torus (hence we can perform JSJ decomposition) if and only if f is reducible.

In particular, in the third case, there exists a collection of essential simple closed curves in Σ so that the suspension of these curves can be glued up by a map isotopic to f to give a collection of essential tori and Klein bottles which are the collection of JSJ tori and Klein bottles. In the following lemma, we formalize some facts about the geometry and topology of surface bundles which will be very useful.

Lemma 3.6 *Let $M_f = \Sigma \times I/(s,1) \sim (f(s),0)$ be a mapping torus where Σ is a compact orientable surface and f is an orientation-preserving homeomorphism on Σ. Then*

(1) *M_f is an irreducible 3–manifold and every JSJ piece of M_f is either hyperbolic or Seifert;*

(2) *every JSJ torus T of M_f corresponds to an essential simple closed curve c in Σ which is periodic up to isotopy under f.*

Now, we come back to observing some topological information about N.

Lemma 3.7 *N is an irreducible 3–manifold such that*

(1) *every JSJ piece of N is either hyperbolic or Seifert;*

(2) *the JSJ diagram of N is a path;*

(3) *every Seifert piece is homeomorphic to $S(0,2;q/p)$ where p and q $(0<q<p)$ are coprime and $S(0,2;q/p)$ represents the Seifert manifold whose base orbifold is a 2–punctured sphere with a (q/p)–singularity.*

Proof Recall that N can be defined to be the mapping torus of (Σ, b), where Σ is an n–punctured compact disk and b is a homeomorphism on Σ. Then by item (1) of Lemma 3.6, N is an irreducible 3–manifold such that every JSJ piece is either hyperbolic or Seifert.

By item (2) of Lemma 3.6, every JSJ torus T of N corresponds to an essential simple closed curve c in Σ which is periodic up to isotopy under b. On the one side, notice that every simple closed curve in Σ is separating, and so every JSJ torus of N is separating. This implies that the JSJ diagram of N is a tree. On the other side, ∂N is the union of two tori, T^{out} and T^{in}. Combing these two observations, one could immediately obtain that the JSJ diagram of N is a path.

Let N_0 be a Seifert piece of N. Then N_0 is homeomorphic to a solid torus minus a small open tubular neighborhood of a closed braid $\widetilde{b}_0$. Since N_0 is Seifert, b_0 should be a periodic braid. Since every periodic homeomorphism on a disk is conjugate to a rotation (see Constantin and Kolev [5]), up to conjugacy, b_0 should be a twisted braid. This implies that N_0 is homeomorphic to a Seifert manifold $S(0,2;q/p)$. □

Recall that $M = N \setminus x \sim \varphi(x)$ $(x \in T^{\text{out}}N)$. By Lemma 3.7, the gluing map φ glues the two JSJ pieces corresponding to the two ends of the JSJ diagram of N (notice that the two JSJ pieces might be the same), and the two JSJ pieces should belong to one of the following three cases:

(1) Both of them are hyperbolic.

(2) One of them is hyperbolic and the other one is Seifert.

(3) Both of them are Seifert.

In the first two cases, it is obvious that the glued torus T is a JSJ torus in M. In the third case, since $\varphi(m_1) = m_2$, one can easily check that up to isotopy, $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ doesn't map a regular fiber on T^{out} to a regular fiber on T^{in} induced by the associated Seifert pieces. Therefore, T is also a JSJ torus in M. Now naturally we have the following corollary.

Corollary 3.8 *Let M be a closed orientable 3–manifold which admits a Hirsch foliation. Then every incompressible torus T embedded in M is a JSJ torus and the JSJ diagram of M is cyclic.*

Lemma 3.9 *Let M be a closed 3–manifold which admits an affine Hirsch foliation. We have the following conclusions.*

(1) *M is the union of n JSJ pieces $M_1, M_2, \dots, M_n$ by the gluing maps*

$$\varphi_1\colon \partial^{\text{out}} M_1 \to \partial^{\text{in}} M_2, \quad \dots, \quad \varphi_{n-1}\colon \partial^{\text{out}} M_{n-1} \to \partial^{\text{in}} M_n \quad \text{and} \quad \varphi_n\colon \partial^{\text{out}} M_n \to \partial^{\text{in}} M_1.$$

Here the union of T_i^{out} and T_i^{in} is the boundary of M_i ($i \in \{1, \dots, n\}$).

(2) *Let $\{T_1, \dots, T_n\}$ be a union of the maximal pairwise disjoint, pairwise nonparallel JSJ tori of M and $\mathcal{H}$ be a Hirsch foliation on M. Then $\mathcal{H}$ can be isotopically leaf-conjugate to $\mathcal{H}'$ so that every T_i ($i \in \{1, \dots, n\}$) is transverse to $\mathcal{H}'$.*

Proof Item (1) of the lemma is a direct consequence of Corollary 3.8. We only need to prove item (2).

Without loss of generality, we can suppose that $T_i = \partial^{\text{out}} M_i$ ($i \in \{1, \dots, n\}$) and $\mathcal{H}$ is transverse to T_n. Let N be the union of $M_1, M_2, \dots, M_n$ by the gluing maps $\varphi_1, \dots, \varphi_{n-1}$. The Hirsch foliation $\mathcal{H}$ restricted to N is an m–punctured disk fibration, denoted by F. Since N admits an m–punctured disk fibration F, by Corollary 3.8, every incompressible torus T in the interior of N is a JSJ torus. Moreover, by item (2) of Lemma 3.6, T can be isotopic to T' relative to ∂N so that T' is transverse to F. Then by an easy inductive argument, $T_1, \dots, T_{n-1}$ in N can be isotopic to $T_1', T_2', \dots, T_{n-1}'$ relative to ∂N, respectively, so that every T_i' is transverse to F. Equivalently, we can perturb F in N relative to ∂N to F' which is transverse to every T_i. Then F' naturally induces a foliation $\mathcal{H}'$ in M such that

- $\mathcal{H}'$ is isotopically leaf-conjugate to $\mathcal{H}$;
- $\mathcal{H}'$ is transverse to every T_i. □

Lemma 3.10 *Let M be a closed 3–manifold which admits an affine Hirsch foliation $\mathcal{H}$. Let T_1 and T_2 be two incompressible tori in M each of which is transverse to $\mathcal{H}$. We denote the path closure of $M - T_i$ by N_i ($i = 1, 2$) and denote the restriction of $\mathcal{H}$ to N_i, which is an n_i–punctured disk fibration on N_i, by $\mathcal{F}_i$. Then $n_1 = n_2$.*

Proof Without loss of generality, we can suppose that T_1 and T_2 are disjoint and nonparallel. The path closure of $M - T_1 \cup T_2$ is the union of two compact 3–manifolds W_1 and W_2. Actually, $N_1 = W_1 \cup_{T_2} W_2$ and $N_2 = W_2 \cup_{T_1} W_1$. We denote $\mathcal{H}$ restricted to W_i ($i = 1, 2$), which is an m_i–punctured disk fibration on W_i, by H_i. Notice that every fiber of $\mathcal{F}_1$ is the union of one fiber of H_1 and m_1 fibers of H_2. Therefore, every fiber of $\mathcal{F}_1$ is an $m_1 \cdot m_2$–punctured disk. Equivalently, $n_1 = m_1 \cdot m_2$. Similarly, $n_2 = m_2 \cdot m_1$. In summary, $n_1 = n_2$. □

Definition 3.11 Let M be a closed 3–manifold which admits an affine Hirsch foliation $\mathcal{F}$ and T be an incompressible torus which is transverse to $\mathcal{F}$. We denote by N

the path closure of $M-T$ and denote $\mathcal{F}$ restricted to N, which is an n–punctured disk fibration, by F. We call n the *strand number* of $\mathcal{F}$.

Remark 3.12 Lemma 3.9 and Lemma 3.10 imply that the strand number of $\mathcal{F}$ doesn't depend on the choice of T. Furthermore, by Lemma 3.2, the strand number of an affine Hirsch foliation is invariant under isotopic leaf-conjugacy.

The following lemma explains that the "affine" property of an affine foliation is independent of the choices of T and the foliations which are isotopically leaf-conjugate to the original affine foliation.

Lemma 3.13 *Let M be a closed 3–manifold which admits an affine Hirsch foliation $\mathcal{H}_1$. Let $\mathcal{H}_2$ be a Hirsch foliation such that*

- *$\mathcal{H}_2$ is isotopically leaf-conjugate to $\mathcal{H}_1$;*
- *$\mathcal{H}_2$ is transverse to an incompressible torus T in M and N is the path closure of $M-T$.*

Let $\mathcal{F}_2$ be the punctured disk fibration on N and F_2 be the circle fibration of $\mathcal{H}_2$ restricted to T. Then the projective holonomy map of $\mathcal{F}$ relative to an embedded torus T transverse to $\mathcal{H}$ is topologically conjugate to the map z^n on S^1 where n is the strand number of $\mathcal{H}_1$ and $\mathcal{H}_2$.

To show Lemma 3.13, by item (2) of Lemma 3.9, we only need to prove the following claim.

Claim 3.14 *Let M be a closed 3–manifold which admits an affine Hirsch foliation $\mathcal{H}$. Let T_1 and T_2 be two incompressible tori in M. Let N_i $(i=1,2)$ be the path closure of $M-T_i$, let $\mathcal{F}_i$ be the punctured disk fibration on N_i, and let F_i be the circle fibration of $\mathcal{H}$ restricting to T_i. Suppose that φ_2^1, the projective holonomy map of $\mathcal{F}_1$ relative to T_1, is topologically conjugate to the map z^n on S^1. Then φ_2^2, the projective holonomy map of $\mathcal{F}_2$ relative to T_2, is also topologically conjugate to the map z^n on S^1.*

Proof By Lemma 3.9, we can suppose that T_1 and T_2 are disjoint and nonparallel. Let the path closure of $M-T_1\cup T_2$ be the union of two compact 3–manifolds M_1 and M_2 such that

(1) $\partial M_i=\partial^{\text{out}}M_i\cup\partial^{\text{in}}M_i$ for $i=1,2$;

(2) M is the union of M_1 and M_2 by the gluing maps $\varphi^1\colon\partial^{\text{out}}M_1\to\partial^{\text{in}}M_2$ and $\varphi^2\colon\partial^{\text{out}}M_2\to\partial^{\text{in}}M_1$;

(3) $\partial^{\text{out}}M_1$ and $\partial^{\text{in}}M_2$ correspond to T_1 and $\partial^{\text{out}}M_2$ and $\partial^{\text{in}}M_1$ correspond to T_2.

Under this notation, $N_1 = M_2 \cup_{\varphi_2} M_1$ and $N_2 = M_1 \cup_{\varphi_1} M_2$. We denote by $P_i\colon N_i \to S^1_i$ the quotient map of the fiber quotient space of $\mathcal{F}_i$.

We can define an m–covering map $\pi\colon S^1_2 \to S^1_1$ ($m \in \mathbb{N}$) as follows. For every $z_2 \in S^1_2$, since S^1_2 can be regarded as the quotient space of the circle fibration F_2 on $\partial^{\mathrm{out}} M_2$, we can regard z_2 as a fiber of F_2. Also notice that $\partial^{\mathrm{out}} M_2$ is embedded into N_1, and so the fiber z_2 is in some punctured disk fiber of $\mathcal{F}_1$. Therefore, the quotient map $P_1\colon N_1 \to S^1_1$ naturally induces a map $\pi\colon S^1_2 \to S^1_1$. One can easily check that π is an m–covering map.

We claim that $\varphi^1_2 \circ \pi = \pi \circ \varphi^2_2$, which is the key observation for the proof. Now let's check this claim. For every point $x_i \in N_i$ $(i = 1, 2)$, we denote by $\langle x_i \rangle_i \in S^1_i$ the fiber of $\mathcal{F}_i$ where x_i lies. Let x_2 be a point in $\partial^{\mathrm{out}} M_2 \subset N_2$ and x_1 be a point in $\partial^{\mathrm{out}} M_1 \subset N_1$ such that $\langle x_1 \rangle_1 = \pi(\langle x_2 \rangle_2)$. Then one can easily show that $P_1 \circ \varphi^1(x_1) = \pi \circ P_2 \circ \varphi^2(x_2)$ by following the definitions of P_i and φ^i $(i = 1, 2)$ and π. Note $P_1 \circ \varphi^1(x_1) = \varphi^1_2(\langle x_1 \rangle_1)$ and $\pi \circ P_2 \circ \varphi^2(x_2) = \pi \circ \varphi^2_2(\langle x_2 \rangle_2)$. By these equalities, we have $\varphi^1_2 \circ \pi(\langle x_2 \rangle_2) = \pi \circ \varphi^2_2(\langle x_2 \rangle_2)$ for every $\langle x_2 \rangle_2 \in S^1_2$.

Since φ^1_2 is affine, we can endow S^1_1 with a suitable metric such that $S^1_1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\varphi^1_2 = z^n$ for some $n \in \mathbb{N}$ $(n \geq 2)$. Since $\pi\colon S^1_2 \to S^1_1$ is an m–covering map, we also can endow S^1_2 with a metric such that $S^1_2 = \{z \in \mathbb{C} : |z| = 1\}$ and $\pi(z) = z^m$ for every $z \in S^1_2$. Furthermore, by the fact that $\pi \circ \varphi^2_2 = \varphi^1_2 \circ \pi$, we have $\varphi^2_2 = z^n\colon S^1_2 \to S^1_2$. □

Proposition 3.15 *Let T be an incompressible torus on a closed 3–manifold M. We denote by N the path closure of $M - T$, so that ∂N is the union of T^{out} and T^{in}. Then up to isotopic leaf-conjugacy, there exists at most one affine Hirsch foliation $\mathcal{H}$ such that $\mathcal{H}$ is transverse to T and $\mathcal{H}|_N$ is a punctured disk fibration such that each fiber of $\mathcal{H}|_N$ intersects T^{out} in one connected component.*

Proof We assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two affine Hirsch foliations on M which satisfy the conditions in the proposition. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the punctured disk fibrations induced on N by $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose $\varphi\colon T^{\mathrm{out}} \to T^{\mathrm{in}}$ is the gluing map so that $M = N \setminus x \sim \varphi(x)$ $(x \in T^{\mathrm{out}})$.

$F^{\mathrm{out}}_i = \mathcal{F}_i \cap T^{\mathrm{out}}$ $(i = 1, 2)$ is an S^1–fibration on T^{out}. Similarly, $F^{\mathrm{in}}_i = \mathcal{F}_i \cap T^{\mathrm{in}}$ is an S^1–fibration on T^{in}. We denote a fiber of F^{out}_1 (resp. F^{in}_1, F^{out}_2, F^{in}_2) by m_1 (resp. m_2, c_1, c_2). Then, up to isotopy, $\varphi(m_1) = m_2$ and $\varphi(c_1) = c_2$. By Lemma 3.4, c_1 is isotopic to m_1 in T^{out} and c_2 is isotopic to m_2 in T^{in}. Then we can suppose that $\mathcal{H}_1 \cap T = \mathcal{H}_2 \cap T$, which we denote by F. Here F is a circle fibration on T.

Since each of $\mathcal{H}_1$ and $\mathcal{H}_2$ is an affine Hirsch foliation, by Lemma 3.13, the projective holonomy maps $\varphi^1_2\colon S^1 \to S^1$ of $\mathcal{H}_1$ and $\varphi^2_2\colon S^1 \to S^1$ of $\mathcal{H}_2$ relative to T

are conjugated by an orientation-preserving homeomorphism $g\colon S^1 \to S^1$, that is, $\varphi_2^2 \circ g = g \circ \varphi_2^1$.

Recall that $P\colon N \to S^1$ is a natural quotient map where S^1 is the fiber quotient space of F. One can lift g to a homeomorphism $G_T\colon T \to T$ such that

- G_T is isotopic to the identity map on T;
- $P \circ G_T = g \circ P$.

Since G_T is isotopic to the identity map on T, we can extend G_T to a homeomorphism $G\colon M \to M$ which is isotopic to the identity map on M. Assume that $\mathcal{H}_1' = G(\mathcal{H}_1)$ is also an affine Hirsch foliation on N. Let $\mathcal{F}_1'$ be the punctured disk fibrations induced by $\mathcal{H}_1'$ on N. By $P \circ G_T = g \circ P$ and $\varphi_2^2 \circ g = g \circ \varphi_2^1$, one can quickly check that the boundaries of $\mathcal{F}_1'$ and $\mathcal{F}_2$ are coherent, ie for every fiber $\Sigma_1 \subset \mathcal{F}_1'$, there exists a fiber Σ_2 such that $\partial\Sigma_1 = \partial\Sigma_2$. Then by Lemma 3.5, one can build a homeomorphism $\phi\colon N \to N$ such that

- ϕ is isotopic to the identity map on N relative to ∂N;
- $\phi(\mathcal{F}_1') = \mathcal{F}_2$.

ϕ can automatically induce a homeomorphism Φ on M such that

- $\Phi(x) = \phi(x)$ for every x in the interior of N;
- Φ is isotopic to the identity map on M;
- $\Phi(\mathcal{H}_1') = \mathcal{H}_2$.

In summary, $\Phi \circ G$ is a homeomorphism on M such that

- $\Phi \circ G$ is isotopic to the identity map on M;
- $\Phi \circ G(\mathcal{H}_1) = \mathcal{H}_2$. □

Now we can finish the proof of Theorem 1.2, ie we can show that up to isotopic leaf-conjugacy, a closed orientable 3–manifold admits at most two affine Hirsch foliations.

Proof of Theorem 1.2 Let $\mathcal{H}$ be an affine Hirsch foliation and T be an incompressible torus in M. By Lemma 3.9, we can suppose that $\mathcal{H}$ is transverse to T. We denote the path closure of $M - T$ by N and the boundary of N by the union of T^{out} and T^{in}. Then $F = \mathcal{H}|_N$ is a punctured disk fibration on N. There are two possibilities for F:

(1) Each leaf of F intersects T^{out} in one connected component.

(2) Each leaf of F intersects T^{in} in one connected component.

In both cases, by Proposition 3.15, up to isotopic leaf-conjugacy, there exists at most one affine Hirsch foliation. The conclusion of the theorem follows. □

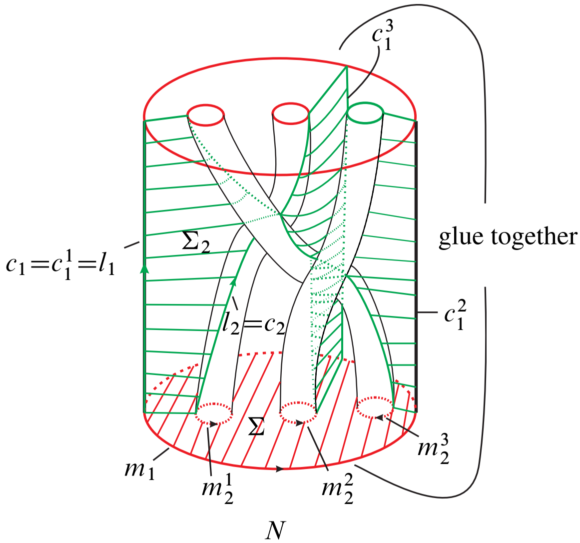

Figure 3: Notation in the case of N associated to the braid $\sigma_1\sigma_2^{-1}$

4 Hirsch manifolds and exchangeably braided links

In this section, we will focus on the study of Hirsch manifolds, ie the closed 3–manifolds which admit two nonisotopically leaf-conjugate affine Hirsch foliations. First, we will introduce or recall some useful notation (see Figure 3 as an illustration[5]):

$\mathcal{H}_1$	an affine Hirsch foliation transverse to T in M
$N, T^{\text{out}}, T^{\text{in}}, \varphi$	$M = N \setminus x \sim \varphi(x)$, $\partial N = T^{\text{out}} \cup T^{\text{in}}$, and $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ is the gluing homeomorphism
$m_1, l_1; m_2, l_2$	$\mathcal{H}_1$ induces oriented simple closed curves m_1, l_1 in T^{out} and m_2, l_2 in T^{in} which are defined at the beginning of Section 3
c_2	$p_2m_2 + q_2l_2$ $(q_2 > 0)$, an oriented simple closed curve in T^{in}
c_1	$p_1m_1 + q_1l_1$, an oriented simple closed curve in T^{out}
$c_1^1, \dots, c_1^s$	s pairwise disjoint oriented simple closed curves which are parallel to c_1 in T^{out}
Σ_2	an oriented punctured disk in N such that $\partial\Sigma_2$ is the union of $c_1^1, \dots, c_1^s$ and c_2
F_2	an oriented punctured disk fibration on N with a fiber Σ_2
$\varphi\colon T^{\text{out}} \to T^{\text{in}}$	$\varphi(m_1) = m_2$ and $\varphi(l_1) = l_2 + km_2$

[5] In the case of the figure, $c_1 = c_1^1 = l_1$. To avoid misunderstanding, we should point out that generally we can think $c_1 = c_1^1$, but l_1 may not be isotopic to c_1.

4.1 Homology and Hirsch manifolds

In this subsection, for every two nonzero integers m and n, we will use $[m,n]$ to represent their greatest positive common divisor.

Lemma 4.1 *Suppose F_2 also induces an affine Hirsch foliation $\mathcal{H}_2$ on M under φ. Then $p_1 = k/[n^2-1,k]$, $q_1 = (n^2-1)/[n^2-1,k]$, $p_2 = n^2 p_1$, $q_2 = q_1$, and $s = n$.*

Proof Since $\partial(\Sigma_2) = c_2 \cup c_1^1 \cup \cdots \cup c_1^s$, we have $[c_2] = s[c_1]$ in $H_1(N)$. Equivalently, $p_2[m_2] + q_2[l_2] = sp_1[m_1] + sq_1[l_1]$. Recall that $[m_1] = n[m_2]$ and $[l_2] = n[l_1]$, and so $(snp_1 - p_2)[m_2] + (nq_2 - sq_1)[l_1] = 0$. Recall that $H_1(N) = \langle [m_2], [l_1] \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Then

$$p_2 = snp_1 \quad \text{and} \quad nq_2 = sq_1. \tag{1}$$

By filling a solid torus to N along T^{out}, we obtain a new compact 3–manifold V so that c_1 bounds a disk in V, and then V is homeomorphic to a solid torus. Following the gluing surgery, we have

$$\begin{aligned} H_1(V) &= \langle [m_2], [l_1] : p_1[m_1] + q_1[l_1] = 0 \rangle \\ &= \langle [m_2], [l_1] : np_1[m_2] + q_1[l_1] = 0 \rangle \\ &\cong \mathbb{Z}. \end{aligned}$$

Then we have

$$np_1 \text{ and } q_1 \text{ are coprime.} \tag{2}$$

Define $\varphi_\star\colon H_1(T^{\text{out}}) \to H_1(T^{\text{in}})$ to be the homomorphism induced by $\varphi\colon T^{\text{out}} \to T^{\text{in}}$. Notice that F_2 also induces an affine Hirsch foliation $\mathcal{H}_2$ on M. Then, on the one hand,

$$\begin{aligned} \varphi_\star([c_1]) &= [c_2] \\ &= p_2[m_2] + q_2[l_2], \end{aligned}$$

and on the other hand,

$$\begin{aligned} \varphi_\star([c_1]) &= \varphi_\star(p_1[m_1] + q_1[l_1]) \\ &= p_1[m_2] + q_1(k[m_2] + [l_2]) \\ &= (p_1 + q_1 k)[m_2] + q_1[l_2]. \end{aligned}$$

Therefore,

$$p_1 + q_1 k = p_2 \quad \text{and} \quad q_1 = q_2. \tag{3}$$

Now, the lemma is a direct consequence of (1), (2), (3) and the fact that p_i and q_i $(i = 1, 2)$ are coprime. □

We can use the strand number of a braid which builds an (affine) Hirsch foliation on M to be an invariant of M. The strand number of M is well defined. Let us explain a little bit more. Suppose there are two braids which build the same Hirsch manifold M; by Lemma 3.10, Definition 3.11 and Lemma 4.1, the strand numbers of the braids are the same.

Definition 4.2 Let M be a Hirsch manifold. The strand number of a braid b which builds an (affine) Hirsch foliation on M is called the *strand number* of M.

Proposition 4.3 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two affine Hirsch foliations defined as above on a Hirsch manifold M. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are not isotopically leaf-conjugate.*

Proof Otherwise, we assume that there exists a homeomorphism $h\colon M \to M$ which maps every leaf of $\mathcal{H}_1$ to a leaf of $\mathcal{H}_2$ and is isotopic to the identity map on M. One can check that every leaf on $\mathcal{H}_1$ is homeomorphic to either a sphere minus a Cantor set or a torus minus a Cantor set. We choose a leaf ℓ_1 on $\mathcal{H}_1$ which is homeomorphic to a sphere minus a Cantor set. We denote $f(\ell_1)$, which is a leaf on $\mathcal{H}_2$, by ℓ_2.

Let $Q\colon N \to M$ be the natural quotient map. By the construction of $\mathcal{H}_1$, without loss of generality, we can assume that $b_1 = Q(m_1) = Q(m_2)$ is an oriented simple closed curve on ℓ_1. The curve b_1 is homotopically nontrivial in M because of the compressibility of T in M. Since h is isotopic to the identity map on M, we also have that $b_2 = h(b_1) \subset \ell_2$ is homotopically nontrivial in M. By the construction of $\mathcal{H}_2$, b_2 is homotopic to λc in ℓ_2 for some nonzero integer λ. Here $c = Q(c_1) = Q(c_2)$ is a simple closed curve in T. We choose an oriented closed curve c_λ in T which is homotopic to λc in T. Then b_1 and c_λ are homotopic in M. This means that there exists an immersion map $F\colon A = S^1 \times [0,1] \to M$ and an orientation on A such that

- $F(S^1 \times \{0\}) = l_1$ and $F(S^1 \times \{1\}) = c_\lambda$, where $S^1 \times \{0\}$ and $S^1 \times \{1\}$ are oriented consistently with the orientation of A;
- $F(\mathrm{int}(A))$ is transverse to T, where $\mathrm{int}(A)$ is the interior of A.

Moreover, under some perturbation of F close to ∂A if necessary, we can assume there exists a neighborhood of ∂A, denoted by $N(A)$, satisfying $F^{-1}(T) \cap N(\partial A) = \partial A$. Then $F^{-1}(T) \cap \mathrm{int}(A)$ is the union of finitely many pairwise disjoint oriented simple closed curves $s_0, s_1, \dots, s_m$ where $s_0 = S^1 \times \{0\}$ and $s_m = S^1 \times \{1\}$. Here the orientation of s_i $(i \in \{0, 1, \dots, m\})$ is consistent with the orientation of s_0 in A. We can assume that m is minimal in the following sense: let $F\colon A \to M$ be an immersion which satisfies the conditions above; then $F^{-1}(T) \cap \mathrm{int}(A)$ contains at least m connected components. If some s_i is inessential in A, then s_i bounds a disk D_i in A. This means that $F(s_i)$ is homotopically trivial in M. Since $F(s_i) \subset T$ and T is incompressible in M, we have that $F(s_i)$ is homotopically trivial in T. Then by some standard surgery, one can build another $F'\colon A \to M$ which satisfies the conditions above and whose intersection circle number is less than m. This contradicts the assumption for m. Therefore, from now on, we can suppose that each s_i is essential in A. Since A is an annulus, $s_0, s_1, \dots, s_m$ are pairwise parallel in A.

Without loss of generality, we can assume that the union of $s_0, s_1, \dots, s_m$ cuts A into m open annuli $A_1, \dots, A_m$ such that

- $\partial A_i = s_{i-1} \cup s_i$ for $i \in \{1, \dots, m\}$;
- $A_i \cap F^{-1}(T) = \varnothing$.

Therefore, $F(s_{i-1})$ and $F(s_i)$ are homotopic in N for every $i \in \{1, \dots, m\}$. We choose a very small tubular neighborhood of s_0 in A. Then $F(N(s_0))$ belongs to one of the two sides of T in M. The two cases for the position of $F(N(s_0))$ and the relations above induce two kinds of "homotopy chain relations". We denote $Q^{-1}(c_\lambda)$ by $c_\lambda^1 \cup c_\lambda^2$ where $c_\lambda^1 \subset T^{\mathrm{in}}$ and $c_\lambda^2 \subset T^{\mathrm{out}}$. In both cases, we can assume there exist $2m$ oriented closed curves $s_1^1, s_2^1, \dots, s_m^1$ in T^{out} and $s_0^2, s_1^2, \dots, s_{m-1}^2$ in T^{in} such that

- $Q(s_i^1) = Q(s_i^2) = F(s_i)$ and $s_i^2 = \varphi(s_i^1)$ for $i \in \{1, \dots, m-1\}$;
- s_{i-1}^2 and s_i^1 are homotopic in N for $i \in \{1, \dots, m\}$.

In one case, $s_0^2 = c_\lambda^2$ and $s_m^1 = m_1$; in the other case, $s_0^2 = m_2$ and $s_m^1 = c_\lambda^1$.

We will get contradictions in both cases by using homology theory. For every oriented closed curve α in N, we will use $[\alpha]$ to represent the corresponding homological element in $H_1(N)$. Recall that $H_1(N) \cong \mathbb{Z} \oplus \mathbb{Z}$, and it is generated by $[l_1]$ and $[m_2]$ (Lemma 3.1). Moreover, $[l_2] = n[l_1]$ and $[m_1] = n[m_2]$. These facts will be used several times in the following.

In the first case, on the one hand, since $s_0^2 = m_2$ and s_1^1 are homotopic in N, we have $[m_2] = [s_1^1]$ in $H_1(N)$; on the other hand, since s_1^1 is an oriented closed curve in T^{out}, we have $[s_1^1] = r[m_1] + t[l_1]$ for two integers r and t. These two sides imply that $[s_1^1] = nr[m_2] + t[l_1] = [m_2]$ in $H_1(N)$. Notice that $n > 1$, and so the equality is impossible. Therefore, we obtain a contradiction.

In the second case, since s_{i-1}^2 and s_i^1 are homotopic in N ($i \in \{1, \dots, m\}$), we have $[s_{i-1}^2] = [s_i^1]$. In particular, $[s_{m-1}^2] = [s_m^1] = [m_1]$. Since s_{m-1}^2 is an oriented closed curve in T^{in}, we have $[s_{m-1}^2] = r_{m-1}[m_2] + t_{m-1}[l_2]$ for two integers r_{m-1} and t_{m-1}. We also have $[s_{m-1}^2] = r_{m-1}[m_2] + nt_{m-1}[l_1] = n[m_2]$. Therefore, $r_{m-1} = n$ and $t_{m-1} = 0$. This implies that s_{m-1}^2 and nm_2 are homotopic in T^{in}. Notice that $s_{m-1}^1 = \varphi^{-1}(s_{m-1}^2)$ and

$\varphi(m_1)=m_2$, and so s^1_{m-1} and nm_1 are homotopic in T^{out}. By some similar arguments, we have that s^1_i and $n^{m-i}m_1$ are homotopic in N for every $i \in \{1,\dots,m-1\}$. Also notice that s^1_1, s^2_0 and c_λ are pairwise homotopic, and so $n^{m-1}m_1$ and c_λ are homotopic in N. This implies that $n^{m-1}[m_1]=[c_\lambda]=n^{m-1+1}[m_2]=n^m[m_2]$. By Lemma 4.1, $[c_\lambda]=\lambda[c_2]=\lambda\big(n^2(k/[n^2-1,k])[m_2]+((n^2-1)/[n^2-1,k])[l_2]\big)$. Since $((n^2-1)/[n^2-1,k])[l_2]=(n(n^2-1)/[n^2-1,k])[l_1]$ is nonzero, $[c_\lambda]\neq n^m[m_2]$. We obtain a contradiction. Then the proposition follows. □

Remark 4.4 By Definition 2.1 and Definition 2.2, we can see that for a given 3–manifold M,

- on the one hand, every Hirsch foliation can be obtained from a unique affine Hirsch foliation by replacing the projective holonomy map $\varphi_2=z^n$ on S^1 by another degree-n endomorphism φ_2' on S^1;
- on the other hand, for every affine Hirsch foliation and every degree-n endomorphism φ_2' on S^1, one can build a Hirsch foliation with the projective holonomy map φ_2'.

Moreover, by Proposition 4.3 and Theorem 1.2, one can classify all of the affine Hirsch foliations on a given 3–manifold M.

Therefore, our results reduce the question of classifying all Hirsch foliations to a classical problem in one-dimensional dynamical systems: classifying degree-n $(n\geq 2)$ endomorphisms on S^1 up to conjugacy.

4.2 DEBL Hirsch manifolds

To aid understanding of the materials in this subsection, we suggest the reader look at Figure 4.

Let $L=K_1\cup K_2$ be an exchangeably braided link in S^3. We choose two disjoint small open tubular neighborhoods V_1 and V_2 of K_1 and K_2, respectively. N is defined to be $S^3-V_1\cup V_2$. Its boundary ∂N satisfies $\partial N=T^{\text{out}}\cup T^{\text{in}}$ with $T^{\text{out}}=\partial\overline{V_1}$ and $T^{\text{in}}=\partial\overline{V_2}$. The linking number of K_1 and K_2 is denoted by n. K_1 is a closed n–braid $\widetilde{b}_1$ relative to K_2, and K_2 is a closed n–braid $\widetilde{b}_2$ relative to K_1.

Up to isotopy, there is a unique way to choose a simple closed curve m_1 in T^{out} and n simple closed curves $m_2^1,\dots,m_2^n$ in T^{in} such that

- $m_2^1,\dots,m_2^n$ each bound a disk in $\overline{V_2}$ and m_1 is isotopic to K_1 in $\overline{V_1}$;
- $m_2^1,\dots,m_2^n$ and m_1 bound an n–punctured disk Σ_1 in N.

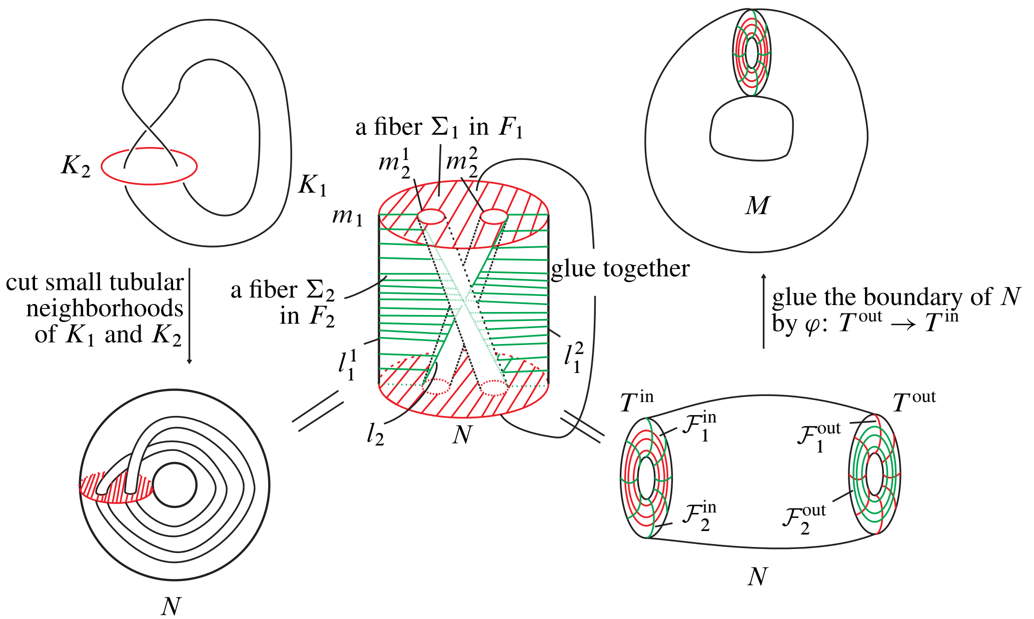

Figure 4: An example DEBL Hirsch manifold

Similarly, up to isotopy, there is a unique way to choose a simple closed curve l_2 in T^{in} and n simple closed curves $l_1^1, \dots, l_1^n$ in T^{out} such that

- $l_1^1, \dots, l_1^n$ each bound a disk in $\overline{V_1}$ and l_2 is isotopic to K_2 in $\overline{V_2}$;
- $l_1^1, \dots, l_1^n$ and l_2 bound an n–punctured disk Σ_2 in N.

Moreover, up to isotopy, we can extend Σ_i $(i = 1, 2)$ to an n–punctured disk fibration F_i on N such that

- $\mathcal{F}_i^{\text{out}} = F_i \cap T^{\text{out}}$ and $\mathcal{F}_i^{\text{in}} = F_i \cap T^{\text{in}}$ are two S^1–fibrations on T^{out} and T^{in};
- $\mathcal{F}_1^{\text{out}}$ and $\mathcal{F}_2^{\text{out}}$ transversely intersect everywhere on T^{out};
- $\mathcal{F}_1^{\text{in}}$ and $\mathcal{F}_2^{\text{in}}$ transversely intersect everywhere on T^{in}.

We can suppose that the intersection number of m_1 and l_1^i is 1 and the intersection number of m_2^i and l_2, for every $i \in \{1, \dots, n\}$, is 1. Similar to the beginning of Section 2 and Section 3, we would like to provide some orientations on these objects:

- We give N an orientation.
- T^{in} and T^{out} are oriented by the orientation of N.

- We give the leaves of F_1 and F_2 orientations continuously so that
  (1) every fiber of $\mathcal{F}_i^{\mathrm{out}}$ and $\mathcal{F}_i^{\mathrm{in}}$ $(i=1,2)$ is oriented as induced by the orientation of F_1 and F_2;
  (2) the orientation of T^{out} is consistent with that of (m_1, l_1^1) and the orientation of T^{in} is consistent with that of (m_2^1, l_2).

Then one can build an orientation-preserving homeomorphism $\varphi\colon T^{\mathrm{out}} \to T^{\mathrm{in}}$ such that

- $\varphi(m_1) = m_2^1$ and $\varphi(l_1^1) = l_2$;
- φ maps every fiber of $\mathcal{F}_1^{\mathrm{out}}$ to a fiber of $\mathcal{F}_1^{\mathrm{in}}$;
- φ maps every fiber of $\mathcal{F}_2^{\mathrm{out}}$ to a fiber of $\mathcal{F}_2^{\mathrm{in}}$;
- F_1 and F_2 induce two affine Hirsch foliations $\mathcal{H}_1$ and $\mathcal{H}_2$ on M under φ where $M = N \setminus x \sim \varphi(x)$ $(x \in T^{\mathrm{out}})$.

Notice that to ensure the glued manifold M can admit two Hirsch foliations induced by F_1 and F_2, up to isotopy, we can suppose that $\varphi(m_1) = m_2^1$ and $\varphi(l_1^1) = l_2$. This implies that under this restriction, the glued manifold M is unique up to homeomorphism. Therefore, we can say that an exchangeably braided link determines a unique Hirsch manifold. Every Hirsch manifold built in this way is called a *Hirsch manifold derived from an exchangeably braided link* (abbreviated as a *DEBL Hirsch manifold*).

By the second item of Corollary 2.4, we have the following consequence.

Corollary 4.5 *For every $n \in \mathbb{N}$, there are only finitely many DEBL Hirsch manifolds with strand number n.*

4.3 A virtual property of Hirsch manifolds

Let M be a Hirsch manifold. By the definition of Hirsch manifold and Lemma 3.9, there exist two affine Hirsch foliations $\mathcal{H}_1$ and $\mathcal{H}_2$ on M and a JSJ torus T in M which satisfy the propositions in Lemma 3.9. Let N be the path closure of $M - T$. Then:

- N admits two n–punctured disk fibrations F_1 and F_2 and we can parametrize some subsets of N as in Section 2.
- The two Hirsch foliations $\mathcal{H}_1$ and $\mathcal{H}_2$ can be induced, respectively, by F_1 and F_2 and a gluing map $\varphi\colon T^{\mathrm{out}} \to T^{\mathrm{in}}$.

$H_1(N) \cong \mathbb{Z} \oplus \mathbb{Z}$, and it is generated by $[m_2]$ and $[l_1]$. We denote the abelianization homomorphism from $\pi_1(N)$ to $H_1(N)$ by ψ_1 and denote by ψ_2 the quotient homomorphism from $H_1(N)$ to $\mathbb{Z}_{q_2}$ such that $\psi_2([m_2]) = 0$ and $\psi_2([l_1]) = 1$. The kernel of $\psi_N = \psi_2 \circ \psi_1 \colon \pi_1(N) \to \mathbb{Z}_{q_2}$, which we denote by G, is a normal subgroup of $\pi_1(N)$. Here, the definitions and properties of m_i and l_i $(i = 1, 2)$ and q_2 can be found in the beginning of Section 4 and Lemma 4.1.

As a subgroup of $\pi_1(N)$, the kernel G induces a q_2–covering space of N by a covering map $P \colon \widetilde{N} \to N$. We collect some useful properties in the following proposition, which one can prove by some routine checks. We omit the details here.

Proposition 4.6 *The following properties hold for $i = 1, 2$:*

(1) $P^{-1}(F_i) = \widetilde{F}_i$ *is an n–punctured disk fibration on* $\widetilde{N}$.

(2) *Let $\widetilde{\Sigma}_1$ be a connected component of $P^{-1}(\Sigma_1)$. Then $P \colon \widetilde{\Sigma}_1 \to \Sigma_1$ is a homeomorphism satisfying $P(\widetilde{m}_i) = m_i$.*

(3) $P \colon \widetilde{l}_1 \to l_1$ *is a q_2–covering map.*

(4) *Let $\widetilde{\Sigma}_2$ be a connected component of $P^{-1}(\Sigma_2)$. Then $P \colon \widetilde{\Sigma}_2 \to \Sigma_2$ is a homeomorphism satisfying $P(\widetilde{c}_i) = c_i$.*

(5) $\widetilde{c}_i$ *intersects* $\widetilde{m}_i$ *at one point.*

Lemma 4.7 *There is a homeomorphism $\widetilde{\varphi} \colon \widetilde{T}^{\text{out}} \to \widetilde{T}^{\text{in}}$ such that*

(1) $P \circ \widetilde{\varphi} = \varphi \circ P \colon \widetilde{T}^{\text{out}} \to T^{\text{in}}$;

(2) $\widetilde{\varphi}(\widetilde{m}_1) = \widetilde{m}_2$ *and* $\widetilde{\varphi}(\widetilde{c}_1) = \widetilde{c}_2$.

Proof By Proposition 4.6, $\widetilde{m}_i \cap \widetilde{c}_i$ $(i = 1, 2)$ is one point, which we will denote by $\widetilde{x}_i$. Denoting $P(\widetilde{x}_i)$ by x_i, we have $\varphi(x_1) = x_2$,

$$(\varphi \circ P)_\star(\pi_1(\widetilde{T}^{\text{out}}, \widetilde{x}_1)) = \langle \varphi_\star([m_1]), \varphi_\star([c_1]) \rangle = \langle [m_2], [c_2] \rangle \lhd \pi_1(T^{\text{in}}, x_2),$$

and

$$P_\star(\pi_1(\widetilde{T}^{\text{in}}, \widetilde{x}_2)) = \langle [m_2], [c_2] \rangle \lhd \pi_1(T^{\text{in}}, x_2).$$

Then, by the classical homotopy lifting theorem, we can construct a unique map $\widetilde{\varphi} \colon \widetilde{T}^{\text{out}} \to \widetilde{T}^{\text{in}}$ such that

(1) $\widetilde{\varphi}(\widetilde{x}_1) = \widetilde{x}_2$ and $\widetilde{\varphi}(\widetilde{m}_1) = \widetilde{m}_2$;

(2) $P \circ \widetilde{\varphi} = \varphi \circ P \colon \widetilde{T}^{\text{out}} \to T^{\text{in}}$.

Now it is routine to check that $\widetilde{\varphi}$ is a homeomorphism. □

We denote $\widetilde{N} \setminus y \sim \widetilde{\varphi}(y)$ $(y \in \widetilde{T}^{\text{out}})$ by $\widetilde{M}$. The corresponding quotient maps are $Q \colon \widetilde{N} \to \widetilde{M}$ and $q \colon N \to M$.

Lemma 4.8 *There is a unique map $\pi\colon \widetilde{M} \to M$ so that $\pi \circ Q(y) = q \circ P(y)$ for every $y \in \widetilde{N}$. Furthermore, $\pi\colon \widetilde{M} \to M$ is a q_2–covering map.*

Proof For every $\tilde{x} \in \widetilde{M}$, we can define $\pi(\tilde{x})$ as follows. Since $Q\colon \widetilde{N} \to \widetilde{M}$ is surjective, there exists $\tilde{y} \in \widetilde{N}$ so that $\tilde{x} = Q(\tilde{y})$. Define $\pi(\tilde{x}) = q \circ P(\tilde{y})$. The first item of Lemma 4.7 ensures that π is well defined. π is unique because it has no freedom in $\widetilde{M} - \widetilde{T}$ where $\widetilde{T} = \pi^{-1}(T)$.

Finally, since $P\colon \widetilde{N} \to N$ is a q_2–covering map, $\pi\colon \widetilde{M} \to M$ is also a q_2–covering map. □

Lemma 4.9 *$\widetilde{M}$ is a Hirsch manifold which admits two affine Hirsch foliations $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$ such that $\widetilde{\mathcal{H}}_i$ $(i = 1, 2)$ is induced by $\mathcal{H}_i$ under π, ie π maps each leaf of $\widetilde{\mathcal{H}}_i$ to a leaf of $\mathcal{H}_i$.*

Proof Assume that $\mathcal{F}_i^{\text{out}} = F_i \cap T^{\text{out}}$ and $\mathcal{F}_i^{\text{in}} = F_i \cap T^{\text{in}}$ are two S^1–fibrations on T^{out} and T^{in}, respectively. Since F_i induces $\mathcal{H}_i$ on M under the gluing homeomorphism $\varphi\colon T^{\text{out}} \to T^{\text{in}}$, we know φ maps every fiber of $\mathcal{F}_i^{\text{out}}$ to a fiber of $\mathcal{F}_i^{\text{in}}$.

Suppose $\widetilde{\mathcal{F}}_i^{\text{out}}$ and $\widetilde{\mathcal{F}}_i^{\text{in}}$ are the lifted fibrations of $\mathcal{F}_i^{\text{out}}$ and $\mathcal{F}_i^{\text{in}}$ on $\widetilde{T}^{\text{out}}$ and $\widetilde{T}^{\text{in}}$ under the covering map P, respectively. $\widetilde{\varphi}\colon \widetilde{T}^{\text{out}} \to \widetilde{T}^{\text{in}}$ is the lifted map of $\varphi\colon T^{\text{out}} \to T^{\text{in}}$, ie $P \circ \widetilde{\varphi} = \varphi \circ P\colon \widetilde{T}^{\text{out}} \to T^{\text{in}}$. Therefore, $\widetilde{\varphi}$ maps every fiber of $\widetilde{\mathcal{F}}_i^{\text{out}}$ to a fiber of $\widetilde{\mathcal{F}}_i^{\text{in}}$. Then $\widetilde{F}_i$ induces a Hirsch foliation $\widetilde{\mathcal{H}}_i$ on $\widetilde{M}$.

To finish the proof, now we only need to check that $\widetilde{\mathcal{H}}_i$ is an affine Hirsch foliation. This actually is a consequence of the following facts:

- $\widetilde{\varphi}$ is the lifted map of φ.
- $\mathcal{H}_i$ is an affine Hirsch foliation.
- Every expanding map on S^1 is topologically conjugate to an affine map on S^1 with the same degree. □

Lemma 4.10 *$\widetilde{M}$ is a DEBL Hirsch manifold.*

Proof We glue two solid tori $\widetilde{V}_1$ and $\widetilde{V}_2$ to $\widetilde{N}$ along its boundary $\widetilde{T}^{\text{in}} \cup \widetilde{T}^{\text{out}}$ by the gluing maps $\phi_1\colon \partial\widetilde{V}_1 \to \widetilde{T}^{\text{in}}$ and $\phi_2\colon \partial\widetilde{V}_2 \to \widetilde{T}^{\text{out}}$, respectively, so that $\widetilde{m}_2$ bounds a disk in $\widetilde{V}_2$ and $\widetilde{c}_1$ bounds a disk in $\widetilde{V}_1$. Then the glued manifold is homeomorphic to S^3.

Let K_i $(i = 1, 2)$ be a simple closed curve in $\widetilde{V}_i$ such that $\widetilde{V}_i$ is a tubular neighborhood of K_i.

Since $\widetilde{F}_2$ is a punctured disk fibration structure on $\widetilde{N}$ and $\widetilde{m}_2$ bounds a disk in $\widetilde{V}_2$, the union of $\widetilde{V}_2$ and $\widetilde{N}$, denoted by $\widetilde{U}_2$, is also homeomorphic to a solid torus. Obviously, K_2 is a closed braid in $\widetilde{U}_2$. Since $S^3 = \widetilde{V}_2 \cup \widetilde{N} \cup \widetilde{V}_2$, automatically, K_2 is a closed braid relative to K_1, ie K_2 is a closed braid in $S^3 - K_1$.

Similarly, one can show that K_1 is a closed braid relative to K_2. Therefore, $L = K_1 \cup K_2$ is an exchangeably braided link. Now it is routine to build the Hirsch manifold derived from L and check that the Hirsch manifold is homeomorphic to $\widetilde{M}$. □

Proof of Theorem 1.5 The first part of Theorem 1.5 is a direct consequence of Lemma 4.8 and Lemma 4.10. Moreover, by Lemma 4.1, q_2 can be divided by n^2-1. □

4.4 Finiteness of Hirsch manifolds with strand number *n*

We will use the following theorem of Wang [15].

Theorem 4.11 *Let M be a closed irreducible 3–manifold which is nonorientable or Seifert fibered or has a nontrivial torus decomposition (ie there is a JSJ torus). Then M covers infinitely many nonhomeomorphic 3–manifolds if and only if M is an orientable Seifert fiber space with nonzero Euler number.*

Proof of Proposition 1.6 On the one hand, by Proposition 1.4, an n–strand Hirsch manifold is an irreducible orientable closed 3–manifold with some JSJ tori. By Theorem 4.11, for a given DEBL Hirsch manifold $\widetilde{M}$, there are only finitely many Hirsch manifolds with $\widetilde{M}$ as a finite covering space.

On the other hand, Corollary 2.4 says that for a positive integer n, up to isotopy, there are only finitely many exchangeably braided links with strand number n. Recall that an exchangeably braided link determines a DEBL Hirsch manifold. Therefore, there are only finitely many DEBL Hirsch manifolds with strand number n.

Let M be a Hirsch manifold with strand number n. Then, by Theorem 1.5, $\widetilde{M}$, a finite covering space of M, is a DEBL Hirsch manifold with strand number n. Combining the two sides above, up to homeomorphism, there are only finitely many Hirsch manifolds with strand number n. □

5 Proof of Proposition 1.7

In this section, we will construct an example to prove Proposition 1.7, which says there exists a 3–manifold which admits an affine Hirsch foliation but is not a Hirsch manifold. We will use the following inequality by Bennequin [2]:

Lemma 5.1 (Bennequin inequality) *Let L be a nonseparating link of μ components, presented by a closed braid with l strands and c_+ (c_-) positive (negative) crossings. Then $g(L)$, the genus of L, is bounded as follows:*

$$\frac{|c_+ - c_-| - l - \mu}{2} + 1 \le g(L) \le \frac{|c_+ + c_-| - l - \mu}{2} + 1.$$

Proof of Proposition 1.7 Let $b = (\sigma_1\sigma_2^{-1})^2$ be a 3–strand braid. Now we can follow the beginning of Section 2 to build an affine Hirsch foliation $\mathcal{H}$ on a closed 3–manifold M. We briefly recall the construction here:

- b also can be used to represent a diffeomorphism on a 3–punctured disk Σ, and we denote the mapping torus of (Σ, b) by N.
- $F = \{\Sigma \times \{\star\}\}$ provides a 3–punctured disk fibration on N, which provides T^{in} and T^{out} two S^1–fibration structures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.
- After carefully choosing orientations on the objects above, we can build an orientation-preserving homeomorphism $\varphi\colon T^{\text{out}} \to T^{\text{in}}$ which maps every fiber of $\mathcal{F}_1$ to a fiber of $\mathcal{F}_2$ and preserves the corresponding orientations.
- Let $M = N \setminus x \sim \varphi(x)$ $(x \in T^{\text{out}}N)$. Then F naturally induces a Hirsch foliation $\mathcal{H}$ on M by φ. If we choose φ suitably, $\mathcal{F}$ is an affine Hirsch foliation.

Now we assume that M is a Hirsch manifold. Following the arguments in Section 4.3, there exists some integer p so that the braid $b^{q_2}\tau^p$ is an exchangeable braid where τ is a 3–strand full-twist braid. This means that the knot $K = \widehat{b^{q_2}\tau^p}$, the closed braid of $b^{q_2}\tau^p$, is a trivial knot. In the following, we will show that $g(K)$, the genus of K, is nonzero. Then K isn't a trivial knot. We obtain a contradiction. Then M isn't a Hirsch manifold.

Using the notation of Lemma 5.1 in our case, $L = K = \widehat{b^{q_2}\tau^p}$, $l = 3$, $\mu = 1$ and $|c_+ - c_-| = 6|p|$. By Lemma 5.1, $g(K) \geq 3|p| - 1$. Therefore, if $g(K) = 0$, then $p = 0$. In the case $p = 0$, we have $K = \widehat{b^{q_2}}$. By Lemma 4.1, q_2 is nonzero. Actually, it is well known that in this case, $\widehat{b^{q_2}}$ is a genus-1 fiber knot (see, for instance, Rolfsen [11, Chapter 10]). Therefore, K isn't a trivial knot. □

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