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Symplectic embeddings of four-dimensional ellipsoids into integral polydiscs

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In previous work, the second author and Müller determined the function $c(a)$ giving the smallest dilate of the polydisc $P(1,1)$ into which the ellipsoid $E(1,a)$ symplectically embeds. We determine the function of two variables $c_b(a)$ giving the smallest dilate of the polydisc $P(1,b)$ into which the ellipsoid $E(1,a)$ symplectically embeds for all integers $b \geqslant 2$.

It is known that, for fixed b, if a is sufficiently large then all obstructions to the embedding problem vanish except for the volume obstruction. We find that there is another kind of change of structure that appears as one instead increases b: the number-theoretic "infinite Pell stairs" from the $b=1$ case almost completely disappears (only two steps remain) but, in an appropriately rescaled limit, the function $c_b(a)$ converges as b tends to infinity to a completely regular infinite staircase with steps all of the same height and width.

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1 Introduction and result

1.1 Introduction

Since Gromov's classic paper [15], it has been known that symplectic embedding problems are intimately related to many phenomena in symplectic geometry, Hamiltonian dynamics, and other fields. The smallest interesting dimension is four, and all our results are in this dimension. So consider the standard four-dimensional symplectic vector space $(\mathbb{R}^4, \omega)$, where $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Open subsets in $\mathbb{R}^4$ are endowed with the same symplectic form. Given two such sets U and V, a symplectic embedding of U into V is a smooth embedding $\varphi\colon U \to V$ that preserves the symplectic form: $\varphi^*\omega = \omega$. We write $U \overset{s}{\hookrightarrow} V$ if there exists a symplectic embedding $U \to V$. Deciding whether $U \overset{s}{\hookrightarrow} V$ is very hard in general.

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One thus looks at simple sets, such as the open ball $B^4(a)$ of radius $\sqrt{a}$, or polydiscs $P(a,b) = B^2(a) \times B^2(b) \subset \mathbb{R}^2(x_1, y_1) \times \mathbb{R}^2(x_2, y_2)$, or ellipsoids

$$E(a,b) := \left\{ \frac{x_1^2 + y_1^2}{a} + \frac{x_2^2 + y_2^2}{b} < 1 \right\}.$$

In four dimensions, Gromov's nonsqueezing theorem states that

$$B^4(a) \overset{s}{\hookrightarrow} B^2(b) \times \mathbb{R}^2(x_2, y_2)$$

only if $a \leqslant b$. In other words, one cannot do better than the identity mapping. After this rough rigidity result, the "fine structure of symplectic rigidity" was investigated by looking at other embedding problems. The first important results were on the "packing problem", where U is a disjoint union of balls; see Biran [2; 3], Gromov [15] and McDuff and Polterovich [25]. Further understanding on the fine structure came with the study of embeddings of ellipsoids; see Choi, Cristofaro-Gardiner, Frenkel, Hutchings and Ramos [9], Frenkel and Müller [14], Hutchings [18], McDuff [23; 24], McDuff and Schlenk [26] and Schlenk [27; 28]. Note that $E(a,b) \overset{s}{\hookrightarrow} V$ if and only if $E\left(1, \frac{b}{a}\right) \overset{s}{\hookrightarrow} (1/\sqrt{a})V$. We can thus take $E(1,a)$ with $a \geqslant 1$ as U. Encode the embedding problems $E(1,a) \overset{s}{\hookrightarrow} B^4(b)$ and $E(1,a) \overset{s}{\hookrightarrow} P(b,b) =: C^4(b)$ in the functions

$$c_B(a) := \inf\{\lambda > 0 \mid E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda)\},$$
$$c_C(a) := \inf\{\lambda > 0 \mid E(1,a) \overset{s}{\hookrightarrow} C^4(\lambda)\}.$$

Since symplectic embeddings are volume-preserving, $c_B(a) \geqslant \sqrt{a}$ and $c_C(a) \geqslant \sqrt{\frac{a}{2}}$. The functions $c_B(a)$ and $c_C(a)$ were computed in [26; 14]:

The function $c_B(a)$ has three parts: On $[1, \tau^4]$, with $\tau = \frac{1}{2}(1+\sqrt{5})$ the golden ratio, c_B is given by the "Fibonacci stairs", namely an infinite stairs each of whose steps is made of a segment on a line going through the origin and a horizontal segment, with feet (endpoints) on the volume constraint $\sqrt{a}$, and both the feet and the edge determined by Fibonacci numbers. Then there is one step over $\left[\tau^4, 7\frac{1}{9}\right]$, whose left part over $[\tau^4, 7]$ is affine but nonlinear: $c_B(a) = \frac{a+1}{3}$. Finally, for $a \geqslant 7\frac{1}{9}$ the graph of $c_B(a)$ is given by eight strictly disjoint steps made of two affine segments, and $c_B(a) = \sqrt{a}$ for $a \geqslant 8\frac{1}{36}$. See Figures 1–6 later in the introduction for similar pictures in our setting.

The function $c_C(a)$ has a similar structure: On $[1, \sigma^2]$, with $\sigma = 1+\sqrt{2}$ the silver ratio, c_C is given by the "Pell stairs", namely an infinite stairs each of whose steps is made of a segment on a line going through the origin and a horizontal segment, with feet on the volume constraint $\sqrt{\frac{a}{2}}$, and both the feet and the edge determined by Pell numbers. Then there is one step over $\left[\sigma^2, 6\frac{1}{8}\right]$, whose left part over $[\sigma^2, 6]$ is affine but nonlinear: $c_C(a) = \frac{a+1}{4}$. Finally, for $a \geqslant 6\frac{1}{8}$ the graph of $c_C(a)$ is given by six strictly disjoint steps made of two affine segments, and $c_C(a) = \sqrt{\frac{a}{2}}$ for $a \geqslant 7\frac{1}{32}$.

1.2 Result

We are interested in understanding what happens with the rich structure of the functions c_B and c_C if we take as targets "longer" sets. To this end, we look at the embedding problems $E(1,a) \overset{s}{\hookrightarrow} P(b,c)$ for $c = kb$ with $k \geqslant 2$ an integer, which we encode in the functions

$$c_b(a) := \inf\{\lambda > 0 \mid E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)\}, \quad b \in \mathbb{N}_{\geqslant 2}. \tag{1-1}$$

Note that $c_1 = c_C$. The volume constraint is now $c_b(a) \geqslant \sqrt{\frac{a}{2b}}$. To formulate our result, we define for $b \in \mathbb{N}_{\geqslant 2}$ and for $k \in \{0, 1, 2, \dots, \lfloor\sqrt{2b}\rfloor\}$ the numbers

$$u_b(k) := \frac{(2b+k)^2}{2b} = 2b + 2k + \frac{k^2}{2b}, \quad v_b(k) := 2b\left(\frac{2b+2k+1}{2b+k}\right)^2$$

and

$$\alpha_b := \frac{1}{b}\left(b^2 + 2b + \sqrt{(b^2+2b)^2 - 1}\right), \quad \beta_b := 2b + 4 + \frac{1}{2b(b+1)^2}.$$

Note that $u_b(k) \leqslant 2b + 2k + 1 \leqslant v_b(k)$ with strict inequalities for $k^2 < 2b$ and equalities for $k^2 = 2b$, and that

$$2b + 2k < u_b(k) \leqslant v_b(k) < 2b + 2k + 2 \quad \text{for } k \geqslant 1.$$

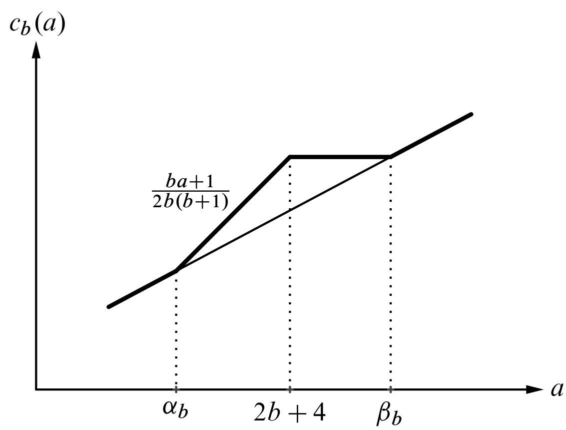

Figure 1: The affine step in our result

Further, $v_b(1) < \alpha_b < 2b+4 < \beta_b < u_b(2)$. The intervals $I_b(k) := [u_b(k), v_b(k)]$ thus have positive length except for $k^2 = 2b$, and the intervals

$$I_b(0),\quad I_b(1),\quad [\alpha_b, \beta_b],\quad I_b(2),\quad \dots,\quad I_b(\lfloor\sqrt{2b}\rfloor)$$

are in the right order and are disjoint except that $I_b(0)$ touches $I_b(1)$.

Theorem 1.1 *For every integer $b \geqslant 2$ the function $c_b(a)$ describing the symplectic embedding problem $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ is given by the volume constraint $\sqrt{\frac{a}{2b}}$ except for the following $\lceil\sqrt{2b}\,\rceil + 2$ intervals:*

(i) $c_b(a) = 1$ *if* $a \in [1, 2b]$.

(ii) *For* $k \in \{0, 1, 2, \dots, \lfloor\sqrt{2b}\rfloor\}$ *and on the interval* $I_b(k)$,

$$c_b(a) = \begin{cases} \dfrac{a}{2b+k} & \text{if } a \in [u_b(k), 2b+2k+1], \\[2ex] \dfrac{2b+2k+1}{2b+k} & \text{if } a \in [2b+2k+1, v_b(k)]. \end{cases}$$

(iii) *On the interval* $[\alpha_b, \beta_b]$,

$$c_b(a) = \begin{cases} \dfrac{ba+1}{2b(b+1)} & \text{if } a \in [\alpha_b, 2b+4], \\[2ex] 1 + \dfrac{2b+1}{2b(b+1)} & \text{if } a \in [2b+4, \beta_b]. \end{cases}$$

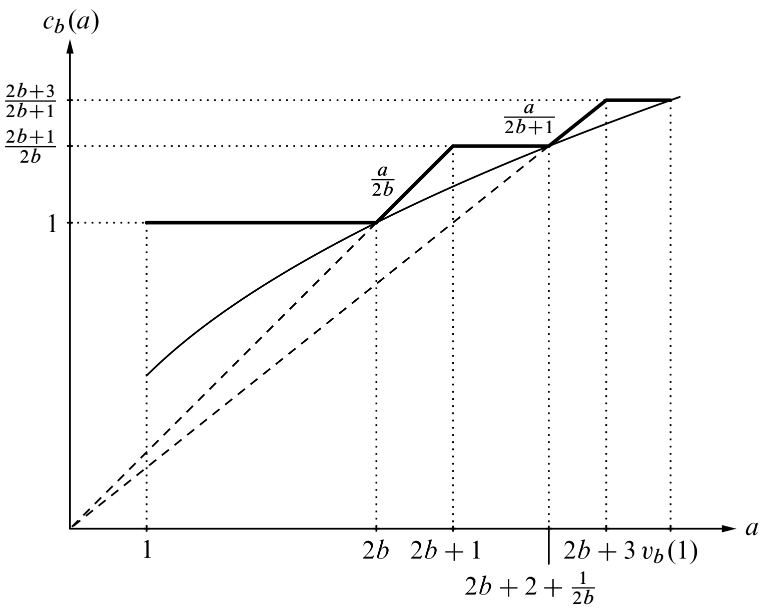

Figure 2: The graph of $c_b(a)$ on $[1, v_b(1)]$

Remarks 1.2 (1) Theorem 1.1 also solves the problem $E(1,a) \overset{s}{\hookrightarrow} E(\lambda, \lambda 2b)$ for integers $b \geqslant 2$, since, for every integer b,

$$E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b) \iff E(1,a) \overset{s}{\hookrightarrow} E(\lambda, \lambda 2b). \tag{1-2}$$

This has been shown by Frenkel and Müller [14, Corollary 1.6] for $b = 1$ by using that ECH capacities provide a complete set of invariants for the embedding problem $E(1,a) \overset{s}{\hookrightarrow} P(b,c)$, and this proof generalizes to all $b \in \mathbb{N}$. In Section 4.1 we shall prove (1-2) by using the "reduction method" (Method 2 of Section 2.2).

(2) One can replace the infimum in definition (1-1) by the minimum. This follows from the previous remark and from the fact that $E(1,a) \overset{s}{\hookrightarrow} E(\lambda, \lambda 2b)$ also for $\lambda = c_b(a)$; see McDuff [23, Corollary 1.6] and also Cristofaro-Gardiner [11, Corollary 1.6] for a generalization. Altogether, we see that

$$E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b) \iff E(1,a) \overset{s}{\hookrightarrow} E(\lambda, \lambda 2b) \iff \lambda \geqslant c_b(a).$$

Geometric description of the result We proceed with describing the functions $c_b(a)$ given in Theorem 1.1 more geometrically. The left part of the steps described in part (ii) of the theorem lie on a line passing through the origin, while the left part of the step described in part (iii) lies on a line crossing the y–axis at $\frac{1}{2b(b+1)}$. We call the steps in (ii) the "linear steps" and the step in (iii) the "affine step".

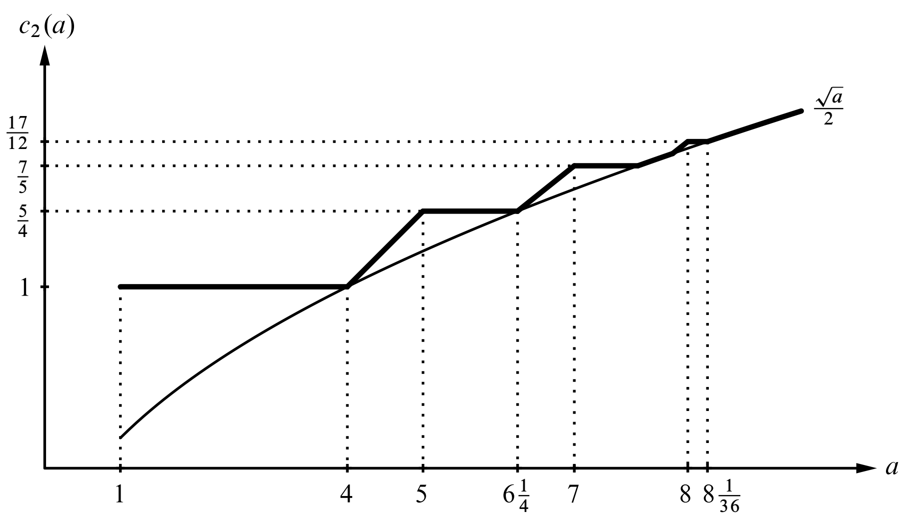

Figure 3: The graph of $c_2(a)$

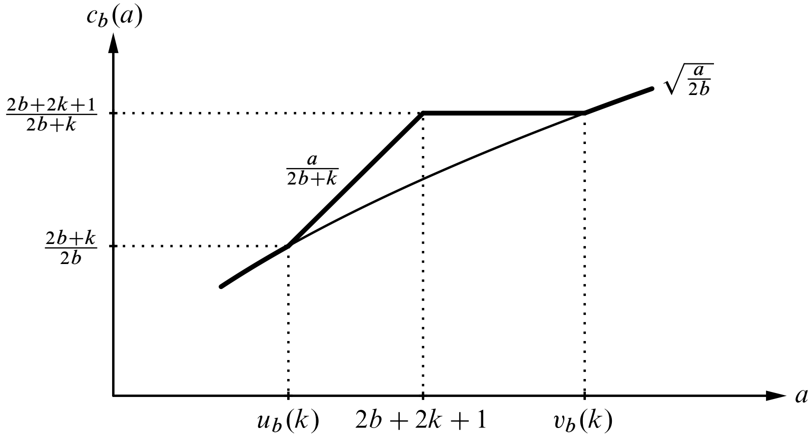

Figure 4: One of the $\lceil\sqrt{2b}\,\rceil$ linear steps

The graph of $c_b(a)$ on $[1, v_b(1)]$ is given by

$$c_b(a) = \begin{cases} 1 & \text{if } a \in [1, 2b], \\ \dfrac{a}{2b} & \text{if } a \in [2b, 2b+1], \\ \dfrac{2b+1}{2b} & \text{if } a \in \left[2b+1, 2b+2+\dfrac{1}{2b}\right], \\ \dfrac{a}{2b+1} & \text{if } a \in \left[2b+2+\dfrac{1}{2b}, 2b+3\right], \\ \dfrac{2b+3}{2b+1} & \text{if } a \in \left[2b+3, 2b+4-\dfrac{4}{(2b+1)^2}\right]; \end{cases}$$

see Figure 2. This part of the graph touches the volume constraint only in three points. Then follows a "volume interval", and then the affine step described in part (iii) and Figure 1. For $b = 2$ there are no further obstructions (Figure 3), but for $b \geqslant 3$ there are $\lceil\sqrt{2b}\,\rceil - 2$ more linear steps, that are strictly disjoint and made of a linear and a horizontal segment (Figures 4 and 5).

The length of the affine step is

$$\beta_b - \alpha_b < \beta_b - v_b(1) = \frac{1}{2b(b+1)^2} + \frac{4}{(2b+1)^2},$$

and hence this step becomes very small for b large. The length of the k^{th} linear step is

$$\ell_b(k) := v_b(k) - u_b(k) = (2b - k^2)\frac{8b^2 + k^2 + (2+8k)b}{2b(2b+k)^2}.$$

For fixed b, the function $\ell_b(k)$ is strictly decreasing, with $\ell_b(\sqrt{2b}) = 0$. For fixed k, however, $\lim_{b\to\infty} \ell_b(k) = 2$. More precisely, $\ell_b(0)$ is strictly decreasing to 2, and

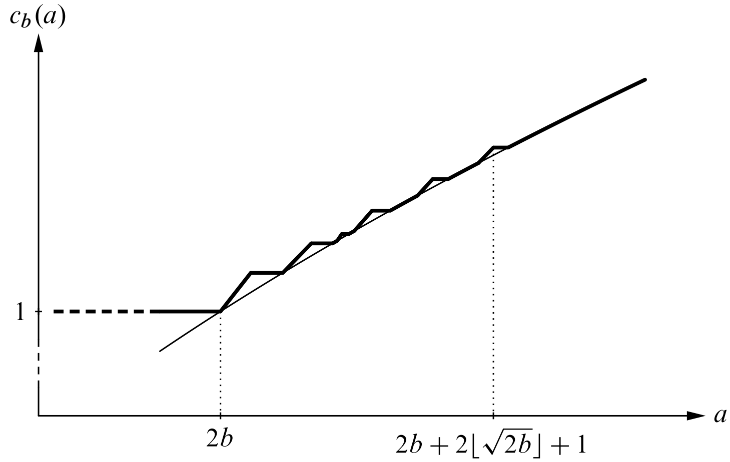

Figure 5: The graph of $c_9(a)$

$\ell_b(k)$ is strictly increasing to 2 for every $k \geqslant 1$. Since the edge of the k^{th} step is at $2b+2k+1$, we see that, for $b \to \infty$, an arbitrarily large (but fixed) part of the graph of $c_b(a)$ consists of linear steps of length almost 2, which almost form a connected staircase (Figure 6).

We reformulate this behaviour of $c_b(a)$ for large b in terms of a rescaled limit function: Consider the rescaled functions

$$\hat{c}_b(a) = 2bc_b(a+2b) - 2b, \quad a \geqslant 0,$$

which are obtained from $c_b(a)$ by first forgetting about the horizontal line $c_b(a)=1$ over $[1, 2b]$ that comes from the nonsqueezing theorem, then vertically rescaling by $2b$, and finally translating the graph by the vector $(-2b, -2b)$. Further, consider the

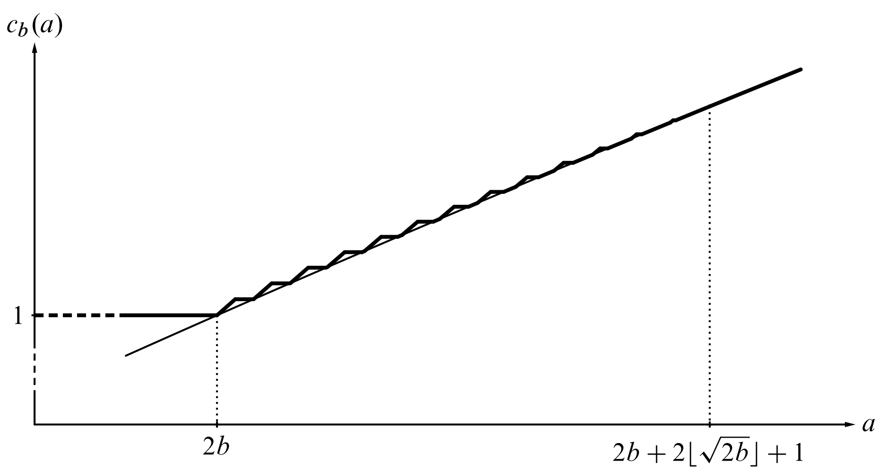

Figure 6: The graph of $c_{85}(a)$

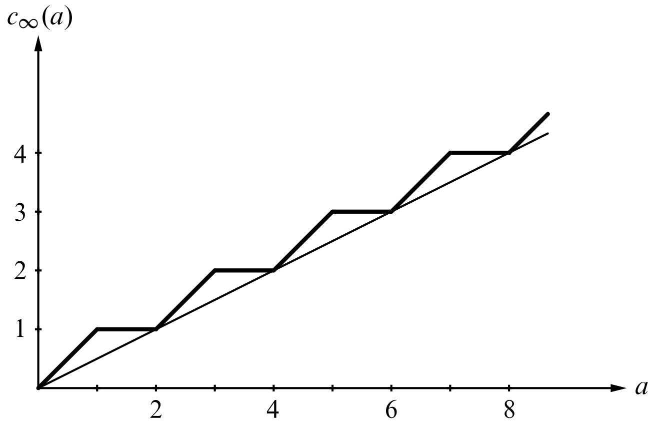

Figure 7: The graph of the rescaled limit function $c_\infty(a)$

function $c_\infty\colon [0,\infty) \to \mathbb{R}$ drawn in Figure 7; its graph consists of infinitely many steps of width 2 and slope 1 that are based at the line $\frac{a}{2}$. Then

$$\lim_{b\to\infty} \widehat{c}_b(a) = c_\infty(a), \quad a \in [0,\infty), \tag{1-3}$$

uniformly on bounded sets. Indeed, applying the same rescaling to $\sqrt{\frac{a}{2b}}$ yields $2b\sqrt{(a+2b)/2b}-2b$, which is $\frac{a}{2}+O\big(\frac{a^2}{2b}\big)$ for $b \geqslant a$. One can also check that $\widehat{c}_b(a)$ is increasing to $c_\infty(a)$ for all a.

1.3 Interpretation

Recall from the introduction that the graph of $c_C(a)$ has three parts: First the infinite Pell stairs, then one affine step, and then six more steps.

If we take $b=1$ in the above description of $c_b(a)$ on $[1, v_b(1)]$, we exactly obtain $c_C(a)$ on $[1, v_1(1)] = \big[1, \frac{50}{9}\big]$. Further, if we take $b=1$ in the description (iii) of the affine step of $c_b(a)$, we exactly obtain the affine step of $c_C(a)$ over $\big[\sigma^2, 6\frac{1}{8}\big]$. Hence $c_b(a)$ generalizes $c_C(a)$ on the first two steps and on the affine step. This is not a coincidence. Indeed, the two exceptional classes giving rise to the first two steps of the Pell stairs are the first two in the sequence (1-4) of exceptional classes E_n giving rise to all the linear steps of $c_b(a)$, and the exceptional class giving rise to the affine step of $c_C(a)$ is the first in a sequence of exceptional classes F_b giving rise to the affine step in $c_b(a)$; see Section 3.

On the other hand, the remaining infinitely many steps of the Pell stairs have no counterpart for $b \geqslant 2$. Similarly, the linear steps described in Theorem 1.1(ii) are more regular than the affine steps on the right part of $c_C(a)$, none of which consists of a linear and a horizontal segment. We thus see that the first two steps and the affine step of $c_C(a) = c_1(a)$ are stable under the deformations of b we consider, while the other steps are not.

By Theorem 1.1, $c_b(a)$ equals the volume constraint $\sqrt{\frac{a}{2b}}$ for $a \geqslant v_b(\lfloor\sqrt{2b}\rfloor) = 2b + O(\sqrt{b})$, that is, there are no packing obstructions for the embedding problem $E(1,a) \stackrel{s}{\hookrightarrow} P(\lambda, \lambda b)$ for a sufficiently large. This is not a surprise. Indeed, this phenomenon was already observed for the embedding problems $E(1,a) \stackrel{s}{\hookrightarrow} B^4(b)$ and $E(1,a) \stackrel{s}{\hookrightarrow} C^4(b)$, and it fits well with previous results: It is known for many closed connected symplectic manifolds (M,ω) that there is a number $N(M,\omega)$ such that (M,ω) admits a full symplectic packing by k equal balls for every $k \geqslant N(M,\omega)$ ("packing stability"; see Biran [2; 3], Buse and Hind [5; 6], Buse, Hind and Opshtein [7] and Buse and Pinsonnault [8]). Similarly, an explicit construction implies that for any connected symplectic manifold (M,ω) of finite volume, the proportion of the volume that can be filled by a dilate of the ellipsoid $E(1,\dots,1,a)$ tends to 1 as $a \to \infty$; see Schlenk [28, Section 6]: The packing obstruction tends to zero as the *domain* is more and more elongated.

Theorem 1.1 exhibits a different phenomenon: If in the problem $E(1,a) \stackrel{s}{\hookrightarrow} P(\lambda, \lambda b)$ the *target* is elongated ($b \to \infty$), then the regular Pell stairs in the graph of $c_1(a)$ first almost disappears (only two linear steps and the affine step remain), but then for large b the graph of $c_b(a)$ reorganizes to a staircase that asymptotically is infinite and completely regular.

1.4 Stabilization and connection with symplectic folding

Let a, $b \geqslant 1$ be real numbers. Following Cristofaro-Gardiner and Hind [12] we consider for each $N \geqslant 3$ the stabilized problem

$$c_b^N(a) := \inf\{\lambda > 0 \mid E(1,a) \times \mathbb{C}^{N-2} \stackrel{s}{\hookrightarrow} P(\lambda, \lambda b) \times \mathbb{C}^{N-2}\}.$$

Then $c_b^N(a) \leqslant c_b(a)$.

Lemma 1.3 *For every $N \geqslant 3$ and all real numbers a, $b \geqslant 1$,*

$$c_b^N(a) \leqslant f_b(a) := \frac{2a}{a+2b-1}.$$

Proof Set $\mu = \frac{a(2b-1)}{a+2b-1}$ and $\lambda = 2\big(1 - \frac{\mu}{a}\big)$. Then $\mu + \frac{\lambda}{2} = b\lambda$. Since $b \geqslant 1$ we have $\mu \geqslant \frac{\lambda}{2}$. Note that $\frac{\lambda}{2} = 1 - \frac{\mu}{a}$ is the area of a z_2–disc in $E(a,1)$ over a point z_1 on the boundary of the disc $D(\mu)$ of area μ. Applying Hind's folding construction in [16, Section 2] with μ — instead of $\frac{S}{S+1}$ — we obtain for every $\varepsilon > 0$ a symplectic embedding

$$E(1,a)\times\mathbb{C} \overset{s}{\hookrightarrow} P\Big(\mu+\frac{\lambda}{2}+\varepsilon, 2\frac{\lambda}{2}+\varepsilon\Big)\times\mathbb{C}.$$

Now recall that $\mu+\frac{\lambda}{2}=b\lambda$ and note that $\lambda = f_b(a)$. □

In view of the above proof, we call the graph of $f_b(a)$ the folding curve. Now note that

$$f_b(2b+2k+1) = \frac{2b+2k+1}{2b+k}, \qquad k \geqslant 0.$$

For $b\in\mathbb{N}$ this is also the value of c_b at the edge points of the k^{th} linear step. In other words, the linear steps oscillate between the volume constraint $\sqrt{\frac{a}{2b}}$ and the folding curve; see Figures 4 and 8.

Conjecture 1.4 *The edge points of the linear steps are stable, in the sense that at these points we have $c_b^N = c_b$ for all $N\geqslant 3$.*

This conjecture is based on the main result of [12], where it is shown that the edge points of the Fibonacci stairs for the problem $E(a,1)\overset{s}{\hookrightarrow} B^4(\lambda)$ are stable. It is likely that one can prove it by a similar method as in [12]; see also the discussion at the end of the next section. A proof of Conjecture 1.4 is not the concern of the present work, but a positive answer would imply that the folding construction in the proof of Lemma 1.3 is sharp at the edge points of the linear steps.

Recall that $c_b(a)=1$ for $a\in[1,2b]$. As we shall see in Proposition 3.5(ii), $c_b(a)=\sqrt{\frac{a}{2b}}$ for all $a\geqslant(\sqrt{2b}+1)^2$ and all real $b\geqslant 2$. Now notice that $f_b(a)\geqslant\sqrt{\frac{a}{2b}}$ if and only if $a\in[(\sqrt{2b}-1)^2,(\sqrt{2b}+1)^2]$. It follows that

$$c_b^N(a) < c_b(a) \quad \text{if } a\notin[2b-1,(\sqrt{2b}+1)^2]$$

for all $b\geqslant 2$ and $N\geqslant 3$.

We finally notice that under the rescaling yielding the limit function $c_\infty(a)$, we have $\widehat{f}_b(a) = 2bf_b(a+2b)-2b = \frac{2b(a+1)}{a+4b-1}$, and so

$$f_\infty(a) := \lim_{b\to\infty}\widehat{f}_b(a) = \frac{a+1}{2}.$$

This means that also the limit function c_∞ oscillates, between the limit function $\frac{a}{2}$ of the volume constraint $\sqrt{\frac{a}{2b}}$ and the limit function $\frac{a+1}{2}$ of the folding curve.

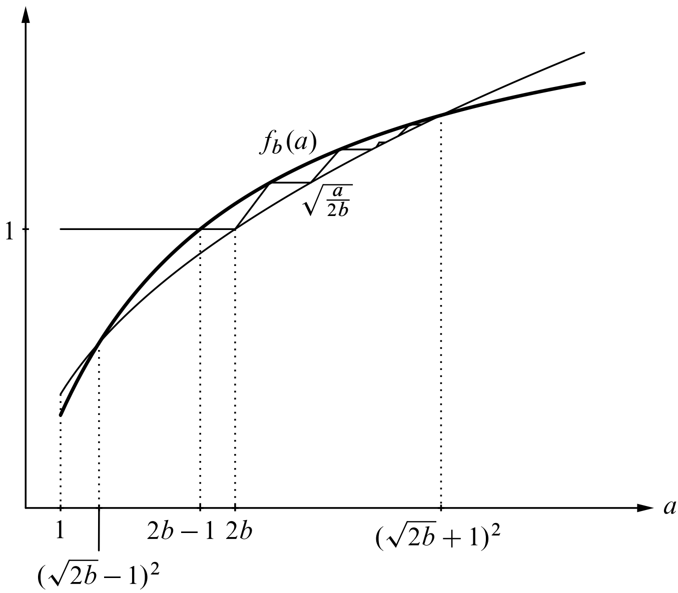

Figure 8: The volume constraint, $c_b(a)$, and the folding curve for $b = 5$

1.5 Method

In principle, there are two methods to prove Theorem 1.1: The first method (Method 1 in Section 2.2, which was used by Frenkel and Müller [14] and McDuff and Schlenk [26]) is to find the strongest obstruction for the embedding problem $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ coming from exceptional classes (ie homology classes in a certain multiple blow-up of $\mathbb{C}P^2$ represented by embedded J–holomorphic -1 spheres). The second method (Method 2 in Section 2.2, that was first used by Buse and Pinsonnault [8]) is a cohomological version of the first method: One associates to a hypothetical embedding $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ a cohomology class, and checks whether this class transforms to a "reduced vector" under Cremona transforms. While the first method is sufficient for solving the problems $E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda)$ and $E(1,a) \overset{s}{\hookrightarrow} C^4(\lambda)$ — see [14; 26] — it does not lead to a proof of the entire Theorem 1.1, because the known upper bound for the number of obstructive exceptional classes tends to infinity with b. On the other hand, Method 2 does yield a proof of Theorem 1.1, as will become clear from our proof. We shall not follow such a puristic approach, however, but an opportunistic one, which uses both methods: Given b, we first write down a finite set of exceptional classes that yield embedding obstructions, namely $E_0 = (1,0;1)$ and

$$\begin{aligned} E_n &:= (n,1;1^{\times(2n+1)}), \quad n = b,\dots,b+\lfloor\sqrt{2b}\rfloor, \\ F_b &:= (b(b+1), b+1; b+1, b^{\times(2b+3)}) \end{aligned} \tag{1-4}$$

(see Section 2.2 for the notation), and then use Method 2 to show that the obstruction $h_b(a)$ given by these classes is complete. In other words, we use Method 1 to show that $c_b(a) \geqslant h_b(a)$ and Method 2 to show that $c_b(a) \leqslant h_b(a)$ — with the exception that for a large and for $b = 2$ and $a \in \left[8\frac{1}{36}, 9\right]$ we use Method 1 to show that $c_b(a)$ equals the volume constraint $\sqrt{\frac{a}{2b}}$.

This hybrid approach yields the shortest proof of Theorem 1.1 we know. Further, knowing a set of exceptional classes that provide all embedding obstructions is interesting for at least two reasons: First, the holomorphic spheres underlying these classes provide a geometric explanation of the graphs of the functions $c_b(a)$. Second, one should be able to use these holomorphic spheres to prove Conjecture 1.4; it is probably the case that one can find the needed obstructions by stretching these spheres and then "stabilizing" as in Cristofaro-Gardiner and Hind [12] and Hind and Kerman [17].

1.6 Outlook

Our ultimate goal is to see the *continuous* film of graphs $c_b(a)$ for $b \geqslant 1$ real. It would be particularly interesting to understand this film for $b \in [1, 2]$, or just for $b \in [1, 1+\varepsilon]$ for some $\varepsilon > 0$, namely to understand how the Pell stairs disappear. In Burkhart, Panescu and Timmons [4], ECH capacities are used to compute $c_b(a)$ for $b = \frac{13}{2}$ and to get an idea of this film. In accordance with Theorem 1.1, Conjecture 6.3 in [4] and further investigations we make:

Conjecture 1.5 *For any real $b \geqslant 2$ the function $c_b(a)$ is given by the maximum of the volume constraint $\sqrt{\frac{a}{2b}}$ and the obstructions coming from the exceptional classes E_n and F_n in (1-4).*

The obstructions given by the exceptional classes E_n and F_n are readily computed; see Section 3.3: While the classes E_n again give rise to a finite staircase with linear steps, the classes F_n give an obstruction only for $b \in \left(n - \frac{n}{(n+1)^2}, n + \frac{1}{n+2}\right)$. While our proof of Theorem 1.1 should extend to a proof of Conjecture 1.5, the analysis is more involved, since fractional parts arise, that are harder to estimate.

Our only definite result for b real is that for every real $b \geqslant 2$ we have $c_b(a) = \sqrt{\frac{a}{2b}}$ for all $a \geqslant (\sqrt{2b} + 1)^2$; see Proposition 3.5(ii).

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2 Methods of proof

In this section we describe the methods we will use in the proof of Theorem 1.1. For more details we refer to the surveys [10; 19; 29].

2.1 Translation to a ball packing problem

Fix $b \geqslant 1$. Since the function $c_b(a)$ is continuous in a, it suffices to compute $c_b(a)$ for $a \geqslant 1$ rational. The *weight expansion* $\boldsymbol{w}(a)$ of such an a is the finite decreasing sequence

$$\boldsymbol{w}(a) := (\underbrace{1,\dots,1}_{\ell_0}, \underbrace{w_1,\dots,w_1}_{\ell_1}, \dots, \underbrace{w_N,\dots,w_N}_{\ell_N}) \equiv (1^{\times \ell_0}, w_1^{\times \ell_1}, \dots, w_N^{\times \ell_N}) \tag{2-1}$$

such that $w_1 = a - \ell_0 < 1$, $w_2 = 1 - \ell_1 w_1 < w_1$, and so on. For example, $a = \frac{25}{9}$ has weight expansion $\boldsymbol{w}(a) = \big(1, 1, \frac{7}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}\big) \equiv \big(1^{\times 2}, \frac{7}{9}, \frac{2}{9}^{\times 3}, \frac{1}{9}^{\times 2}\big)$.

Write $B(\boldsymbol{w}(a))$ for the disjoint union of balls $B(1) \amalg \cdots \amalg B(w_N)$ whose weights are those appearing in $\boldsymbol{w}(a)$, with multiplicities. Based on [23] it was shown in [14, Proposition 1.4] that $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ if and only if

$$B(\boldsymbol{w}(a)) \amalg B(\lambda) \amalg B(\lambda b) \overset{s}{\hookrightarrow} B(\lambda(b+1)); \tag{2-2}$$

compare the moment map picture on the left of Figure 9.

2.2 Three translations to a combinatorial problem

In order to reformulate problem (2-2), we look at the general ball packing problem

$$\coprod_{i=1}^{n} B(a_i) \overset{s}{\hookrightarrow} B(\mu). \tag{2-3}$$

We shall describe three combinatorial solutions of (2-3).

Denote by X_n the n–fold complex blow-up of $\mathbb{C}\mathrm{P}^2$, endowed by the orientation induced by the complex structure. Its homology group $H_2(X_n;\mathbb{Z})$ has the canonical basis $\{L, E_1, \dots, E_n\}$, where $L = [\mathbb{C}\mathrm{P}^1]$ and the E_i are the classes of the exceptional divisors. The Poincaré duals of these classes are denoted by $\ell, e_1, \dots, e_n$. Let $K := -3L + \sum_{i=1}^n E_i$ be the Poincaré dual of $-c_1(X_n)$, and consider the K–symplectic cone $\mathcal{C}_K(X_n) \subset H^2(X_n;\mathbb{R})$, namely the set of cohomology classes that can be represented by symplectic forms ω on X_n that are compatible with the orientation of X_n and have first Chern class $c_1(\omega) = c_1(X_n) = \mathrm{PD}(-K)$. Denote by $\bar{\mathcal{C}}_K(X_n)$ its closure in $H^2(X_n;\mathbb{R})$.

McDuff and Polterovich [25] proved that an embedding (2-3) exists if and only if

$$\mu\ell - \sum_{i=1}^{n} a_i e_i \in \bar{\mathcal{C}}_K(X_n).$$

We thus need to describe $\bar{\mathcal{C}}_K(X_n)$. For this consider the set $\mathcal{E}_K(X_n) \subset H_2(X_n;\mathbb{Z})$ of classes E with $-K\cdot E = c_1(E) = 1$ and $E\cdot E = -1$ that can be represented by smoothly embedded spheres. Li and Liu [22] characterized $\bar{\mathcal{C}}_K(X_n)$ as

$$\bar{\mathcal{C}}_K(X_n) = \{\alpha \in H^2(X_n;\mathbb{R}) \mid \alpha^2 \geqslant 0 \text{ and } \alpha(E) \geqslant 0 \text{ for all } E \in \mathcal{E}_K(X_n)\}. \tag{2-4}$$

We thus need to describe $\mathcal{E}_K(X_n)$. For this define, for $n \geqslant 3$, the *Cremona transform* $\mathrm{Cr}\colon \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ as the linear map taking $(x_0; x_1, \dots, x_n)$ to

$$(2x_0 - x_1 - x_2 - x_3;\ x_0 - x_2 - x_3,\ x_0 - x_1 - x_3,\ x_0 - x_1 - x_2,\ x_4,\ \dots,\ x_n). \tag{2-5}$$

A vector $(x_0; x_1, \dots, x_n)$ is *ordered* if $x_1 \geqslant \cdots \geqslant x_n$. The *standard Cremona move* takes an ordered vector $(x_0; \boldsymbol{x})$ to the vector obtained by ordering $\mathrm{Cr}(x_0; \boldsymbol{x})$. More generally, a *Cremona move* is a Cremona transform followed by any permutation of the components of $\boldsymbol{x}$.

For later use we recall the geometric origin of Cr and of Cremona moves. For any nonzero vector u in an inner-product space, the map $r_u(x) = x - 2\frac{\langle u,x\rangle}{\langle u,u\rangle} u$ is the reflection about u, and hence an involution. Similarly, for a class $A \in H_2(X_n;\mathbb{R})$ with $A\cdot A \neq 0$ the map $r_A(B) = B - 2\frac{A\cdot B}{A\cdot A}A$ is an involution of $H_2(X_n;\mathbb{R})$. For $|A\cdot A| \in \{1,2\}$, this map is also an automorphism of $H_2(X_n;\mathbb{Z})$. Now take the classes $A_0 = L - E_1 - E_2 - E_3$ and $A_{ij} = E_i - E_j$ for $1 \leqslant i < j \leqslant n$. Their self-intersection number is -2 and so, for these classes,

$$r_A(B) = B + (A\cdot B)A. \tag{2-6}$$

With respect to the basis $\{L, E_1, \dots, E_n\}$ we have that r_{A_0} is given by (2-5), that is, $r_{A_0} = \mathrm{Cr}\colon \mathbb{Z}^{1+n} \to \mathbb{Z}^{1+n}$ takes the integral vector $(d;\boldsymbol{m}) = (d; m_1, \dots, m_n)$ to

$$(2d - m_1 - m_2 - m_3;\ d - m_2 - m_3,\ d - m_1 - m_3,\ d - m_1 - m_2,\ m_4,\ \dots,\ m_n), \tag{2-7}$$

and $r_{A_{ij}}$ is the transposition τ_{ij} interchanging the i^{th} and j^{th} coordinate. These involutions of $H_2(X_n;\mathbb{Z})$ are induced by orientation-preserving diffeomorphisms of X_n. This is clear for τ_{ij} (lift to X_n an isotopy of $\mathbb{C}\mathrm{P}^2$ interchanging holomorphically small discs around the i^{th} and j^{th} blow-up points), and it holds for all classes A_0 and A_{ij} because each of them can be represented by a smoothly embedded sphere S, and the smooth version of the Dehn–Seidel twist along S [30] is a diffeomorphism inducing (2-6), in view of the Picard–Lefschetz formula [1, page 26]. Since the maps

Cr and τ_{ij} preserve both the intersection product on $H_2(X_n;\mathbb{Z})$ and the class K, they preserve the set $\mathcal{E}_K(X_n)$.

Based on [21; 22] it was shown in [26, Proposition 1.2.12] that a homology class $E = dL - \sum_{i=1}^n m_i E_i$ belongs to $\mathcal{E}_K(X_n)$ if and only if the vector $(d;\boldsymbol{m}) = (d;m_1,\dots,m_n)$ is equal to $(0;-1,0,\dots,0)$ up to a permutation of the m_i, or if $(d;\boldsymbol{m}) \in \mathbb{N} \cup (\mathbb{N}\cup\{0\})^n$ satisfies the Diophantine system

$$\sum_{i=1}^n m_i = 3d-1, \quad \sum_{i=1}^n m_i^2 = d^2+1, \tag{2-8}$$

and reduces to $(0;-1,0,\dots,0)$ under repeated standard Cremona moves. Summarizing, we find:

Method 1 (obstructive classes) *An embedding (2-3) exists if and only if* $\sum_{i=1}^n a_i^2 \leqslant \mu^2$ *and* $\sum_{i=1}^n a_i m_i \leqslant \mu d$ *for all vectors* $(d;\boldsymbol{m})$ *of nonnegative integers satisfying (2-8) and reducing to* $(0;-1,0,\dots,0)$ *under repeated standard Cremona moves.*

Remark 2.1 It is shown in [23] (see also [19]) that (2-3) is also equivalent to $\sum_{i=1}^n a_i m_i \leqslant \mu d$ *for all vectors* $(d;\boldsymbol{m})$ *of nonnegative integers satisfying the Diophantine system (2-8).* It follows that if we use exceptional classes only to give *lower* bounds for $c_b(a)$ (as we do in this paper), then we do not need to show that these classes reduce to $(0;-1,0,\dots,0)$ under repeated standard Cremona moves. We shall nevertheless perform these reductions, since they are readily done (see Section 3.2) and since we wish to know explicit exceptional classes responsible for the embedding obstructions beyond the volume constraint.

In view of (2-2) we find that $E(1,a) \overset{s}{\hookrightarrow} P(\lambda,\lambda b)$ if and only if $\lambda \geqslant \sqrt{\frac{a}{2b}}$ and

$$\lambda(b+1) \geqslant \lambda(bm_1+m_2) + m_3 w_1 + \cdots + m_{k+2} w_k \tag{2-9}$$

for all vectors $(d;\boldsymbol{m})$ of nonnegative integers satisfying (2-8) with $n = k+2$ and reducing to $(0;-1,0,\dots,0)$ under repeated standard Cremona moves.

Condition (2-9) is not handy, since λ appears on both sides. We thus better work directly in $P(\lambda,\lambda b)$ or in its compactification $S^2\times S^2$ endowed with the product symplectic form of the same volume. Let Y_{k+1} be the complex blow-up of $S^2\times S^2$ in $k+1$ points. Then the classes $S_1 = [S^2\times \mathrm{pt}]$ and $S_2 = [\mathrm{pt}\times S^2]$ and the classes $F_1,\dots,F_{k+1}$ of the exceptional divisors form a basis of $H_2(Y_{k+1})$. As one can guess from the picture on the right of Figure 9, there exists a diffeomorphism $\psi\colon Y_{k+1}\to X_{k+2}$ such that the induced map $\psi_*\colon H_2(Y_{k+1}) \to H_2(X_{k+2})$ is given by

$$\begin{aligned} S_1 &\mapsto L - E_1, \\ S_2 &\mapsto L \qquad - E_2, \\ F_1 &\mapsto L - E_1 - E_2, \\ F_i &\mapsto \qquad\qquad - E_{i+1}, \quad i \geqslant 2. \end{aligned}$$

If we write $(d, e; m_1, \dots, m_{k+1})$ for $dS_1 + eS_2 - m_1 F_1 - \dots - m_{k+1} F_{k+1}$, we thus have

(2-10) $$\psi_*(d, e; \boldsymbol{m}) = (d + e - m_1;\, d - m_1,\, e - m_1,\, m_2,\, \dots, m_{k+1}).$$

Given vectors $\boldsymbol{u} \in \mathbb{R}^{n_1}$ and $\boldsymbol{v} \in \mathbb{R}^{n_2}$ we write $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{\max(n_1,n_2)} u_i v_i$. In the basis S_1, S_2, $F_1, \dots, F_{k+1}$, we can reformulate Method 1 as:

Method 1′ (obstructive classes) *An embedding $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ exists if and only if $\lambda \geqslant \sqrt{\frac{a}{2b}}$ and*

(2-11) $$\lambda \geqslant \frac{\langle \boldsymbol{m}, \boldsymbol{w}(a) \rangle}{d + be} =: \mu_b(d, e; \boldsymbol{m})(a)$$

for all vectors $(d, e; \boldsymbol{m})$ of nonnegative integers that satisfy the Diophantine system

(2-12) $$\sum m_i = 2(d+e) - 1, \qquad \sum m_i^2 = 2de + 1$$

and for which $\psi_*(d, e; \boldsymbol{m})$ reduces to $(0; -1, 0, \dots, 0)$ under repeated standard Cremona moves.

For the detailed translation of Method 1 to Method 1′ we refer to the proof of [14, Proposition 3.9]. As we shall see in Section 3, the obstructions to embeddings $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ beyond the volume (that is, the steps in the graphs $c_b(a)$) are all given by the two series of exceptional classes $(d, e, \boldsymbol{m})$

(2-13) $$\begin{aligned} E_n &:= (n, 1; 1^{\times(2n+1)}), \\ F_n &:= (n(n+1), n+1; n+1, n^{\times(2n+3)}). \end{aligned}$$

In Method 1, the Cremona moves acted on integral homology classes $(d; \boldsymbol{m})$. The second method applies Cremona moves to real cohomology classes α, and verifies by a finite algorithm whether $\alpha \in \overline{\mathcal{C}}_K(X_n)$.

For convenience, we write $(\mu; a_1, \dots, a_n)$ instead of $\mu\ell - \sum_{i=1}^n a_i e_i$. Recall that the Cremona transform Cr on $H_2(X_n; \mathbb{Z})$ is induced by an orientation-preserving diffeomorphism φ of X_n. Since $\mathrm{Cr} = \varphi_*$ is an involution, the map φ^* induced on cohomology $H^2(X_n;\mathbb{R})$ is also given by formula (2-5) with respect to the Poincaré dual basis $\{\ell, e_1, \dots, e_n\}$, that is, $\varphi^* = \mathrm{Cr}\colon \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ takes the vector $(\mu; a_1, \dots, a_n)$ to

$$(2\mu - a_1 - a_2 - a_3;\ \mu - a_2 - a_3,\ \mu - a_1 - a_3,\ \mu - a_1 - a_2,\ a_4,\ \dots,\ a_n). \tag{2-14}$$

Call an ordered vector $(\mu; a_1, \dots, a_n)$ *reduced* if $\mu \geqslant a_1 + a_2 + a_3$. Using the characterisation (2-4) and building on [21; 22], Buse and Pinsonnault [8, Section 2.3] and Karshon and Kessler [20, Section 6.3] designed the following algorithm to decide whether an embedding (2-3) exists:

Method 2 (reduction at a point) *Let $\alpha = (\mu; a_1, \dots, a_n)$ be an ordered vector with $\mu \geqslant 0$ and $\alpha^2 \geqslant 0$. The sequence obtained from applying to α standard Cremona moves contains a reduced vector. Let $(\widehat{\mu}; \widehat{a}_1, \dots, \widehat{a}_n)$ be the first reduced vector in this sequence. Then $\alpha \in \overline{\mathcal{C}}_K(X_n)$ if and only if $\widehat{a}_1, \dots, \widehat{a}_n \geqslant 0$.*

We shall only need the "if" part of this equivalence. In fact, we shall use a version thereof that will permit us to avoid finding the reordering after each Cremona transform:

Proposition 2.2 *Let $\alpha = (\mu; a_1, \dots, a_n)$ be a vector with $\mu \geqslant 0$ and $\alpha^2 \geqslant 0$, and assume that there is a sequence $\alpha = \alpha_0, \alpha_1, \dots, \alpha_m$ of vectors such that α_{j+1} is obtained from α_j by a Cremona move. If $\alpha_m = (\widehat{\mu}; \widehat{a}_1, \dots, \widehat{a}_n)$ is reduced and $\widehat{a}_1, \dots, \widehat{a}_n \geqslant 0$, then $\alpha \in \overline{\mathcal{C}}_K(X_n)$.*

Proof According to [22, Proposition 4.9(3)], a reduced vector with nonnegative coefficients belongs to $\overline{\mathcal{C}}_K(X_n)$. Hence $\alpha_m \in \overline{\mathcal{C}}_K(X_n)$. By assumption we have $\alpha_m = (\pi \circ \mathrm{Cr})(\alpha_{m-1})$, where π is a coordinate permutation of $\mathbb{R}^n$. Write π as a product $\tau_s \circ \cdots \circ \tau_1$ of transpositions. Since Cr and τ_i are involutions,

$$\alpha_{m-1} = (\mathrm{Cr} \circ \tau_1 \circ \cdots \circ \tau_s)(\alpha_m).$$

Recall that Cr and τ_i preserve the set $\mathcal{E}_K(X_n)$. By (2-4), these maps also preserve $\overline{\mathcal{C}}_K(X_n)$. Thus $\alpha_{m-1} \in \overline{\mathcal{C}}_K(X_n)$. Iterating this argument yields $\alpha = \alpha_0 \in \overline{\mathcal{C}}_K(X_n)$. □

It turns out that for transforming a (reducible) vector to a reduced vector by Cremona moves, it is best to reorder every vector in the process. In our reduction schemes in Sections 5–8 we will usually do this, but not always, to avoid distinguishing even more cases. The point of Proposition 2.2 is that even when we do restore the order of a vector, we do not need to prove this, except for the head of the last vector: All we need to make sure is that we eventually arrive at a vector $(\widehat{\mu}; \widehat{a}_1, \widehat{a}_2, \widehat{a}_3, \widehat{a}_4, \dots)$ that is reduced and has $\widehat{a}_j \geqslant 0$ for all j, ie is such that

$$\min\{\widehat{a}_1, \widehat{a}_2, \widehat{a}_3\} \geqslant \max\{\widehat{a}_4, \dots, \widehat{a}_n\}, \qquad \widehat{\mu} \geqslant \widehat{a}_1 + \widehat{a}_2 + \widehat{a}_3, \qquad \widehat{a}_j \geqslant 0 \quad \text{for all } j.$$

On the other hand, we will always immediately check in each step that the new coefficients are nonnegative, since otherwise we may easily forget checking a coefficient at the end.

Recall that an embedding $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ exists if and only if an embedding (2-2) exists. Together with Proposition 2.2 we find the following recipe:

Proposition 2.3 *An embedding $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ exists if there exists a finite sequence of Cremona moves that transforms the vector (4-1) to an ordered vector with nonnegative entries and defect $\delta \geqslant 0$.*

In our applications of this proposition we will have $\lambda \in (1,2)$. The first Cremona transform thus maps

$$((b+1)\lambda;\ b\lambda,\ \lambda,\ 1^{\times \lfloor a \rfloor},\ w_1^{\times \ell_1}, \dots)$$

with $\delta = -1$ to the vector

$$((b+1)\lambda - 1;\ b\lambda - 1,\ \lambda - 1,\ 0,\ 1^{\times (\lfloor a \rfloor - 1)},\ w_1^{\times \ell_1}, \dots),$$

which reorders to

$$((b+1)\lambda - 1;\ b\lambda - 1,\ 1^{\times (\lfloor a \rfloor - 1)} \parallel \lambda - 1,\ w_1^{\times \ell_1}, \dots).$$

The action of this Cremona move on the balls

$$B(\boldsymbol{w}(a)) \amalg B(\lambda) \amalg B(b\lambda) \overset{s}{\hookrightarrow} B((b+1)\lambda)$$

with $B(\boldsymbol{w}(a)) \overset{s}{\hookrightarrow} P(\lambda, b\lambda)$ is illustrated in Figure 9.

Notation 2.4 The symbol $\parallel$ indicates that the terms before $\parallel$ are ordered, while the terms after $\parallel$ are possibly not ordered, and that all terms before $\parallel$ are not less than the terms after $\parallel$.

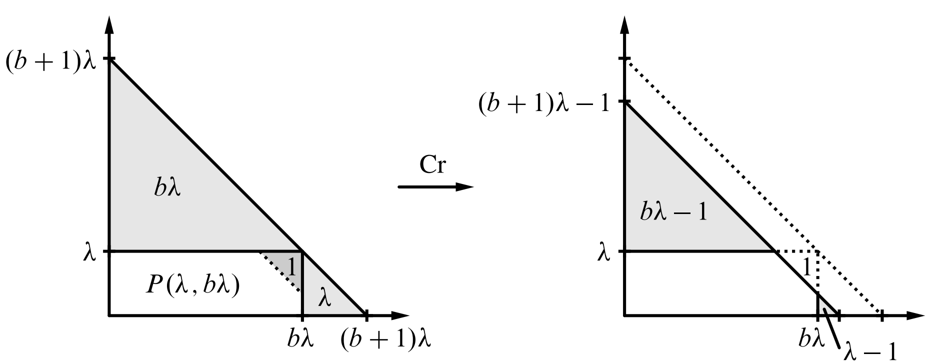

Figure 9: The effect of the first Cremona move

Method 3 (ECH capacities) *Hutchings [18] used his embedded contact homology to associate with every bounded starlike domain $U \subset \mathbb{R}^4$ a sequence of symplectic capacities $c_1(U) \leqslant c_2(U) \leqslant \cdots$. For an ellipsoid $E(a,b)$, this sequence is given by arranging the numbers of the form $ma+nb$ with m, $n \geqslant 0$ in nondecreasing order, with multiplicities. For instance,*

$$(c_k(E(1,1))) = (1,1,2,2,2,3,3,3,3,4,\dots).$$

McDuff [24] showed that ECH capacities provide a complete set of invariants for the embedding problem $E(a,b) \overset{s}{\hookrightarrow} E(c,d)$:

$$E(a,b) \overset{s}{\hookrightarrow} E(c,d) \quad \iff \quad c_k(E(a,b)) \leqslant c_k(E(c,d)) \quad \text{for all } k \geqslant 1.$$

Since the embedding problems $E(1,a) \overset{s}{\hookrightarrow} E(\lambda, \lambda 2b)$ and $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ are equivalent, it follows that

$$c_b(a) = \sup_{k \geqslant 1} \left\{ \frac{c_k(E(1,a))}{c_k(E(1,2b))} \right\}. \tag{2-15}$$

It is not clear, though, how to derive from this description of $c_b(a)$ the graphs given in Theorem 1.1.

We say that an exceptional class $E = (d,e;\boldsymbol{m}) \in \mathcal{E}_K(X_n)$ is *b–obstructive* if there is some $a \geqslant 1$ such that the obstruction function (2-11) is larger than the volume constraint:

$$\mu_b(d,e;\boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}.$$

According to Method 1, it suffices to find all b–obstructive classes: the graph of $c_b(a)$ is given as the supremum of the constraints of the b–obstructive classes and of the volume constraint. Since exceptional classes are represented by holomorphic spheres, this method gives insight into the nature of the obstruction to a full embedding. It is also useful for guessing the graph of $c_b(a)$, by first guessing a relevant set of b–obstructive classes (see Section 3). On the other hand, it is sometimes hard to find all b–obstructive classes for a point a. Method 2 is very efficient at a given point a, at least if one has an idea what $c_b(a)$ should be. However, the reduction scheme often depends rather subtly on the point a; see Sections 5–8. The reduction method is thus quite "local in a". While it is usually impossible to compute $c_b(a)$ by Method 3 (see however [4; 13]), this method is very useful for guessing the graph of $c_b(a)$, since, using (2-15) and a computer, one gets good lower bounds for $c_b(a)$.

Accordingly, we have found Theorem 1.1 as follows: We first found the exceptional classes E_n and F_n in (2-13), then used ECH capacities to convince ourselves that there are no further constraints besides the volume, and then proved this by the reduction method. This seems to be a convenient procedure for solving symplectic embedding problems for which ECH capacities are known to form a complete set of invariants, such as those studied in [11].

3 Applications of Method 1

Fix a real number $b \geqslant 1$. As in (2-11) we associate with every solution $(d,e;\boldsymbol{m})$ of the Diophantine system (2-12) the obstruction function

$$\mu_b(d,e;\boldsymbol{m})(a) = \frac{\langle \boldsymbol{m}, \boldsymbol{w}(a)\rangle}{d+be}, \tag{3-1}$$

where as before $\boldsymbol{w}(a)$ is the weight expansion of $a \geqslant 1$. Further, define the error vector $\varepsilon := \varepsilon(a)$ by

$$\boldsymbol{m} = \frac{d+be}{\sqrt{2ba}}\boldsymbol{w}(a) + \varepsilon.$$

(Here, we add zeros to $\boldsymbol{m}$ or $\boldsymbol{w}(a)$ if they do not have the same length.)

3.1 Recollections

The following proposition generalizes [14, Lemma 4.8]:

Proposition 3.1 *Fix a real number $b \geqslant 1$. Given a nonnegative solution $(d,e;\boldsymbol{m})$ of (2-12) and $a \geqslant 1$, we have:*

(i) $\mu_b(d,e;\boldsymbol{m})(a) \leqslant \sqrt{2de+1}\sqrt{a}/(d+be)$.

(ii) $\mu_b(d,e;\boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}$ *if and only if* $\langle \varepsilon, \boldsymbol{w}(a)\rangle > 0$.

(iii) *If* $\mu_b(d,e;\boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}$, *then* $d = be+h$ *with* $|h| < \sqrt{2b}$ *and* $\langle \varepsilon,\varepsilon\rangle < 1-\frac{h^2}{2b}$.

Proof By the Cauchy–Schwarz inequality and since $\sum w_i^2 = a$,

$$(d+be)\mu_b(d,e;\boldsymbol{m})(a) = \langle \boldsymbol{m}, \boldsymbol{w}(a)\rangle \leqslant \|\boldsymbol{m}\|\|\boldsymbol{w}(a)\| = \sqrt{2de+1}\sqrt{a},$$

proving (i). Assertion (ii) is immediate. To prove (iii), we compute

$$\begin{aligned} 2(be+h)e+1 = 2de+1 = \langle \boldsymbol{m},\boldsymbol{m}\rangle &= \left\langle \frac{2be+h}{\sqrt{2ba}}\boldsymbol{w}(a)+\varepsilon, \frac{2be+h}{\sqrt{2ba}}\boldsymbol{w}(a)+\varepsilon \right\rangle \\ &= \frac{(2be+h)^2}{2ba}a + 2\frac{2be+h}{\sqrt{2ba}}\langle \boldsymbol{w}(a),\varepsilon\rangle + \langle\varepsilon,\varepsilon\rangle. \end{aligned}$$

The first of the three summands is $2be^2 + 2eh + \frac{h^2}{2b}$, and so

$$1 = \frac{h^2}{2b} + 2\frac{2be + h}{\sqrt{2ba}} \langle \boldsymbol{w}(a), \varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle.$$

Hence, if $\mu_b(d, e; \boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}$, then, by (ii), $\langle \boldsymbol{w}(a), \varepsilon \rangle > 0$, whence $0 \leqslant \langle \varepsilon, \varepsilon \rangle < 1 - \frac{h^2}{2b}$. This also shows $|h| < \sqrt{2b}$. □

3.2 Two sequences of exceptional classes and their constraints

In our analysis of the functions $c_b(a)$, two sequences of exceptional homology classes will play a role. For each $n \in \mathbb{N}$ we define the classes

$$E_n := (n, 1; 1^{\times(2n+1)}),$$
$$F_n := (n(n+1),\ n+1;\ n+1,\ n^{\times(2n+3)}).$$

Notice that E_n is a perfect class at $a = 2n + 1$, in the sense that $\boldsymbol{m}$ is a multiple of $\boldsymbol{w}(a)$. Similarly, F_n is nearly perfect at $a = 2n + 4$. While the constraints of the classes $E_b, E_{b+1}, \dots, E_{b+\lfloor\sqrt{2b}\rfloor}$ will give the $\lceil\sqrt{2b}\,\rceil$ linear steps in the graph of $c_b(a)$ centred at $2b + 2k + 1$, the constraint of F_b will give the affine step of $c_b(a)$ centred at $2b + 4$.

Lemma 3.2 *The classes E_n and F_n satisfy the Diophantine system (2-12) and their image under ψ_* reduces to $(0; -1, 0, \dots, 0)$ under repeated standard Cremona moves.*

Proof One readily checks that the classes E_n and F_n satisfy the Diophantine system (2-12).

For the sequel it is useful to rewrite the Cremona transform Cr as follows: Define the *defect* of a vector $(d; \boldsymbol{m}) = (d; m_1, \dots, m_k)$ by $\delta := d - m_1 - m_2 - m_3$. Then (2-7) can be written as

$$\operatorname{Cr}(d; \boldsymbol{m}) = (d + \delta;\ m_1 + \delta,\ m_2 + \delta,\ m_3 + \delta,\ m_4,\ \dots,\ m_k).$$

The isomorphism ψ_* from (2-10) maps $E_n = (n, 1; 1^{\times(2n+1)})$ to $(n;\ n-1,\ 1^{\times 2n})$, which under one standard Cremona move is mapped to $(n-1;\ n-2,\ 1^{\times 2(n-1)})$, and thence, under n such moves, to $(0; -1)$. Next, ψ_* maps F_1 to the class $(2; 1^{\times 5})$, which reduces to $(0; -1)$ under two standard Cremona moves, Further, for $n \geqslant 2$,

$$\psi_*(F_n) = (n^2 + n;\ n^2 - 1,\ n^{\times(2n+3)}).$$

Under n standard Cremona moves with $\delta = -n + 1$ this vector reduces to

$$(2n;\ n^{\times 3},\ n-1,\ 1^{\times 2n}).$$

Applying one more standard Cremona move with $\delta = -n$ yields $(n;\, n-1,\, 1^{\times 2n})$, which reduces in n steps to $(0;-1)$, as we have seen above. □

We next compute the constraints given by the classes E_n and F_n. In view of definition (3-1) and the definition of these classes,

$$\mu_b(E_{b+k})(a) = \frac{\langle 1^{\times(2b+2k+1)}, \boldsymbol{w}(a)\rangle}{2b+k}, \quad \mu_b(F_b)(a) = \frac{\langle (b+1, b^{\times(2b+3)}), \boldsymbol{w}(a)\rangle}{2b(b+1)}.$$

From this we readily find:

Lemma 3.3 *Fix an integer $b \geqslant 2$.*

(i) *For $k \in \{0,1,2,\dots,\lfloor\sqrt{2b}\rfloor\}$,*

$$\mu_b(E_{b+k})(a) = \begin{cases} \dfrac{a}{2b+k} & \text{if } a \in [2b+2k,\, 2b+2k+1], \\[2mm] \dfrac{2b+2k+1}{2b+k} & \text{if } a \geqslant 2b+2k+1. \end{cases}$$

(ii)
$$\mu_b(F_b)(a) = \begin{cases} \dfrac{ba+1}{2b(b+1)} & \text{if } a \in [2b+3,\, 2b+4], \\[2mm] 1 + \dfrac{2b+1}{2b(b+1)} & \text{if } a \geqslant 2b+4. \end{cases}$$

We in particular see that the class E_{b+k} gives rise to the linear step over $I_b(k)$ and F_b gives rise to the affine step over $[\alpha_b, \beta_b]$.

3.3 The constraints of E_n and F_n for real $b \geqslant 2$

In this subsection we compute the obstructions to the problem $E(1,a) \to P(\lambda, \lambda b)$ given by the exceptional classes E_n and F_n for all real $b \geqslant 2$. This is not used in the proof of Theorem 1.1, but supports Conjecture 1.5.

Let $b \geqslant 2$ be a real number. Recall that for $a \geqslant 1$ every exceptional class $E = (d, e; \boldsymbol{m})$ yields the constraint

$$\mu_b(E)(a) = \frac{\langle \boldsymbol{m}, \boldsymbol{w}(a)\rangle}{d+be}.$$

For $E_0 = (1,0;1)$ we have

$$\mu_b(E_0)(a) = 1 \tag{3-2}$$

and for $E_n = (n, 1; 1^{\times(2n+1)})$ with $n \geqslant 1$ we have

$$\mu_b(E_n)(a) = \begin{cases} \dfrac{a}{n+b} & \text{if } a \in [2n,\, 2n+1], \\[2mm] \dfrac{2n+1}{n+b} & \text{if } a \geqslant 2n+1. \end{cases}$$

The class E_n is b–obstructive on $[2n,\infty)$ only if $\frac{2n+1}{n+b} > \sqrt{\frac{2n+1}{2b}}$, and in view of (3-2) we can also assume that $\frac{2n+1}{n+b} > 1$, or $n > b-1$. The relevant values of n are thus

$$n \in \{\lfloor b \rfloor, \dots, \lfloor b+\sqrt{2b} \rfloor\},$$

where $\lfloor b \rfloor$ is the largest integer not greater than b. The constraint 1 of E_0 meets the first linear step, given by $E_{\lfloor b \rfloor}$, at $a = b+\lfloor b \rfloor$, and is thus strictly above $\sqrt{\frac{a}{2b}}$ if $b \notin \mathbb{N}$. For $n \geqslant \lfloor b \rfloor$ the step of E_n meets the step of E_{n+1} at $a = (2n+1)(n+b+1)/(n+b)$, which is above $\sqrt{\frac{a}{2b}}$ if and only if $b-n \geqslant (b-n)^2$. The step of $E_{\lfloor b \rfloor}$ thus meets the one of $E_{\lfloor b \rfloor+1}$ above the volume constraint, with equality if and only if $b \in \mathbb{N}$, and all other linear steps are strictly disjoint.

Next, let $\boldsymbol{b}$ be the "integer closest to b", namely $b = \boldsymbol{b}+\varepsilon$ with $\varepsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Then

$$\mu_b(F_{\boldsymbol{b}})(a) = \begin{cases} \dfrac{\boldsymbol{b}a+1}{(\boldsymbol{b}+b)(\boldsymbol{b}+1)} & \text{if } a \in [2\boldsymbol{b}+3, 2\boldsymbol{b}+4], \\ \dfrac{2\boldsymbol{b}^2+4\boldsymbol{b}+1}{(\boldsymbol{b}+b)(\boldsymbol{b}+1)} & \text{if } a \geqslant 2\boldsymbol{b}+4. \end{cases}$$

But notice that this constraint is stronger than $\sqrt{\frac{a}{2b}}$ only if

$$\mu_b(F_{\boldsymbol{b}})(2\boldsymbol{b}+4) = \frac{2\boldsymbol{b}^2+4\boldsymbol{b}+1}{(2\boldsymbol{b}+\varepsilon)(\boldsymbol{b}+1)} > \sqrt{\frac{\boldsymbol{b}+2}{\boldsymbol{b}+\varepsilon}}$$

or, equivalently, $\varepsilon \in \left(-\frac{\boldsymbol{b}}{(\boldsymbol{b}+1)^2}, \frac{1}{\boldsymbol{b}+2}\right)$. One readily checks that the affine step defined by $\mu_b(F_{\boldsymbol{b}})$ is strictly disjoint from the two neighbouring linear steps given by $E_{\boldsymbol{b}+1}$ and $E_{\boldsymbol{b}+2}$.

For $a \geqslant 1$ and $b \geqslant 2$ let $d_b(a)$ be the maximum of the volume constraint $\sqrt{\frac{a}{2b}}$ and the obstructions $\mu_b(E_n)(a)$ and $\mu_b(F_{\boldsymbol{b}})$ discussed above. Then $d_b(a) \geqslant c_b(a)$ of course, and Conjecture 1.5 claims that $d_b(a) = c_b(a)$ for all real $b \geqslant 2$.

3.4 $c_b(a)$ at $a = 2b+2+\frac{1}{2b}$

Set $a_b := 2b+2+\frac{1}{2b}$. We will show in Section 4.2 by the reduction method that $c_b(a_b) = \frac{2b+1}{2b}$. (Notice that this value equals the volume constraint $\sqrt{a_b/2b}$.) Here we show this by using positivity of intersection with the class

$$G_b := (b(2b+1), 2b+1; (2b)^{\times(2b+2)}, 1^{\times(2b+1)}), \quad b \in \mathbb{N}.$$

The $\boldsymbol{m}$ of G_b is obtained from $2b\boldsymbol{w}(a_b)$ by adding one 1, whence G_b is nearly perfect at a_b. One readily checks that G_b satisfies the Diophantine system (2-12) and that its image under ψ_* reduces to $(0;-1,0,\dots,0)$ under repeated standard Cremona moves. Hence G_b is an exceptional class. Its obstruction at a_b is

$$\mu_b(G_b)\Big(2b+2+\frac{1}{2b}\Big) = \frac{2b(2b+2)+1}{2b(2b+1)} = \frac{2b+1}{2b}.$$

Write $G_b = (b(2b+1), 2b+1; \boldsymbol{m}_b, 1)$ with $\boldsymbol{m}_b := ((2b)^{\times(2b+2)}, 1^{\times 2b}) = 2b\boldsymbol{w}(a_b)$. Recall that exceptional classes are represented by embedded J–holomorphic spheres, whence, by positivity of intersection, $E \cdot E' \geqslant 0$ for any two different exceptional classes $E \neq E'$. Applying this to G_b and any different exceptional class $(d, e; \boldsymbol{m})$, we obtain

$$(be+d)(2b+1) = b(2b+1)e + (2b+1)d \geqslant \langle \boldsymbol{m}, (\boldsymbol{m}_b, 1)\rangle \geqslant \langle \boldsymbol{m}, \boldsymbol{m}_b\rangle = 2b\langle \boldsymbol{m}, \boldsymbol{w}(a_b)\rangle.$$

Hence

$$\mu_b(d, e; \boldsymbol{m})(a_b) = \frac{\langle \boldsymbol{m}, \boldsymbol{w}(a_b)\rangle}{be+d} \leqslant \frac{2b+1}{2b},$$

as we wished to show.

Remarks 3.4 (i) The classes E_1 and E_2 also give rise to the first two steps of $c_C(a) = c_1(a)$, and the class F_1 gives rise to the affine step of $c_C(a)$; see [14]. This is the "holomorphic reason" why the first two steps of the Pell stairs and the affine step of $c_C(a)$ survive to all functions $c_b(a)$ for $b \geqslant 2$. On the other hand, none of the classes E_n with $n \geqslant 3$ and F_n with $n \geqslant 2$ is obstructive for the problem $E(1,a) \overset{s}{\hookrightarrow} C^4(\lambda)$, and none of the classes giving rise to the other steps of the Pell stairs, nor any of the classes giving rise to the six exceptional steps of $c_C(a)$, gives an obstruction for the problems $E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ with $b \geqslant 2$.

Similarly, G_1 is the first of the sequence of exceptional classes $E(\alpha_n)$ in [14] that imply, via positivity of intersection, that at the feet of the Pell stairs there is no embedding obstruction beyond the volume constraint.

(ii) We do not know all b–obstructive classes. However, using positivity of intersection and the analogues of Lemmas 3.8 and 3.11 we checked that $\mu_b(E)(2b+2k+1) < \frac{2b+2k+1}{2b+k}$ for any exceptional class $E \neq E_{b+k}$, and that $\mu_b(E)(2b+4) \leqslant \sqrt{\frac{2b+4}{2b}}$ for any exceptional class $E \neq F_b$, that is, F_b is the only b–obstructive class at $2b+4$. For F_2 this is carried out in Lemma 3.10.

3.5 $c_b(a)$ for a large

For $b \in \mathbb{N}_{\geqslant 2}$ we abbreviate

$$v_b^+ := v_b(\lfloor\sqrt{2b}\rfloor) = 2b\left(\frac{2b+2\lfloor\sqrt{2b}\rfloor+1}{2b+\lfloor\sqrt{2b}\rfloor}\right)^2.$$

Assertion (ii) of the following proposition improves [4, Theorem 1.1].

Proposition 3.5 (i) *For every* $b \in \mathbb{N}_{\geqslant 2}$,

$$c_b(a) = \begin{cases} \dfrac{2b+2\lfloor\sqrt{2b}\rfloor+1}{2b+\lfloor\sqrt{2b}\rfloor} & \text{if } a \in [2b+2\lfloor\sqrt{2b}\rfloor+1, v_b^+], \\ \sqrt{\dfrac{a}{2b}} & \text{if } a \geqslant v_b^+. \end{cases}$$

(ii) *For every real* $b \geqslant 2$ *we have* $c_b(a) = \sqrt{\frac{a}{2b}}$ *for all* $a \geqslant (\sqrt{2b}+1)^2$.

Notice that the length of the interval $[2b+2\lfloor\sqrt{2b}\rfloor+1, v_b^+]$ in (i) is

$$\frac{(2b+2\lfloor\sqrt{2b}\rfloor+1)(2b-\lfloor\sqrt{2b}\rfloor^2)}{(2b+\lfloor\sqrt{2b}\rfloor)^2}$$

and hence positive if and only if $\lfloor\sqrt{2b}\rfloor < \sqrt{2b}$, ie $2b$ is not a perfect square.

Proof Assume that $(d,e;\boldsymbol{m})$ is a nonnegative solution of (2-12). If $e=0$, then $(d,e;\boldsymbol{m}) = (1,0;1)$, and so $\mu_b(d,e;\boldsymbol{m})(a) = 1$ is smaller than the values of $c_b(a)$ claimed in (i) and (ii). We can thus assume that $e \geqslant 1$.

Suppose that $\mu_b(d,e;\boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}$ for some $a \geqslant 1$. Then, by Proposition 3.1(iii), $d < be + \sqrt{2b}$. We estimate

$$\mu_b(d,e;\boldsymbol{m})(a) = \frac{\langle \boldsymbol{m}, \boldsymbol{w}(a)\rangle}{be+d} \leqslant \frac{\sum m_i}{be+d} = \frac{2(d+e)-1}{be+d} =: f_{b,e}(d). \tag{3-3}$$

The function $d \mapsto f_{b,e}(d)$ is increasing. We can thus further estimate

$$\mu_b(d,e;\boldsymbol{m})(a) \leqslant f_{b,e}(be+\sqrt{2b}) = \frac{2(be+\sqrt{2b}+e)-1}{2be+\sqrt{2b}} =: L(b,e). \tag{3-4}$$

Claim 1 $\frac{\partial}{\partial e} L(b,e) \leqslant 0$.

Proof We compute

$$\frac{\partial}{\partial e} L(b,e) = \frac{2(b+1)(2be+\sqrt{2b}) - 2b(2(be+\sqrt{2b}+e)-1)}{(2be+\sqrt{2b})^2},$$

which is nonpositive if and only if the numerator is nonpositive. Expanding the numerator, we see that this holds if and only if $b+\sqrt{2b} \leqslant b\sqrt{2b}$, which holds true because $b \geqslant 2$. □

Proof of Proposition 3.5(ii) Assume that $(d,e;\boldsymbol{m})$ is an exceptional class with $e \geqslant 1$ and such that $\mu_b(d,e;\boldsymbol{m})(a) > \sqrt{\frac{a}{2b}}$ for some $a \geqslant (\sqrt{2b}+1)^2$. By (3-4) and Claim 1,

$$\mu_b(d,e;\boldsymbol{m})(a) \leqslant L(b,e) \leqslant L(b,1) = \frac{\sqrt{2b}+1}{\sqrt{2b}} \leqslant \sqrt{\frac{a}{2b}},$$

a contradiction.

Proof of Proposition 3.5(i) Assume from now on that $b \in \mathbb{N}_{\geqslant 2}$. If $e=1$, then (2-12) becomes

$$\sum m_i = \sum m_i^2 = 2d+1$$

and so $(d,e;\boldsymbol{m})$ is the exceptional class $E_d = (d,1;1^{\times(2d+1)})$. Recall that on the interval $[2d, 2d+2]$ the obstruction function

$$\mu_b(E_d)(a) = \frac{\langle \boldsymbol{w}(a), 1^{\times(2d+1)}\rangle}{b+d}$$

gives a linear step with edge at $2d+1$. If $\lfloor\sqrt{2b}\rfloor < \sqrt{2b}$, then the largest k for which E_{b+k} yields a constraint strictly stronger than the volume is $k = \lfloor\sqrt{2b}\rfloor$, because

$$\frac{2b+2k+1}{2b+k} > \sqrt{\frac{2b+2k+1}{2b}} \iff 2b > k^2.$$

We are left with showing that for $e \geqslant 2$ we have $\mu_b(d,e;\boldsymbol{m})(a) \leqslant \sqrt{\frac{a}{2b}}$ for all solutions $(d,e;\boldsymbol{m})$ of (2-12) and all $a \geqslant v_b^+$. Assume first that $e \geqslant 3$. Then (3-4) and Claim 1 yield

$$\mu_b(d,e;\boldsymbol{m})(a) \leqslant L(b,e) \leqslant L(b,3).$$

Claim 2 *$L(b,3) \leqslant \sqrt{\frac{a}{2b}}$ for all $b \in \mathbb{N}_{\geqslant 2}$ and $a \geqslant v_b^+$.*

Proof It suffices to prove the claim for $a = v_b^+$. We have

$$L(b,3) - 1 = \frac{\sqrt{2b}+5}{6b+\sqrt{2b}} \quad \text{and} \quad \sqrt{\frac{v_b^+}{2b}} - 1 = \frac{\lfloor\sqrt{2b}\rfloor + 1}{2b + \lfloor\sqrt{2b}\rfloor}.$$

For $b \in \{2,3,4\}$ the inequality

$$\frac{\lfloor\sqrt{2b}\rfloor + 1}{2b + \lfloor\sqrt{2b}\rfloor} \geqslant \frac{\sqrt{2b}+5}{6b+\sqrt{2b}}$$

is readily verified. For $b \geqslant 5$ we use that $x \mapsto \frac{x+1}{2b+x}$ is increasing, and estimate

$$\sqrt{\frac{v_b^+}{2b}} - L(b,3) \geqslant \frac{(\sqrt{2b}-1)+1}{2b+(\sqrt{2b}-1)} - \frac{\sqrt{2b}+5}{6b+\sqrt{2b}}.$$

The right-hand side multiplied with the product of the denominators equals $f(b) := 4b\sqrt{2b} - 10b - 4\sqrt{2b} + 5$. Since $bf'(b) = 6b\sqrt{2b} - 2\sqrt{2b} - 10b \geqslant 0$ for $b \geqslant 2$ and $f(5) > 0$, the claim follows. □

Assume now that $e = 2$. We first treat the case $b \geqslant 5$. In view of (3-4) it suffices to show that $L(b,2) \leqslant \sqrt{v_b^+/2b}$, or

$$\frac{\sqrt{2b}+3}{4b+\sqrt{2b}} \leqslant \frac{\lfloor\sqrt{2b}\rfloor+1}{2b+\lfloor\sqrt{2b}\rfloor}.$$

This inequality is readily verified for $b = 5$. For $b \geqslant 6$ the stronger inequality

$$\frac{\sqrt{2b}+3}{4b+\sqrt{2b}} \leqslant \frac{(\sqrt{2b}-1)+1}{2b+(\sqrt{2b}-1)}$$

holds true. Indeed, this inequality is equivalent to $g(b) := 2b\sqrt{2b} - 6b - 2\sqrt{2b} + 3 \geqslant 0$, which holds true since $bg'(b) = 3b\sqrt{2b} - \sqrt{2b} - 6b \geqslant 0$ for $b \geqslant 6$ and $g(6) \geqslant 0$.

Assume now that $b \in \{2,3,4\}$. Then $\sqrt{v_b^+/2b} = 1 + \frac{3}{2b+2}$. Using (3-3) this time with $d \leqslant \lfloor be + \sqrt{2b}\rfloor$, we find

$$\mu_b(d,2;\boldsymbol{m})(a) \leqslant f_{b,2}(\lfloor 2b+\sqrt{2b}\rfloor) = \frac{2\lfloor 2b+\sqrt{2b}\rfloor+3}{2b+\lfloor 2b+\sqrt{2b}\rfloor}.$$

For $b \in \{2,3,4\}$ the right-hand side is at most $1 + \frac{3}{2b+2}$. Proposition 3.5 is proven. □

3.6 The interval $\left[8\frac{1}{36}, 9\right]$ for $b = 2$

Proposition 3.6 $c_2(a) = \frac{1}{2}\sqrt{a}$ *for* $a \in \left[8\frac{1}{36}, 9\right]$.

Proof The arguments in this section are close to those in [26, Section 5.3] and [14, Section 7.3]. In fact, the last step of $c_B(a)$ and of $c_2(a)$ both end at $8\frac{1}{36}$ and are given by the class F_2. There are some differences, however, and so we give a complete exposition for the convenience of the reader.

Fix a rational number $a = \frac{p}{q} \in (8,9)$, with $\frac{p}{q}$ in reduced form, with weight expansion

(3-5) $$(1^{\times \ell_0}, w_1^{\times \ell_1}, \dots, w_N^{\times \ell_N}).$$

Then $w_N = \frac{1}{q}$ and $\sum_{j=0}^N \ell_j w_j = a + 1 - \frac{1}{q}$ by [26, Lemma 1.2.6]. Set $M := \ell(a) := \sum_{j=0}^N \ell_j$ and $L = \sum_{j=1}^N \ell_j = \ell(a) - 8$. Then $q \geqslant L$ by [26, Sublemma 5.1.1].

For $b = 2$ the error vector ε of an exceptional class $(d,e;\boldsymbol{m})$ at a is

(3-6) $$\boldsymbol{m} = \frac{d+2e}{2\sqrt{a}}\boldsymbol{w}(a) + \varepsilon.$$

Define the partial error sums

$$\sigma := \sum_{i=\ell_0+1}^{M} \varepsilon_i^2 \quad \text{and} \quad \sigma' := \sum_{i=\ell_0+1}^{M-\ell_N} \varepsilon_i^2 \leqslant \sigma.$$

Recall from Proposition 3.1(iii) that for an obstructive class $(d,e;\boldsymbol{m})$ we have $d = 2e+h$ with $h \in \{-1,0,1\}$, and $\sigma < 1$ if $h=0$ and $\sigma < \frac{3}{4}$ if $|h|=1$. For the function

$$y(a) := a - 3\sqrt{a} + 1$$

we have $y\left(\frac{p}{q}\right) > \frac{1}{q}$ for all $\frac{p}{q} \in (8,9)$. Write $\ell(\boldsymbol{m})$ for the number of positive entries in $\boldsymbol{m}$.

Lemma 3.7 *Let $(d,e;\boldsymbol{m})$ be an exceptional class such that there exists $a = \frac{p}{q} \in (8,9)$ with $\ell(a) = \ell(\boldsymbol{m})$ and $\mu_2(d,e;\boldsymbol{m})(a) > \frac{1}{2}\sqrt{a}$. Set $v_M := (d+2e)/(2q\sqrt{a})$. Then:*

(i) $\left|\sum \varepsilon_i\right| \leqslant \sqrt{\sigma L}$.

(ii) *If $v_M < 1$, then* $\left|\sum \varepsilon_i\right| \leqslant \sqrt{\sigma' L}$.

(iii) *If $v_M \leqslant \frac{1}{2}$, then $v_M > \frac{1}{3}$ and $\sigma' \leqslant \frac{1}{2}$. If $v_M \leqslant \frac{2}{3}$, then $\sigma' \leqslant \frac{7}{9}$.*

(iv) *With $\delta := y(a) - \frac{1}{q}$ we have*

$$4e+h \leqslant \frac{2\sqrt{a}}{\delta}\left(\sqrt{\sigma q} - \left(1-\frac{h}{2}\right)\right) \leqslant \frac{2\sqrt{a}}{\delta}\left(\frac{\sigma}{\delta v_M} - \left(1-\frac{h}{2}\right)\right).$$

If $v_M < 1$, then σ can be replaced by σ'.

Proof The proofs of (i), (ii) and (iii) are as for [26, Lemma 5.1.2]. To prove (iv) we compute

$$\begin{aligned} -\sum_{i=1}^{M} \varepsilon_i = \frac{d+2e}{2\sqrt{a}} \sum_{j=0}^{N} \ell_j w_j - \sum_{i=1}^{M} m_i &= \frac{d+2e}{2\sqrt{a}}\left(a+1-\frac{1}{q}\right) - (2d+2e-1) \\ &= \frac{4e+h}{2\sqrt{a}}\left(a+1-\frac{1}{q}\right) - (6e+2h-1) \\ &= \frac{4e+h}{2\sqrt{a}}\left(y(a)-\frac{1}{q}\right) + \left(1-\frac{h}{2}\right), \end{aligned}$$

where we have used (3-6) and (2-12). Then, using $q \geqslant L$ and (i), we find

$$\sqrt{\sigma q} \geqslant \sqrt{\sigma L} \geqslant \frac{4e+h}{2\sqrt{a}}\left(y(a)-\frac{1}{q}\right) + \left(1-\frac{h}{2}\right) = \frac{4e+h}{2\sqrt{a}}\delta + \left(1-\frac{h}{2}\right) > \delta v_M q.$$

Thus $\sqrt{q} < \sqrt{\sigma}/\delta v_M$, and so

$$4e+h \leqslant \frac{2\sqrt{a}}{\delta}\Big(\sqrt{\sigma q} - \Big(1-\frac{h}{2}\Big)\Big) < \frac{2\sqrt{a}}{\delta}\Big(\frac{\sigma}{\delta v_M} - \Big(1-\frac{h}{2}\Big)\Big).$$

If $v_M < 1$, the same arguments go through when replacing σ by σ'. □

The following lemma is proven as in [26, Lemma 2.1.7]:

Lemma 3.8 *Let $(d,e;\boldsymbol{m})$ be an exceptional class such that $\mu_2(d,e;\boldsymbol{m})(a) > \frac{1}{2}\sqrt{a}$ for some $a \in [8,9)$. Then:*

(i) *The vector $(m_1,\dots,m_8)$ is of the form*

$$(m,\dots,m) \quad \text{or} \quad (m,\dots,m,m-1) \quad \text{or} \quad (m+1,m,\dots,m).$$

(ii) *If $m_1 \neq m_8$, then $\sum_{i=1}^{8} \varepsilon_i^2 \geqslant \frac{7}{8}$.*

Lemma 3.9 *There is no exceptional class $(d,e;\boldsymbol{m})$ such that $\mu_2(d,e;\boldsymbol{m})(a) > \frac{1}{2}\sqrt{a}$ for some $a \in (8,9)$ with $\ell(a) = \ell(\boldsymbol{m})$.*

Proof Assume that $(d,e;\boldsymbol{m})$ is an exceptional class such that $\mu_2(d,e;\boldsymbol{m})(a) > \frac{1}{2}\sqrt{a}$ for some $a \in (8,9)$ with $\ell(a) = \ell(\boldsymbol{m})$.

We first show that $m_1 = \dots = m_8$. Assume the contrary. By Lemma 3.8, $\langle \varepsilon,\varepsilon\rangle \geqslant \frac{7}{8}$ and $\sigma \leqslant \frac{1}{8}$. The inequality $\langle \varepsilon,\varepsilon\rangle \geqslant \frac{7}{8}$ and Proposition 3.1(iii) show that $h=0$. Since $M>8$ and $\sigma \leqslant \frac{1}{8}$, we find $v_M \geqslant 1 - \frac{1}{\sqrt{8}} > \frac{1}{2}$. Further, since $a \geqslant 8\frac{1}{q}$,

$$\delta = y(a) - \tfrac{1}{q} \geqslant y\big(8\tfrac{1}{q}\big) - \tfrac{1}{q} = 9 - 3\sqrt{8\tfrac{1}{q}} \geqslant 9 - 3\sqrt{8\tfrac{1}{2}} \geqslant \tfrac{1}{4}.$$

Altogether, $\frac{\sigma}{\delta v_M} < 1$, in contradiction with Lemma 3.7(iv).

We are now going to show that e must be small. For this we first notice that, by Lemma 3.7(iii),

$$v_M \in \Big[\frac{1}{3},\frac{1}{2}\Big] \implies \frac{\sigma'}{v_M} \leqslant \frac{1/2}{1/3} = \frac{3}{2},$$
$$v_M \in \Big[\frac{1}{2},\frac{2}{3}\Big] \implies \frac{\sigma'}{v_M} \leqslant \frac{7/9}{1/2} = \frac{14}{9},$$
$$v_M \geqslant \frac{2}{3} \implies \frac{\sigma}{v_M} \leqslant \frac{3}{2}.$$

For fixed q and h, the functions

$$F(a,q,h) := \frac{2\sqrt{a}}{\delta}\Big(\sqrt{q} - \Big(1-\frac{h}{2}\Big)\Big),$$
$$G(a,q,h) := \frac{2\sqrt{a}}{\delta}\Big(\frac{14}{9}\frac{1}{\delta} - \Big(1-\frac{h}{2}\Big)\Big)$$

are strictly decreasing for $a \in (8,9)$. Since $a \geqslant 8\frac{1}{q}$, we see from Lemma 3.7(iv) that

$$4e+h \leqslant f(q,h),\ g(q,h),$$

where $f(q,h) := F\big(8\frac{1}{q}, q, h\big)$ and $g(q,h) := G\big(8\frac{1}{q}, q, h\big)$. Explicitly,

$$f(q,h) := \frac{2\sqrt{8\frac{1}{q}}}{\delta(q)}\Big(\sqrt{q} - \Big(1 - \frac{h}{2}\Big)\Big),$$

$$g(q,h) := \frac{2\sqrt{8\frac{1}{q}}}{\delta(q)}\Big(\frac{14}{9}\frac{1}{\delta(q)} - \Big(1 - \frac{h}{2}\Big)\Big),$$

where $\delta(q) := y(8\frac{1}{q}) - \frac{1}{q} = 9 - 3\sqrt{8\frac{1}{q}}$. We have that $\frac{\partial f}{\partial q}(q,h) > 0$ for $q \geqslant 3$ and $\frac{\partial g}{\partial q}(q,h) < 0$ for $q \geqslant 2$, and $f(q,h) < g(q,h)$ for $q \in \{2,3\}$. In fact, $f(q,h) = g(q,h)$ if and only if $\sqrt{q} = \frac{14}{9}\frac{1}{\delta(q)}$, which happens at $q \approx 11.1$. One readily checks that

$$f(11,-1),\ g(12,-1) < 23, \quad f(11,0),\ g(12,0) < 29, \quad f(11,1),\ g(12,1) < 35.$$

It follows that

$$4e+h \leqslant 22, 28, 34 \quad \text{for } h = -1, 0, 1, \text{ respectively},$$

and so

$$e \leqslant 5 \quad \text{if } h=-1, \qquad e \leqslant 7 \quad \text{if } h = 0, \qquad e \leqslant 8 \quad \text{if } h = 1. \tag{3-7}$$

However, one readily checks that there are no solutions $(2e+h, e; \boldsymbol{m})$ of (2-12) satisfying (3-7) and $m_1 = \cdots = m_8$. To illustrate the computation, we take $e = 8$ and $h = 1$. The Diophantine system then becomes

$$\sum_{i \geqslant 1} m_i = 49, \quad \sum_{i \geqslant 1} m_i^2 = 273.$$

Since $m := m_1 = \cdots = m_8$, we must have $m \leqslant 5$. For $m = 5$ we get

$$\sum_{i \geqslant 9} m_i = 9, \quad \sum_{i \geqslant 9} m_i^2 = 73,$$

which has no solution for $m_i \leqslant 5$. Similarly there are no solutions for $m \in \{1,2,3,4\}$. □

Lemma 3.10 *The only exceptional class* $(d,e;\boldsymbol{m})$ *with* $\mu_2(d,e;\boldsymbol{m})(8) > \frac{1}{2}\sqrt{8}$ *is* $F_2 = (6,3;3,2^{\times 7})$.

Proof Consider an exceptional class with $\mu_2(d,e;\boldsymbol{m})(8) > \frac{1}{2}\sqrt{a}$. By Lemma 3.11 below, $\ell(\boldsymbol{m}) \leqslant 8$. If $\ell(\boldsymbol{m}) \leqslant 7$, Lemma 3.8(i) shows that $\boldsymbol{m} = (1^{\times 7})$; but the only solution of (2-12) with this $\boldsymbol{m}$ is $(3,1;1^{\times 7})$, and $\mu_2(3,1;1^{\times 7})(8) = \frac{7}{5} < \frac{1}{2}\sqrt{8}$. We can thus assume that $\ell(\boldsymbol{m}) = 8$. By Lemma 3.8, the vector $\boldsymbol{m}$ has the form

$$\boldsymbol{m} = (m^{\times 8}) \quad \text{or} \quad \boldsymbol{m} = (m^{\times 7}, m-1) \quad \text{or} \quad \boldsymbol{m} = (m+1, m^{\times 7})$$

for some $m \in \mathbb{N}$.

If $\boldsymbol{m} = (m^{\times 8})$, then the linear of the Diophantine equations yields $8m = 2(d+e)-1$, which is impossible since $8m$ is even and $2(d+e)-1$ is odd.

In the two other cases, Proposition 3.1(iii) and Lemma 3.8(ii) show that $d = 2e$.

If $\boldsymbol{m} = (m^{\times 7}, m-1)$, the Diophantine system becomes

$$8m = 6e, \quad 8m^2 - 2m = 4e^2.$$

Inserting $e = \frac{4}{3}m$ into the second equation leads to $4m^2 = 9m$, which has no solution in $\mathbb{N}$.

If $\boldsymbol{m} = (m+1, m^{\times 7})$, the Diophantine system becomes

$$8m + 2 = 6e, \quad 8m^2 + 2m = 4e^2.$$

Inserting $e = \frac{1}{3}(4m+1)$ into the second equation leads to $4m^2 - 7m - 2 = 0$, whose only integral solution is $m = 2$. Hence $(d,e;\boldsymbol{m}) = (6,3;3,2^{\times 7}) = F_2$. □

The following lemma is a version of [26, Lemma 2.1.3]:

Lemma 3.11 *Let $(d,e;\boldsymbol{m})$ be an exceptional class, and suppose that I is a maximal nonempty open interval such that $\frac{1}{2}\sqrt{a} < \mu_2(d,e;\boldsymbol{m})(a)$ for all $a \in I$. Then there is a unique $a_0 \in I$ such that $\ell(a_0) = \ell(\boldsymbol{m})$. Moreover, $\ell(a) \geqslant \ell(\boldsymbol{m})$ for all $a \in I$.*

Here, the last assertion is proven as follows: If $\ell(a) < \ell(\boldsymbol{m})$, then $\sum_{i \leqslant \ell(a)} m_i^2 < 2de+1$, so that $\langle \boldsymbol{w}(a), \boldsymbol{m}\rangle \leqslant \|\boldsymbol{w}(a)\|\sqrt{2de} = \sqrt{a}\sqrt{2de}$. Hence

$$\mu_2(d,e;\boldsymbol{m})(a) \leqslant \frac{\sqrt{2de}\sqrt{a}}{d+2e} \leqslant \frac{\sqrt{a}}{2}.$$

End of the proof of Proposition 3.6 Suppose to the contrary that $\mu_2(d,e;\boldsymbol{m})(a) > \frac{1}{2}\sqrt{a}$ for some $a \in \left[8\frac{1}{36}, 9\right)$. By Lemma 3.11 we may choose a_0 with $\ell(a_0) = \ell(\boldsymbol{m})$ in the interval I containing a on which this inequality holds.

Assume that $a_0 \leqslant 8$. Then $a_0 \leqslant 8 < a$, and so $8 \in I$. Then Lemma 3.10 shows that $(d,e;\boldsymbol{m}) = F_2$. But F_2 is not obstructive for $a \geqslant 8\frac{1}{36}$.

Hence $a_0 > 8$. We already know from Proposition 3.5 that $c_2(a) = \frac{1}{2}\sqrt{a}$ for $a \geqslant 9$. Hence $a_0 \in (8,9)$. Hence Lemma 3.9 applies, and yields the desired contradiction. □

4 First applications of the reduction method

In this section we first use the reduction method to prove the equivalence (1-2). We then use this method to prove that the obstructions given by the exceptional classes E_n are sharp at their edges, and then to compute $c_b(a)$ at the end points of the first linear step.

As in Section 3.2 we define the *defect* of $(\mu;\boldsymbol{a})=(\mu;a_1,\dots,a_k)$ by $\delta:=\mu-a_1-a_2-a_3$. Then the Cremona transform (2-14) can be written as

$$\mathrm{Cr}(\mu;\boldsymbol{a}) = (\mu+\delta;\ a_1+\delta,\ a_2+\delta,\ a_3+\delta,\ a_4,\ \dots,\ a_k).$$

4.1 Proof of the equivalence (1-2)

By continuity we can assume that a is rational. Recall that $E(1,a) \overset{s}{\hookrightarrow} P(\lambda,\lambda b)$ if and only if there exists an embedding (2-2). By the nonsqueezing theorem we must have $\lambda \geqslant 1$. Hence Method 2 formulated in Section 2.2 shows that an embedding (2-2) exists if and only if $\lambda \geqslant \sqrt{\frac{a}{2b}}$ and if the first reduced vector in the orbit of

$$(\lambda(b+1);\lambda b,\lambda,\boldsymbol{w}(a)) \tag{4-1}$$

under standard Cremona moves has no negative entries.

The weight decomposition of the ellipsoid $E((2b-1)\lambda,2b\lambda)$ is $((2b-1)\lambda,\lambda^{\times(2b-1)})$. The main result of [23] thus shows that $E(1,a) \overset{s}{\hookrightarrow} E(\lambda,2b\lambda)$ if and only if

$$B(\boldsymbol{w}(a)) \amalg B((2b-1)\lambda) \amalg \coprod_{2b-1} B(\lambda) \overset{s}{\hookrightarrow} B(2b\lambda).$$

Method 2 shows that such an embedding exists if and only if $\lambda \geqslant \sqrt{\frac{a}{2b}}$ and if the first reduced vector in the orbit of

$$(2b\lambda;(2b-1)\lambda,\lambda^{\times(2b-1)},\boldsymbol{w}(a)) \tag{4-2}$$

under standard Cremona moves has no negative entries. Applying $b-1$ standard Cremona moves with defect $\delta=-\lambda$ to the vector (4-2) we reach the vector (4-1). □

In the rest of this paper we will show that besides for the volume constraint $\sqrt{\frac{a}{2b}}$ there are no other obstructions to the embedding problem $E(1,a) \overset{s}{\hookrightarrow} P(\lambda,\lambda b)$ than those given by the exceptional classes E_n and F_n. For this it suffices to show that if we take for λ the value claimed for $c_b(a)$ in Theorem 1.1, then there exists an embedding $E(1,a) \overset{s}{\hookrightarrow} P(\lambda,\lambda b)$. This problem, in turn, we solve by the recipe formulated in Proposition 2.3.

4.2 $c_b(a)$ at $a = 2b + 2k + 1$, and at $a = 2b$ and $a = 2b + 2 + \frac{1}{2b}$

Lemma 4.1 $c_b(2b + 2k + 1) \leqslant \frac{2b+2k+1}{2b+k}$ *for* $k \in \{0, 1, 2, \dots, \lfloor\sqrt{2b}\rfloor\}$.

Proof Set $\lambda = \frac{2b+2k+1}{2b+k} = 1 + \frac{k+1}{2b+k} \in (1, 2)$. Then one standard Cremona move with $\delta = -1$ takes the vector $(\lambda(b+1); \lambda b, \lambda, 1^{\times(2b+2k+1)})$ to

$$(\lambda(b+1) - 1; \lambda b - 1, 1^{\times(2b+2k)}, \lambda - 1).$$

Since $\lambda b - 1 + (b + k)(\lambda - 2) = 0$, applying $b + k$ Cremona moves with $\delta = \lambda - 2$ to this vector yields the vector $(\lambda; (\lambda - 1)^{\times(2b+2k+1)})$, which is reduced, since $\delta = 3 - 2\lambda = \frac{2b-k-2}{2b+k} \geqslant 0$ for $k \leqslant \sqrt{2b}$ and $b \geqslant 2$. □

Lemma 4.2 $c_b(2b) = 1$ *and* $c_b\big(2b + 2 + \frac{1}{2b}\big) = \frac{2b+1}{2b}$.

Proof In view of the volume constraint $c_b(a) \geqslant \sqrt{\frac{a}{2b}}$, it suffices to show the inequalities $c_b(2b) \leqslant 1$ and $c_b\big(2b + 2 + \frac{1}{2b}\big) \leqslant \frac{2b+1}{2b}$.

Set $\lambda = 1$. Then b Cremona moves with $\delta = -1$ take the vector $(b + 1; b, 1^{\times(2b+1)})$ to $(1; 1)$, which is reduced.

Set $\lambda = \frac{2b+1}{2b} = 1 + \frac{1}{2b}$. Then one standard Cremona move with $\delta = -1$ takes the vector $\big(\lambda(b + 1); \lambda b, \lambda, 1^{\times(2b+2)}, \big(\frac{1}{2b}\big)^{\times 2b}\big)$ to

$$\left(\lambda(b + 1) - 1; \lambda b - 1, 1^{\times(2b+1)}, \left(\frac{1}{2b}\right)^{\times(2b+1)}\right).$$

Since $\lambda b - 1 + b(\lambda - 2) = 0$, applying b Cremona moves with $\delta = \lambda - 2$ yields the vector $\big(\lambda; 1, \big(\frac{1}{2b}\big)^{\times(4b+1)}\big)$. Applying $2b$ Cremona moves with $\delta = \frac{1}{2b}$ yields $\big(\frac{1}{2b}; \frac{1}{2b}\big)$, which is reduced. □

Corollary 4.3 *Theorem 1.1 holds for* $a \in [1, 2b + 3]$.

Proof By Gromov's nonsqueezing theorem, $E(1, 1) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$ implies $\lambda \geqslant 1$. (In our language this reads $\mu_b(E_0)(1) = 1$ for $E_0 := (1, 0; 1)$.) Since the function c_b is monotone increasing, this and $c_b(2b) = 1$ show that $c_b(a) = 1$ for $a \in [1, 2b]$.

The functions c_b have the scaling property

$$\frac{c_b(\lambda a)}{\lambda a} \leqslant \frac{c_b(a)}{a} \quad \text{for all } \lambda \geqslant 1;$$

see [26, Lemma 1.1.1] for the easy proof. Therefore:

Lemma 4.4 *If for two values $a_0 < a_1$ the points $(a_0, c_b(a_0))$ and $(a_1, c_b(a_1))$ lie on a line through the origin, then the whole segment between these two points belongs to the graph of c_b, that is, c_b is linear on $[a_0, a_1]$.*

Lemmas 3.3(i), 4.1 and 4.2 thus show that the graph of c_b on $[1, 2b+3]$ is as in Figure 2. □

4.3 Organization of the proof of Theorem 1.1

We order the rest of the proof by increasing difficulty.

For $b \in \mathbb{N}_{\geqslant 5}$ and $k = 2, \dots, \lfloor\sqrt{2b}\rfloor - 1$, the intervals $I_b(k)$ and $I_b(k+1)$ enclose the interval $[v_b(k), u_b(k+1)]$, that contains the point $2b+2k+2$. We first show that $c_b(a) = \sqrt{\frac{a}{2b}}$ on this interval. More precisely, we subdivide this interval into its left and right part,

$$L_b(k) := [v_b(k), 2b+2k+2] \quad \text{and} \quad R_b(k) := [2b+2k+2, u_b(k+1)],$$

and show in Section 5 and Section 6 that $c_b(a) = \sqrt{\frac{a}{2b}}$ on $L_b(k)$ and $R_b(k)$, respectively. Theorem 1.1 then follows for all $a \geqslant 2b+5$. Indeed, together with Lemmas 3.3(i) and 4.1, we now know that for $k \geqslant 2$ the edge point and the two end points of the linear steps lie on the graph of $c_b(a)$, and hence by Lemma 4.4 these linear steps belong to $c_b(a)$ entirely. Further, by Proposition 3.5(i), Theorem 1.1 holds for $a \geqslant v_b(\lfloor\sqrt{2b}\rfloor)$.

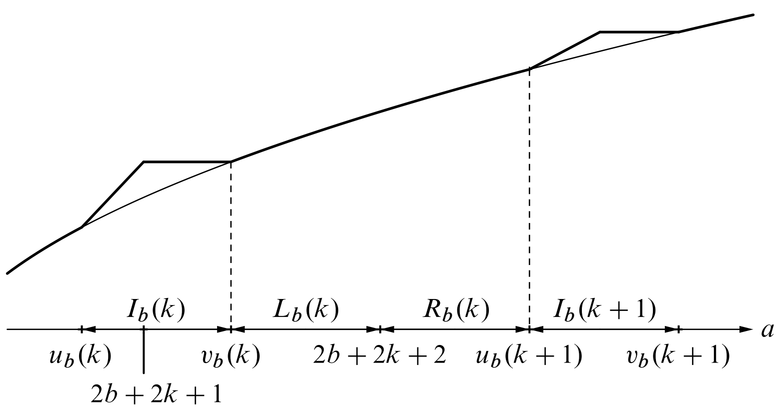

Figure 10

We already know from Corollary 4.3 that Theorem 1.1 holds for $a \leqslant 2b+3$. We are thus left with the interval $[2b+3, 2b+5]$. It suffices to treat the subinterval $[v_b(1), u_b(2)]$. Indeed, we then know that $c_b(2b+3) = c_b(v_b(1))$, whence the second linear step is established, and we already know that the third linear step, which begins at $u_b(2)$, belongs to $c_b(a)$. (Note that for $b=2$ there is no third linear step, but then $u_b(2)=2b+5=9$.) Recall that

$$v_b(1) < \alpha_b < 2b+4 < \beta_b < u_b(2).$$

We shall treat the interval $[v_b(1), 2b+4]$ in Section 7. The case $b=2$ is then complete, since $c_2(8)=\frac{17}{12}=c_2\big(8\frac{1}{36}\big)$ and in view of Proposition 3.6. The interval $[2b+4, u_b(2)]$ for $b \geqslant 3$ is treated in Section 8. Showing $c_b(a)=\sqrt{\frac{a}{2b}}$ on the intervals $[v_b(1), \alpha_b]$ and $[\beta_b, u_b(2)]$ is the hardest part of the paper, since on these intervals the reduction algorithm is rather intricate. On the other hand, establishing the affine segment over $[\alpha_b, 2b+4]$ will be easier, and it turns out that the reduction method establishes the affine steps of $c_B(a)$ and $c_C(a)$ much faster than the positivity of intersection argument used in [26; 14].

Since the embedding functions $c_b(a)$ are continuous, it suffices to compute them on a dense set. In the rest of the paper we shall assume that $a \geqslant 1$ is rational. Hence a has a finite weight expansion $\boldsymbol{w}(a)=(1^{\times \lfloor a \rfloor}, w_1^{\times \ell_1}, w_2^{\times \ell_2}, \dots)$. Sometimes it will be convenient to assume also that $\ell_1 \geqslant 1$ or $\ell_2 \geqslant 1$ or $\ell_3 \geqslant 2$, which holds for a dense set of rational a.

5 The intervals $L_b(k)=[v_b(k), 2b+2k+2]$

Recall that

$$v_b(k)=2b\Big(\frac{2b+2k+1}{2b+k}\Big)^2.$$

Theorem 5.1 *Assume that $b \in \mathbb{N}_{\geqslant 5}$ and $k \in \{2, \dots, \lfloor \sqrt{2b} \rfloor - 1\}$. Then $c_b(a)=\sqrt{\frac{a}{2b}}$ for $a \in L_b(k)$.*

Proof The weight expansion at $a \in L_b(k)$ is

$$\boldsymbol{w}(a)=(1^{\times 2(b+k)+1}, w_1^{\times \ell_1}, w_2^{\times \ell_2}, \dots).$$

Define the numbers λ, z_1 and z_2 by

$$\lambda=\sqrt{\frac{a}{2b}}=\sqrt{\frac{2(b+k)+1+w_1}{2b}}=:1+z_1,$$

$$z_2:=(2b+k)\lambda-(2b+2k+1)=\sqrt{\frac{2(b+k)+1+w_1}{2b}}(2b+k)-(2b+2k+1).$$

Lemma 5.2 (i) $2z_1 \leqslant 1+z_2$.

(ii) $z_2 \geqslant 0$ *and* $z_2 \leqslant w_1$.

(iii) *For* $k \geqslant 3$ *and* $\ell_1=1$ *we have* $w_2+z_2-z_1 \geqslant 0$.

Proof (i) We wish to show that

$$2b+2k \leqslant 2+(2b+k-2)\lambda.$$

We show that this inequality even holds if $w_1 \leqslant 0$ in λ is set to zero, ie that

$$2b+2k \leqslant 2+\sqrt{\frac{2(b+k)+1}{2b}}(2b+k-2).$$

After solving for the root, squaring and multiplying with $2b(2b+k-2)^2$, we find that this inequality is equivalent to

$$4b^2+(2k+1)(k-2)^2+2b(k^2-2k-4) \geqslant 0,$$

which holds true since $k \geqslant 2$ and $b \geqslant 2$.

(ii) Note that $z_2 = 0$ at the left boundary $v_b(k)$ of $L_b(k)$. Since z_2 is increasing on $L_b(k)$, we see that $z_2 \geqslant 0$.

At $v_b(k)$ we have $w_1 \geqslant 0 = z_2$. In order to show that $z_2 \leqslant w_1$ on $L_b(k)$, it thus suffices to check that the derivative of the function $f_{b,k}(w_1) = w_1 - z_2(b,k,w_1)$ is nonnegative, ie

$$f'_{b,k}(w_1) = 1-\frac{1}{2}\sqrt{\frac{2b}{2(b+k)+1+w_1}}\frac{2b+k}{2b} \geqslant 0.$$

This holds if it holds for $w_1 = 0$, ie if

$$\frac{4b}{2b+k} \geqslant \sqrt{\frac{2b}{2b+2k+1}}.$$

This is equivalent to

$$8b(2b+2k+1) \geqslant 4b^2+4bk+k^2,$$

which holds true since $k^2 \leqslant 2b$.

(iii) Fix $k \geqslant 3$ and $b \geqslant 2$. Define the function $f_{b,k}$ on $[v_b(k)-\lfloor v_b(k)\rfloor, 1]$ by

(5-1) $$f_{b,k}(w_1) := w_2+z_2-z_1 = -w_1+(2b+k-1)\lambda-(2b+2k)+1.$$

Then $f'_{b,k} \leqslant 0$. Indeed, this is equivalent to

$$2b+k-1 \leqslant 2\sqrt{2b(2b+2k+1+w_1)},$$

which follows from

$$2b+k \leqslant 2\sqrt{2b(2b+2k+1)}.$$

It therefore suffices to show that $f_{b,k}(1) \geqslant 0$, ie

$$\sqrt{\frac{b+k+1}{b}} \geqslant \frac{2b+2k}{2b+k-1}.$$

Squaring and multiplying by $b(2b+k-1)^2$ this becomes

$$(1+k)((k-3)b+(k-1)^2) \geqslant 0,$$

which holds true since $k \geqslant 3$. □

In view of Proposition 2.3 we wish to transform the vector

$$((b+1)\lambda; b\lambda, \lambda, \boldsymbol{w}(a))$$

to a reduced vector by a finite sequence of Cremona moves. One Cremona move yields

$$\big((b+1)\lambda-1; b\lambda-1, 1^{\times 2(b+k)} \,\|\, z_1, w_1^{\times \ell_1}, w_2^{\times \ell_2}, \dots\big).$$

Here and in the sequel we use the notation explained in Notation 2.4. Next, $b+k$ Cremona moves with $\delta = \lambda - 2 = z_1 - 1$ yield

$$(\lambda + z_2; z_2, w_1^{\times \ell_1}, z_1^{\times 2(b+k)+1}, w_2^{\times \ell_2}, \dots). \tag{5-2}$$

Assume that $z_1 \geqslant w_1$. Since $z_2 \leqslant w_1$, the vector (5-2) reorders to

$$(\lambda + z_2; z_1^{\times 2(b+k)+1}, w_1^{\times \ell_1} \,\|\, z_2, w_2^{\times \ell_2}, \dots). \tag{5-3}$$

Then $\delta = \lambda + z_2 - 3z_1 = 1 + z_2 - 2z_1 \geqslant 0$ by Lemma 5.2(i). Since all entries of (5-3) are nonnegative, this vector is reduced.

From now on we thus assume that $w_1 \geqslant z_1$. Then the vector (5-2) becomes

$$(\lambda + z_2; w_1^{\times \ell_1} \,\|\, z_1^{\times 2(b+k)+1}, z_2, w_2^{\times \ell_2}, \dots). \tag{5-4}$$

If $\ell_1 \geqslant 3$, then $\delta = 1 + z_1 + z_2 - 3w_1 \geqslant z_1 + z_2 \geqslant 0$. If $\ell_1 = 2$, then

$$\delta = 1 + z_1 + z_2 - 2w_1 - (z_1 \text{ or } z_2 \text{ or } w_2) \geqslant 1 - (2w_1 + w_2) \geqslant 0.$$

So assume that $\ell_1 = 1$, that is, the vector (5-4) is

$$(\lambda + z_2; w_1 \,\|\, z_1^{\times 2(b+k)+1}, z_2, w_2^{\times \ell_2}, \dots).$$

Case 1 $(z_1 \geqslant z_2, w_2)$ Then the vector at hand is

$$(\lambda + z_2; w_1, z_1^{\times 2(b+k)+1} \,\|\, z_2, w_2^{\times \ell_2}, \dots).$$

Hence $\delta = 1 + z_1 + z_2 - w_1 - 2z_1 = w_2 + z_2 - z_1$. For $k \geqslant 3$ this number is nonnegative by Lemma 5.2(iii). Assume now that $k = 2$ and that $\delta = w_2 + z_2 - z_1 < 0$. We reduce the above vector $b+2$ times by δ and get

$$(w_2+z_1+z_2+*;\, * = w_1+(b+2)(w_2+z_2-z_1), z_1, (w_2+z_2)^{\times(2b+4)} \,\|\, z_2, w_2^{\times \ell_2}, \dots).$$

The order is right by the assumption $w_2 + z_2 \leqslant z_1$ and the following lemma. For this vector, $\delta = 0$.

Lemma 5.3 $$w_1 + (b+2)(w_2 + z_2 - z_1) \geqslant z_1.$$

Proof Define the function f_b on $[v_b(2) - \lfloor v_b(2) \rfloor, 1]$ by

$$f_b(w_1) := w_1 + (b+2)(w_2 + z_2 - z_1) - z_1. \tag{5-5}$$

We compute

$$f_b(w_1) = -(b+1)w_1 + (2b^2+5b+1)\lambda - (2b^2+7b+5), \quad \text{where } \lambda = \sqrt{\frac{2b+5+w_1}{2b}}.$$

We wish to show that $f_b(w_1) \geqslant 0$. We estimate

$$f_b'(w_1) = -(b+1) + \frac{2b^2+5b+1}{2\sqrt{2b(2b+5+w_1)}} \leqslant -(b+1) + \frac{2b^2+5b}{4b} \leqslant 0.$$

Hence $f_b(w_1) \geqslant f_b(1) = -(b+1) + (2b^2+5b+1)\sqrt{\frac{b+3}{b}} - (2b^2+7b+5)$. The right-hand side is nonnegative if and only if

$$\sqrt{\frac{b+3}{b}} \geqslant \frac{2(b^2+4b+3)}{2b^2+5b+1}.$$

Squaring and multiplying by $b(2b^2+5b+1)^2$ we find that this is equivalent to the inequality $(b+3)(b-1)^2 \geqslant 0$, which holds true. □

Case 2 $(z_2 \geqslant z_1, w_2)$ Then $\delta = 1 + z_1 - w_1 - (z_1 \text{ or } w_2) \geqslant 0$.

Case 3 $(w_2 \geqslant z_1, z_2)$ The vector at hand is

$$(\lambda + z_2; w_1, w_2^{\times \ell_2} \,\|\, z_1^{\times 2(b+k)+1}, z_2, w_3^{\times \ell_3}, w_4^{\times \ell_4}, \dots).$$

Subcase 3(a) $(\ell_2 \geqslant 2)$ Then $\delta = z_1 + z_2 - w_2$. Assume that $\delta < 0$, ie $w_2 > z_1 + z_2$.

If $\ell_2 = 2m_2 \geqslant 2$ is even, we reduce m_2 times by δ and get

$$\big(z_1 + z_2 + w_2 + *;\, * = w_1 + m_2(z_1 + z_2 - w_2) \,\| \\ (z_1+z_2)^{\times m_2}, z_1^{\times 2(b+k)+1}, z_2, w_3^{\times \ell_3}, w_4^{\times \ell_4}, \dots\big).$$

Here, $* \geqslant z_1 + z_2$ and $* \geqslant w_3 = w_1 - \ell_2 w_2$ because $m_2 w_2 \leqslant \ell_2 w_2 \leqslant w_1$.

If $z_1 + z_2 \geqslant w_3$, then

$$\delta = w_2 - (z_1 + z_2 \text{ or } z_1 \text{ or } z_2 \text{ or } w_3) \geqslant w_2 - (z_1 + z_2) > 0.$$

If $w_3 \geqslant z_1 + z_2$, then

$$\delta = z_1 + z_2 + w_2 - w_3 - (w_3 \text{ if } \ell_3 \geqslant 2 \text{ and } z_1 + z_2 \text{ or } w_4 \text{ if } \ell_3 = 1).$$

In the first case, $\delta \geqslant 0$ since $w_2 = \ell_3 w_3 + w_4 \geqslant 2w_3$, and in the second case, $\delta = w_2 - w_3 \geqslant 0$ or $\delta = z_1 + z_2 \geqslant 0$.

If $\ell_2 = 2m_2 + 1 \geqslant 3$ is odd, we again reduce m_2 times by δ and get

$$\big(z_1 + z_2 + w_2 + *;\ * = w_1 + m_2(z_1 + z_2 - w_2),\ w_2 \,\|\, (z_1 + z_2)^{\times m_2}, z_1^{\times 2(b+k)+1}, z_2, w_3^{\times \ell_3}, w_4^{\times \ell_4}, \dots\big).$$

If $z_1 + z_2 \geqslant w_3$, then $\delta = 0$. If $w_3 > z_1 + z_2$, then $\delta = z_1 + z_2 - w_3 < 0$. The vector at hand is

$$(z_1 + z_2 + w_2 + *;\ * = w_1 + m_2(z_1 + z_2 - w_2),\ w_2,\ w_3^{\times \ell_3} \,\|\, (z_1 + z_2)^{\times m_2},\ w_4^{\times \ell_4},\ \dots),$$

and applying one more Cremona transform yields the vector

$$(z_1 + z_2 + w_2 + *;\ *,\ w_2 + z_1 + z_2 - w_3,\ w_3^{\times \ell_3 - 1} \,\|\, (z_1 + z_2)^{\times m_2 + 1},\ w_4^{\times \ell_4},\ \dots),$$

where now $* = w_1 + m_2(z_1 + z_2 - w_2) + (z_1 + z_2 - w_3)$. The ordering holds since if $\ell_3 \geqslant 2$ then $w_2 + z_1 + z_2 - w_3 \geqslant w_2 - w_3 \geqslant w_3$, and if $\ell_3 = 1$ then $w_2 + z_1 + z_2 - w_3 = z_1 + z_2 + w_4$. Now

$$\delta = w_3 - (w_3 \text{ or } z_1 + z_2 \text{ or } w_4) \geqslant 0.$$

Subcase 3(b) $(\ell_2 = 1)$ Then $\delta = 1 + z_1 + z_2 - w_1 - w_2 - x = z_1 + z_2 - x$ with $x \in \{z_1, z_2, w_3\}$. If $x \in \{z_1, z_2\}$ then $\delta \in \{z_2, z_1\} \geqslant 0$. If $x = w_3$, then the vector at hand is

$$(\lambda + z_2; w_1, w_2, w_3^{\times \ell_3} \,\|\, z_1^{\times 2(b+k)+1}, z_2, w_4^{\times \ell_4}, \dots).$$

Notice that $w_2 = 1 - w_1$ and $w_3 = w_1 - w_2$. We have $\delta = z_1 + z_2 - w_3$. If $w_3 > z_1 + z_2$, we apply one more Cremona transform and obtain

$$(z_1 + z_2 + w_2 + *;\ * = z_1 + z_2 + w_2,\ z_1 + z_2 + w_2 - w_3,\ w_3^{\times(\ell_3 - 1)} \,\|\, z_1 + z_2,\ z_1^{\times 2(b+k)+1},\ z_2,\ w_4^{\times \ell_4},\ \dots).$$

The ordering is right since if $\ell_3 \geqslant 2$ then $w_2 \geqslant 2w_3$, and if $\ell_3 = 1$ then $w_2 - w_3 = w_4$.

If $\ell_3 \geqslant 2$ then $\delta = 0$.

If $\ell_3 = 1$ then $\delta = w_3 - (z_1 + z_2) > 0$ or $\delta = w_3 - w_4 \geqslant 0$.

The proof of Theorem 5.1 is complete. □

6 The intervals $R_b(k) = [2b + 2k + 2\,, u_b(k+1)]$

Theorem 6.1 *Assume that $b \in \mathbb{N}_{\geqslant 5}$ and that $k \in \{2, \dots, \lfloor\sqrt{2b}\rfloor - 1\}$. Then $c_b(a) = \sqrt{\frac{a}{2b}}$ for $a \in R_b(k)$.*

Proof For notational convenience we shift the index k by one, and prove that $c_b(a) = \sqrt{\frac{a}{2b}}$ for $a \in R_b(k-1)$ and $k \in \{3, \dots, \lfloor\sqrt{2b}\rfloor\}$.

We start with three inequalities that will be useful later on.

Lemma 6.2 *We have:*

(i) $$\frac{2b+2k}{2b} \geqslant \left(\frac{2b+2k-2}{2b+k-2}\right)^2 \quad \text{for } k \geqslant 4.$$

(ii) $$\sqrt{\frac{2b+2k}{2b}} \geqslant \frac{2b+2k}{2b+k}.$$

(iii) $$\frac{2b+k}{2b} \leqslant \frac{2b+2k}{2b+k-1} \qquad \text{if } k^2 \leqslant 2b.$$

Proof (i) This is equivalent to

$$\frac{(2b+2k)(2b+k-2)^2 - 2b(2b+2k-2)^2}{2b(2b+k-2)^2} \geqslant 0,$$

which holds true for $k \geqslant 4$ because the numerator of the left-hand side can be written as $2k(b(k-4) + (k-2)^2)$.

(ii) This follows from $(2b+k)^2 - 2b(2b+2k) = k^2$.

(iii) This follows from

$$(2b+2k)(2b) - (2b+k-1)(2b+k) = 2b + k - k^2,$$

since $2b + k - k^2 \geqslant k > 0$ by assumption. □

Except possibly for the right endpoint, which we can neglect, the weight expansion at $a \in R_b(k-1) = \left[2b+2k, 2b+2k+\frac{k^2}{2b}\right]$ is

$$\boldsymbol{w}(a) = (1^{\times 2b+2k}, w_1^{\times \ell_1}, w_2^{\times \ell_2}, \dots).$$

Set $\lambda = \sqrt{\frac{a}{2b}}$. We wish to transform the vector

$$((b+1)\lambda; b\lambda, \lambda, \boldsymbol{w}(a)) \tag{6-1}$$

to a reduced vector by a sequence of Cremona moves. Define the numbers

$$\begin{aligned} z_1 &:= \lambda - 1, \\ y_1 &:= (2b+k)\lambda - (2b+2k-1), \\ z_2 &:= y_1 - \lambda. \end{aligned}$$

Then $z_1, y_1 \geqslant 0$ and $z_2 \in [0,1]$. Indeed, as we have seen in Lemma 5.2(ii), $z_2 = 0$ at the left endpoint of $L_b(k-1)$, and $z_2 \leqslant 1$ since $\lambda \leqslant \frac{2b+k}{2b}$, using also Lemma 6.2(iii). Applying one Cremona move to (6-1) we obtain

$$((b+1)\lambda - 1;\ b\lambda - 1,\ 1^{\times 2b+2k-1} \parallel z_1,\ w_1^{\times \ell_1},\ \dots).$$

Applying $b+k-1$ Cremona transforms with $\delta = \lambda - 2$ and reordering we obtain

$$(y_1; 1 \parallel z_1^{\times 2b+2k-1}, z_2, w_1^{\times \ell_1}, \dots). \tag{6-2}$$

6.1 The case $z_1 \geqslant w_1$

For $z_1 \geqslant w_1$, assume first that $k \geqslant 4$, or that $k = 3$ and $z_2 \geqslant z_1$. If $z_2 \geqslant z_1$, then the vector (6-2) reorders to

$$(y_1; 1, z_2, z_1^{\times 2b+2k-1}, w_1^{\times \ell_1}, \dots).$$

This vector has defect $\delta = y_1 - 1 - z_2 - z_1 = 0$ and hence is reduced. If $z_1 \geqslant z_2$, then the vector (6-2) reorders to the vector

$$(y_1; 1, z_1^{\times 2b+2k-1} \parallel z_2, w_1^{\times \ell_1}, \dots), \tag{6-3}$$

which for $k \geqslant 4$ is reduced, since then $\delta = y_1 - 1 - 2z_1 = y_1 - (2\lambda - 1) \geqslant 0$ by Lemma 6.2(i) and the fact that $\lambda \geqslant \sqrt{\frac{2b+2k}{2b}}$.

Assume now that $k = 3$ and $z_1 \geqslant z_2$. If $\widehat{\delta} := y_1 - 1 - 2z_1 \geqslant 0$, the vector (6-3) is reduced. Otherwise, we apply $b+2$ Cremona moves to obtain

$$(y_1 + (b+2)\widehat{\delta};\ 1 + (b+2)\widehat{\delta},\ z_1,\ z_2^{\times 2b+5} \parallel w_1^{\times \ell_1},\ \dots). \tag{6-4}$$

The ordering is right by the following claim, and the defect is $y_1 - 1 - z_1 - z_2 = 0$, whence this vector is reduced.

Claim *Assume that $k = 3$. Then:*

(i) $1 + (b+2)\widehat{\delta} \geqslant z_1$.

(ii) *If $z_1 \geqslant w_1$, then $z_2 \geqslant w_1$.*

Proof Inequality (i) is equivalent to

$$(2b^2+5b+1)\lambda \geqslant (2b^2+8b+6).$$

It suffices to check this inequality for $\lambda = \sqrt{\frac{2b+6}{2b}}$, where it is equivalent to the inequality $3b^2+10b+3 \geqslant 0$, which holds true for all $b \geqslant 1$.

For (ii), we know that $\lambda - 1 \geqslant w_1$, ie

$$a \leqslant \lambda + (2b+5). \tag{6-5}$$

Since $a = 2b\lambda^2$, this is equivalent to

$$\lambda \geqslant \frac{1+\sqrt{1+8b(2b+5)}}{4b}. \tag{6-6}$$

We wish to show that $z_2 - 2 \geqslant w_1$, ie $a \leqslant 1 + (2b+2)\lambda$. In view of (6-5), this will hold if $\lambda + (2b+5) \leqslant 1 + (2b+2)\lambda$, ie

$$\frac{2b+4}{2b+1} \leqslant \lambda. \tag{6-7}$$

By (6-6), this would follow from

$$\frac{2b+4}{2b+1} \leqslant \frac{1+\sqrt{1+8b(2b+5)}}{4b}.$$

Isolating the root and squaring, this becomes the true inequality $72b/(2b+1)^2 \geqslant 0$. □

6.2 The case $w_1 \geqslant z_1$

Assume now that

$$w_1 \geqslant z_1. \tag{6-8}$$

The vector (6-2) in question is

$$(y_1; 1 \parallel w_1^{\times \ell_1}, z_1^{\times 2b+2k-1}, z_2, \dots). \tag{6-9}$$

Define

$$z_3 := y_1 - 1 - w_1.$$

Note that $z_3 \geqslant 0$ on our interval, and $z_3 = 0$ at the right endpoint $a = \frac{(2b+k)^2}{2b}$. The significance of z_3 and of the following lemma will become clear later.

Lemma 6.3 *If $z_3 \geqslant w_1$, then the vector (6-9) is reduced.*

Proof For $\delta := y_1 - 1 - z_2 - w_1$ we have $z_2 + \delta = z_3$ and $w_1 + \delta = z_1$. Applying one Cremona move to

$$(y_1; 1 \parallel z_2, w_1^{\times \ell_1}, z_1^{\times 2b+2k-1}, \dots)$$

we thus obtain

$$(y_1 + \delta;\ 1 + \delta,\ z_3 \parallel w_1^{\times \ell_1 - 1},\ z_1^{\times 2b+2k},\ \dots). \tag{6-10}$$

The ordering is right because $z_3 \geqslant w_1 \geqslant z_1$ by assumption and by (6-8). The defect of (6-10) is thus

$$y_1 - 1 - z_3 - (w_1 \text{ or } z_1 \text{ or } w_2) \geqslant y_1 - 1 - z_3 - w_1 = 0. \qquad \square$$

From now on we thus assume that

$$w_1 \geqslant z_3. \tag{6-11}$$

Lemma 6.4 $z_2 \geqslant z_1, z_3.$

Proof The inequality $z_2 \geqslant z_1$ translates to

$$(2b + k - 1)\lambda - (2b + 2k - 1) \geqslant \lambda - 1,$$

or, equivalently,

$$\lambda \geqslant \frac{2b + 2k - 2}{2b + k - 2}. \tag{6-12}$$

But we know that $\lambda \geqslant \sqrt{\frac{2b+2k}{2b}}$, whence in the case $k \geqslant 4$ the inequality (6-12) follows from Lemma 6.2(i). In the case $k = 3$, (6-12) is (6-7).

The inequality $z_2 \geqslant z_3$ is $-\lambda \geqslant -1 - w_1$. This is equivalent to $\lambda - 1 \leqslant w_1$, which follows from (6-8). $\square$

The rest of the proof of Theorem 6.1 is divided into the cases $\ell_1 = 2m$ even and $\ell_1 = 2m + 1$ odd.

Case I ($\ell_1 = 2m$ even) We can assume by continuity that $\ell_1 > 0$, so that $m \geqslant 1$. By applying m Cremona transforms to the vector (6-9) with $\delta = y_1 - 1 - 2w_1$ we obtain

$$(y_2 + y_1 - 1; y_2 \parallel z_1^{\times 2b+2k-1}, z_2, z_3^{\times \ell_1}, w_2^{\times \ell_2}, \dots), \tag{6-13}$$

where $y_2 := 1 + m(y_1 - 1 - 2w_1)$. The ordering is right by the previous and the next lemma.

Lemma 6.5 $y_2 \geqslant z_2, w_2.$

Proof The inequality $y_2 \geqslant z_2$ is equivalent to

$$1 + m(y_1 - 1 - 2w_1) \geqslant y_1 - \lambda.$$

Since $\ell_1 w_1 \leqslant 1$ and $\lambda \geqslant 1$, it suffices to show that $(m-1)(y_1-1) \geqslant 0$. This follows since $y_1 \geqslant 1$, by Lemma 6.2(ii).

The inequality $y_2 \geqslant w_2$ is equivalent to

$$1 + m(y_1 - 1 - 2w_1) \geqslant w_2.$$

Since $\ell_1 w_1 \leqslant 1$, it suffices to show that $m(y_1 - 1) \geqslant w_2$. For this, it suffices to show that $y_1 - 1 \geqslant w_1$, ie $a \leqslant \lambda(2b+k)$. This follows from the fact that $a \leqslant \frac{(2b+k)^2}{2b}$. □

Lemma 6.6 *If $z_3 \geqslant w_2$, then the vector (6-13) is reduced.*

Proof Assume that $z_3 \geqslant w_2$. If $z_1 \geqslant z_3$, then (6-13) is

$$(y_2 + y_1 - 1; y_2, z_2, z_1^{\times 2b+2k-1}, z_3^{\times \ell_1}, w_2^{\times \ell_2} \parallel \dots),$$

which is reduced. Hence we can assume that $z_3 \geqslant z_1$. In this case, we apply one Cremona transform to

$$(y_2 + y_1 - 1;\ y_2,\ z_2,\ z_3^{\times \ell_1} \parallel z_1^{\times 2b+2k-1},\ w_2^{\times \ell_2}, \dots)$$

with $\delta = z_1 - z_3$ and obtain

$$(y_2 + y_1 - 1 + \delta;\ y_2 + \delta,\ z_2 + \delta,\ z_3^{\times \ell_1 - 1} \parallel z_3 + \delta,\ z_1^{\times 2b+2k-1},\ w_2^{\times \ell_2}, \dots)$$

since $\ell_1 \geqslant 2$. First note that $z_3 + \delta = z_1 \geqslant 0$. To see that the ordering is right, we need to check that $z_2 + \delta \geqslant z_3$. This is equivalent to $y_1 - 1 \geqslant 2z_3$, which is equivalent to $y_1 - 1 - 2w_1 \leqslant 0$, which holds by (6-11). Since the defect vanishes, this vector is reduced. □

From now on we thus assume that

$$w_2 \geqslant z_3. \tag{6-14}$$

Lemma 6.7 *If $z_1 \geqslant w_2$, then the vector (6-13) is reduced.*

Proof Assume that $z_1 \geqslant w_2$. Then the vector (6-13) is

$$(y_2 + y_1 - 1;\ y_2,\ z_2,\ z_1^{\times 2b+2k-1},\ w_2^{\times \ell_2},\ z_3^{\times \ell_1} \parallel \dots)$$

with defect $y_1 - 1 - z_2 - z_1 = 0$. □

From now on we thus assume that

$$w_2 \geqslant z_1. \tag{6-15}$$

By now, our vector is

$$(y_2+y_1-1;\ y_2,\ w_2^{\times \ell_2} \parallel z_2,\ z_1^{\times 2b+2k-1},\ z_3^{\times \ell_1},\ w_3^{\times \ell_3},\ \dots) \quad \text{if } w_2 \geqslant z_2, \tag{6-16}$$

$$(y_2+y_1-1;\ y_2,\ z_2,\ w_2^{\times \ell_2} \parallel z_1^{\times 2b+2k-1},\ z_3^{\times \ell_1},\ w_3^{\times \ell_3},\ \dots) \quad \text{if } z_2 \geqslant w_2. \tag{6-17}$$

Subcase 1 $(\ell_2 \geqslant 2)$ In case (6-16) we have $\delta \geqslant y_1-1-w_1$, since $2w_2 \leqslant w_1$. Since $y_1-1-w_1 = z_3 \geqslant 0$, the vector is reduced.

In case (6-17) we have $\delta = z_1 - w_2 < 0$. Applying one Cremona transform yields

$$(y_2+y_1-1+\delta;\ y_2+\delta,\ z_2+z_1-w_2,\ w_2^{\times \ell_2-1} \parallel z_1^{\times 2b+2k},\ z_3^{\times \ell_1},\ w_3^{\times \ell_3},\ \dots). \tag{6-18}$$

The ordering is right since $z_2+z_1 \geqslant 2w_2$. Indeed, this is equivalent to $y-1 \geqslant 2w_2$. Since $w_1 \geqslant 2w_2$, this follows from $y_1-1 \geqslant w_1$, which holds because $y_1-1-w_1 = z_3 \geqslant 0$. The defect of (6-18) vanishes.

Subcase 2 $(\ell_2 = 1)$ We again distinguish two cases.

Assume first that $w_3 \geqslant z_2$. We are then in case (6-16), and, since $z_2 \geqslant z_1$ and $z_2 \geqslant z_3$, the vector at hand is

$$(y_2+y_1-1;\ y_2,\ w_2,\ w_3^{\times \ell_3},\ z_2 \parallel \dots).$$

This vector is reduced, since $w_1 = w_2+w_3$ and hence $\delta = y_1-1-w_2-w_3 = z_3$.

Assume now that either $w_2 \geqslant z_2 \geqslant w_3$ or $z_2 \geqslant w_2$. Since also $z_2 \geqslant z_1$ and $z_2 \geqslant z_3$, in both (6-16) and (6-17) we have $\delta = z_1 - w_2$. Further, $w_2 = w_1 - w_3$ since $\ell_2 = 1$, and so $z_2 + \delta = z_2 + z_1 - w_2 = w_3 + z_3$. Hence both vectors transform to

$$(y_2+y_1-1+\delta;\ y_2+\delta \parallel w_3+z_3,\ z_1^{\times 2b+2k},\ z_3^{\times \ell_1},\ w_3^{\times \ell_3},\ \dots).$$

This vector is reduced after reordering: if $w_3+z_3 \geqslant z_1$, then

$$\delta = z_1+z_2-w_3-z_3-(z_1 \text{ or } z_3 \text{ or } w_3) = w_2-(z_1 \text{ or } z_3 \text{ or } w_3) \geqslant 0$$

by (6-14) and (6-15), and if $z_1 \geqslant w_3+z_3$, then $\delta = z_1+z_2-2z_1 = z_2-z_1 \geqslant 0$.

Case II $(\ell_1 = 2m+1 \text{ odd})$ We start from the vector (6-9). By applying $m \geqslant 0$ Cremona transforms with $\delta = y_1-1-2w_1$ we obtain

$$(\hat{y}_2+y_1-1;\ \hat{y}_2,\ z_3^{\times(\ell_1-1)},\ w_1,\ z_1^{\times 2b+2k-1},\ z_2,\ w_2^{\times \ell_2},\ \dots),$$

where $\hat{y}_2 := 1+m(y_1-1-2w_1)$.

Now apply another Cremona transform to the partially reordered vector

$$(\widehat{y}_2+y_1-1;\ \widehat{y}_2,\ w_1,\ z_2,\ z_1^{\times 2b+2k-1},\ z_3^{\times(\ell_1-1)},\ w_2^{\times \ell_2},\ \dots).$$

With $\delta = y_1-1-w_1-z_2 = z_1-w_1$ we obtain

$$(\widehat{y}_2+y_1-1+\delta;\ \widehat{y}_2+\delta \parallel z_1^{\times 2b+2k},\ z_3^{\times \ell_1},\ w_2^{\times \ell_2},\ \dots) \tag{6-19}$$

since $w_1+\delta = z_1$ and $z_2+\delta = z_3$. We are again assuming, by continuity, that $\ell_2 \geqslant 1$. The ordering is right in view of the following lemma:

Lemma 6.8 (i) $\widehat{y}_2+\delta \geqslant z_1$.

(ii) $\widehat{y}_2+\delta \geqslant z_3$.

(iii) $\widehat{y}_2+\delta \geqslant w_2$.

Proof Using $1 = \ell_1 w_1 + w_2$ and $y_1 - 1 = z_1 + z_2$, we compute

$$\widehat{y}_2+\delta = (m+1)(z_1+z_2)-z_2+w_2.$$

Assertions (i) and (iii) follow at once. Assertion (ii) follows at once for $m \geqslant 1$, and for $m=0$ it also holds, since then $w_1+w_2 = 1 \geqslant z_2$. □

We now show that the vector (6-19) is reduced, or can be transformed in one step to a reduced vector. (We will need to transform the vector in only one case). In view of Lemma 6.8, we just have to consider the various possibilities for the orderings of z_1, z_3 and w_2. Denote by δ_* the defect of the reordering of (6-19).

Case 1 $(z_1 \geqslant z_3, w_2)$ Then $\delta_* = y_1-1-2z_1 = z_2-z_1 \geqslant 0$ by Lemma 6.4.

Case 2 $(z_3 \geqslant z_1, w_2)$ Then $\delta_* \geqslant y_1-1-2z_3 = w_1-z_3 \geqslant 0$ by (6-11).

Case 3 $(w_2 \geqslant z_1, z_3)$ Then the vector (6-19) is

$$(\widehat{y}_2+y_1-1+\delta;\ \widehat{y}_2+\delta,\ w_2^{\times \ell_2} \parallel z_1^{\times 2b+2k},\ z_3^{\times \ell_1},\ w_3^{\times \ell_3},\ \dots). \tag{6-20}$$

Subcase $(\ell_2 \geqslant 2)$ Then (6-20) is reduced if $y_1-1 \geqslant 2w_2$. We know that $2w_2 \leqslant w_1$. Hence it suffices to show that $y_1-1 \geqslant w_1$, which follows from the fact that $z_3 \geqslant 0$.

Subcase $(\ell_2 = 1)$ We distinguish three cases.

Assume first that $w_3 \geqslant z_1, z_3$. Then (6-20) is reduced, since

$$\delta_* = y_1-1-(w_2+w_3) = y_1-1-w_1 = z_3.$$

Assume next that $z_3 \geqslant z_1, w_3$. Then (6-20) is reduced, since

$$\delta_* = y_1-1-w_2-z_3 = w_1-w_2.$$

Assume finally that $z_1 \geqslant z_3, w_3$. Then the vector in question is

$$(\widehat{y}_2 + y_1 - 1 + \delta;\ \widehat{y}_2 + \delta,\ w_2,\ z_1^{\times 2b+2k} \parallel z_3^{\times \ell_1},\ w_3^{\times \ell_3}, \dots).$$

If $\widehat{\delta} := y_1 - 1 - w_2 - z_1 = z_2 - w_2 \geqslant 0$, this vector is reduced. Otherwise, we apply one Cremona transform and obtain

$$(\widehat{y}_2 + y_1 - 1 + \delta + \widehat{\delta};\ \widehat{y}_2 + \delta + \widehat{\delta},\ w_2 + \widehat{\delta},\ z_1 + \widehat{\delta},\ z_1^{\times 2b+2k-1}, \dots). \tag{6-21}$$

Note that $z_1 + \widehat{\delta} = y_1 - 1 - w_2 \geqslant y_1 - 1 - w_1 = z_3 \geqslant 0$ and that $w_2 + \widehat{\delta} = z_2 \geqslant z_1$ by Lemma 6.4. Hence (6-21) reorders to the vector

$$(\widehat{y}_2 + y_1 - 1 + \delta + \widehat{\delta};\ \widehat{y}_2 + \delta + \widehat{\delta},\ z_2,\ z_1^{\times 2b+2k-1}, \dots)$$

which is reduced, since its defect is $y_1 - 1 - z_2 - z_1 = 0$.

The proof of Theorem 6.1 is finally complete. □

7 The interval $[v_b(1), 2b+4]$

Recall that for $b \in \mathbb{N}_{\geqslant 2}$ we defined $v_b(1) := 2b\left(\frac{2b+3}{2b+1}\right)^2$ and

$$\alpha_b := \frac{1}{b}\left(b^2 + 2b + \sqrt{(b^2+2b)^2 - 1}\right) \in \,]v_b(1), 2b+4[.$$

Theorem 7.1 *For every $b \in \mathbb{N}_{\geqslant 2}$ we have*

$$c_b(a) = \begin{cases} \sqrt{\dfrac{a}{2b}} & \text{if } a \in [v_b(1), \alpha_b], \\[2ex] \dfrac{ba+1}{2b(b+1)} & \text{if } a \in [\alpha_b, 2b+4]. \end{cases}$$

In particular, $c_b(\alpha_b) = \sqrt{\frac{\alpha_b}{2b}}$ and $c_b(2b+4) = 1 + \frac{2b+1}{2b(b+1)}$.

Proof Let $a \in [v_b(1), 2b+4]$ be a rational number. For $w_1(b) = v_b(1) - (2b+3)$ we compute $w_1'(b) = 16/(2b+1)^3$. Hence $w_1(b) \geqslant w_1(2) = \frac{21}{25} > \frac{5}{6}$ for $b \geqslant 2$, and so $\ell_1 = 1$ and $\ell_2 \geqslant 5$. The weight expansion of a thus has the form

$$\boldsymbol{w}(a) = (1^{\times(2b+3)}, w_1, w_2^{\times \ell_2}, \dots, w_N^{\times \ell_N}).$$

We wish to show that for $\lambda = c_b(a)$ as in the theorem, the vector $((b+1)\lambda; b\lambda, \lambda, \boldsymbol{w}(a))$ can be reduced to a reduced vector.

7.1 The interval $[v_b(1), \alpha_b]$

Assume that $a \in [v_b(1), \alpha_b]$. Then $\lambda = \sqrt{\frac{a}{2b}}$. Define the numbers

$$\begin{aligned} z_1 &:= \lambda - 1, \\ z_2 &:= (2b+1)\lambda - (2b+3), \\ z_3 &:= (2b+1)\lambda - (a-1), \\ z_4 &:= b(z_3 - z_1) + w_1, \\ z_5 &:= 2b(b+1)\lambda - (ba+1), \\ z_6 &:= b(2z_5 + z_1 - z_4 - 2z_3) + z_4. \end{aligned}$$

In the following, the symbol $\overset{\mathrm{e}}{=}$ means that an identity is readily checked by expanding the relevant z_i as polynomials of degree two in λ with coefficients polynomials in b. For instance,

$$z_3 = 1 + z_2 - w_1 \overset{\mathrm{e}}{=} z_1 + z_5 - z_4, \tag{7-1}$$

$$z_6 \overset{\mathrm{e}}{=} b(2b(b+1)-1)\lambda - (b^2 a - w_1). \tag{7-2}$$

In this section, all newly created numbers will be one of $z_1, \dots, z_6$ or 0, and we shall write down each z_i of every vector. In other words, the dots $\dots$ in any vector are either w_j or 0.

7.1.1 Inequalities

Lemma 7.2 *On the interval $[v_b(1), \alpha_b]$ the following inequalities hold true:*

(i) $b\lambda - 1 \geqslant 1$ *and* $w_1 \geqslant z_1 \geqslant w_2$.

(ii) $w_1 \geqslant 1 - z_1 + z_2 \geqslant z_1 \geqslant z_2$.

(iii) $z_1 \geqslant z_3 \geqslant z_2, w_2$.

(iv) $z_1 \geqslant z_5$. *Moreover,* $z_5 \geqslant z_3$ *is equivalent to* $z_4 \geqslant z_1$.

(v) $z_4 \geqslant z_3$.

(vi) $z_6 \geqslant z_2, z_5, w_2$.

(vii) *If* $b \geqslant 3$, *then* $z_1 - z_4 + 2z_5 - 2w_2 \geqslant 0$.

(viii) $z_i \geqslant 0$ *for all* $i \in \{1, \dots, 6\}$.

Proof (i) We have $b\lambda - 1 \geqslant b - 1 \geqslant 1$. In order to prove $w_1 \geqslant z_1$, we show that the function

$$f_b(a) := w_1 - z_1 = a - (2b+2) - \sqrt{\frac{a}{2b}}$$

is nonnegative. Since $f_b'(a) = 1 - \frac{1}{4b}\sqrt{\frac{2b}{a}} > 0$, it suffices to see that $f_b(v_b(1)) = (4b^2-5)/(2b+1)^2 \geqslant 0$, which holds true for $b \geqslant 2$.

To prove $z_1 \geqslant w_2$, define the function $f_b(a) := z_1 - w_2 = \sqrt{\frac{a}{2b}} + a - (2b+5)$. Since $f_b'(a) = \frac{1}{4b}\sqrt{\frac{2b}{a}} + 1 > 0$, it suffices to see that $f_b(v_b(1)) = (4b-2)/(2b+1)^2 \geqslant 0$, which holds true for $b \geqslant 2$.

(ii) We compute

$$1 - z_1 + z_2 = 2b(\lambda - 1) - 1 \geqslant \lambda - 1 = z_1.$$

This proves the second inequality, and that the first inequality $w_1 \geqslant 1 - z_1 + z_2$ is equivalent to $2b\lambda^2 - 2b\lambda - 2 \geqslant 0$. Since the left-hand side is increasing for $\lambda \geqslant 1$, it suffices to check this inequality at $\lambda(v_b(1)) = \frac{2b+3}{2b+1}$, where it becomes $(4b-2)/(2b+1)^2 \geqslant 0$.

The third inequality $z_1 \geqslant z_2$ is equivalent to $\sqrt{2ab} \leqslant 2b+2$. Squaring this leads to $a \leqslant 2b + 4 + \frac{2}{b}$, which is verified for $a \leqslant \alpha_b < 2b+4$.

(iii) The inequality $z_1 \geqslant z_3$ is equivalent to $w_1 \geqslant 1 - z_1 + z_2$, hence true. The other two inequalities follow from $z_3 = z_2 + w_2$.

(iv) The inequality $z_1 \geqslant z_5$ is equivalent to $a \geqslant (2b^2+2b-1)^2/2b^3$. This inequality is satisfied since $(2b^2+2b-1)^2/2b^3 \leqslant v_b(1)$ is equivalent to $8b^3 + 12b^2 - 1 \geqslant 0$, which is true for $b \geqslant 2$.

The inequality $z_5 \geqslant z_3$ is equivalent to $z_4 \geqslant z_1$ since $z_3 = z_1 + z_5 - z_4$.

(v) Define the function $f_b(\lambda) := z_4 - z_3 \overset{\text{e}}{=} \lambda(2b^2 - 2b - 1) - (b-2)2b\lambda^2 - 4$. For $b = 2$ we compute $f_2(\lambda) = 3\lambda - 4 \geqslant f_2(\lambda(v_2(1))) = \frac{1}{5} > 0$. For $b \geqslant 3$ we have

$$f_b'(\lambda) = 2b^2 - 2b - 1 - 4b(b-2)\lambda \leqslant -2b^2 + 6b - 1 \leqslant -1$$

since $\lambda \geqslant 1$. It thus suffices to show that $f_b(\lambda) > 0$ at $\lambda = \sqrt{\frac{2b+4}{2b}}$, that is,

$$\sqrt{\frac{2b+4}{2b}}(2b^2 - 2b - 1) \geqslant 2b^2 - 4.$$

Squaring both sides leads to $4b^2 - 7b + 2 \geqslant 0$ which is verified for $b \geqslant 3$.

(vi) The first inequality means that the function

$$f_b(a) = z_6 - z_2 \overset{\text{e}}{=} (2b^3 + 2b^2 - 3b - 1)\lambda + (1 - b^2)a$$

is nonnegative for $a \in [v_b(1), \alpha_b]$. Equivalently,

$$\frac{1}{\sqrt{2b}}(2b^3 + 2b^2 - 3b - 1) \geqslant \sqrt{a}\,(b^2 - 1).$$

It suffices to show this inequality for $a = 2b+4$, ie

$$\frac{1}{2b}(2b^3+2b^2-3b-1)^2 \geqslant (2b+4)(b^2-1)^2.$$

This is equivalent to $(b-1)^2 \geqslant 0$, which holds true.

We next show that the function

$$f_b(a) = z_6 - z_5 \stackrel{\mathrm{e}}{=} -2-2b+(2b^3-3b)\lambda+(1+b-b^2)a$$

is nonnegative for $a \in [v_b(1), \alpha_b]$.

If $b=2$, then $f_b(a) = -a+5\sqrt{a}-6 > 0$ on $[2b+3, 2b+4] = [7,8]$.

For $b \geqslant 3$ we compute that

$$f_b'(a) = (2b^3-3b)\lambda_b'(a)+(1+b-b^2)$$

is negative on $[v_b(1), \alpha_b]$, since $\lambda_b'(a) = 1/(2\sqrt{2ab})$ is decreasing and since $f_b'(2b) = \frac{1}{4}+b-\frac{b^2}{2} < 0$ for $b \geqslant 3$. It thus suffices to show that

$$f_b(2b+4) = 2(1+2b-b^2-b^3)+(2b^3-3b)\sqrt{\frac{b+2}{b}}$$

is positive. This is equivalent to $b^2+2b-4 \geqslant 0$, which holds true.

We finally show that the function

$$f_b(a) = z_6 - w_2 \stackrel{\mathrm{e}}{=} -7-4b+b(2b^2+2b-1)\lambda+(2-b^2)a$$

is nonnegative for $a \in [v_b(1), \alpha_b]$.

If $b=2$, then $f_b(a) = -2a+11\sqrt{a}-15 > 0$ on $[2b+3, 2b+4] = [7,8]$.

For $b \geqslant 3$ we compute that

$$f_b'(a) = b(2b^2+2b-1)\lambda_b'(a)+2-b^2$$

is negative on $[v_b(1), \alpha_b]$, since $f_b'(2b) = \frac{1}{4}(2b^2+2b-1)+2-b^2 < 0$ for $b \geqslant 3$. It thus suffices to show that

$$f_b(2b+4) = 1-2b^2(b+2)+b(2b^2+2b-1)\sqrt{\frac{b+2}{b}}$$

is positive. This is equivalent to $b^2+2b-1 \geqslant 0$, which holds true.

(vii) We compute

$$\delta_b(a) := z_1 - z_4 + 2z_5 - 2w_2 \stackrel{\text{e}}{=} -8 - 4b + (1+4b+2b^2)\lambda + (1-b)a$$

and

$$\delta_b'(a) = 1 - b + \frac{2b^2+4b+1}{2\sqrt{2}\sqrt{ab}}.$$

Assume first that $b=3$. Then $\delta_3(a) = -20 + \frac{31}{\sqrt{6}}\sqrt{a} - 2a$. Since

$$\delta_3'(a) = -2 + \frac{31}{2\sqrt{6}\sqrt{a}}$$

is positive for $a \in [2b+3, 2b+4] = [9, 10]$, and since $\delta_3(v_3(1)) = \frac{1}{49} > 0$, the function $\delta_3(a)$ is positive on $[v_3(1), \alpha_3]$.

Assume now that $b=4$. Then

$$\delta_4(a) = -24 + \frac{49\sqrt{a}}{2\sqrt{2}} - 3a.$$

Hence $\delta_4(2b) = \delta_4(8) = 1$ and $\delta_4(2b+4) = \delta_4(12) = -60 + 49\sqrt{\frac{3}{2}} > 0$, and so $\delta_4(a) > 0$ for all $a \in [2b, 2b+4]$.

Assume finally that $b \geqslant 5$. Then $\delta_b'(a) < 0$ for $a \in [2b, 2b+4]$. Indeed, $\delta_b'(a)$ is decreasing and $\delta_b'(2b) = 1 - b + (2b^2+4b+1)/4b < 0$. We are left with showing that

$$\delta_b(2b+4) = -(4+6b+2b^2) + (1+4b+2b^2)\sqrt{\frac{b+2}{b}}$$

is positive, which is true since equivalent to $\frac{b+2}{b} > 0$.

(viii) We show that z_2, $z_5 \geqslant 0$. The other inequalities then follow from the previous items. The inequality $z_2 \geqslant 0$ is equivalent to $\lambda \geqslant \frac{2b+3}{2b+1}$, which holds true. Moreover, $z_5 \geqslant 0$ is equivalent to

$$\lambda \geqslant \frac{ba+1}{2b(b+1)}, \tag{7-3}$$

which means that the line $a \mapsto \frac{ba+1}{2b(b+1)}$ of the affine step is below the volume constraint $\sqrt{\frac{a}{2b}}$. This holds true on $[2b, \alpha_b]$, since $\sqrt{\frac{a}{2b}}$ is convex and since (7-3) is an equality at α_b and a strict inequality at $2b$. □

7.1.2 Reductions Reducing the vector $((b+1)\lambda; b\lambda, \lambda, 1^{\times(2b+3)}, w_1, w_2^{\times \ell_2} \parallel \dots)$ with $\delta = -1$ yields

$$((b+1)\lambda - 1; b\lambda - 1, \underbrace{\lambda - 1}_{=z_1}, 0, 1^{\times(2b+2)}, w_1, w_2^{\times \ell_2}, \dots).$$

By Lemma 7.2(i) this vector reorders to

$$((b+1)\lambda-1; b\lambda-1, 1^{\times(2b+2)}, w_1, z_1, w_2^{\times\ell_2} \parallel \dots, 0).$$

Applying b Cremona transforms with $\delta=\lambda-2$ and regrouping the produced z_1's, we get

$$(\underbrace{(2b+1)\lambda-(2b+1)}_{=z_2+2}; \underbrace{2b\lambda-(2b+1)}_{=1-z_1+z_2}, 1^{\times 2}, w_1, z_1^{\times(2b+1)}, w_2^{\times\ell_2}, \dots).$$

By Lemma 7.2(ii), this vector reorders to

$$(z_2+2; 1^{\times 2}, w_1, 1-z_1+z_2, z_1^{\times(2b+1)}, w_2^{\times\ell_2} \parallel \dots).$$

Applying one Cremona transform with $\delta=z_2-w_1$ yields the vector

$$(2z_2+2-w_1; \underbrace{(1+z_2-w_1)}_{=z_3 \text{ by (7-1)}}{}^{\times 2}, z_2, 1-z_1+z_2, z_1^{\times(2b+1)}, w_2^{\times\ell_2}, \dots),$$

which by Lemma 7.2(iii) reorders to

$$(2z_2+2-w_1;\ 1-z_1+z_2,\ z_1^{\times(2b+1)},\ z_3^{\times 2} \parallel z_2,\ w_2^{\times\ell_2}, \dots).$$

Applying $b-1$ Cremona transforms with $\delta=z_3-z_1$ and regrouping the produced z_3's, we get

$$\text{(7-4)} \qquad (\underbrace{(b-1)(z_3-z_1)+2z_2+2-w_1}_{\stackrel{e}{=} 2z_1+z_5};\ \underbrace{b(z_3-z_1)+w_1}_{=z_4},\ z_1^{\times 3},\ z_3^{\times 2b},\ z_2,\ w_2^{\times\ell_2}, \dots).$$

We now distinguish the cases $z_4 \geqslant z_1$ and $z_1 \geqslant z_4$.

Case 1 $(z_4 \geqslant z_1)$ The ordered vector is then

$$(2z_1+z_5; z_4, z_1^{\times 3}, z_3^{\times 2b} \parallel z_2, w_2^{\times\ell_2}, \dots).$$

One more Cremona transform with $\delta=z_5-z_4$ yields

$$(2(z_1+z_5)-z_4; z_5, \underbrace{(z_1+z_5-z_4)}_{=z_3 \text{ by (7-1)}}{}^{\times 2}, z_1, z_3^{\times 2b}, z_2, w_2^{\times\ell_2}, \dots),$$

which by Lemma 7.2(iv) reorders to

$$(2(z_1+z_5)-z_4; z_1, z_5, z_3^{\times(2b+2)} \parallel z_2, w_2^{\times\ell_2}, \dots).$$

We already know that all entries of this vector are nonnegative, and its defect is $\delta=z_1+z_5-z_4-z_3=0$. Hence this vector is reduced.

Case 2 $(z_1 \geqslant z_4)$ Reorder the vector (7-4) as

$$(2z_1 + z_5; z_1^{\times 3}, z_4, z_3^{\times 2b} \parallel z_2, w_2^{\times \ell_2}, \dots).$$

Recall from Lemma 7.2(iv) that $z_1 \geqslant z_5$. Apply one Cremona transform with $\delta = z_5 - z_1$ to obtain

$$(2z_5 + z_1; z_5^{\times 3}, z_4, z_3^{\times 2b}, z_2, w_2^{\times \ell_2}, \dots).$$

Since $z_3 \geqslant z_5$ by Lemma 7.2(iv), this vector reorders to

$$(2z_5 + z_1; z_4, z_3^{\times 2b} \parallel z_2, z_5^{\times 3}, w_2^{\times \ell_2}, \dots).$$

Applying b Cremona transforms with $\delta = 2z_5 + z_1 - z_4 - 2z_3$ and regrouping the produced z_5's, we obtain the vector

$$(\underbrace{(b+1)(2z_5+z_1)-b(z_4+2z_3)}_{=:\mu}; \underbrace{b(2z_5+z_1-z_4-2z_3)+z_4}_{=z_6}, z_2, z_5^{\times(2b+3)}, w_2^{\times \ell_2}, \dots),$$

which by Lemma 7.2(vi) reorders to

$$(\mu; z_6 \parallel z_2, z_5^{\times(2b+3)}, w_2^{\times \ell_2}, \dots). \tag{7-5}$$

Notice that this vector does not contain z_1, z_3 or z_4.

Proposition 7.3 *Assume that $a \leqslant \alpha_b$ and $z_1 \geqslant z_4$. If $b = 2$, also assume that $w_2 \leqslant \max\{z_2, z_5\}$. Then the vector (7-5) is reduced.*

Proof We already know that all entries of (7-5) are nonnegative. Using (7-1) we compute

$$\mu - z_6 = z_1 - z_4 + 2z_5 = z_3 + z_5. \tag{7-6}$$

Subcase 1 $(z_5 \geqslant w_2)$ Then

$$\delta = \mu - z_6 - z_5 - (z_2 \text{ or } z_5) = z_3 - (z_2 \text{ or } z_5) \geqslant 0,$$

where in the last step we have used Lemma 7.2(iii)–(iv).

Subcase 2 $(z_2 \geqslant w_2 \geqslant z_5)$ Then

$$\delta = \mu - z_6 - (z_2 + w_2) = z_3 + z_5 - z_3 = z_5 \geqslant 0.$$

Subcase 3 $(w_2 \geqslant z_2,\ z_5)$ This is the case where we assume that $b \geqslant 3$. Recall that $\ell_2 \geqslant 2$. Hence

$$\delta_b(a) = \mu - z_6 - 2w_2 = z_1 - z_4 + 2z_5 - 2w_2$$

is nonnegative by Lemma 7.2(vii). □

In view of Proposition 7.3 we can assume that $b=2$ and that $w_2 \geqslant \max\{z_2, z_5\}$. The vector at hand then is

$$(\mu; z_6, w_2^{\times \ell_2} \parallel z_2, z_5^{\times(2b+3)}, \dots). \tag{7-7}$$

We set $z_7 := z_2 + z_5$ and compute

$$\delta = \mu - z_6 - 2w_2 \overset{(7\text{-}6)}{=} z_3 + z_5 - 2w_2 \overset{(7\text{-}1)}{=} 1 + z_2 - w_1 + z_5 - 2w_2 = z_7 - w_2.$$

If $\delta \geqslant 0$ we are done. So assume that $\delta = z_7 - w_2 < 0$, and set $m := \lfloor \frac{1}{2}\ell_2 \rfloor$ and $\hat{\mu} := \mu + m\delta$, $\hat{z}_6 := z_6 + m\delta$. Applying m Cremona transforms and swapping the position of w_2 and $z_7^{\times 2m}$ if ℓ_2 is odd, we obtain

$$(\hat{\mu}; \hat{z}_6, z_7^{\times 2m}, z_2, z_5^{\times(2b+3)}, w_3^{\times \ell_3}, \dots) \quad \text{if } \ell_2 = 2m, \tag{7-8}$$

$$(\hat{\mu}; \hat{z}_6, w_2, z_7^{\times 2m}, z_2, z_5^{\times(2b+3)}, w_3^{\times \ell_3}, \dots) \quad \text{if } \ell_2 = 2m+1. \tag{7-9}$$

Proposition 7.4 *After reordering, the vector (7-8) is reduced. After reordering, the vector (7-9) is reduced if $z_7 \geqslant w_3$, and transforms to a reduced vector by one Cremona move if $w_3 > z_7$.*

Proof We first show the inequalities

$$\hat{z}_6 \geqslant w_2 \geqslant z_7 \geqslant z_2, z_5. \tag{7-10}$$

Then also $\hat{z}_6$, $z_7 \geqslant 0$. We have $w_2 - z_7 = -\delta > 0$ and $z_7 = z_2 + z_5 \geqslant z_2, z_5$. We are thus left with proving $\hat{z}_6 \geqslant w_2$. For $m \in \mathbb{N}$ we compute

$$f_m(a) := \hat{z}_6 - w_2 = z_6 + m z_7 - (m+1) w_2 = -(m+2)a + \left(\tfrac{17}{2}m + 11\right)\sqrt{a} - (16m+15).$$

Then $f_m'(a) = -(m+2) + \left(\frac{17}{2}m+11\right)/(2\sqrt{a}) > 0$ for all $m \in \mathbb{N}$ and $a \in [2b+3, 2b+4] = [7,8]$, since this holds true for $a = 8$. Recall that $\ell_2 \geqslant 5$. Since $\ell_2 = \lfloor w_1/w_2 \rfloor = \left\lfloor \frac{-7+a}{8-a} \right\rfloor$ and $\ell_2(\alpha_2) = 30$, we can assume that $2 \leqslant m \leqslant 15$. If the multiplicity of w_2 is ℓ_2, then $w_1 \in [\ell_2/(\ell_2+1), (\ell_2+1)/(\ell_2+2)[$. Thus $\hat{z}_6 - w_2$ is given by f_m for $a \in \left[7 + \frac{2m}{2m+1}, 7 + \frac{2m+2}{2m+3}\right[\cap [v_2(1), \alpha_2]$. Since each f_m is increasing on $[7,8]$, it now suffices to note that $f_2(v_2(1)) = f_2\left(\left(\frac{14}{5}\right)^2\right) = \frac{1}{25} > 0$ and to check that $f_m\left(7 + \frac{2m}{2m+1}\right) \geqslant 0$ for $m \in \{3, \dots, 15\}$, which is readily done, for instance by noticing that $m \mapsto f_m\left(7 + \frac{2m}{2m+1}\right)$ is increasing.

Case 1 ($z_7 \geqslant w_3$) The part $(\mu; a_1, a_2, a_3)$ of the ordered vectors is then as in (7-8) and (7-9). Therefore, $\hat{\delta} = \mu - z_6 - 2z_7 = \delta - 2(z_7 - w_2) = -\delta > 0$ if ℓ_2 is even, and

$\widehat{\delta} = \mu - z_6 - w_2 - z_7 = \delta - (z_7 - w_2) = 0$ if ℓ_2 is odd. Hence the vectors (7-8) and (7-9) are reduced.

Case 2 $(w_3 > z_7)$ In this case, the vectors at hand are

(7-11) $$(\widehat{\mu}; \widehat{z}_6, w_3^{\times \ell_3} \,\|\, z_7^{\times 2m}, w_4^{\times \ell_4}, z_2, z_5^{\times(2b+3)}, \dots) \quad \text{if } \ell_2 = 2m,$$

(7-12) $$(\widehat{\mu}; \widehat{z}_6, w_2, w_3^{\times \ell_3} \,\|\, z_7^{\times 2m}, w_4^{\times \ell_4}, z_2, z_5^{\times(2b+3)}, \dots) \quad \text{if } \ell_2 = 2m+1.$$

Assume first that ℓ_2 is even. If $\ell_3 = 1$, then (7-10) shows that

$$\widehat{\delta} = \mu - z_6 - w_3 - (z_7 \text{ or } w_4) = w_2 + z_7 - w_3 - (z_7 \text{ or } w_4) = (w_2 - w_3 \text{ or } z_7) \geqslant 0.$$

If $\ell_3 \geqslant 2$, then $\widehat{\delta} = \mu - z_6 - 2w_3 = w_2 + z_7 - 2w_3 \geqslant z_7 \geqslant 0$.

Assume now that ℓ_2 is odd. Then $\widehat{\delta} = \mu - z_6 - w_2 - w_3 = z_7 - w_3 < 0$. Applying one more Cremona move to the vector (7-12) yields

$$(\widehat{\mu} + \widehat{\delta};\ \widehat{z}_6 + \widehat{\delta},\ w_2 + \widehat{\delta},\ w_3^{\times \ell_3 - 1} \,\|\, z_7^{\times 2m+1},\ w_4^{\times \ell_4},\ z_2,\ z_5^{\times(2b+3)}, \dots).$$

The ordering is right because if $\ell_3 = 1$, then $w_2 + \widehat{\delta} = w_2 + z_7 - w_3 = z_7 + w_4$, and if $\ell_3 \geqslant 2$, then $w_2 + \widehat{\delta} = w_2 + z_7 - w_3 \geqslant w_3$.

If $\ell_3 = 1$, then the defect is now $\widetilde{\delta} = \mu - z_6 - w_2 - \widehat{\delta} - (z_7 \text{ or } w_4) = w_3 - (z_7 \text{ or } w_4) > 0$, and if $\ell_3 \geqslant 2$, then $\widetilde{\delta} = w_3 - w_3 = 0$. □

This completes the proof of Theorem 7.1 for $a \leqslant \alpha_b$.

7.2 The interval $[\alpha_b, 2b+4]$

It turns out that the reduction process for $a \in [\alpha_b, 2b+4]$ is the same as for $a \in [v_b(1), \alpha_b]$ in Case 2. Set $\lambda = \frac{ba+1}{2b(b+1)}$ and define $z_1, \dots, z_6$ as in Section 7.1. Applying the same Cremona moves (ie the same sequence of Cremona transforms and reorderings) as in Case 2, we obtain the vector (7-5), namely

(7-13) $$(\mu; z_6 \,\|\, z_2, z_5^{\times(2b+3)}, w_2^{\times \ell_2}, \dots).$$

It suffices to prove the following statement:

Proposition 7.5 *If $a \geqslant \alpha_b$, then the vector (7-13) is reduced.*

Proof The identity $\lambda = \frac{ba+1}{2b(b+1)}$ is equivalent to $z_5 = 0$. We now show $z_6, z_2 \geqslant w_2$, implying $z_6, z_2 \geqslant 0$. Using (7-2) we find that the inequality $z_6 \geqslant w_2$ is equivalent to the inequality

$$w_1 \geqslant \frac{3b+3}{3b+4},$$

which is satisfied since $\frac{3b+3}{3b+4} \leqslant \alpha_b - (2b+3)$ for all $b \geqslant \frac{2}{3}(-1+\sqrt{7})$. The inequality $z_2 \geqslant w_2$ is equivalent to the inequality

$$w_1 \geqslant \frac{4b^2+3b-1}{4b^2+3b},$$

which is satisfied since $\frac{4b^2+3b-1}{4b^2+3b} \leqslant \alpha_b - (2b+3)$ for all $b \geqslant \frac{5}{4}$.

The ordered vector is thus

$$(\mu; z_6, z_2, w_2^{\times \ell_2}, \dots, 0^{\times(2b+3)}).$$

(The inequality $z_6 \geqslant z_2$ holds true, but there is no need to prove it). Using again $\mu - z_6 = z_1 - z_4 + 2z_5$ and $z_1 + z_5 - z_4 = 1 + z_2 - w_1$ from (7-1) we find, since $z_5 = 0$,

$$\delta = (\mu - z_6) - (z_2 + w_2) = (z_1 - z_4) - (z_2 + 1 - w_1) = 0.$$

Hence the vector (7-13) is reduced. □

8 The interval $[2b+4, u_b(2)]$ for $b \geqslant 3$

Recall that $\gamma_b := u_b(2) = \frac{(2b+2)^2}{2b} = 2b+4+\frac{2}{b}$ and that

$$\beta_b := \frac{(2b^2+4b+1)^2}{2b(b+1)^2} = 2b+4+\frac{1}{2b(b+1)^2} \in \,]2b+4, \gamma_b[.$$

Throughout this section we assume that $b \geqslant 3$.

Theorem 8.1 *For $b \geqslant 3$ we have*

$$c_b(a) = \begin{cases} 1 + \dfrac{2b+1}{2b(b+1)} & \textit{if } a \in [2b+4, \beta_b], \\ \sqrt{\dfrac{a}{2b}} & \textit{if } a \in [\beta_b, \gamma_b]. \end{cases}$$

Proof In view of Theorem 7.1 it suffices to prove that $c_b(a) = \sqrt{\frac{a}{2b}}$ on $[\beta_b, \gamma_b]$. Let $a \in [\beta_b, \gamma_b]$ be a rational number with weight expansion

$$\boldsymbol{w}(a) = (1^{\times(2b+4)}, w_1^{\times \ell_1}, w_2^{\times \ell_2}, \dots, w_n^{\times \ell_n}).$$

8.1 Inequalities

Set $\lambda = \sqrt{\frac{a}{2b}}$. We wish to show that the vector $((b+1)\lambda; b\lambda, \lambda, \boldsymbol{w}(a))$ can be reduced to a reduced vector. Notice that

$$\lambda(\beta_b) = 1 + \frac{2b+1}{2b(b+1)}, \quad \lambda(\gamma_b) = 1 + \frac{1}{b}.$$

Define the numbers

$$\begin{aligned} z_1 &:= \lambda - 1, \\ z_2 &:= (2b+1)\lambda - (2b+3), \\ z_3 &:= 1 + b(z_2 - z_1), \\ z_4 &:= 1 + (b+1)(z_2 - z_1), \\ z_5 &:= z_1 + z_2 - w_1 \end{aligned}$$

and $m = \lfloor \frac{1}{2}\ell_1 \rfloor$, where $\ell_1 = \lfloor \frac{1}{w_1} \rfloor$.

Lemma 8.2 *On the interval $[\beta_b, \gamma_b]$ the following inequalities hold true:*

(i) $1 - z_1 + z_2 \geqslant z_1 \geqslant z_2 \geqslant 0$.

(ii) $1 - z_1 + z_2 \geqslant w_1$.

(iii) $z_3 \geqslant z_2, z_4, w_1$ *and* $z_4 \geqslant 0$.

(iv) $z_3 + b(z_4 - z_2) \geqslant z_2$.

(v) $z_2 + z_4 - w_1 \geqslant w_1$.

(vi) $2z_2 \geqslant w_1$ *and* $z_2 \geqslant w_3$.

(vii) $1 - z_1 + z_2 + m(z_1 + z_2 - 2w_1) \geqslant w_1$.

In particular, $z_i \geqslant 0$ for all i.

Proof (i) The inequality $z_1 \geqslant z_2$ was already shown in the proof of Lemma 7.2(ii).

The inequality $z_2 \geqslant 0$ is equivalent to $(2b+1)\lambda \geqslant 2b+3$. Since λ is increasing, it suffices to verify this in $a = \beta_b$, that is, that

$$(2b+1)\left(1 + \frac{2b+1}{2b(b+1)}\right) \geqslant 2b+3,$$

or, equivalently, $(2b+1)^2 \geqslant 4b(b+1)$, which holds true.

The inequality $1 - z_1 + z_2 \geqslant z_1$ is equivalent to $(2b-1)\lambda \geqslant 2b$. It suffices to verify this in $a = \beta_b$, that is, that

$$(2b-1)\left(1 + \frac{2b+1}{2b(b+1)}\right) \geqslant 2b,$$

or, equivalently, $2b^2 \geqslant 2b+1$, which holds true.

(ii) This is equivalent to $a-3 \leqslant 2b\lambda$. Since the slope of $2b\lambda = \sqrt{2ba}$ is $\sqrt{\frac{b}{2a}} < 1$, it suffices to check this inequality at $a=\gamma_b$, ie that $2b+2 \geqslant a-3$, which holds true.

(iii) The inequality $z_3 \geqslant z_2$ is equivalent to $(2b^2-2b-1)\lambda \geqslant 2b^2-4$. It suffices to verify this at $a=\beta_b$, that is, that

$$(2b^2-2b-1)\left(1+\frac{2b+1}{2b(b+1)}\right) \geqslant 2b^2-4,$$

or, equivalently, $2b \geqslant 1$, which holds true.

The inequality $z_3 \geqslant z_4$ follows from $z_1 \geqslant z_2$.

The inequality $z_3 \geqslant w_1$ is equivalent to $2b^2\lambda \geqslant 2b^2+a-5$ or, using $a=2b\lambda^2$, to

$$f_b(\lambda) := -2b\lambda^2+2b^2\lambda-2b^2+5 \geqslant 0.$$

Since $b \geqslant 3$ we have $f_b'(\lambda)=2b(b-2\lambda)>0$, and $f_b(\lambda(\beta_b)) = \frac{2b^2+2b-1}{2b(b+1)^2} > 0$.

The inequality $z_4 \geqslant 0$ is equivalent to $2b(b+1)\lambda \geqslant 2b^2+4b+1$, which holds true, since this is an equality at $a=\beta_b$.

(iv) This is equivalent to $(2b^2+4b+1)\lambda \geqslant 2(b^2+3b+2)$. At $a=\beta_b$, this inequality is equivalent to

$$(2b^2+4b+1)(2b+1) \geqslant 2b(b+1)(2b+3),$$

which in turn simplifies to $1 \geqslant 0$.

(v) This is equivalent to $(2b^2+4b+1)\lambda \geqslant 2(a+b^2+b-2)$, or, using $a=2b\lambda^2$, to

$$f_b(\lambda) := 4b\lambda^2-(2b^2+4b+1)\lambda+2(b^2+b-2) \leqslant 0 \tag{8-1}$$

on $[\beta_b,\gamma_b]$. Its derivative is $f_b'(\lambda)=8b\lambda-(2b^2+4b+1)$.

Assume first that $b=3$. Then $f_b'(\lambda)=24\lambda-31 \geqslant 0$ since this holds true in $\lambda(\beta_b)=\frac{31}{24}$. Hence (8-1) follows from $f_b(\lambda(\gamma_b)) = f_3\left(\frac{4}{3}\right)=0$.

Assume now that $b \geqslant 4$. Then $f_b'(\lambda(\gamma_b)) = 8(b+1)-(2b^2+4b+1) \leqslant 0$, whence $f_b'(\lambda) \leqslant 0$. Thus (8-1) follows from $f_b(\lambda(\beta_b)) = -\frac{b-1}{2b(b+1)^2} \leqslant 0$.

(vi) This is equivalent to

$$f_b(\lambda) := b\lambda^2-(2b+1)\lambda+(b+1) \leqslant 0 \tag{8-2}$$

on $[\beta_b,\gamma_b]$. Since $f_b'(\lambda)=2b\lambda-(2b+1) \geqslant 2b\lambda(\beta_b)-(2b+1) = \frac{b}{b+1} \geqslant 0$ on $[\beta_b,\gamma_b]$, the inequality (8-2) follows from $f_b(\lambda(\gamma_b))=0$.

Further, $z_2 \geqslant \frac{1}{2}w_1 \geqslant w_3$ since $w_1 = \ell_2 w_2 + w_3 \geqslant w_2 + w_3 \geqslant 2w_3$.

(vii) Recall that $1 = \ell_1 w_1 + w_2$. If $\ell_1 = 2m+1$, then (vii) becomes

$$w_2 - z_1 + z_2 + m(z_1 + z_2) \geqslant 0,$$

which holds true. If $\ell_1 = 2m$, then (vii) becomes $w_2 - z_1 + z_2 + m(z_1 + z_2) \geqslant w_1$. This holds true since it holds true for $m = 1$ by assertion (vi). □

The following lemma will be very useful:

Lemma 8.3 *If $w_2 \geqslant z_2$, then $\ell_2 = 1$.*

Proof Recall that we can assume $\ell_3 \geqslant 1$, that is, $w_3 > 0$. If $\ell_2 \geqslant 2$, then $w_1 = \ell_2 w_2 + w_3 > 2w_2 \geqslant 2z_2 \geqslant w_1$, by Lemma 8.2(vi). □

8.2 Reductions

Applying one Cremona transform to

$$((b+1)\lambda; b\lambda, \lambda, 1^{\times(2b+4)}, w_1^{\times \ell_1}, \dots)$$

with $\delta = -1$ yields

$$((b+1)\lambda - 1; b\lambda - 1, \underbrace{\lambda - 1}_{=z_1}, 0, 1^{\times(2b+3)}, w_1^{\times \ell_1}, \dots),$$

which we reorder to

$$((b+1)\lambda - 1; b\lambda - 1, 1^{\times(2b+3)} \parallel z_1, w_1^{\times \ell_1}, \dots, 0).$$

Applying b Cremona transforms with $\delta = \lambda - 2$ we obtain

$$(\underbrace{(2b+1)\lambda - (2b+1)}_{=z_2+2}; \underbrace{2b\lambda - (2b+1)}_{=1-z_1+z_2}, 1^{\times 3}, z_1^{\times(2b+1)}, w_1^{\times \ell_1}, \dots, 0),$$

which by Lemma 8.2 reorders to

$$(z_2 + 2; 1^{\times 3}, 1 - z_1 + z_2 \parallel z_1^{\times(2b+1)}, w_1^{\times \ell_1}, \dots, 0).$$

Applying one Cremona transform with $\delta = z_2 - 1$ yields

$$(2z_2 + 1; z_2^{\times 3}, 1 - z_1 + z_2, z_1^{\times(2b+1)}, w_1^{\times \ell_1}, \dots, 0),$$

which we reorder to

(8-3) $$(2z_2 + 1; 1 - z_1 + z_2 \parallel z_1^{\times(2b+1)}, z_2^{\times 3}, w_1^{\times \ell_1}, \dots, 0).$$

We now distinguish several cases, according to the order of $z_1 \geqslant z_2$ and w_1.

Case 1 $(z_1 \geqslant z_2, w_1)$ Applying $b-1$ Cremona moves to the vector (8-3) with $\delta = z_2 - z_1$ we get the vector

$$(z_1+z_2+z_3; z_3, z_1^{\times 3}, z_2^{\times(2b+1)}, w_1^{\times \ell_1}, \dots). \tag{8-4}$$

Case 1(a) $(z_1 \geqslant z_2 \geqslant w_1)$ If $z_3 \geqslant z_1$, we apply one more Cremona move with $\delta = z_2 - z_1$ and obtain

$$(2z_2+z_3; \underbrace{z_3+z_2-z_1}_{=z_4}, z_1, z_2^{\times(2b+3)}, w_1^{\times \ell_1}, \dots).$$

The assumption $z_3 \geqslant z_1$ is equivalent to $z_4 \geqslant z_2$. Hence this vector is ordered up to possibly swapping z_4 and z_1, and in either case $\delta = 0$, whence this vector is reduced. We can thus assume for the rest of Case 1(a) that

$$z_1 \geqslant z_3 \quad \text{and} \quad z_2 \geqslant z_4. \tag{8-5}$$

By Lemma 8.2(iii) the vector (8-4) reorders to

$$(z_1+z_2+z_3; z_1^{\times 3}, z_3 \parallel z_2^{\times(2b+1)}, w_1^{\times \ell_1}, \dots). \tag{8-6}$$

One Cremona transform with $\delta = z_4 - z_1$ yields the vector

$$(2z_4+z_1; z_4^{\times 3}, z_3, z_2^{\times(2b+1)}, w_1^{\times \ell_1}, \dots),$$

which by (8-5) reorders to

$$(2z_4+z_1; z_3, z_2^{\times(2b+1)} \parallel z_4^{\times 3}, w_1^{\times \ell_1}, \dots).$$

Under b Cremona transforms with $\delta = z_4 - z_2$ this vector becomes

$$(2z_4+z_1+b(z_4-z_2); z_3+b(z_4-z_2), z_2 \parallel z_4^{\times(2b+3)}, w_1^{\times \ell_1}, \dots),$$

where the ordering follows from Lemma 8.2(iv). Then

$$\delta = z_4 - (z_4 \text{ or } w_1).$$

If $z_4 \geqslant w_1$ we are done. If $w_1 \geqslant z_4$, one more Cremona transform with $\delta = z_4 - w_1$ yields the vector

$$(2z_4+z_1+b(z_4-z_2)+\delta;\ z_3+b(z_4-z_2)+\delta,\ z_2+z_4-w_1,\ w_1^{\times(\ell_1-1)} \parallel z_4^{\times(2b+4)}, \dots),$$

which is ordered by Lemma 8.2(v) and has defect 0.

Case 1(b) $(z_1 \geqslant w_1 \geqslant z_2)$ Assume first that $z_1 \geqslant z_3$. The vector (8-4) then reorders to

(8-7) $$(z_1+z_2+z_3; z_1^{\times 3}, z_3, w_1^{\times \ell_1} \parallel z_2^{\times(2b+1)}, w_2^{\times \ell_2}, \dots).$$

Since $z_4 \leqslant z_3 \leqslant z_1$, we also have $z_4 \leqslant z_1$, and so $\delta = z_4 - z_1 \leqslant 0$. One Cremona transform yields

$$(2z_4+z_1; z_4^{\times 3}, z_3, w_1^{\times \ell_1}, z_2^{\times(2b+1)}, w_2^{\times \ell_2}, \dots).$$

Since $z_1+z_4 = z_2+z_3$ and $z_1 \geqslant z_3$, we have $z_4 \leqslant z_2$, whence this vector reorders to

$$(2z_4+z_1; z_3, w_1^{\times \ell_1} \parallel z_2^{\times(2b+1)}, z_4^{\times 3}, w_2^{\times \ell_2}, \dots).$$

By Lemma 8.2(v) we can estimate

$$\delta = (z_4+z_2-w_1) - (w_1 \text{ or } z_2 \text{ or } z_4 \text{ or } w_2) \geqslant w_1 - w_1 = 0.$$

For the rest of Case 1(b) we can thus assume that

$$z_3 \geqslant z_1 \quad \text{and} \quad z_4 \geqslant z_2.$$

The vector (8-4) then reorders to

$$(z_1+z_2+z_3; z_3, z_1^{\times 3}, w_1^{\times \ell_1} \parallel z_2^{\times(2b+1)}, w_2^{\times \ell_2}, \dots).$$

Applying one Cremona transform with $\delta = -z_1+z_2$ yields

$$(2z_2+z_3; z_4 \leftrightarrow z_1, w_1^{\times \ell_1} \parallel z_2^{\times(2b+3)}, w_2^{\times \ell_2}, \dots).$$

The ordering is right up to possibly swapping $z_4 \leftrightarrow z_1$, since $z_4 \geqslant w_1$ by Lemma 8.2(v). Abbreviate

$$* := z_2+z_4-w_1 \quad \text{and} \quad z_5 := z_1+z_2-w_1.$$

Then $z_5 \geqslant z_2$. Applying one Cremona transform with $\delta = z_2 - w_1$ we obtain

(8-8) $$(*+z_1+z_2; *, z_5, w_1^{\times(\ell_1-1)}, z_2^{\times(2b+4)}, w_2^{\times \ell_2}, \dots).$$

By Lemma 8.2(v) we have $* \geqslant w_1$. If also $z_5 \geqslant w_1$, then

$$\delta = w_1 - (w_1 \text{ or } z_2 \text{ or } w_2) \geqslant 0.$$

So assume that $z_5 \leqslant w_1$. Then the vector (8-8) reorders to

(8-9) $$(z_1+z_2+*; *, w_1^{\times(\ell_1-1)} \parallel z_5, z_2^{\times(2b+4)}, w_2^{\times \ell_2}, \dots).$$

Subcase 1 ($\ell_1 = 2m+1$ with $m \geqslant 0$) Applying m Cremona transforms with $\delta_* := z_5 - w_1$ we get

$$(z_1 + z_2 + * + m\delta_*;\ * + m\delta_*,\ z_5^{\times \ell_1},\ z_2^{\times(2b+4)} \leftrightarrow w_2^{\times \ell_2}, \dots). \tag{8-10}$$

We claim that this vector is reduced after reordering.

Assume that $z_5 \geqslant w_2$. Then the ordering in (8-10) is right by Lemma 8.4(i) below, and

$$\delta = w_1 - (z_5 \text{ or } z_2 \text{ or } w_2) \geqslant 0.$$

Assume that $w_2 \geqslant z_5$. Recall that $z_5 = z_1 + z_2 - w_1 \geqslant z_2 \geqslant w_3$. By Lemma 8.3 we have $\ell_2 = 1$, and so by Lemma 8.4(i) the vector (8-10) reorders to

$$(z_1 + z_2 + * + m\delta_*;\ * + m\delta_* \leftrightarrow w_2,\ z_5^{\times \ell_1},\ z_2^{\times(2b+4)} \parallel \dots).$$

Now $\delta = z_1 + z_2 - w_2 - z_5 = w_1 - w_2 \geqslant 0$.

Subcase 2 ($\ell_1 = 2m$ with $m \geqslant 1$) Applying $m-1$ Cremona transforms to (8-9) with $\delta_* = z_5 - w_1$ we get

$$(z_1 + z_2 + * + (m-1)\delta_*;\ * + (m-1)\delta_*,\ w_1,\ z_5^{\times(\ell_1 - 1)},\ z_2^{\times(2b+4)},\ w_2^{\times \ell_2}, \dots). \tag{8-11}$$

Assume that $z_5 \geqslant w_2$. Then Lemma 8.4(ii) shows that (8-11) reorders to

$$(z_1 + z_2 + * + (m-1)\delta_*;\ * + (m-1)\delta_* \leftrightarrow w_1,\ z_5^{\times(\ell_1 - 1)} \parallel z_2^{\times(2b+4)},\ w_2^{\times \ell_2}, \dots),$$

and $\delta = 0$.

Assume that $w_2 \geqslant z_5$. Then $\ell_2 = 1$ by Lemma 8.3, and we reorder (8-11) to

$$(z_1 + z_2 + * + (m-1)\delta_*;\ * + (m-1)\delta_*,\ w_1,\ w_2,\ z_5^{\times(\ell_1 - 1)},\ z_2^{\times(2b+4)}, \dots).$$

One Cremona transform with $\widehat{\delta} = z_5 - w_2$ yields the vector

$$(z_1 + z_2 + * + (m-1)\delta_* + \widehat{\delta};\ * + (m-1)\delta_* + \widehat{\delta},\ z_1 + z_2 - w_2,\ z_5^{\times \ell_1},\ z_2^{\times(2b+4)}, \dots).$$

Recall that $z_1 + z_2 - w_2 \geqslant z_5 \geqslant z_2 \geqslant w_3$ (by Lemma 8.2(vi)) and note that

$$* + (m-1)\delta_* + \widehat{\delta} \geqslant z_5 + \widehat{\delta} = 2z_1 + 2z_2 - 2w_1 - w_2 \geqslant 0$$

by Lemma 8.4(ii), the assumption $z_1 \geqslant w_1$ and Lemma 8.2(vi).

If $* + (m-1)\delta_* + \widehat{\delta} \geqslant z_5$, then $\delta = w_2 - z_5 \geqslant 0$.

If $* + (m-1)\delta_* + \widehat{\delta} \leqslant z_5$, then $\delta = * + (m-1)\delta_* - z_5 \geqslant 0$.

Lemma 8.4 *Assume that $z_1 \geqslant w_1 \geqslant z_5$.*

(i) *If $\ell_1 = 2m+1$, then $* + m\delta_* \geqslant z_5$.*

(ii) *If $\ell_1 = 2m$, then $* + (m-1)\delta_* \geqslant z_5$.*

The proof is given in Section 8.3.

Case 2 $(w_1 \geqslant z_1 \geqslant z_2)$ Then $z_1 \geqslant z_2 \geqslant z_5$. Recall from Lemma 8.2(vi) that $z_2 \geqslant w_3$. We shall therefore not display $w_3^{\times \ell_3}$ in the vectors below. The vector (8-3) reorders to

$$(2z_2+1;\ 1-z_1+z_2,\ w_1^{\times \ell_1} \,\|\, z_1^{\times(2b+1)},\ z_2^{\times 3},\ w_2^{\times \ell_2}, \dots). \tag{8-12}$$

Case 2(a) ($\ell_1 = 2m+1$ is odd) Applying m Cremona transforms with $\delta_* = z_5 - w_1 \leqslant 0$ we obtain the vector

$$(2z_2+1+m\delta_*;\ 1-z_1+z_2+m\delta_*,\ w_1,\ z_5^{\times(\ell_1-1)},\ z_1^{\times(2b+1)},\ z_2^{\times 3},\ w_2^{\times \ell_2}, \dots).$$

By assumption, $z_1 \geqslant z_2 \geqslant z_5$. By Lemma 8.2(vii) this vector reorders to

$$(2z_2+1+m\delta_*;\ 1-z_1+z_2+m\delta_*,\ w_1 \,\|\, z_1^{\times(2b+1)},\ z_2^{\times 3},\ z_5^{\times(\ell_1-1)},\ w_2^{\times \ell_2}, \dots). \tag{8-13}$$

Subcase 1 $(z_1 \geqslant w_2)$ Applying one Cremona move with $\delta = z_2 - w_1$ we obtain

$$(3z_2+1-w_1+m\delta_*;\ 1-z_1+2z_2-w_1+m\delta_*,\ z_1^{\times 2b},\ z_2^{\times 4},\ z_5^{\times \ell_1},\ w_2^{\times \ell_2}, \dots).$$

Applying b Cremona transforms with $\delta = z_2 - z_1$ and setting

$$*_1 := 1+m\delta_*+(b+1)(z_2-z_1)+z_2-w_1,$$

we obtain

$$(*_1+z_1+z_2; *_1, z_2^{\times(2b+4)}, z_5^{\times \ell_1}, w_2^{\times \ell_2}, \dots). \tag{8-14}$$

We claim that this vector is reduced after reordering. To see this, assume first that $z_2 \geqslant w_2$. If $*_1 \geqslant z_2$ then $\delta = z_1 - z_2 \geqslant 0$, and if $z_2 \geqslant *_1$ then $\delta = *_1 + z_1 - 2z_2 \geqslant 0$ by Lemma 8.5. Assume now that $w_2 \geqslant z_2$. Then $\ell_2 = 1$ by Lemma 8.3. If $*_1 \geqslant z_2$ then $\delta = z_1 - w_2 \geqslant 0$, and if $z_2 \geqslant *_1$ then $\delta = *_1 + z_1 - z_2 - w_2 \geqslant 0$, by Lemma 8.5.

Subcase 2 $(w_2 \geqslant z_1)$ Then $\ell_2 = 1$ by Lemma 8.3, and

$$w_1 \geqslant w_2 \geqslant z_1 \geqslant z_2 \geqslant z_1+z_2-w_2 \geqslant z_1+z_2-w_1 = z_5. \tag{8-15}$$

The vector (8-13) becomes

$$(2z_2+1+m\delta_*;\ 1-z_1+z_2+m\delta_*,\ w_1,\ w_2,\ z_1^{\times(2b+1)},\ z_2^{\times 3},\ z_5^{\times(\ell_1-1)}, \dots).$$

Applying one Cremona move with $\delta = z_1 + z_2 - w_1 - w_2$ we obtain

$$(*+z_1+z_2; *, z_1^{\times(2b+1)}, z_2^{\times 3}, z_1+z_2-w_2, z_5^{\times \ell_1}, \dots),$$

where $* := 1+2z_2+m\delta_*-w_1-w_2$. Applying b Cremona transforms with $\delta = z_2 - z_1$ we obtain the vector

$$(*_2+z_1+z_2; *_2, z_1, z_2^{\times(2b+3)}, z_1+z_2-w_2, z_5^{\times \ell_1}, \dots),$$

where

$$*_2 := 1 + m\delta_* + b(z_2 - z_1) + 2z_2 - w_1 - w_2 = *_1 + z_1 - w_2.$$

This vector is reduced after reordering. Indeed, if $*_2 \geqslant z_2$ then $\delta = 0$, and if $z_2 \geqslant *_2$ then $\delta = *_2 - z_2 = *_1 + z_1 - z_2 - w_2 \geqslant 0$, by Lemma 8.5.

Lemma 8.5 *Assume that $w_1 \geqslant z_1 \geqslant z_2 \geqslant z_5$ and that $\ell_1 = 2m+1$. Then*

$$*_1 \geqslant 2z_2 - z_1,\ w_2 + z_2 - z_1.$$

The proof is given in Section 8.3.

Case 2(b) ($\ell_1 = 2m$ is even) Applying to the vector (8-12) m Cremona transforms with $\delta_* = z_5 - w_1 \leqslant 0$ we obtain the vector

$$(2z_2 + 1 + m\delta_*;\ 1 - z_1 + z_2 + m\delta_*,\ z_5^{\times \ell_1},\ z_1^{\times(2b+1)},\ z_2^{\times 3},\ w_2^{\times \ell_2}, \dots).$$

By Lemma 8.2(vii) this vector reorders to

$$(2z_2 + 1 + m\delta_*;\ 1 - z_1 + z_2 + m\delta_* \parallel z_1^{\times(2b+1)},\ z_2^{\times 3},\ z_5^{\times \ell_1},\ w_2^{\times \ell_2}, \dots). \tag{8-16}$$

Subcase 1 ($z_1 \geqslant w_2$) Applying b Cremona transforms with $\delta = z_2 - z_1$ and setting

$$*_3 := 1 + m\delta_* + (b+1)(z_2 - z_1)$$

we obtain

$$(*_3 + z_1 + z_2; *_3, z_1, z_2^{\times(2b+3)}, z_5^{\times \ell_1}, w_2^{\times \ell_2}, \dots). \tag{8-17}$$

If $z_2 \geqslant w_2$, then Lemma 8.6 shows that the ordering is

$$(*_3 + z_1 + z_2;\ *_3 \leftrightarrow z_1,\ z_2^{\times(2b+3)} \parallel z_5^{\times \ell_1},\ w_2^{\times \ell_2}, \dots),$$

and this vector is reduced since $\delta = 0$. So assume that $z_1 \geqslant w_2 \geqslant z_2$. Then $\ell_2 = 1$ by Lemma 8.3, and we reorder the vector (8-17) to

$$(*_3 + z_1 + z_2; *_3, z_1, w_2, z_2^{\times(2b+3)}, z_5^{\times \ell_1}, \dots).$$

Applying one Cremona transform with $\delta = z_2 - w_2$ we obtain

$$(*_3 + z_2 - w_2 + z_1 + z_2;\ *_3 + z_2 - w_2 \leftrightarrow z_1 + z_2 - w_2,\ z_2^{\times(2b+4)},\ z_5^{\times \ell_1}, \dots).$$

Note that $z_1 + z_2 - w_2 \geqslant z_2$ by assumption. If the ordering is right, then $\delta = w_2 - z_2 \geqslant 0$. Otherwise, $z_2 > *_3 + z_2 - w_2$, and then $\delta = *_3 - z_2 \geqslant 0$ by Lemma 8.6.

Subcase 2 ($w_2 \geqslant z_1$) By Lemma 8.3 we have $\ell_2 = 1$, and the vector (8-16) becomes

$$(2z_2 + 1 + m\delta_*;\ 1 - z_1 + z_2 + m\delta_*,\ w_2,\ z_1^{\times(2b+1)},\ z_2^{\times 3} \parallel z_5^{\times \ell_1}, \dots).$$

Applying one more Cremona move with $\delta = z_2 - w_2$ we obtain

$$(*_4 + z_1 + z_2; *_4, z_1^{\times 2b}, z_2^{\times 4}, z_1 + z_2 - w_2, z_5^{\times \ell_1}, \dots),$$

where $*_4 := 1 + m\delta_* - z_1 + 2z_2 - w_2$. Applying b Cremona transforms with $\delta = z_2 - z_1$ we obtain the vector

$$(*_4 + b(z_2 - z_1) + z_1 + z_2;\ *_4 + b(z_2 - z_1),\ z_2^{\times(2b+4)},\ z_1 + z_2 - w_2,\ z_5^{\times \ell_1}, \dots).$$

We claim that this vector is reduced after reordering. Indeed, if the ordering is right, then $\delta = z_1 - z_2 \geqslant 0$. Otherwise, $z_2 > *_4 + b(z_2 - z_1)$, and then

$$\delta = *_4 + b(z_2 - z_1) + z_1 - 2z_2 = *_3 + z_1 - z_2 - w_2 \geqslant 0$$

in view of Lemma 8.6.

Lemma 8.6 *Assume that $w_1 \geqslant z_1 \geqslant z_2 \geqslant z_5$ and that $\ell_1 = 2m$. Then*

$$*_3 \geqslant z_2,\ w_2 + z_2 - z_1.$$

8.3 Proof of Lemmas 8.4, 8.5 and 8.6

In this section we prove Lemmas 8.4, 8.5 and 8.6. Recall that $\delta_* = z_1 + z_2 - 2w_1$ and

$$* = z_2 + z_4 - w_1 = 1 + (b+1)(z_2 - z_1) + z_2 - w_1.$$

Hence

$$* + m\delta_* = *_1 = 1 + m\delta_* + (b+1)(z_2 - z_1) + z_2 - w_1,$$
$$*_3 = 1 + m\delta_* + (b+1)(z_2 - z_1).$$

Note that $\delta_* \leqslant 0$ in all three lemmas. The proofs are along the following lines. All inequalities are, roughly, of the form

$$1 + m\delta_* + b(z_2 - z_1) \geqslant 0 \tag{8-18}$$

or, using $1 = (2m(+1))w_1 + w_2$,

$$m(z_1 + z_2) + b(z_2 - z_1) \geqslant 0. \tag{8-19}$$

In Lemma 8.4, the assumption $z_1 \geqslant w_1$ translates, roughly, to $m \succcurlyeq \frac{b}{2}$. Further, $w_1 \geqslant z_5$ translates to $3z_2 \geqslant z_1$, which together with (8-19) implies Lemma 8.4 for $m \succcurlyeq \frac{b}{2} + 1$. For the remaining one or two $m \approx \frac{b+1}{2}$ we prove the lemma using (8-18) and $\delta_* \leqslant 0$.

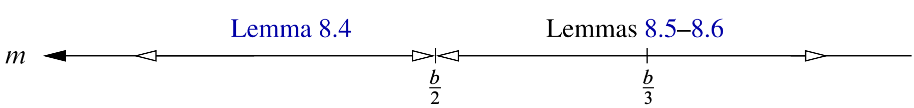

Figure 11

Lemmas 8.5 and 8.6 are proven similarly: the case $m \succcurlyeq \frac{b}{3}$ is settled using $2z_2 \geqslant z_1$ and (8-19), and the case $m \preccurlyeq \frac{b}{3} - 1$ is settled using (8-18) and $\delta_* \leqslant 0$.

Proof of Lemma 8.4 The inequality $z_1 \geqslant w_1$ implies that

$$\ell_1 \geqslant b. \tag{8-20}$$

Indeed, $z_1 \geqslant w_1$ is equivalent to $\sqrt{\frac{a}{2b}} \geqslant a - (2b+3)$, or

$$a \leqslant 2b + 3 + \frac{1 + \sqrt{16b^2 + 24b + 1}}{4b},$$

which in turn translates to

$$\frac{1}{w_1} \geqslant \frac{4b}{1 + \sqrt{16b^2 + 24b + 1} - 4b}.$$

Since the right-hand side is larger than b, inequality (8-20) follows.

We next observe that $w_1 \geqslant z_5$ implies that

$$3z_2 \geqslant z_1. \tag{8-21}$$

Indeed, $(3z_2 - z_1) - (w_1 - z_5) = 2(2z_2 - w_1) \geqslant 0$ by Lemma 8.2(vi). This is the main ingredient for proving the next two claims.

Claim 1 *Lemma 8.4(i) holds for $m \geqslant \frac{b}{2} + 1$.*

Proof This follows from $* + m\delta_* \geqslant z_1$, and, since $1 = (2m+1)w_1 + w_2$, this inequality follows from

$$(b+2)(z_2 - z_1) + m(z_1 + z_2) \geqslant 0.$$

Using (8-21) we estimate

$$(b+2)(z_2 - z_1) + m(z_1 + z_2) = (-b+m-2)z_1 + (b+m+2)z_2 \geqslant (-b+2m-2)\tfrac{2}{3}z_1,$$

which is nonnegative if $m \geqslant \frac{b}{2} + 1$. □

Claim 2 *Lemma 8.4(ii) holds for $m \geqslant \frac{b}{2} + \frac{3}{2}$.*

Proof This follows from $* + (m-1)\delta_* \geqslant w_1$, and since $1 = 2mw_1 + w_2$, this inequality follows from

$$(b+1)(z_2 - z_1) + (m-1)(z_1 + z_2) + z_2 \geqslant 0.$$

Using (8-21) we estimate

$$(b+1)(z_2-z_1)+(m-1)(z_1+z_2)+z_2 = (-b+m-2)z_1+(b+m+1)z_2 \\ \geqslant (-2b+4m-5)\tfrac{1}{3}z_1,$$

which is nonnegative if $m \geqslant \frac{b}{2}+\frac{5}{4}$. □

Proof of Lemma 8.4(i) In view of (8-20) and Claim 1 we can assume that m is in $\left[\frac{b-1}{2}, \frac{b+1}{2}\right]$. We wish to show that for these m (of which there are one or two) we have $* + m\delta_* \geqslant z_5$. Since $\delta_* \leqslant 0$, this follows if $* + \frac{b+1}{2}\delta_* \geqslant z_5$, that is,

$$f_b(\lambda) := -2b(b+1)\lambda^2 + b(3b+4)\lambda - (b^2+b-2) \geqslant 0$$

for $a \in \left[2b+4+\frac{1}{2m+2}, 2b+4+\frac{1}{2m+1}\right]$ and $m \in \left[\frac{b-1}{2}, \frac{b+1}{2}\right]$. Since $f_b'(\lambda) \leqslant -b^2 < 0$ and $m \geqslant \frac{b-1}{2}$, it suffices to show that $f_b(\lambda) \geqslant 0$ at

$$\lambda = \sqrt{\frac{2b+4+1/b}{2b}},$$

that is,

$$1+\frac{2}{b}+\frac{1}{2b^2} \geqslant \left(\frac{3b^2+7b+3+1/b}{3b^2+4b}\right)^2.$$

Subtracting 1 and multiplying by $2b^2(3b^2+4b)^2$ this inequality becomes

$$3b^4 - 8b^3 - 30b^2 - 12b - 2 \geqslant 0,$$

which holds true for $b \geqslant 5$.

To deal with the cases $b \in \{3,4\}$ we return to $* + m\delta_* \geqslant z_5$, ie

$$1+(b+1)(z_2-z_1)+m(z_1+z_2-2w_1)-z_1 \geqslant 0. \tag{8-22}$$

Assume that $b=4$. Then $m=2$, and (8-22) becomes

$$7z_2+1 \geqslant 4z_1+4w_1 \quad \text{on } I := \left[12+\tfrac{1}{6}, 12+\tfrac{1}{5}\right],$$

ie $f(a) := -a + \frac{59}{8}\sqrt{\frac{a}{2}} - 6 \geqslant 0$ on I. This holds true since $f'(a) < 0$ on I and $f\left(12+\frac{1}{5}\right) > 0$. Finally, if $b=3$, then $m \in \{1,2\}$. For $m=2$, (8-22) becomes

$$-2a + \frac{13}{2}\sqrt{\frac{3a}{2}} - 5 \geqslant 0$$

on $\left[10+\frac{1}{6}, 10+\frac{1}{5}\right]$, which holds true; and for $m=1$, (8-22) becomes $-a+\frac{31}{2}\sqrt{\frac{a}{6}}-10 \geqslant 0$ on $\left[10+\frac{1}{4}, 10+\frac{1}{3}\right]$, which holds true too. □

Proof of Lemma 8.4(ii) In this case, (8-20) and Claim 2 show that we can assume that $m \in \left[\frac{b}{2}, \frac{b}{2}+1\right]$. We wish to show that for these m we have $* + (m-1)\delta_* \geqslant z_5$. Since $\delta_* \leqslant 0$, this follows if $* + \frac{b}{2}\delta_* \geqslant z_5$, that is,

$$f_b(\lambda) := -2b^2\lambda^2 + (3b^2+3b-1)\lambda - b(b+2) \geqslant 0$$

for $a \in \left[2b+4+\frac{1}{2m+1}, 2b+4+\frac{1}{2m}\right]$ and $m \in \left[\frac{b}{2}, \frac{b}{2}+1\right]$. Since $f_b'(\lambda) \leqslant -b^2+3b-1 < 0$ and since $m \geqslant \frac{b}{2}$, it suffices to show that $f_b(\lambda) \geqslant 0$ at

$$\lambda = \sqrt{\frac{2b+4+1/b}{2b}},$$

that is,

$$1 + \frac{2}{b} + \frac{1}{2b^2} \geqslant \left(\frac{3b^2+6b+1}{3b^2+3b-1}\right)^2.$$

Subtracting 1 and multiplying by $2b^2(3b^2+3b-1)^2$ this becomes

$$3b^4 - 6b^3 - 21b^2 - 2b + 1 \geqslant 0,$$

which holds true for $b \geqslant 4$.

Assume that $b=3$. Then $m=2$, and $*+(m-1)\delta_* \geqslant z_5$ becomes $-a+\frac{31}{2}\sqrt{\frac{a}{6}}-10 \geqslant 0$ on $\left[10+\frac{1}{5}, 10+\frac{1}{4}\right]$, which holds true. □

Proof of Lemma 8.5 This is equivalent to

$$1 + m\delta_* + b(z_2 - z_1) + z_2 - w_1 \geqslant z_2, w_2. \tag{8-23}$$

Since $1 = (2m+1)w_1 + w_2$, this is equivalent to

$$m(z_1+z_2) + b(z_2-z_1) + z_2 + w_2 \geqslant z_2, w_2,$$

which follows if

$$m(z_1+z_2) + b(z_2-z_1) \geqslant 0. \tag{8-24}$$

Claim 1 *(8-24) holds for* $m \geqslant \frac{b}{3}$.

Indeed, since $2z_2 \geqslant w_1 \geqslant z_1$ by Lemma 8.2 and by assumption,

$$m(z_1+z_2) + b(z_2-z_1) = (m-b)z_1 + (m+b)z_2 \geqslant (3m-b)\frac{z_1}{2}.$$

Claim 2 *(8-23) holds for* $m \leqslant \frac{b}{3} - 1$.

Proof Since $\delta_* \leqslant 0$ and $w_1 \geqslant z_2,\ w_2$, it suffices to show that

$$1 + \left(\frac{b}{3} - 1\right)\delta_* + b(z_2 - z_1) + z_2 - w_1 \geqslant w_1, \tag{8-25}$$

or, equivalently, that

$$f_b(\lambda) := -4b^2\lambda^2 + (8b^2 + 2b - 3)\lambda - 2(2b^2 + b - 3) \geqslant 0. \tag{8-26}$$

Note that $f_b'(\lambda) = -8b^2\lambda + (8b^2 + 2b - 3) < 0$ for $\lambda \geqslant \lambda(\beta_b)$ since $(b+1) f_b'(\lambda(\beta_b)) = -(6b^2 + 5b + 3) < 0$. Hence (8-26) follows from $b f_b(\lambda(\gamma_b)) = b - 3 \geqslant 0$. □

Claim 3 *(8-23) holds for* $m \leqslant \frac{b-1}{3}$ *if* $b \geqslant 7$.

Proof It suffices to show that

$$1 + \frac{b-1}{3}\delta_* + b(z_2 - z_1) + z_2 - w_1 \geqslant w_1,$$

or, equivalently, that

$$g_b(\lambda) := -(4b^2 + 8b)\lambda^2 + (8b^2 + 6b + 1)\lambda - 4b^2 + 2b + 14 \geqslant 0. \tag{8-27}$$

Since $g_b'(\lambda) < 0$ for $\lambda \geqslant 1$, (8-27) follows from $b g_b(\lambda(\gamma_b)) = b - 7$. □

In view of the three claims above we are left with showing the lemma for $b \in \{4, 5\}$ and $m = 1$.

Assume that $b = 5$. It suffices to show that $1 + \delta_* + 5(z_2 - z_1) + z_2 \geqslant 2w_1$ for $a \in [\beta_b, \gamma_b]$, that is,

$$f(\lambda) := -40\lambda^2 + 73\lambda - 30 \geqslant 0 \quad \text{for } a \in [\beta_b, \gamma_b].$$

This holds true since $f'(\lambda) < 0$ for $\lambda \geqslant 1$ and $f(\lambda(\gamma_b)) = 0$.

Assume that $b = 4$. Then $*_1 = 1 + \delta_* + 5(z_2 - z_1) + z_2 - w_1$. The inequality $*_1 \geqslant 2z_2 - z_1$ becomes $1 + 5z_2 \geqslant 3w_1 + 3z_1$, or

$$f(\lambda) := -8\lambda^2 + 14\lambda - 5 \geqslant 0,$$

which holds true since $f'(\lambda) < 0$ for $\lambda \geqslant 1$ and $f(\lambda(\gamma_b)) = 0$. The inequality $*_1 \geqslant w_2 + z_2 - z_1 = 1 - 3w_1 + z_2 - z_1$ becomes $6z_2 \geqslant 3z_1$, which holds true. □

Proof of Lemma 8.6 This is equivalent to

$$\zeta := 1 + m\delta_* + (b+1)(z_2 - z_1) \geqslant z_2,\ w_2 + z_2 - z_1. \tag{8-28}$$

Since $1 = 2mw_1 + w_2$, this is equivalent to $m(z_1+z_2)+b(z_2-z_1) \geqslant z_1 - w_2, 0$, which follows if

$$m(z_1+z_2)+b(z_2-z_1) \geqslant z_1,\ 0. \tag{8-29}$$

Claim 1 *(8-29) holds for* $m \geqslant \frac{b+2}{3}, \frac{b}{3}$.

Claim 2 *(8-28) holds for* $m \leqslant \frac{b}{3}, \frac{b-2}{3}$.

Proof For $m \leqslant \frac{b}{3}$, the inequality $\zeta \geqslant z_2$ in (8-28) follows from the inequality $1 + \frac{b}{3}\delta_* + (b+1)(z_2-z_1) \geqslant z_2$, which is equivalent to (8-25). For $m \leqslant \frac{b-2}{3}$, the inequality $\zeta \geqslant w_2 + z_2 - z_1$ in (8-28) follows from $1 + \frac{b-2}{3}\delta_* + b(z_2-z_1) \geqslant w_1$, or

$$f_b(\lambda) := (-4b^2+2b)\lambda^2 + (8b^2-2b-4)\lambda - 4b^2 + 7 \geqslant 0. \tag{8-30}$$

Note that $f_b'(\lambda) < 0$ for $\lambda \geqslant \lambda(\beta_b)$ since $(b+1)f_b'(\lambda(\beta_b)) = -2(3b^2+b+1) < 0$. Hence (8-30) follows from $bf_b(\lambda(\gamma_b)) = b-2$. □

Claim 3 *(8-28) holds for* $m = \frac{b+1}{3}$ *if* $b \geqslant 5$*, and for* $m = \frac{b-1}{3}$ *if* $b \geqslant 4$.

Proof The first assertion is that $1 + \frac{b+1}{3}\delta_* + (b+1)(z_2-z_1) \geqslant z_2$ for $b \geqslant 5$, or

$$g_b(\lambda) := (-4b^2-4b)\lambda^2 + (8b^2+4b-1)\lambda - 4b^2 + 10 \geqslant 0. \tag{8-31}$$

Since $g_b'(\lambda) < 0$ for $\lambda \geqslant 1$, (8-31) follows from $bg_b(\lambda(\gamma_b)) = b-5$.

The second assertion follows if $1 + \frac{b-1}{3}\delta_* + b(z_2-z_1) \geqslant w_1$ for $b \geqslant 4$, that is,

$$h_b(\lambda) := (-4b^2-2b)\lambda^2 + (8b^2-2)\lambda - 4b^2 + 2b + 11 \geqslant 0. \tag{8-32}$$

Since $h_b'(\lambda) < 0$ for $\lambda \geqslant 1$, (8-32) follows from $bh_b(\lambda(\gamma_b)) = b-4$. □

The three claims above imply Lemma 8.6. □

Remark 8.7 One can use the reduction method also for showing that $c_2(a) = \frac{1}{2}\sqrt{a}$ on $[\beta_2, u_2(2)] = \left[8\frac{1}{36}, 9\right]$, of course. Contrary to all other assertions in Lemma 8.2, assertion (v) does not hold for $b = 2$ if $a \geqslant 8.0831$, however. The reduction scheme for $b = 2$ on $[\beta_b, u_b(2)]$ is therefore quite different from the one for $b \geqslant 3$, in particular in Case 1(b).

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