

Horowitz–Randol pairs of curves in q –differential metrics

ANJA BANKOVIC

The Euclidean cone metrics coming from q –differentials on a closed surface of genus $g \geq 2$ define an equivalence relation on homotopy classes of closed curves, where two classes are equivalent if they have the equal length in every such metric. We prove an analogue of the result of Randol for hyperbolic metrics (building on the work of Horowitz): for every integer $q \geq 1$, the corresponding equivalence relation has arbitrarily large equivalence classes. In addition, we describe how these equivalence relations are related to each other.

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1 Introduction

The works of Horowitz [11] and Randol [21] provide examples of the following: for every $n > 0$ there exist n distinct homotopy classes of curves $\gamma_1, \dots, \gamma_n$ on a compact oriented surface S such that for every hyperbolic metric m , $l_m(\gamma_i) = l_m(\gamma_j)$ for all i, j , where $l_m(\gamma)$ represents the length of the geodesic representative of γ in the metric m . In [14], Leininger studies this and related phenomena for curves on S . He also asks whether other families of metrics exhibit similar behavior.

Question 1.1 [14] Do there exist pairs of distinct homotopy classes of curves γ and γ' that have the same length with respect to every metric in a given family of path metrics?

For an arbitrary family of metrics, one expects the answer to be no. Here we study this question for the metrics coming from q –differentials on a closed oriented surface S , for all $q \geq 1$. More precisely, let $\text{Flat}(S)$ denote the set of non-positively curved Euclidean cone metrics on S and $\text{Flat}(S, q)$ those that come from q –differentials. (See Section 2 for more details.) Let $\mathcal{C}(S)$ denote the set of homotopy classes of homotopically nontrivial closed curves on S . For every $q \in \mathbb{Z}_+$, define an equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_q \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every $m \in \text{Flat}(S, q)$.

In [14] Leininger answers [Question 1.1](#) in the affirmative for metrics in $\text{Flat}(S, 2)$. In fact, writing $\gamma \equiv_h \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every hyperbolic metric m , he proves:

Theorem 1.2 (Leininger [14]) *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_2 \gamma'$.*

Consequently there are arbitrary large \equiv_2 -equivalence classes. In this paper we resolve [Question 1.1](#) for all families $\text{Flat}(S, q)$, $q \geq 1$, proving \equiv_q is nontrivial. In fact there are arbitrarily large \equiv_q -classes of curves.

Theorem 5.1 *For every $q_0, k \in \mathbb{Z}_+$, there are k distinct homotopy classes of curves $\gamma_1, \dots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j$ for all i, j and for all $q \leq q_0$. Thus for every $q \in \mathbb{Z}_+$, the relation \equiv_q is non-trivial.*

However, in the limit this phenomenon disappears. To describe this, define $\gamma \equiv_\infty \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every $m \in \text{Flat}(S, q)$ and for every $q \in \mathbb{Z}_+$.

Theorem 4.1 *The equivalence relation \equiv_∞ is trivial.*

In fact, the argument proves something stronger.

Theorem 4.5 *Let $\{q_i\}_{i=1}^\infty$ be an infinite sequence of distinct positive integers. If $\gamma \equiv_{q_i} \gamma'$ for every $i = 1, 2, \dots$, then $\gamma = \gamma'$ in $\mathcal{C}(S)$.*

Therefore, not all \equiv_q are the same equivalence relations. However we have:

Theorem 3.1 *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$.*

Leininger [14] also shows that the implication in [Theorem 1.2](#) cannot be reversed. As a consequence of our construction we see that a similar statement is true for any $q \in \mathbb{Z}_+$.

Theorem 5.5 *For every $q \in \mathbb{Z}_+$, there exist $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$ but $\gamma \not\equiv_h \gamma'$.*

Although we will work exclusively with closed surfaces, there are versions of the theorems for punctured surfaces. The proofs of these require slightly different arguments. So for the sake of simplifying the exposition, the main body of the paper treats only closed surfaces. We will discuss modifications to the statements and proofs for punctured surfaces in [Section 6](#).

The outline of the paper is as follows. In [Section 2](#) we define Euclidean cone and flat metrics, give standard definitions and state known theorems. [Section 3](#) contains the

proof of [Theorem 3.1](#). In [Section 4](#), we show that for every metric $m \in \text{Flat}(S)$ there is a sequence of metrics in $\bigcup_{q \in \mathbb{Z}_+} \text{Flat}(S, q)$ that converge to m , and using this, we prove that the equivalence relation \equiv_∞ is trivial [[Theorem 4.1](#)]. Then in [Section 5](#) we describe constructions of curves reflecting properties of the metrics in $\text{Flat}(S, q)$ and we prove [Theorem 5.1](#). Finally, in [Section 6](#) we sketch proofs of these theorems for punctured surfaces.

Remark The construction of Horowitz and Randol is studied in Anderson [[2](#)], where a number of variants are also surveyed. These include analogous results for hyperbolic 3-manifolds by Masters [[15](#)] and for metric graphs by Kapovich, Levitt, Schupp and Shpilrain [[13](#)].

Acknowledgments

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2 Background

Let S denote a closed oriented surface of genus at least 2.

2.1 Euclidean cone metrics

A metric m on S is called a *Euclidean cone metric* if it satisfies the following properties:

- (i) m is a geodesic metric (not necessarily uniquely geodesic): the distance between 2 points is the length of a geodesic path between them.
- (ii) There is a finite set $X \subset S$ such that m on $S \setminus X$ is Euclidean, that is, locally isometric to \mathbb{R}^2 with the Euclidean metric.
- (iii) For every $x \in X$, there exists $\epsilon > 0$ so that $B_\epsilon(x)$ is isometric to some cone. More precisely, $B_\epsilon(x)$ is isometric to the metric space obtained by gluing together some (finite) number of sectors of ϵ -balls about 0 in \mathbb{R}^2 . Each x therefore has a well-defined cone angle $c(x) \in \mathbb{R}_+$, which is the sum of the angles of the sectors used in construction. See [Figure 1](#). For any $x \in S \setminus X$ we define $c(x) = 2\pi$.

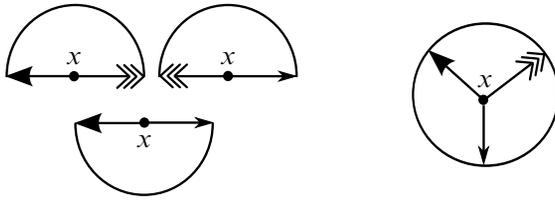


Figure 1: An example of a cone angle $c(x) = 3\pi$

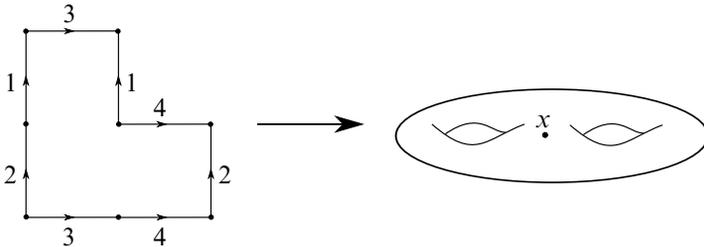


Figure 2: An example of a surface with Euclidean cone metric obtained by gluing sides of a polygon in \mathbb{C} by translations as indicated. The result is a surface of genus 2 with a single cone point x with $c(x) = 6\pi$.

See Figure 2 for an example of a Euclidean cone metric on a genus-2 surface.

The holonomy homomorphism associated to m at any point $y \in S \setminus X$ is a homomorphism

$$\rho_y: \pi_1(S \setminus X, y) \rightarrow O(T_y(S \setminus X)),$$

where $O(T_y(S \setminus X))$ is a group of orthogonal transformations of the tangent space of $S \setminus X$ at y . This is obtained by parallel translating a vector in $T_y(S \setminus X)$ along a loop in $S \setminus X$ based at y . Since our surface is oriented, the image is a subgroup of the group of rotations $SO(T_y(S \setminus X))$.

An orientation preserving isometry $\phi: T_y(S \setminus X) \rightarrow \mathbb{C}$ determines an isomorphism $SO(T_y(S \setminus X)) \rightarrow SO(2)$ independent of the choice of isometry ϕ . We therefore view the holonomy homomorphism as a homomorphism to $SO(2)$. For a Euclidean cone metric m , define $\text{Hol} = \text{Hol}(m) \leq SO(2)$ to be the image of the holonomy homomorphism.

We will construct Euclidean cone surfaces by gluing sides of polygons by maps $\{\rho_i \circ \tau_i\}_{i=1}^k$, which are compositions of translations τ_i and rotations ρ_i . Given $\gamma \in \pi_1(S \setminus X, y)$, $\rho_y(\gamma)$ is given by the composition of the rotations for the side gluings of the sides of the polygons crossed by γ . Therefore $\text{Hol} \leq \langle \rho_1, \rho_2, \dots, \rho_k \rangle$. See Figure 3.

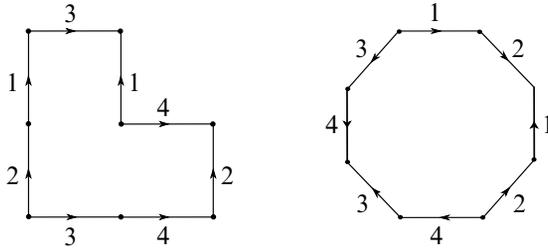


Figure 3: Genus 2 surfaces with $\text{Hol} = \text{Id}$ (on the left) and $\text{Hol} = \langle \rho_{\pi/2} \rangle$ (on the right)

In fact, we can obtain any Euclidean cone surface by gluing sides of a single *generalized Euclidean polygon* immersed in \mathbb{C} . To explain, consider a triangulation of S by Euclidean triangles for which the vertex set is precisely the set of cone points. Such a triangulation exists by Masur and Smillie [16, Theorem 4.4] (this is actually a Δ -complex structure as by Hatcher [9] instead of a proper triangulation, but the distinction is unimportant). We get a dual graph of the triangulation constructed by defining a vertex for each triangle and an edge for each pair of triangles that share an edge. See Figure 4. This graph has a maximal tree. By cutting along the edges of the triangles whose dual edges do not belong in the maximal tree, we get a simply connected surface that is a union of Euclidean triangles, which isometrically immerses in the plane. This is the generalized Euclidean polygon, which we denote P . The surface S can be reconstructed from P by gluing pairs of edges. See Figure 5. If we glue P by translations and rotations $\{\rho_i \circ \tau_i\}_{i=1}^k$, then $\text{Hol} = \langle \rho_1, \rho_2, \dots, \rho_k \rangle$.

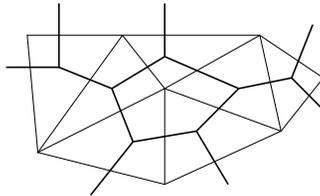


Figure 4: The dual graph of a triangulation

Another important tool for us is the following well-known fact:

Proposition 2.1 (*Gauss–Bonnet formula*) *Let R be a closed surface of genus $g \geq 0$ equipped with a Euclidean cone metric m . Then*

$$2\pi \chi(R) = \sum_{x \in X} (2\pi - c(x)).$$

where X is the set of cone points.

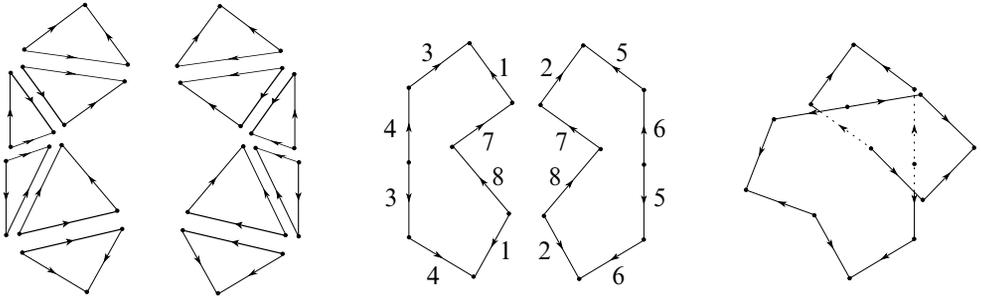


Figure 5: A genus-2 surface obtained by gluing triangles (on the left), two polygons (in the middle) and by gluing an immersed polygon (on the right)

The proof of the Gauss–Bonnet formula for compact surfaces with geometric structure of constant curvature can be found in Neumann’s article [18].

2.2 Flat metrics

A Euclidean cone metric is called NPC (non-positively curved) if it is locally $CAT(0)$. By Gromov’s link condition this is equivalent to $c(x) \geq 2\pi$ for all $x \in S$ (see Bridson and Haefliger [5]). For example the surface in Figure 2 has one cone point and its angle is 6π , and so is NPC.

Define

$$\text{Flat}(S) = \{m \mid m \text{ is a NPC Euclidean cone metric on } S\}.$$

By a metric on S we mean a metric inducing the given topology.

We are interested in the following class of metrics: For any $q \in \mathbb{Z}_+$ define

$$\text{Flat}(S, q) = \{m \in \text{Flat}(S) \mid \text{Hol}(m) \leq \langle \rho_{2\pi/q} \rangle\},$$

where ρ_θ is a rotation by angle θ .

Alternatively, $\text{Flat}(S, q)$ is the space of metrics coming from q -differentials on S . For any $q \in \mathbb{Z}_+$, a q -differential is a complex structure and a family of holomorphic functions φ_j on $z_j(U_j)$ for all coordinate neighborhoods (U_j, z_j) of S , so that on $U_j \cap U_i$ they satisfy

$$\varphi_j(z_j) = \varphi_i(z_i) \left(\frac{dz_i}{dz_j} \right)^q.$$

See Farkas and Kra [7, Chapter II], for example. It is customary to denote the q -differential φ in coordinates z_i by $\varphi = \varphi_i(z_i) dz_i^q$.

To explain how to get a metric in $\text{Flat}(S, q)$, suppose we are given a complex structure and a holomorphic q -differential φ . We can pick a small disk neighborhood U of any point $p_0 \in S$ with $\varphi(p_0) \neq 0$, containing no zeros of φ , and define preferred coordinates ζ for φ by

$$\zeta(p) = \int_{p_0}^p \sqrt[q]{\varphi}.$$

In these coordinates, $\varphi = d\zeta^q$. Let ζ_1, ζ_2 be two preferred coordinates, so that on the overlap of their domains we have $d\zeta_1^q = d\zeta_2^q$. Since this is possible if and only if $\zeta_2 = e^{2\pi ik/q} \zeta_1 + w$ for some $k \in \mathbb{Z}_+$ and $w \in \mathbb{C}$, the preferred coordinates give an atlas of charts on $S \setminus \{\text{zeros}(\varphi)\}$ to \mathbb{C} with transition functions of the form $T(z) = e^{2\pi ik/q} z + w$, where $k \in \mathbb{Z}_+$, $w \in \mathbb{C}$. Pulling back the Euclidean metric, we get the Euclidean metric on $S \setminus \{\text{zeros}(\varphi)\}$. The completion of this metric is obtained by adding back in $\{\text{zeros}(\varphi)\}$, and at a zero of order k we have a cone angle $2\pi + 2\pi k/q$. Therefore, the metric lies in $\text{Flat}(S, q)$. Conversely, take any metric in $\text{Flat}(S, q)$ and choose local coordinates away from the singularities that are local isometries, so that the transition functions are translations and rotations by integer multiples of $2\pi/q$. Since these are holomorphic transformations that preserve dz^q , this determines a complex structure and dz^q determines a holomorphic q -differential, and this extends over the singularities (compare with Minsky [17] and Strebel [22] for the case $q = 2$).

To give some idea of how “big” $\text{Flat}(S, q)$ is we calculate its dimension. By the Riemann–Roch theorem the dimension of the space of holomorphic q -differentials on a Riemann surface of genus g is $2(2q - 1)(g - 1)$. The space \mathcal{Q}_q of all holomorphic q -differentials on S is a vector bundle over Teichmüller space, and since every two q -differentials that differ by some rotation define the same metric on S , we see that the real dimension of $\text{Flat}(S, q)$ is

$$\dim(\text{Flat}(S, q)) = \dim(\mathcal{Q}_q) + \dim(\mathcal{T}(S)) - 1 = 2(2q - 1)(g - 1) + 6g - 7.$$

For every $q \in \mathbb{Z}_+$, $\text{Flat}(S, q) \subset \text{Flat}(S)$ and $\lim_{q \rightarrow \infty} \dim(\text{Flat}(S, q)) = \infty$. Thus

$$\dim(\text{Flat}(S)) = \infty.$$

Observe that $q_1 \mid q_2$ if and only if $\rho_{2\pi/q_1} \in \langle \rho_{2\pi/q_2} \rangle$. In fact $q_1 \mid q_2$ if and only if $\text{Flat}(S, q_1) \subseteq \text{Flat}(S, q_2)$.

2.3 Closed curves

Given a metric m and a homotopy class of curves $\gamma \in \mathcal{C}(S)$, we define the length function $l_m(\gamma) = \inf\{l_m(c) \mid c \in \gamma\}$. Every class $\gamma \in \mathcal{C}(S)$ admits a geodesic repre-

sentative on S due to the Arzela–Ascoli theorem. Therefore $l_m(\gamma)$ is the length of the m -geodesic representative of γ .

Proposition 2.2 *For m in $\text{Flat}(S)$, a closed curve γ is an m -geodesic if and only if γ is a closed Euclidean geodesic or a concatenation of Euclidean segments between cone points such that angles between consecutive segments are at least π on each side of the curve γ . (See Figure 6.)*

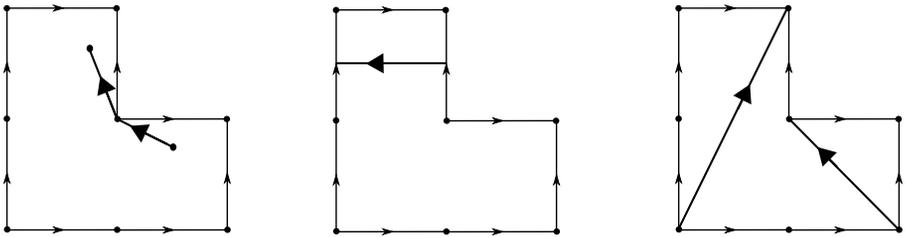


Figure 6: A geodesic through a cone point (on the left), a closed geodesic with no cone points (in the middle) and a closed self-intersecting geodesic containing a cone point (on the right). See Figure 2 for the gluings.

Proof Assume γ is a geodesic. Away from cone points, m is Euclidean, so in the complement of the cone points, geodesics are straight Euclidean segments. If γ enters a cone point x and exits at an angle less than π , we can find a path shorter than γ in the neighborhood of the cone point. Therefore all geodesics have to make an angle greater than equal to π on both sides around a cone point.

Conversely, because of the non-positive curvature, to see that paths satisfying these conditions are geodesics we just need to show that they locally minimize the length. For this we only need to check near the cone points. Let x be a cone point on γ and denote each ray of γ coming out of x inside a small ball B around x containing no other cone point, with γ_- and γ_+ . Construct two different straight line rays starting at x and making angles $\pi/2$ with γ_- on either side, and do the same for γ_+ . See Figure 7. Notice that these rays define two non-intersecting neighborhoods of γ_- and γ_+ since angles on each side of γ at x are greater than or equal to π . Now define a projection inside B in the following way. Every point in the region bounded by the two neighborhoods of γ_- and γ_+ projects orthogonally onto γ , and every other point maps to x . This projection is distance non-increasing since orthogonal projection and projection to a point do not increase distances (and the two projections agree on the overlap of their domains). Therefore γ is a local geodesic. \square

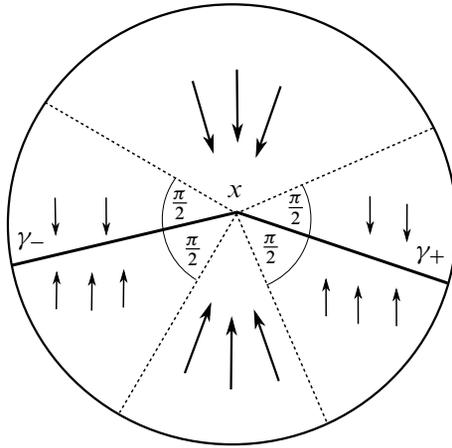


Figure 7: Projection onto γ

By Bridson and Haefliger [5, Theorem II.6.8(4)], if an m -geodesic representative of a curve in $\mathcal{C}(S)$ is not unique in its homotopy class, then the set of geodesic representatives foliates a cylinder in S and each geodesic representative has the same length in m . (See Figure 8.)

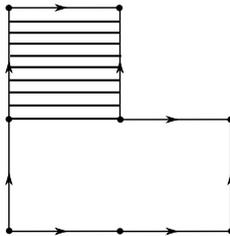


Figure 8: A cylinder on a genus-2 surface foliated by closed geodesics. See Figure 2 for the gluings.

For two curves $\alpha, \beta \in \mathcal{C}(S)$ we can define the geometric intersection number $i(\alpha, \beta)$, which is equal to the minimum number of double points of intersection of any two representatives of α and β . For hyperbolic metrics, geodesic representatives realize the geometric intersection number.

Define $\mathcal{S}(S) \subset \mathcal{C}(S)$ to be the set of homotopy classes of simple closed curves on S .

Combined results of Thurston (see Fathi, Laudenbach and Poénaru [8], and Penner and Harer [19]) and Bonahon [4] yield the following:

Theorem 2.3 Given any separating simple closed curve $\beta \in \mathcal{S}(S)$, there is a sequence $\{(t_n, \beta_n)\}_{n=1}^\infty \subset \mathbb{R}_+ \times \mathcal{S}(S)$ where each β_n is non-separating, so that for all $\alpha \in \mathcal{C}(S)$, $t_n i(\beta_n, \alpha) \rightarrow i(\beta, \alpha)$ as $n \rightarrow \infty$.

We say that γ and $\gamma' \in \mathcal{C}(S)$ are simple intersection equivalent, $\gamma \equiv_{si} \gamma'$, if $i(\gamma, \alpha) = i(\gamma', \alpha)$ for every $\alpha \in \mathcal{S}(S)$. We will also need the next fact.

Theorem 2.4 (Leininger [14]) For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_{si} \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$.

3 Relations between \equiv_1 and \equiv_2

Theorem 3.1 For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$.

Proof Given $\gamma, \gamma' \in \mathcal{C}(S)$, we have $\gamma \equiv_{si} \gamma' \Rightarrow \gamma \equiv_2 \gamma' \Rightarrow \gamma \equiv_1 \gamma'$ by **Theorem 2.4** and the fact $\text{Flat}(S, 1) \subseteq \text{Flat}(S, 2)$. We want to prove $\gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_{si} \gamma'$. We claim that if $\gamma \equiv_1 \gamma'$, then $i(\alpha, \gamma) = i(\alpha, \gamma')$ for every non-separating curve α . Then if β is a separating curve, by **Theorem 2.3** there is a sequence of non-separating curves β_n and positive real numbers t_n such that $t_n i(\gamma, \beta_n) \rightarrow i(\gamma, \beta)$ and $t_n i(\gamma', \beta_n) \rightarrow i(\gamma', \beta)$. Since $t_n i(\beta_n, \gamma) = t_n i(\beta_n, \gamma')$ by the claim, we will have $i(\beta, \gamma) = i(\beta, \gamma')$ for every separating simple closed curve β , hence every $\beta \in \mathcal{S}(S)$, and thus $\gamma \equiv_{si} \gamma'$.

For any $g \geq 2$, we can construct a closed genus- g surface by gluing the arcs in the boundary of a cylinder so that in the resulting surface the core curve is non-separating. Moreover, this construction can be carried out on a Euclidean cylinder so that the resulting Euclidean cone metric has trivial holonomy. See **Figure 9**.

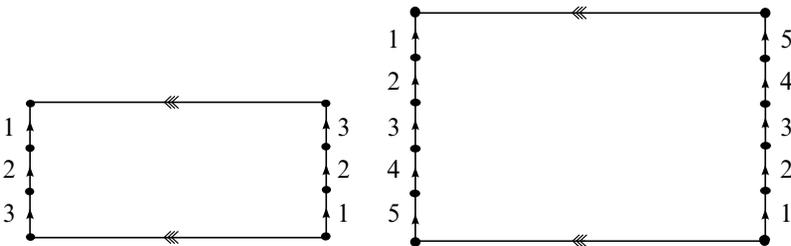


Figure 9: Examples of genus-2 and 3 surfaces. Gluing top sides to the bottom sides results in a cylinder; the rest of the gluing produces the closed surfaces.

Suppose S has genus g and let α be a non-separating curve on S . Let X_g^ϵ be the surface of genus g obtained by gluing a rectangle Y_g^ϵ as in **Figure 9** with horizontal

side length 1 and vertical side length $\epsilon > 0$. Let α_g be the core nonseparating curve of the cylinder obtained by gluing only the horizontal sides of Y_g^ϵ . See Figure 10. We assume that the obvious affine map from Y_g^1 to Y_g^ϵ descends to a homeomorphism $f_\epsilon: X_g^1 \rightarrow X_g^\epsilon$ for all $\epsilon > 0$. Choose a homeomorphism $f: S \rightarrow X_g^1$ so that α is sent to α_g and let $h_\epsilon = f_\epsilon \circ f: S \rightarrow X_g^\epsilon$. Write m_α^ϵ to denote the metric obtained by pulling back the Euclidean metric on X_g^ϵ via h_ϵ . Since the edges of Y_g^ϵ are glued only by translations, we have $m_\alpha^\epsilon \in \text{Flat}(S, 1)$.

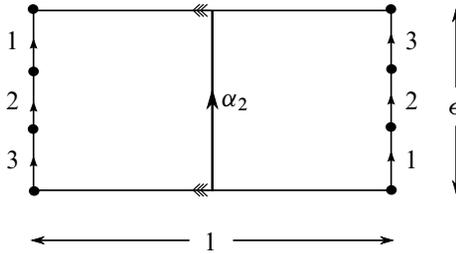


Figure 10: The surface X_2^ϵ obtained by gluings from the rectangle Y_2^ϵ , and curve α_2 .

Let γ be a curve on S . The geodesic representative of γ in m_α^ϵ is sent by h_ϵ to a closed geodesic on X_g^ϵ . On the cylinder obtained by gluing the horizontal sides of Y_g^ϵ , this is a union of straight lines running from one boundary component of the cylinder to another and along the boundary components. See Figure 11.

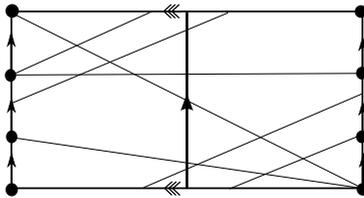


Figure 11: The geodesic representative of γ in m_α^ϵ . See Figure 10 for the gluings.

Thus we have

$$(3-1) \quad l_{m_\alpha^\epsilon}(\gamma) \geq i(\alpha, \gamma) \cdot 1$$

since each geodesic segment that crosses the cylinder contributes 1 to intersection number and at least 1 to the length.

The curve γ is homotopic to a curve $\bar{\gamma}$, which is sent by h_ϵ to a union of straight line segments parallel to the side of length 1 of the rectangle Y_g^ϵ and some segments of the vertical ϵ -length sides as in Figure 12.

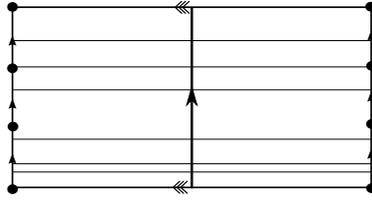


Figure 12: The representative $\bar{\gamma}$ in m_α^ϵ . See Figure 10 for the gluings.

From this we get

$$(3-2) \quad l_{m_\alpha^\epsilon}(\gamma) \leq \text{length}(\bar{\gamma}) \leq i(\alpha, \gamma) \cdot 1 + n_\gamma \epsilon,$$

where n_γ is the number of vertical segments of $\bar{\gamma}$.

Now suppose $\gamma \equiv_1 \gamma'$. Then for every $\epsilon > 0$ we have $l_{m_\alpha^\epsilon}(\gamma) = l_{m_\alpha^\epsilon}(\gamma')$. By (3-1) and (3-2),

$$l_{m_\alpha^\epsilon}(\gamma) - n_\gamma \epsilon \leq i(\alpha, \gamma) \leq l_{m_\alpha^\epsilon}(\gamma),$$

$$l_{m_\alpha^\epsilon}(\gamma') - n_{\gamma'} \epsilon \leq i(\alpha, \gamma') \leq l_{m_\alpha^\epsilon}(\gamma').$$

Letting $\epsilon \rightarrow 0$, we get $i(\alpha, \gamma) = i(\alpha, \gamma')$, proving the claim.

Hence $\gamma \equiv_{si} \gamma'$, completing the proof. □

Corollary 3.2 For every $q \in \mathbb{Z}_+$ and every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_q \gamma' \Rightarrow \gamma \equiv_2 \gamma'$.

Proof Since $\text{Flat}(S, 1) \subset \text{Flat}(S, q)$ for every q , we have

$$\gamma \equiv_q \gamma' \Rightarrow \gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_2 \gamma'. \quad \square$$

4 The equivalence relation \equiv_∞

Theorem 4.1 The equivalence relation \equiv_∞ is trivial.

To prove this, we first prove a weaker statement. Define another equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_{\mathbb{R}} \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every $m \in \text{Flat}(S)$.

Theorem 4.2 The equivalence relation $\equiv_{\mathbb{R}}$ is trivial.

Proof To prove this theorem we will show that for every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \neq \gamma'$, there is an $m \in \text{Flat}(S)$ so that $l_m(\gamma) \neq l_m(\gamma')$.

Let γ and γ' be distinct curves in $\mathcal{C}(S)$. Pick a hyperbolic Riemannian metric g on S . If the geodesic representatives of γ and γ' have the same length in g , let $\epsilon > 0$ be small enough so that γ' does not intersect ϵ -ball around a point $x \in \gamma$. Let $\varphi: S \rightarrow \mathbb{R}$ be a smooth function so that $\varphi \equiv 0$ on $S \setminus B_\epsilon(x)$, $\varphi(x) = 1$ and $\varphi(S) = [0, 1]$. For $\delta > 0$, consider the Riemannian metric $g_\delta = g(1 - \delta\varphi)$. The metric g_δ is negatively curved for δ sufficiently small and the g_δ -length of γ is smaller than the g -length, while the length of γ' is not changed. Thus for any two curves γ and $\gamma' \in \mathcal{C}(S)$ there is a negatively curved Riemannian metric m' such that $l_{m'}(\gamma) \neq l_{m'}(\gamma')$. In fact there are negatively curved metrics for which there are no closed curves of the same length (see Randol [21], Abraham [1], Anosov [3] for more general results).

Being a locally CAT(k) space with $k < 0$ implies also being locally CAT(0). Thus m' is a locally CAT(0) metric. Pick a geodesic triangulation of the surface S with metric m' so that each triangle belongs to a CAT(0) neighborhood on S and so that the m' -geodesic representatives of γ and γ' are unions of edges of triangles. Now build a Euclidean cone metric m on S by replacing the triangles, whose edges are m' -geodesics, with Euclidean triangles with the same length sides. By the CAT(0) property the angles of the Euclidean triangles are larger than the angles in the triangles they replaced. The vertices of the triangles are the only possible cone points. There are finitely many of them, and their angles are therefore at least 2π . Hence $m \in \text{Flat}(S)$.

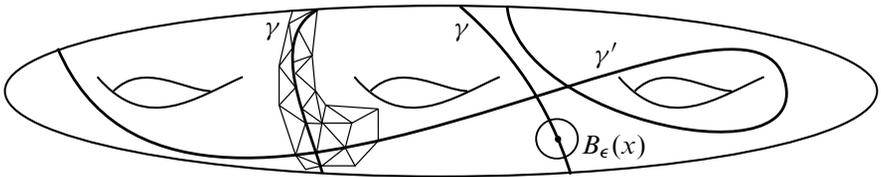


Figure 13: Part of a triangulation of a genus-3 surface with curves γ and γ' as unions of edges of the triangles.

The curves γ and γ' in the new metric become concatenations of Euclidean segments (sides of triangles). Since the sum of angles of a cone point on one side of the curve in the metric m is greater than or equal to the sum of the angles in the metric m' , we get that the angle around every cone point on each side of γ and γ' is at least π . Therefore γ and γ' are also geodesics in the metric m .

We now have

$$l_m(\gamma) = l_{m'}(\gamma) \neq l_{m'}(\gamma') = l_m(\gamma'),$$

and we proved our theorem. □

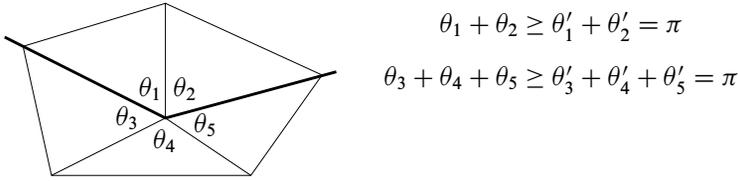


Figure 14: Euclidean triangles and a geodesic segment containing edges of the triangles.

Theorem 4.3

$$\overline{\bigcup_{q \in \mathbb{Z}_+} \text{Flat}(S, q)} = \text{Flat}(S).$$

More precisely, for every $m \in \text{Flat}(S)$, there exists a sequence of metrics $m_n \in \bigcup_{q \in \mathbb{Z}_+} \text{Flat}(S, q)$, so that $\text{id}: (S, m_n) \rightarrow (S, m)$ is K_n -bilipschitz and $K_n \rightarrow 1$ as $n \rightarrow \infty$.

Proof Let $m \in \text{Flat}(S)$. Take a triangulation of S by Euclidean triangles with all vertices being cone points and all cone points being vertices. Using this triangulation, view S as being obtained from a generalized Euclidean polygon P isometrically immersed in \mathbb{C} by gluing the edges in pairs by isometries as described in Section 2. We will construct an isometrically immersed polygon P_ϵ so that the sides of P_ϵ meet the real axis at angles that are rational multiples of π , and $P_\epsilon \rightarrow P$ as $\epsilon \rightarrow 0$. Since two sides of P_ϵ that are glued together make angles with the real axis that are rational multiples of π , they are glued by translating and rotating by an angle in $\mathbb{Q}\pi$. Therefore the holonomy of the flat metric m_ϵ on S , which we get by gluing the sides of the immersed polygon P_ϵ , is generated by rotations by angles in $\mathbb{Q}\pi$. Thus m_ϵ belongs in $\text{Flat}(S, q_\epsilon)$ for some $q_\epsilon \in \mathbb{Z}_+$.

To do this construction, we first orient the edges s_i of the polygon P using the boundary orientation coming from the immersion into \mathbb{C} . Immersion in \mathbb{C} makes each oriented edge s_i into an oriented line segment, which we may view as a vector, or equivalently a complex number, which we denote z_i . Let θ_i be the argument of z_i . We have $z_1 + z_2 + \dots + z_{2n} = 0$. Assume, by rotating if necessary, that $\theta_1 = 0$ and, by relabeling if necessary, that z_{2i-1} and z_{2i} correspond to the edges that are glued together for all $i = 1, \dots, n$. We have $|z_1| = |z_2|, |z_3| = |z_4|, \dots, |z_{2n-1}| = |z_{2n}|$. See Figure 15.

If $z_{2i-1} = -z_{2i}$, for all $i = 1, \dots, n$, then $\text{Hol}(S) = \{\text{Id}\}$ and therefore $m \in \text{Flat}(S, 1)$ and we set $m_n = m$ for all n .

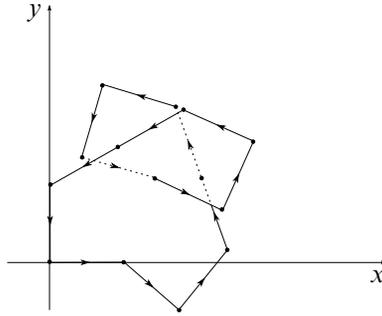


Figure 15: An immersed polygon in \mathbb{R}^2 with one edge on the x -axis

Therefore we assume that there is some i so that $z_{2i-1} \neq -z_{2i}$. After changing indices if necessary we may assume $z_{2n-1} \neq -z_{2n}$. Fix $0 < \epsilon < |z_{2n-1} + z_{2n}|/2$. Denote $\tau = -z_1 - \dots - z_{2n-2}$. Since $\tau = z_{2n-1} + z_{2n}$ it follows that $|\tau| > 2\epsilon$. Let $\tilde{z}_1 = z_1$. For $2 \leq j \leq 2n-4$ choose \tilde{z}_j so that $|z_j| = |\tilde{z}_j|$, $\tilde{\theta}_j \in \mathbb{Q}\pi$ and $|z_j - \tilde{z}_j| < \epsilon/(2n)$. Now choose $\tilde{z}_{2n-3}, \tilde{z}_{2n-2}$ so that $|\tilde{z}_{2n-3}| = |\tilde{z}_{2n-2}|$, $\tilde{\theta}_{2n-3}, \tilde{\theta}_{2n-2} \in \mathbb{Q}\pi$, $|z_i - \tilde{z}_i| < \epsilon/(2n)$ for $i = 2n-3, 2n-2$, and also so that the argument of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. We arrange this in the following way: First construct z'_i , $i = 2n-3, 2n-2$, so that their arguments are in $\mathbb{Q}\pi$ and $|z'_i| = |z_i|$, $|z'_i - z_i| < \epsilon/(4n)$ for $i = 2n-3, 2n-2$. Then construct \tilde{z}_i , $i = 2n-3, 2n-2$, so that $\text{Arg}(z'_i) = \text{Arg}(\tilde{z}_i)$, $|z'_i - \tilde{z}_i| < \epsilon/(4n)$ and the argument of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. See Figure 16.

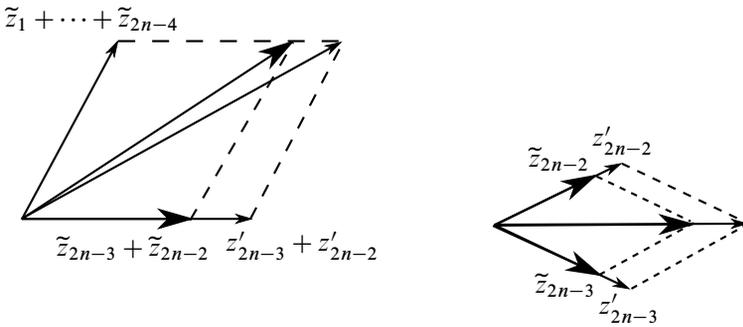


Figure 16: Changing the angle of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$, keeping $|\tilde{z}_{2n-3}| = |\tilde{z}_{2n-2}|$

Denote $\tilde{\tau} = -\tilde{z}_1 - \dots - \tilde{z}_{2n-2}$. By the triangle inequality,

$$|\tau - \tilde{\tau}| \leq \sum_{j=1}^{2n-2} |z_j - \tilde{z}_j| < \epsilon.$$

Observe that $\tilde{\tau} \neq 0$, because if $\tilde{\tau} = 0$ then $2\epsilon < |\tau| < \epsilon$, which is a contradiction.

Since $\tau \neq 0$, the point z_{2n-1} is on the perpendicular bisector l of the vector τ . Let \tilde{l} be the perpendicular bisector of the vector $\tilde{\tau}$. Let $\delta = 2 \text{dist}(z_{2n-1}, \tilde{l})$. Note that as $\epsilon \rightarrow 0, \delta \rightarrow 0$. Now choose $\tilde{z}_{2n-1} \in \tilde{l} \cap B(z_{2n-1}, \delta)$ so that the argument of \tilde{z}_{2n-1} is in $\mathbb{Q}\pi$. See Figure 17. Set $\tilde{z}_{2n} = \tilde{\tau} - \tilde{z}_{2n-1}$. Since \tilde{z}_{2n-1} is on the perpendicular bisector of $\tilde{\tau}$, $|\tilde{z}_{2n-1}| = |\tilde{z}_{2n}|$, and since $\text{Arg}(\tilde{\tau}) = \frac{1}{2}(\theta_{2n-1} + \tilde{\theta}_{2n})$, it follows that $\tilde{\theta}_{2n} \in \mathbb{Q}\pi$.

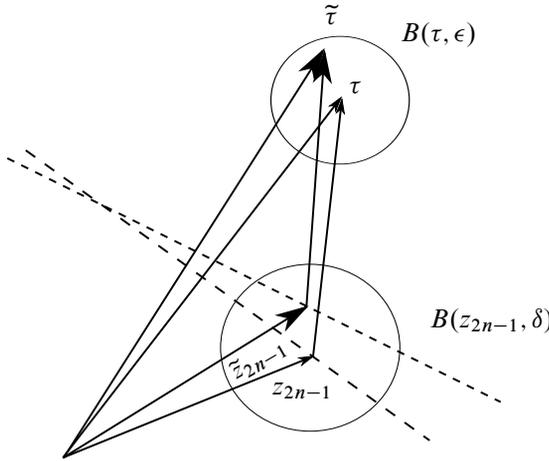


Figure 17: Constructing the vector \tilde{z}_{2n-1}

We have $\tilde{z}_1 + \dots + \tilde{z}_{2n} = 0$ and $\tilde{\theta}_i \in \mathbb{Q}\pi$ for every i . Therefore for $\epsilon > 0$ sufficiently small, $\{\tilde{z}_i\}_{i=1}^{2n}$ determines an isometrically immersed polygon P_ϵ with all sides meeting the real axis at angles that are rational multiples of π .

For sufficiently small ϵ there is a bilipschitz homeomorphism $f_\epsilon: P \rightarrow P_\epsilon$, which is linear on edges. For example, when $\epsilon > 0$ is sufficiently small, we can take a triangulation of P_ϵ with the same combinatorics as the one used to construct P . Map each triangle of P linearly to the corresponding triangle of P_ϵ . That way we get a map f_ϵ that is linear, and thus bilipschitz, on each triangle. It follows that f_ϵ is bilipschitz on P and as $\epsilon \rightarrow 0$, the bilipschitz constant $K(f_\epsilon) \rightarrow 1$. The surface S is obtained by gluing the polygon P , so $S = P/\sim$ where $t \sim \varphi(t)$ and $\varphi: \cup s_i \rightarrow \cup s_i$ is the gluing map. Define $\varphi_\epsilon = f_\epsilon \varphi f_\epsilon^{-1}: \cup f_\epsilon(s_i) \rightarrow \cup f_\epsilon(s_i)$. The map f_ϵ descends to $\bar{f}_\epsilon: P/\sim \rightarrow P_\epsilon/\sim_\epsilon$ where $t \sim_\epsilon \varphi_\epsilon(t)$. Define a Euclidean cone metric m_ϵ on S by pulling back the metric on P_ϵ/\sim_ϵ by \bar{f}_ϵ . The holonomy of m_ϵ is generated by rotations through angles in $\mathbb{Q}\pi$. Therefore there is $q_\epsilon \in \mathbb{Z}_+$ so that $m_\epsilon \in \text{Flat}(S, q_\epsilon)$ and $m_\epsilon \rightarrow m$ as $\epsilon \rightarrow 0$. □

Theorem 4.1 now follows easily from the previous theorem.

Proof of Theorem 4.1 Suppose that we are given two distinct curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_\infty \gamma'$. By Theorem 4.3, for every metric m in $\text{Flat}(S)$ there is a sequence of metrics $\{m_n \in \text{Flat}(S, q_n)\}_{n=1}^\infty$ such that $\text{id}: (S, m_n) \rightarrow (S, m)$ is K_n -bilipschitz and $K_n \rightarrow 1$ as $n \rightarrow \infty$. Thus $\lim_{n \rightarrow \infty} l_{m_n}(\gamma) = l_m(\gamma)$ and $\lim_{n \rightarrow \infty} l_{m_n}(\gamma') = l_m(\gamma')$. Since $l_{m_n}(\gamma) = l_{m_n}(\gamma')$ by assumption then $l_m(\gamma) = l_m(\gamma')$. That implies $\gamma \equiv_{\mathbb{R}} \gamma'$, which is a contradiction by Theorem 4.2. \square

Theorem 4.4 For every infinite sequence of distinct positive integers $\{q_i\}_{i=1}^\infty$,

$$\overline{\bigcup_{i=1}^\infty \text{Flat}(S, q_i)} = \text{Flat}(S).$$

Proof The proof of this theorem follows the proof of Theorem 4.3 if we approximate angles θ_j with angles $\tilde{\theta}_j \in \mathbb{Z} \frac{2\pi}{q_l}$ for appropriately chosen $q_l \gg 0$. \square

Theorem 4.5 Let $\{q_i\}_{i=1}^\infty$ be an infinite sequence of distinct positive integers. If $\gamma \equiv_{q_i} \gamma'$ for every $i = 1, 2, \dots$, then $\gamma = \gamma'$ in $\mathcal{C}(S)$.

Proof This follows from Theorem 4.4 in the same arguments that prove Theorem 4.1 from Theorem 4.3. \square

From Theorem 1.2 we know that h -equivalence implies 2-equivalence (and thus implies 1-equivalence). However from Theorem 4.5 we see that h -equivalence implies q -equivalence for only finitely many $q \in \mathbb{Z}_+$. More precisely, we have the following corollary.

Corollary 4.6 Let $\gamma, \gamma' \in \mathcal{C}(S)$ be distinct curves with $\gamma \equiv_h \gamma'$. Then for all but finitely many $q, \gamma \not\equiv_q \gamma'$.

Proof Assume that there exist infinitely many $q_i \in \mathbb{Z}_+$ so that $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_{q_i} \gamma'$. Then, there exist two distinct curves γ and γ' so that $\gamma \equiv_{q_i} \gamma'$ for some infinite sequence $\{q_i\}_{i=1}^\infty$ of positive integers, which is a contradiction to Theorem 4.5. \square

5 Curves in q -differential metrics and relations \equiv_q

Theorem 5.1 For every $q_0, k \in \mathbb{Z}_+$ there are k distinct homotopy classes of curves $\gamma_1, \dots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j$ for all i, j and for every $q \leq q_0$. Thus for every $q \in \mathbb{Z}_+$, the relation \equiv_q is non-trivial.

The idea of the proof is the following. For every q , we construct a 2-complex Γ , where $\pi_1(\Gamma)$ is a rank-2 free group, and a homotopy class of maps $\Gamma \rightarrow S$. Then for every metric $m \in \text{Flat}(S, q)$, $q \leq q_0$, we will define a metric on the 2-complex Γ and a map $\Gamma \rightarrow S$ in the given homotopy class, so that with respect to the metric on Γ and m on S , the map is locally convex and locally isometric. In particular, the length of a homotopy class of curves in Γ is equal to the length of the image homotopy class in S . The metrics that occur on Γ are very restrictive, and it is easy to construct a set of homotopy classes of curves w_0, w_1, \dots, w_{k-1} in Γ with equal lengths in any metric assigned to Γ from the construction. The image homotopy classes in S then also have equal length for any metric $m \in \text{Flat}(S, q)$, and by a homological argument, we prove that the homotopy classes are all distinct.

Lemma 5.2 *For any $q_0 \in \mathbb{Z}_+$, there is a curve $\gamma \in \mathcal{C}(S)$ such that for every $m \in \text{Flat}(S, q)$, $q \leq q_0$, the geodesic representative γ_m of γ contains a cone point and $[\gamma] \neq 0$ in $H_1(S, \mathbb{Z})$.*

Proof Fix a hyperbolic metric on S and identify the universal cover as the hyperbolic plane $\mathbb{H}^2 \rightarrow S$. We will use the Poincaré disk model for \mathbb{H}^2 throughout the proof. Fix $q_0 \in \mathbb{Z}_+$, and take $q_0 + 1$ bi-infinite geodesics $\alpha_1, \dots, \alpha_{q_0+1}$ in \mathbb{H}^2 meeting at 0. See Figure 18.

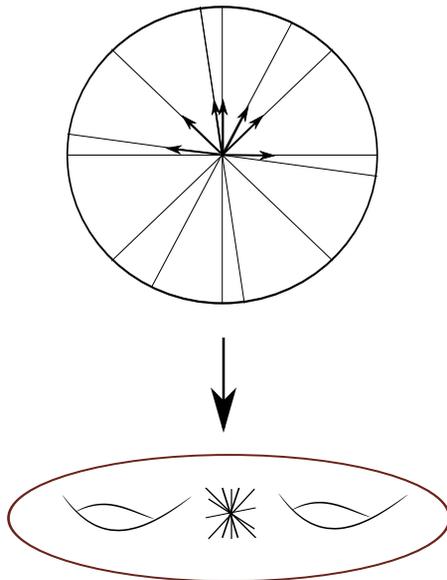


Figure 18: Projecting $\alpha_1, \dots, \alpha_{q_0+1}$ into the surface S

By Hopf [10], by ergodicity of the geodesic flow, there is a dense geodesic on S . Given $\epsilon > 0$, we can therefore construct an ϵ -dense closed geodesic.

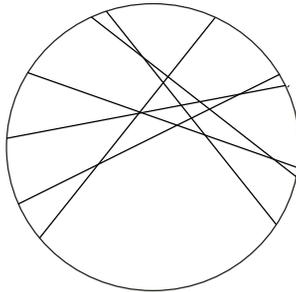


Figure 19: Lifts of γ_m to \mathbb{H}^2

From this it follows that we can construct a closed geodesic γ on S with lifts $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$ having tangent vectors within distance ϵ from the tangent vectors of $\alpha_1, \dots, \alpha_{q_0+1}$ at 0. For ϵ sufficiently small the endpoints of $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$ pairwise link in the same pattern as $\alpha_1, \dots, \alpha_{q_0+1}$. We say that two bi-infinite geodesics in \mathbb{H}^2 link each other if the endpoints of one geodesic separate the circle at infinity S^1_∞ of \mathbb{H}^2 in two connected components so that the two endpoints of the second geodesic belong to different components. In particular, every pair $\tilde{\alpha}_i, \tilde{\alpha}_j$ intersects. For every $m \in \text{Flat}(S, q)$ and m -geodesic representative γ_m of γ in m there are lifts $(\tilde{\gamma}_1)_m, \dots, (\tilde{\gamma}_{q_0+1})_m$ that are quasi-geodesics in \mathbb{H}^2 , with the same endpoints as $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$, and hence with endpoints that all pairwise link. See Figure 19.

If $[\gamma] = 0$ in $H_1(S, \mathbb{Z})$, we replace γ with a different curve as follows. Let δ be any curve with $[\delta] \neq 0$ in $H_1(S, \mathbb{Z})$ that intersects γ . Construct a new curve that runs n times around γ , then at the intersection point switches and runs once around δ . If n is large enough, this constructs a curve that maintains the property of having $q_0 + 1$ lifts with endpoints that pairwise link. This new curve is homologous to δ , and so we replace γ with this curve.

Now assume that there is $m \in \text{Flat}(S, q)$, $q \leq q_0$ so that γ_m does not go through a cone point. Then there is an isometrically immersed Euclidean cylinder $S^1 \times [0, a] \rightarrow S$, for some $a > 0$, such that the image of $S^1 \times \{\frac{a}{2}\}$ is the geodesic γ_m and the image of $S^1 \times [0, a]$ does not contain any cone points. It follows that γ_m has only finitely many transverse self-intersecting points. At any point P of self-intersection of γ_m , one can form the loops based at P by following one of the arcs of γ_m until returning to P . The holonomy around this loop is just the rotation by the angle of self-intersection of γ_m at P . Thus, any tangent vector is rotated by the angle of self-intersection. Because

the holonomy group of m is in $\langle \rho_{2\pi/q} \rangle$, every tangent vector gets rotated by some integer multiple of $2\pi/q$. Therefore, the angles of intersections are integer multiples of $2\pi/q$.

Lifts of γ_m have neighborhoods that are lifted cylinders. These are strips isometric to $\mathbb{R} \times [0, a]$ in the universal cover. We consider the $q_0 + 1$ lifts $(\tilde{\gamma}_1)_m, \dots, (\tilde{\gamma}_{q_0+1})_m$ as above and fix one of the lifts $(\tilde{\gamma}_1)_m$ and a strip about it. Let $P \in (\tilde{\gamma}_1)_m \cap (\tilde{\gamma}_2)_m$ be the point of intersection of $(\tilde{\gamma}_2)_m$ with $(\tilde{\gamma}_1)_m$. For every $j \geq 3$, parallel transport the tangent vector to $\tilde{\gamma}_j$ at the point of intersection $(\tilde{\gamma}_1)_m \cap (\tilde{\gamma}_j)_m$ inside the strip to P . In the complement of the cone points in the universal cover, the holonomy is also in $\langle \rho_{2\pi/q} \rangle$. Thus, parallel transport of tangent vectors along $(\tilde{\gamma}_k)_m, k = 2, \dots, q_0 + 1$, from the point of intersection with another lift of γ_m to P rotates them by some integer multiple of $2\pi/q$. The point P is not a cone point and there are no cone points inside the strip. If $(\tilde{\gamma}_1)_m, (\tilde{\gamma}_i)_m, (\tilde{\gamma}_j)_m$ do not intersect in a common point then they form a triangle in \tilde{S} . Since \tilde{S} is a CAT(0) space the angle sum of a triangle is less than π , and hence $(\tilde{\gamma}_i)_m$ and $(\tilde{\gamma}_j)_m$ cannot make the same angle with $(\tilde{\gamma}_1)_m$. Hence no two tangent vectors get transported to the same vector. See Figure 20.

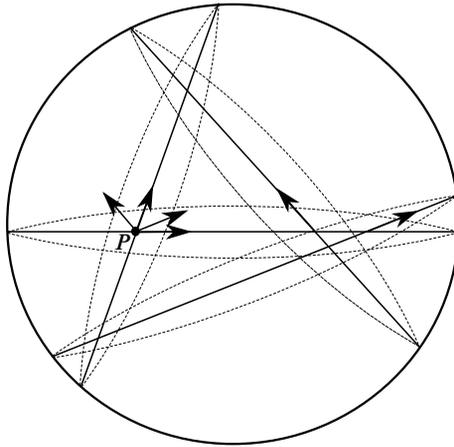


Figure 20: Strips in the universal cover

By the holonomy condition, all angles of intersections are integer multiples of $2\pi/q$, and hence greater than or equal to $2\pi/q_0$. On the other hand we have $q_0 + 1$ vectors based at P ; no two of them equal, therefore there is a pair of vectors with the angle between them less than $2\pi/q_0$. It follows that for some j , $(\tilde{\gamma}_1)_m$ and $(\tilde{\gamma}_j)_m$ make an angle that is not an integer multiple of $2\pi/q$. This is a contradiction and thus proves that γ_m has to contain at least one cone point for every $m \in \text{Flat}(S, q), q \leq q_0$. \square

Lemma 5.3 *Let γ be the curve constructed in Lemma 5.2, and let m be a metric in $\text{Flat}(S, q)$. Given two lifts $\tilde{\gamma}, \tilde{\gamma}_0$ with linking endpoints and stabilizers generated by h and g , respectively, let $w \in \tilde{\gamma} \cap \tilde{\gamma}_0$. Then the m -geodesic from $h^{(q+2)}(w) \in \tilde{\gamma}$ to any point along $\tilde{\gamma}_0$ must run along a positive length segment of $\tilde{\gamma}$. Moreover, this geodesic meets $\tilde{\gamma}_0$ on $[g^{-(q+2)}(w), g^{(q+2)}(w)] \subset \tilde{\gamma}_0$.*

Proof It could happen that $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) > 0$ as in Figure 21. We first claim that $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) < l_m(\gamma)$. To show this, let v, u be cone points at each end of $\tilde{\gamma}_0 \cap \tilde{\gamma}$. If $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) \geq l_m(\gamma)$ then since the translation length of g and h is equal to $l_m(\gamma)$, we have $g(v) = h(v)$ or $g^{-1}(v) = h(v)$, which is a contradiction as $\pi_1(S)$ acts freely on \tilde{S} .

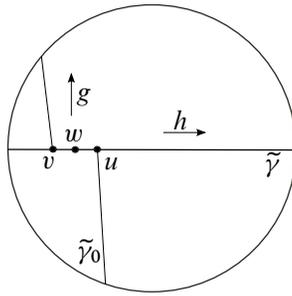


Figure 21: Two lifts of γ to the universal cover

Now let v_1, \dots, v_n be all different cone points along $[h^{(q+1)}w, h^{(q+2)}(w)] \subset \tilde{\gamma}$ and let $\theta_i, i = 1, \dots, n$, be the angles $\tilde{\gamma}$ makes at v_i all on one side of $\tilde{\gamma}$, and let θ'_i be all the angles at v_i on the other side. Observe that there is some i so that $\theta_i > \pi$. Otherwise there is a small neighborhood of one side of $\tilde{\gamma}$ with no cone points and we can find a geodesic homotopic to γ with no cone points by doing a straight line homotopy as shown in Figure 22. Similarly, there is j so that $\theta'_j > \pi$.

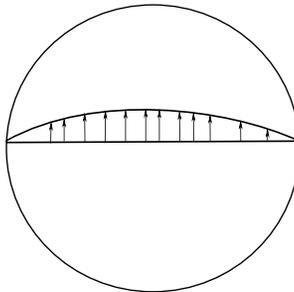


Figure 22: Homotopy

Let $v = v_i$ with $\theta_i > \pi$ and $v' = v_j$ with $\theta'_j > \pi$. Let σ be the continuation of the geodesic segment $[h^{(q+2)}(w), v] \subset \tilde{\gamma}$ to a geodesic ray starting at $h^{(q+2)}(w)$ and at every cone point past v having angle π on the right. See Figure 23. We define σ' similarly as the continuation of $[h^{(q+2)}(w), v'] \subset \tilde{\gamma}$ with angle π on the left.

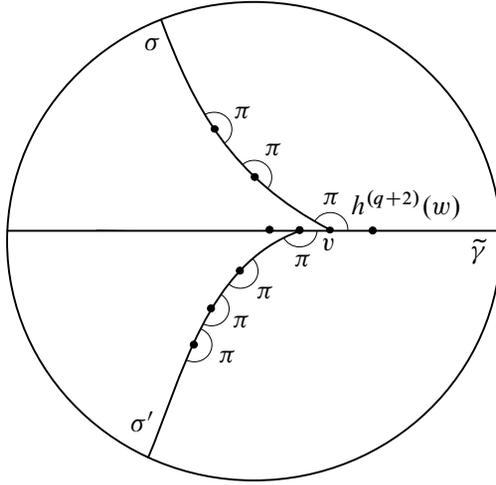


Figure 23: Geodesic rays as continuations of the geodesic segment γ

Claim The rays σ and σ' do not intersect $\tilde{\gamma}_0$.

Proof Assume that $\sigma \cap \tilde{\gamma}_0 \neq \emptyset$. Then $\tilde{\gamma}$, $\tilde{\gamma}_0$ and σ form a triangle in \tilde{S} . Denote the angles of the triangle by β_1 , β_2 and β_3 . See Figure 24.

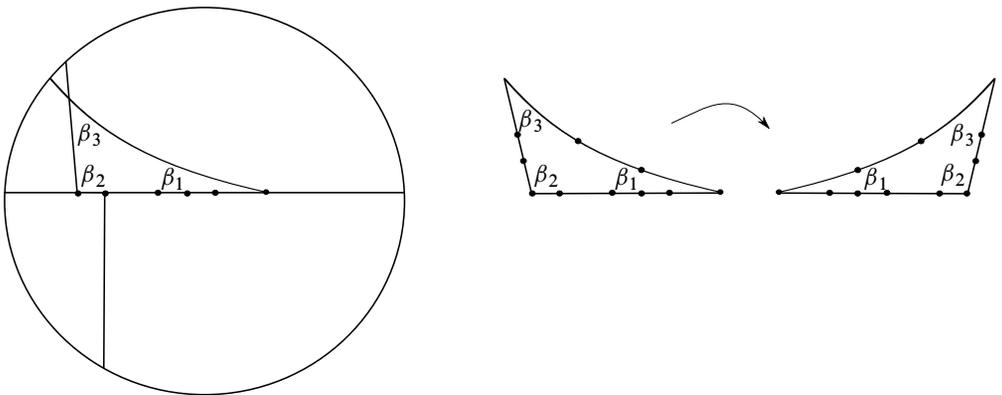


Figure 24: Gluing two triangles to get a sphere

Take two copies of the given triangle and glue them together to get a sphere R with an induced Euclidean cone metric. See Figure 24. The Euler characteristic of R is 2, and by the Gauss–Bonnet formula (see Proposition 2.1)

$$-2\pi\chi(R) = \sum_{x \in X} (c(x) - 2\pi),$$

where X is the set of all cone points of R . Three cone points come from the triangle vertices and they have cone angles $2\beta_l$, $l = 1, 2, 3$. The rest come from cone points along the sides or the inside of the triangles and have cone angles greater than 2π . In particular we have cone points $h^{-1}(v_j), \dots, h^{-q}(v_j)$ with cone angles $2\theta_j$, $j = 1, \dots, n$. Therefore we get the inequality

$$-4\pi = \sum_{x \in X} (c(x) - 2\pi) \geq \sum_{i=1}^q \sum_{j=1}^n (2\theta_j - 2\pi) + (2(\beta_1 + \beta_2 + \beta_3) - 6\pi).$$

It follows that

$$\pi - (\beta_1 + \beta_2 + \beta_3) \geq \sum_{i=1}^q \sum_{j=1}^n (\theta_j - \pi).$$

The holonomy for every metric in $\text{Flat}(S, q)$ is a rotation through an angle of the form $(2\pi/q)k$, $k \in \mathbb{Z}$. Therefore we get $\sum_{j=1}^n (\theta_j - \pi) = (2\pi/q)k$ for some integer $k > 0$, and so $\sum_{j=1}^n (\theta_j - \pi) \geq (2\pi/q)$.

It follows that

$$\sum_{i=1}^q \sum_{j=1}^n (\theta_j - \pi) = q \sum_{j=1}^n (\theta_j - \pi) \geq q \frac{2\pi}{q} = 2\pi,$$

and thus $\pi - (\beta_1 + \beta_2 + \beta_3) \geq 2\pi$, which is a contradiction. Therefore $\sigma \cap \tilde{\gamma}_0 = \emptyset$. The same argument shows $\sigma' \cap \tilde{\gamma}_0 = \emptyset$. This proves the claim. \square

Now we will show that every geodesic from $h^{(q+2)}(w)$ to $\tilde{\gamma}_0$ has to share a positive length geodesic segment with $\tilde{\gamma}$.

There is a unique geodesic between any 2 distinct points in \tilde{S} , since it is a CAT(0) space. Let ζ be a geodesic from $h^{(q+2)}(w)$ to some point on $\tilde{\gamma}_0$. Let $\sigma_0 = \sigma - \tilde{\gamma}$ and $\sigma'_0 = \sigma' - \tilde{\gamma}$. Assume ζ does not share a positive length geodesic segment of $\tilde{\gamma}$. By the previous claim ζ must intersect one of σ_0 or σ'_0 . See Figure 25. Without loss of generality assume that $A \in \zeta \cap \sigma_0$ is such a point. The path from $h^{(q+2)}(w)$ to v following the geodesic segments on $\tilde{\gamma}$ and then from v to A along σ_0 is a geodesic by construction since all cone angles are at least π on both sides of the path. Since the initial arc of ζ to A provides a different geodesic, this contradicts the uniqueness of

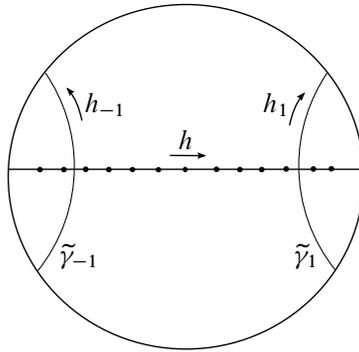


Figure 26: Lifts of γ

that $(\tilde{\gamma}_i)_m \cap (\tilde{\gamma})_m \in [h^{i(q+1)}(z'), h^{i(q+2)}(z')]$, $i = \pm 1$. From Lemma 5.3 we know that every geodesic from z' to $(\tilde{\gamma}_i)_m$, $i = \pm 1$, has to run along positive line segment of $(\tilde{\gamma})_m$ and meet $(\tilde{\gamma}_i)_m$ on $[h_i^{-(q+2)}(z'), h_i^{(q+2)}(z')]$.

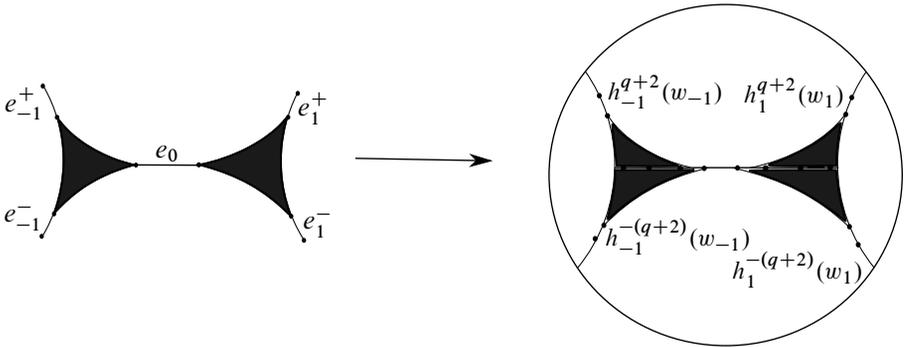


Figure 27: The m -convex hull of $(\tilde{\gamma}_{-1})_m$ and $(\tilde{\gamma}_1)_m$ on the right, and the 2-complex Γ' on the left.

It follows that the m -convex hull of $(\tilde{\gamma}_{-1})_m$ and $(\tilde{\gamma}_1)_m$ consists of $(\tilde{\gamma}_{-1})_m \cup (\tilde{\gamma}_1)_m$ together with an arc of $(\tilde{\gamma})_m$ and two (possibly degenerate) triangles. Choose a point $w_i \in (\tilde{\gamma})_m \cap (\tilde{\gamma}_i)_m$ for $i = \pm 1$ and consider the point $h_i^{\pm(q+2)}(w_i)$ along $(\tilde{\gamma}_i)_m$. Let Γ' be the metric 2-complex with two (possibly degenerate) triangles and five edges determined by $h_i^{\pm(q+2)}(w_i)$ as shown in the Figure 27 and view this as mapping into \tilde{S} . Let Γ denote the metric 2-complex obtained by gluing $h_i^{-(q+2)}(w_i)$ to $h_i^{(q+2)}(w_i)$, for $i = \pm 1$. The inclusion $\Gamma' \rightarrow \tilde{S}$ descends to a locally convex, local isometry $f: \Gamma \rightarrow S$. Identifying $\pi_1(\Gamma) = F(a, b)$ as shown in Figure 26 we have $f_* = \phi$.

By the previous construction, $[f_*(a)] = [f_*(b)] = \pm 2(q+2)[\gamma]$ in $H_1(S, \mathbb{Z})$. It follows that $[f_*(w_j)] = \pm 2(q+2)(2k-2j)[\gamma]$. By construction $[\gamma] \neq 0$, thus $\{f_*(w_j)\}_{j=0}^{k-1}$

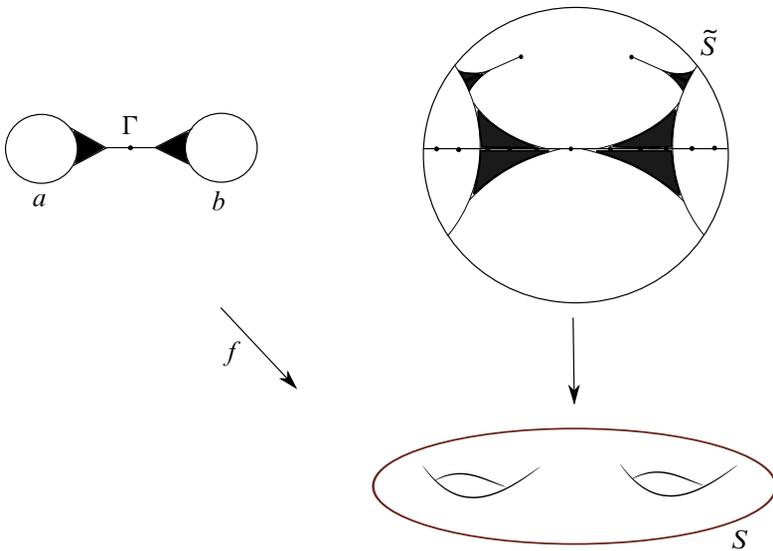


Figure 28: Constructing the 2-complex Γ from the curve γ

are distinct classes of non-null homotopic curves. Since f is a locally convex, local isometry, by measuring lengths on Γ instead of S we can see $l_m(f_*(w_j)) = l_m(f_*(w_i))$ for every $i, j = 0, \dots, k - 1$. □

Corollary 5.4 *Let $q_1, q_2 \in \mathbb{Z}_+$. If $q_1 \mid q_2$ then $\equiv_{q_2} \Rightarrow \equiv_{q_1}$. The reverse implication is not true in general.*

Proof If $q_1 \mid q_2$ then $\text{Flat}(S, q_1) \subset \text{Flat}(S, q_2)$. Thus the first part of the statement follows.

Assume the reverse statement is true for all $q_1 \mid q_2$. Let $q \in \mathbb{Z}_+$. Then $\equiv_q \Rightarrow \equiv_{q^i}$ for every positive integer i . By [Theorem 5.1](#) there are two distinct curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$. By assumption we get $\gamma \equiv_{q^i} \gamma'$, for all i . This is in contradiction to [Theorem 4.4](#) if we take our infinite sequence to be $\{q^i\}_{i=1}^\infty$. □

Theorem 5.5 *For every $q \in \mathbb{Z}_+$, there exist $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$ but $\gamma \not\equiv_h \gamma'$.*

Proof Let γ and γ' be from the construction in [Theorem 5.1](#). If $\gamma \equiv_h \gamma'$ then γ and γ' can be oriented so that they represent the same homology class; see [Leininger \[14\]](#). In our construction γ and γ' have different homology representatives, thus those curves cannot be h -equivalent. □

6 Punctured surfaces

In this section we will describe the main results for punctured surfaces. Some of the results in this case are actually stronger, and in general the proofs go through with little change. The main technical difference is in the structure of geodesics, which is more complicated in this setting.

Let \widehat{S} denote a closed, oriented surface of genus g . Let S be the surface obtained by removing a finite set of points from \widehat{S} .

Define $\text{Flat}(S)$ to be the set of metrics on S with the following properties:

- (i) The metric completion of $m \in \text{Flat}(S)$ is a Euclidean cone metric \widehat{m} on \widehat{S} .
- (ii) Cone points of \widehat{m} contained in $S \subset \widehat{S}$ have cone angles at least 2π .

For $q \in \mathbb{Z}_+$, define $\text{Flat}(S, q)$ to be the set of metrics $m \in \text{Flat}(S)$, such that the holonomy around every loop in $S \setminus \{\text{cone points}\}$ is in $\langle \rho_{2\pi/q} \rangle$. Cone angles of the cone points on \widehat{S} are therefore of the form $k2\pi/q$, $k > 0$. Cone angles of points in S have $k \geq q$, but this is not required for points in $\widehat{S} \setminus S$. One can show that metrics in $\text{Flat}(S, q)$ are precisely those that come from meromorphic q -differentials on \widehat{S} , all of whose poles (if any) are contained in $\widehat{S} \setminus S$ and have order at most $q - 1$. For $k < q$, a cone point with cone angle $k2\pi/q$ is a pole of order $q - k$.

Let $\mathcal{C}(S)$ be a set of all homotopy classes of non-trivial, non-peripheral curves on S . We will define the length of $\gamma \in \mathcal{C}(S)$ in the metric $m \in \text{Flat}(S)$ as the infimum of all lengths of representatives of γ in m :

$$l_m(\gamma) = \inf_{\sigma \in \gamma} \text{length}_m(\sigma).$$

All of the equivalence relations on $\mathcal{C}(S)$ can be defined as before (see [Section 1](#)).

Because S is incomplete, there may not be a geodesic representative in S for every $\gamma \in \mathcal{C}(S)$. On the other hand, given a sequence of closed curves γ_n representing γ , by the Arzela–Ascoli theorem we can extract a limiting curve $\widehat{\gamma}$ in \widehat{S} . Unfortunately, the homotopy class of γ cannot be recovered from $\widehat{\gamma}$, but this can be remedied by working in the universal covering as we now explain.

Let $p: \widetilde{S} \rightarrow S$ denote the universal covering. Write \check{S} to denote the metric completion of \widetilde{S} (with respect to some $m \in \text{Flat}(S, q)$, for some q), which is a CAT(0) space, and note that the action of $\pi_1(S)$ on \widetilde{S} extends to an action of $\pi_1(S)$ on \check{S} . We further observe that the universal covering extends to the completion $p: \check{S} \rightarrow \widehat{S}$, though this is no longer a covering map. Using this projection, we can see that any two metrics

m, m' give rise to metric completions that are homeomorphic by a homeomorphism that is the identity on \tilde{S} , and so we view \check{S} as independent of the metric m .

Given $\gamma \in \mathcal{C}(S)$, we let $h_\gamma \in \pi_1(S)$ denote an element that represents the conjugacy class determined by γ (after arbitrarily choosing an orientation). This element has a geodesic axis $\check{\gamma}$. It may enter and exit a completion point (ie a point of $\check{S} \setminus \tilde{S}$), but when it does so, the two geodesic sub-rays emanating from that point have one well-defined angle which must be greater than or equal to π (see Figure 29). Composing with p produces a geodesic $\hat{\gamma} = p \circ \check{\gamma}$ as in the previous paragraph, but we note that the axis does determine the homotopy class γ uniquely. See Rafi [20] for this discussion in the case $q = 2$.

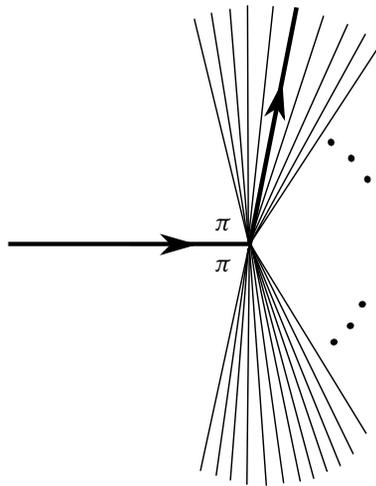


Figure 29: A segment of a geodesic in the universal cover through a completion point. Every other ray emanating from the point in this figure forms another geodesic with the ray merging into that point since the angle is at least π .

It will be useful to choose representatives γ_n of γ constructed from $\check{\gamma}$, which we do as follows. Let $\{\epsilon_n\}_{n=1}^\infty$ be any sequence of positive real numbers limiting to 0. For each point $x \in \hat{S} \setminus S$, consider the ϵ_n -ball $B_{\epsilon_n}(x)$ about x in \hat{S} . We assume that all ϵ_n are sufficiently small so that each ball $B_{\epsilon_n}(x)$ is isometric to some cone with cone angle $c(x) > 0$ as in Section 2.1, and for any two $x, x' \in \hat{S} \setminus S$, $B_{\epsilon_n}(x) \cap B_{\epsilon_n}(x') = \emptyset$. There is an h_γ -invariant path $\tilde{\gamma}_n$ in \tilde{S} that follows $\check{\gamma}$ in the complement of the preimage of these balls, and monotonically traverses the boundary of the preimage the balls between arcs in the complement. See Figure 30. Composing with p , we obtain a sequence of representatives $\gamma_n = p \circ \tilde{\gamma}_n$ in S of γ with length limiting to the translation length of h_γ , which is precisely $l_m(\gamma)$.

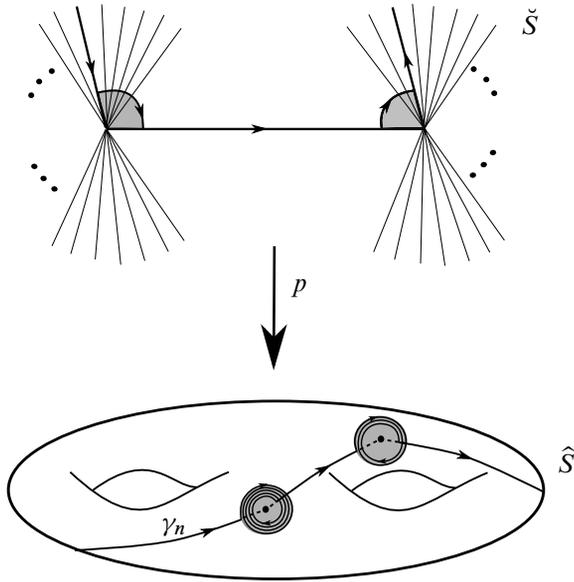


Figure 30: A segment of a curve γ_n in S approximating a geodesic through cone points in \hat{S} and its preimage in \tilde{S}

We can view γ_n as a concatenation of geodesic segments between balls B_{ϵ_n} so that in between those segments γ_n makes some number of turns around the boundaries of B_{ϵ_n} balls. Here we can formally define the *turn* of γ_n as it passes around one of the balls $B_{\epsilon_n}(x)$ to be the positive real number that is the length of the path around the boundary of $B_{\epsilon_n}(x)$, divided by the length of the boundary of $B_{\epsilon_n}(x)$ (which is precisely $c(x) \cdot \epsilon_n$). This is independent of n , and in this way we can think of the turn as a number associated to the geodesic representative. This also provides a method for constructing geodesics: a concatenation of geodesic segments between the boundaries of the balls $\{B_{\epsilon_n}(x)\}$ followed by a monotone path around the boundary of the ball making k turns, so that $k \cdot c(x) \geq \pi$, determines a geodesic by letting $n \rightarrow 0$. The point is that every time the limiting path of the lifting to \tilde{S} in \hat{S} enters a completion point, it makes an angle equal to the turn times $c(x)$, which is therefore at least π . Note that for a metric in $\text{Flat}(S, q)$, every cone angle $c(x)$ is at least $2\pi/q$, therefore having $q/2$ turns around x with q even (and $(q + 1)/2$ turns when q is odd) is a sufficient condition for any curve to have a lift whose geodesic representative in \tilde{S} goes through a completion point in the preimage of x . With this understanding of geodesics, we are ready to explain the extension of the results for punctured surfaces.

In case $q = 1$, all cone angles are greater than or equal to 2π . A Euclidean cone metric with all cone angles at least 2π on a sphere contradicts the Gauss–Bonnet formula

(see [Proposition 2.1](#)). Therefore, there is no $\text{Flat}(S, 1)$ metric on a punctured sphere. However, [Theorem 3.1](#) is true for punctured surfaces of genus $g \geq 1$:

Theorem 6.1 *Let S be a surface with genus $g \geq 1$ and $n > 0$ punctures. For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$.*

Proof The implication $\gamma \equiv_{si} \gamma' \Rightarrow \gamma \equiv_2 \gamma'$ is also true for punctured surfaces, and follows, for example, by Duchin, Leininger and Rafi [\[6\]](#). [Theorem 2.3](#) and proof are equally valid for any punctured surface containing a nonseparating curve (so when the genus is at least 1). To prove $\gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_{si} \gamma'$ we follow the proof of [Theorem 3.1](#). In this case we choose a homeomorphism $f_\epsilon: \widehat{S} \rightarrow X_g^\epsilon$, where X_g^ϵ is obtained by gluing the rectangle Y_g^ϵ , so that the points $f(\widehat{S} \setminus S)$ belong to the image on X_g^ϵ of the vertical ϵ -length edges on Y_g^ϵ . In the case of a punctured torus, Y_1^ϵ is a rectangle, as X_1^ϵ is obtained from Y_1^ϵ by gluing opposite sides together. \square

The equivalence relation \equiv_∞ is trivial on punctured surfaces as well. Most of the proof of [Theorem 4.2](#) goes through, but we require a few modifications. Start with a complete, finite area hyperbolic metric on S with n cusps that correspond to n punctures. Any two distinct homotopy classes of closed curves have distinct closed geodesic representatives in this metric and we denote these γ and γ' . We deform a hyperbolic metric to a hyperbolic cone metric replacing cusps with hyperbolic cones (see Judge [\[12\]](#)), so that γ and γ' do not intersect the cone points. The rest of the proof is the same and we construct a Euclidean cone metric in which geodesic representatives of γ and γ' on \widehat{S} have different lengths and do not contain cone points coming from punctures. The proofs of [Theorem 4.3](#) and [Theorem 4.4](#) are the same for punctured surfaces when we regard the punctures as cone points. Thus we have:

Theorem 6.2 *For every finite area, punctured surface S , the equivalence relation \equiv_∞ on $\mathcal{C}(S)$ is trivial.*

Regarding the equivalence relation \equiv_q , $q \in \mathbb{Z}_+$, a stronger statement than [Theorem 5.1](#) is true for punctured surfaces:

Theorem 6.3 *For every finite area, punctured surface S and for every $q_0 \in \mathbb{Z}_+$ there are infinitely many distinct homotopy classes of curves $\gamma_i \in \mathcal{C}(S)$, $i = 1, 2, \dots$, such that $\gamma_i \equiv_q \gamma_j$, for all i, j and for all $q \leq q_0$.*

Proof Similarly to the proof of [Theorem 5.1](#) we will construct a rank-2 free subgroup of $\pi_1(S)$ and find infinitely many words in this free subgroup corresponding to curves on S that all have the same length for any metric $m \in \text{Flat}(S, q)$.

We start by picking a reference metric $m_0 \in \text{Flat}(S, q_0)$ and an arc $\check{\delta}$ between two distinct completion points in \check{S} that have endpoints projecting to a single point x in S . Let σ be the boundary of the ϵ -ball $B_\epsilon(x)$ for some small $\epsilon > 0$ and let δ be the restriction of $p(\check{\delta})$ to the subpath from σ to itself. Let Γ be a graph consisting of two circles joined by an edge (see Figure 31). Define a map $F: \Gamma \rightarrow S$ by sending the middle arc to δ and the circles to loops that traverse σ $[(q_0 + 1)/2]$ times. Here $[k]$ is the integer part of k .

Picking a basepoint on the middle edge of Γ , we let a and b be generators for $\pi_1(\Gamma)$ represented by loops that run from the basepoint, out to one or the other of the circles, and traverse the circle once before returning to the basepoint. Let $w_n = a^n b$ for $n \in \mathbb{Z}$.

For any metric $m \in \text{Flat}(S, q)$, let $\check{\delta}_m$ be the straightening of $\check{\delta}$ to an m -geodesic between the endpoints in \check{S} , and let $l_m(\delta)$ be the length of δ_m . Note that the geodesic representative of the image of w_n simply traverses the image in \hat{S} of $\check{\delta}_m$ once in both directions. This is because the corresponding approximates in S make $[(q_0 + 1)/2] > [(q + 1)/2]$ turns around x . Thus, for every $m \in \text{Flat}(S, q)$, $l_m(F_*(w_n)) = 2l_m(\delta)$ for every $n \in \mathbb{Z}$. See Figure 31 for lifts of $F_*(w_n)$ to \check{S} for 3 different n .

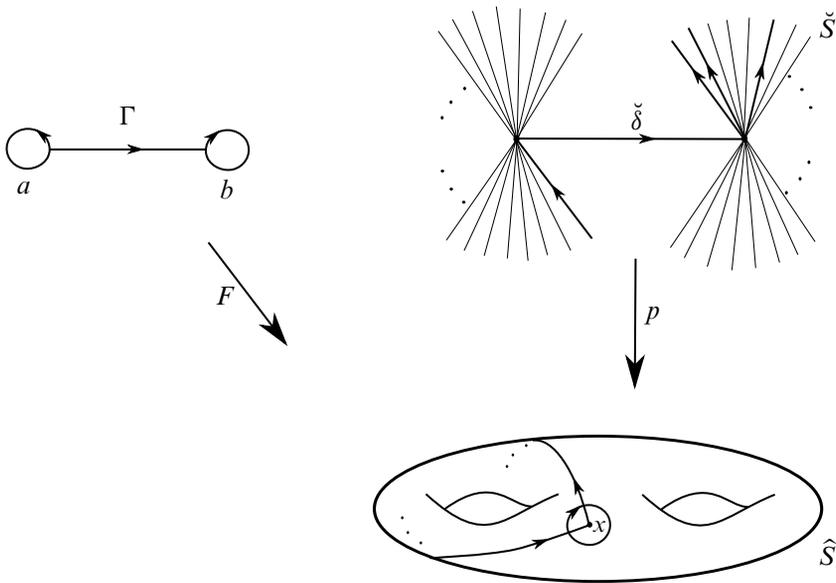


Figure 31: Mapping the graph Γ to S , and some lifts of different w_n to \check{S}

The only thing left to prove now is that there is an infinite subsequence of w_{n_k} of w_n , such that each two $F_*(w_{n_k})$ represent different homotopy classes of curves in $\mathcal{C}(S)$. Let m' be an arbitrary hyperbolic metric on S . The hyperbolic metric is

complete, and thus every closed curve has a geodesic representative. As $n \rightarrow \infty$ the lengths $l_{m'}(F_*(w_{n_k}))$ tend to infinity since they wrap more and more around the cusps. Therefore we can find a subsequence of geodesics whose m' -lengths are all different and thus cannot be homotopic. This completes our proof. \square

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Department of Mathematics, Boston College
Carney Hall, Chestnut Hill, MA 02467-3806, USA

anja.bankovic@bc.edu

<https://www2.bc.edu/anja-bankovic/>

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