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Noninjectivity of the "hair" map

BERTRAND PATUREAU-MIRAND

Kricker constructed a knot invariant Z^{rat} valued in a space of Feynman diagrams with beads. When composed with the "hair" map H, it gives the Kontsevich integral of the knot. We introduce a new grading on diagrams with beads and use it to show that a nontrivial element constructed from Vogel's zero divisor in the algebra Λ is in the kernel of H. This shows that H is not injective.

57M25, 57M27

Introduction

The Kontsevich integral Z is a universal rational finite type invariant for knots (see the Bar-Natan survey [1]). For a knot K, $Z(K)$ lives in the space of Chinese diagrams isomorphic to $\hat{\mathcal{B}}(*)$ (see Section 1.1). Rozansky [5] conjectured and Kricker [3] proved that Z can be organized into a series of "lines" called Z^{rat}. They can be represented by finite $\mathbb{Q}$–linear combinations of diagrams whose edges are labelled, in an appropriate way, with rational functions. Garoufalidis and Kricker [2] directly proved that the map Z^{rat} with values in a space of diagrams with beads is an isotopy invariant and that Z factors through Z^{rat}. For a knot K with trivial Alexander polynomial, $Z(K) = H \circ Z^{\mathrm{rat}}(K)$ where H is the hair map (see Section 1.3). Rozansky, Garoufalidis and Kricker conjectured (see Ohtsuki [4, Conjecture 3.18]) that H could be injective. Theorem 4 gives a counterexample to this conjecture.

1 The hair map

1.1 Classical diagrams

Let X be a finite set. A X–diagram is an isomorphism class of finite unitrivalent graphs K with the following data:

- At each trivalent vertex x of K, we have a cyclic ordering on the three oriented edges starting from x.
- A bijection between the set of univalent vertices of K and the set X.

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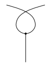
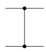
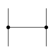
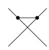
We define $\mathcal{A}(X)$ to be the quotient of the $\mathbb{Q}$–vector space generated by X–diagrams by the relations:

(1) The (AS) relations for “antisymmetry”:

 $+$  $= 0$

(2) The (IHX) relations for three diagrams which differ only in a neighborhood of an edge:

 $=$  $-$ 

These spaces are graded. The degree of an X–diagram is given by half the total number of vertices.

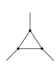
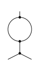
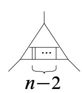
Let $[n] = \{1, 2, \ldots, n\}$ and define F_n to be the subspace of $\mathcal{A}([n])$ generated by connected diagrams with at least one trivalent vertex. The permutation group $\mathfrak{S}(X)$ acts on $\mathcal{A}(X)$. Let $B(*)$ be the coinvariant space for this action:

$$B(*) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}([n]) \otimes_{\mathfrak{S}_n} \mathbb{Q}$$

and let $\widehat{B}(*)$ be the completion of $B(*)$ for the grading.

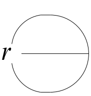
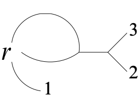
Finally let Λ be Vogel’s algebra generated by totally antisymmetric elements of F_3 (for the action of $\mathfrak{S}_3$).

We recall (see [6]) that Λ acts on the modules F_n and that for this action, F_0 and F_2 are free Λ–modules of rank one. Furthermore, the following elements are in Λ:

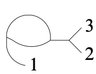
$t =$  $= \frac{1}{2}$ , $\qquad x_n =$ 

Theorem 1 (Vogel [6, Section 8 and Proposition 8.5]) *The element t is a divisor of zero in Λ.*

Corollary 2 *There exists an element $r \in \Lambda \setminus \{0\}$ such that $t \cdot r = 0$. So one has*

r  $\neq 0 \in F_0$ *but* r  $= 0 \in F_3.$

Proof F_0 is a free Λ–module of rank one generated by the diagram Θ and the previous diagram of F_0 is $r \cdot \Theta \neq 0$. The diagram of F_3 of the corollary is the product

$r \cdot$  $= 2tr = 0 \in \Lambda.$ $\square$

Remark Vogel shows that r can be chosen with degree fifteen in Λ (the degree in Λ is the degree in F_3 minus two), and in the algebra generated by the x_n. This element is killed by all the weight systems coming from Lie algebras (but r is not killed by the Lie superalgebras $\mathfrak{D}_{2,1,\alpha}$).

1.2 Diagrams with beads

Diagrams with beads were introduced by Kricker and Garoufalidis [3; 2]. A presentation of $\mathcal{B}$ which uses the first cohomology classes of diagrams is already present in [5]. Vogel explained me this point of view for diagrams with beads.

Let G be the multiplicative group $\{b^n,\ n\in\mathbb{Z}\}\simeq(\mathbb{Z},+)$ and consider its group algebra $R=\mathbb{Q}G=\mathbb{Q}[b,b^{-1}]$. Let $a\mapsto\overline{a}$ be the involution of the $\mathbb{Q}$–algebra R that maps b to b^{-1}.

A diagram with beads in R is an $\varnothing$–diagram with the following supplementary data: The beads form a map $f\colon E\longrightarrow R$ from the set of oriented edges of K such that if $-e$ denotes the same edge than e with opposite orientation, one has $f(-e)=\overline{f(e)}$.

We will represent the beads by some arrows on the edges with label in R. The value of the bead f on e is given by the product of these labels and we will not represent the beads with value 1. So with graphical notation, we have:

$$\xleftarrow{\;f(b)\;} \quad = \quad \xrightarrow{\;\overline{f(b)}\;} \qquad \text{and} \qquad \xrightarrow{\;f(b)\;}\xrightarrow{\;g(b)\;} \quad = \quad \xrightarrow{\;f(b)g(b)\;}$$

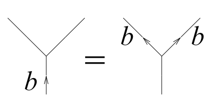
The loop degree of a diagram with beads is the first Betti number of the underlying graph.

Let $\mathcal{A}^R(\varnothing)$ be the quotient of the $\mathbb{Q}$–vector space generated by diagrams with beads in R by the following relations:

(1) (AS)

(2) The (IHX) relations should only be considered near an edge with bead 1.

(3) PUSH:



(4) Multilinearity:

$$\xrightarrow{\;\alpha f(b)+\beta g(b)\;} \quad = \quad \alpha\ \xrightarrow{\;f(b)\;} \quad + \quad \beta\ \xrightarrow{\;g(b)\;}$$

$\mathcal{A}^R(\varnothing)$ is graded by the loop degree:

$$\mathcal{A}^R(\varnothing) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}^R_n(\varnothing)$$

We will prefer another presentation of $\mathcal{A}^R(\varnothing)$:

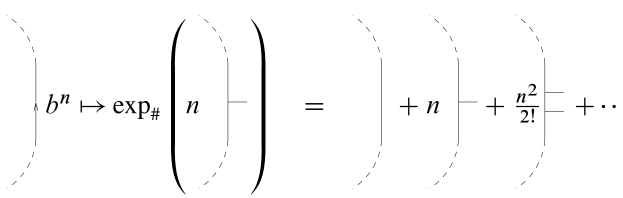
- Note that it is enough to consider diagrams with beads in G and the multilinear relation can be viewed as a notation.
- Next note that for a diagram with beads in G, the map f define a 1–cochain $\tilde{f}$ with values in $\mathbb{Z} \simeq G$ on the underlying simplicial set of K. The elements $\tilde{f}$ are in fact 1–cocycles because of the condition $f(-e) = \overline{f(e)}$ which implies $\tilde{f}(-e) = -\tilde{f}(e)$.
- The "PUSH" relation at a vertex v implies that $\tilde{f}$ is only given up to the coboundary of the 0–cochain with value 1 on v and 0 on the other vertices. Hence $\mathcal{A}^R(\varnothing)$ is also the $\mathbb{Q}$–vector space generated by the pairs (3–valent graph D, $x \in H^1(D,\mathbb{Z})$) quotiented by the relations (AS) and (IHX). With these notation one can describe the (IHX) relations in the following way:
  Let K_I, K_H and K_X be three graphs which appear in a (IHX) relation on an edge e. Let $K_\bullet$ be the graph obtained by collapsing the edge e. The maps $p_?\colon K_? \longrightarrow K_\bullet$ induce three cohomology isomorphisms.
  If $x \in H^1(K_\bullet,\mathbb{Z})$ then the (IHX) relation at e says that

  $$(K_I, p_I^* x) = (K_H, p_H^* x) - (K_X, p_X^* x)$$

  holds in $\mathcal{A}^R(\varnothing)$.

1.3 The hair map

The hair map $H\colon \mathcal{A}^R(\varnothing) \longrightarrow \hat{\mathcal{B}}(*)$ replaces beads by legs (or hair): Just replace a bead b^n by the exponential of n times a leg.



H is well defined (see [2]).

2 Grading on diagrams with beads

Note that for a 3–valent graph K, $H^1(K,\mathbb{Z})$ is a free $\mathbb{Z}$–module. The beads $x \in H^1(K,\mathbb{Z})$ which occur in an (AS) or (IHX) relation are the same up to isomorphisms. We will call $p \in \mathbb{N}$ the bead degree of (K,x) if x is p times an indivisible element of $H^1(K,\mathbb{Z})$.

Theorem 3 *The bead degree is well defined in $\mathcal{A}^R_n(\varnothing)$. Thus we have a grading*

$$\mathcal{A}^R_n(\varnothing) = \bigoplus_{p\in\mathbb{N}} \mathcal{A}^R_{n,p}(\varnothing),$$

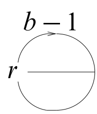
where $\mathcal{A}^R_{n,p}(\varnothing)$ is the subspace of $\mathcal{A}^R_n(\varnothing)$ generated by diagrams with bead degree p. Furthermore, $\mathcal{A}^R_{n,0}(\varnothing) \simeq \mathcal{A}_n(\varnothing)$ and for $p>0$, $\mathcal{A}^R_{n,p}(\varnothing) \simeq \mathcal{A}^R_{n,1}(\varnothing)$.

Proof The second presentation we have given for $\mathcal{A}^R_n(\varnothing)$ implies that this degree is well defined. Indeed, the elements in a IHX relation have the same degree because the set of indivisible elements of the cohomology is preserved by isomorphisms.

Now, the map $\psi\colon R \longrightarrow \mathbb{Q}$ that sends b to 1 induces the isomorphism $\mathcal{A}^R_{n,0}(\varnothing) \simeq \mathcal{A}_n(\varnothing)$ and the group morphism $\phi_p\colon G \longrightarrow G$ that sends b to b^p (or the multiplication by p in $H^1(\cdot,\mathbb{Z})$) induces the isomorphism $\mathcal{A}^R_{n,1}(\varnothing) \simeq \mathcal{A}^R_{n,p}(\varnothing)$. These maps are isomorphisms because they have obvious inverses. $\square$

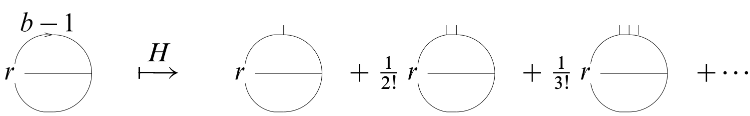
3 A nontrivial element in the kernel of H

Theorem 4 *This nontrivial element of $\mathcal{A}^R(\varnothing)$ is in the kernel of H:*



Thus H is not injective.

Proof This element is not zero because its bead degree zero part is the opposite of the element $r\cdot\Theta$ of Corollary 2. Then, one has



but all these diagrams are zero in $B(*)$ because they contain, as a subdiagram, the element of F_3 of Corollary 2. $\square$

Remark The element of Theorem 4 has a loop degree seventeen.

The hair map is obviously injective on the space of diagrams with bead degree zero. I don't know if the same is true in other degrees.

References

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LMAM, Université de Bretagne-Sud, Université Européenne de Bretagne
BP 573, 56017 Vannes, France

`bertrand.patureau@univ-ubs.fr`

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