

# Quantum invariants of random 3–manifolds

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We consider the  $SO(3)$  Witten–Reshetikhin–Turaev quantum invariants of random 3–manifolds. When the level  $r$  is prime, we show that the asymptotic distribution of the absolute value of these invariants is given by a Rayleigh distribution which is independent of the choice of level. Hence the probability that the quantum invariant certifies the Heegaard genus of a random 3–manifold of a fixed Heegaard genus  $g$  is positive but very small, less than  $10^{-7}$  except when  $g \leq 3$ . We also examine random surface bundles over the circle and find the same distribution for quantum invariants there.

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## 1 Introduction

An important class of 3–manifold invariants arises from the topological quantum field theories (TQFTs) introduced by Witten [26] and then formalized by Atiyah [1; 2]. At the coarsest level, a TQFT associates to each closed 3–manifold  $M$  a complex number  $Z(M)$ . Here, we study the distribution of  $Z(M)$  for a random 3–manifold in the sense of Dunfield and W Thurston [7]. Such manifolds are generated by gluing together two handlebodies of large genus by an extremely complicated element in the mapping class group (see Section 2 for details). For the  $SO(3)$  quantum invariants at a prime level, we show that the distribution of  $|Z(M)|$  is in fact independent of the particular invariant:

**1.1 Theorem** *Let  $Z$  be the  $SO(3)$  TQFT associated to a prime level  $r \geq 5$ . Then for a random 3–manifold  $M$ , the invariant  $|Z(M)|$  is distributed by the Rayleigh distribution with mean  $\sqrt{\pi}/2$ . In particular, the probability that  $|Z(M)|$  is at least some  $x \in [0, \infty)$  is:*

$$P\{|Z(M)| \geq x\} = e^{-x^2}$$

We conjecture that  $Z(M)$  itself is distributed by the standard complex Gaussian, though we were not able to prove this. (See Section 4 for the definitions of these probability distributions.) It is known that for each prime level the values of  $Z(M)$  form a dense set in  $\mathbb{C}$  (see Wong [28]).

While [Theorem 1.1](#) has intrinsic interest, our motivation was a classical topological question: when does the rank of  $\pi_1(M)$  determine the *Heegaard genus* of  $M$ , that is, the minimal genus of a Heegaard splitting of  $M$ ? This was first posed by Waldhausen [[10](#); [25](#)], and in 1984 Boileau and Zieschang gave the first breakthrough examples where the rank and genus differ [[4](#)]. However, when  $M$  is hyperbolic, there are still no known examples where the rank and genus differ (see Souto [[23](#)] for a survey). Garoufalidis [[9](#)] and Turaev [[24](#)] showed that  $|Z(M)|$  gives a lower bound on Heegaard genus (see [Theorem 3.6](#) below), and Wong used this to give an alternate proof that the examples of Boileau and Zieschang have rank smaller than genus [[27](#)]. This is interesting in part because the technique of [[4](#)] cannot be applied to any hyperbolic 3-manifold, whereas quantum invariants have no such apparent restrictions. This led us to try to find hyperbolic 3-manifolds where the rank and genus differ and the genus could be certified by  $|Z(M)|$ .

Unfortunately, in that search we found that quantum invariants typically gave very poor bounds on the Heegaard genus. This can be explained by our results here. In particular, [Theorem 4.1](#) gives the precise distribution of  $|Z(M)|$  for a random 3-manifold of Heegaard genus  $g$ , and we use this to show:

**1.2 Theorem** *Fix a prime level  $r \geq 5$ . Let  $M$  come from a random Heegaard splitting of genus  $g$ . Then the  $SO(3)$  quantum invariant of level  $r$  gives a sharp lower bound on the genus of  $M$  with probability  $(1 - \mu^2)^{d-1}$ , where  $\mu = \frac{2}{\sqrt{r}} \sin \frac{\pi}{r}$ , and  $d$  is the dimension of the Hilbert space of this TQFT for a surface of genus  $g$ .*

Here  $d$  depends on both  $g$  and  $r$  with  $d \sim \mu^{2-2g}$  (see [Lemma 3.11](#)); since  $\mu < 1$ , the dimension  $d$  increases rapidly in both  $r$  and  $g$ . [Table 1.3](#) gives the probabilities when  $r$  and  $g$  are small. For comparison, the far right column gives the corresponding

	$r = 5$	$r = 7$	$r = 11$	$r = 13$	$ H_1(M; \mathbb{F}_2) $
$g = 2$	0.274	0.228	0.206	0.202	0.067
$g = 3$	0.011	$1.6 \times 10^{-5}$	$1.2 \times 10^{-17}$	$4.0 \times 10^{-28}$	0.007
$g = 4$	$1.3 \times 10^{-7}$	$7.5 \times 10^{-42}$	$1.8 \times 10^{-542}$	$1.2 \times 10^{-1442}$	$4.3 \times 10^{-4}$
$g = 5$	$3.6 \times 10^{-25}$	$1.2 \times 10^{-373}$	$4.2 \times 10^{-18437}$	$6.7 \times 10^{-80495}$	$1.3 \times 10^{-5}$

Table 1.3: Probabilities that the  $SO(3)$  quantum invariants of various levels  $r$  give sharp bounds on genus.

probability that the rank of  $H_1(M; \mathbb{F}_2)$  certifies the genus, namely  $\prod_{k=1}^g (2^k + 1)^{-1}$  as

determined by Dunfield and Thurston [7, Section 8.6]. As you can see, the probability that  $Z(M)$  gives a sharp genus bound decreases as either  $g$  or  $r$  increases, and the bound is very rarely sharp except when  $g = 2$  or (marginally) when  $(g, r) = (3, 5)$ . Unfortunately, the case  $g = 2$  isn't interesting for the question of rank versus genus since by Perelman's Geometrization Theorem, any closed 3-manifold where  $\pi_1(M)$  is cyclic must have genus one. Moreover, it seems more likely that hyperbolic examples with rank differing from genus exist when  $g > 2$ . However, Table 1.3 makes it clear that it would be very hard to find such examples where the genus is certified by  $Z(M)$  via a brute force search. Of course, it is possible to systematically generate examples where  $Z(M)$  does certify the genus by always doing certain powers of Dehn twists, but we had no luck finding examples among such manifolds.

One natural class of hyperbolic 3-manifolds is surface bundles over the circle, and one place we looked for possible examples was the Hall–Schleimer census of small genus 2 bundles [11]. For the natural notion of random surface bundle, we show here that the asymptotic distribution of  $|Z(M)|$  is the same as that for all random 3-manifolds, even though general random 3-manifolds are bundles with probability 0. This can be taken as evidence for the naturality and robustness of this notion of random 3-manifold. We also show that for a generic bundle, no  $SO(3)$  quantum invariant certifies its Heegaard genus. For details, see Section 5.

## 1.4 Outline

In Section 2, we review the notion of a random 3-manifold and make precise how we discuss (asymptotically) their various properties. Then in Section 3, we introduce our preferred formalism for  $Z(M)$ , in particular focusing on certain (projective) unitary representations of mapping class groups. Section 4 is the core of the paper and contains the proofs of the main results. The two main tools we use are the work of Freedman, Larsen, and Wang [14; 8] on the density of these unitary representations, and the Ito–Kawada theorem about equidistribution of random walks in compact groups. Finally, Section 5 discusses what happens for surface bundles.

## 1.5 Acknowledgments

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## 2 Random Heegaard splittings

We now review the notion of *random 3-manifold* studied by Dunfield and Thurston [7]. Fix a positive integer  $g$ . Let  $H$  be a genus- $g$  handlebody, and denote  $\partial H$  by  $\Sigma$ . Let  $\Gamma$  be the mapping class group of  $\Sigma$ . For any mapping class  $f \in \Gamma$ , we may associate to it a closed 3-manifold  $M_f$  obtained by gluing together two copies of  $H$  via the homeomorphism  $f$ .

To define a random Heegaard splitting of genus  $g$ , fix generators  $T$  for  $\Gamma$ . A random element  $w$  of  $\Gamma$  of complexity  $\ell$  is defined to be the result of a random walk in the generators  $T$  of length  $\ell$ , that is, a random word of length  $\ell$  in  $T$ . For such a  $w$ , we call  $M_w$  the *manifold of a random Heegaard splitting* of genus  $g$  and complexity  $\ell$ .

We are interested in the properties of such random  $M_w$  as the complexity  $\ell \rightarrow \infty$ . A priori, this might depend on the choice of generators  $T$  for  $\Gamma$ , though it turns out that many properties behave nicely and are independent of such choices. Consider a function

$$F: (\text{closed 3-manifolds}) \rightarrow S,$$

where  $S$  is some set; a typical example is  $F(M) = \dim H_1(M; \mathbb{F}_2)$ . Now let  $\mu_\ell$  be the finitely supported atomic measure on  $S$  defined by taking  $\mu_\ell(\{s\})$  to be the probability that  $F(M) = s$ , where  $M$  is a random Heegaard splitting of genus  $g$  and complexity  $\ell$ . If there is a measure  $\nu_g$  on  $S$  so that, for every choice of generating set of  $\Gamma$ , the measures  $\mu_\ell$  converge weakly to  $\nu_g$  as  $\ell \rightarrow \infty$ , then we say that  $F$  is *distributed by  $\nu_g$  for a random 3-manifold of genus  $g$* . When  $\nu_g$  is atomic, we will say things like “the probability that  $F(M) = s$  is  $\nu_g(s)$ ”, though properly speaking this is a limiting statement as  $\ell \rightarrow \infty$ .

If the  $\nu_g$  themselves have a weak limit  $\nu$  as  $g \rightarrow \infty$ , then we say that  $F$  is *distributed by  $\nu$  for a random 3-manifold*. Of course, not all  $F$  have such limiting distributions  $\nu$ , but for instance  $F(M) = \dim H_1(M; \mathbb{F}_p)$  does [7, Section 8.7]. Another example is that, for fixed  $n$ , the number of epimorphisms from  $\pi_1(M)$  to the alternating group  $A_n$  is Poisson distributed with mean 1 [7, Theorem 7.1]. A geometric example is Maher’s result that  $M$  is hyperbolic with probability 1 [17].

## 3 WRT invariants via unitary representations

In this section, we review the definition of the Witten–Reshetikhin–Turaev TQFT for  $\text{SO}(3)$  in terms of a certain unitary representation of an extension of the mapping class group. Though the WRT TQFT was originally defined using the representation theory of  $U_q\mathfrak{sl}(2)$  in Witten [26] and Reshetikhin–Turaev [22], an equivalent definition

using skein theory was developed by Blanchet, Habegger, Masbaum and Vogel [3] and Lickorish [15], among others. We follow the latter approach as summarized by Lickorish [16], and will also refer to Masbaum and Roberts [18] when necessary.

Fix an odd integer  $r \geq 5$  as the level, and let  $A = ie^{2\pi i/4r}$ . For a closed surface  $\Sigma$  of genus  $g$ , the associated finite-dimensional Hilbert space  $(V, \langle \cdot, \cdot \rangle)$  has the following form. Here  $V$  depends on both  $r$  and  $g$ , and so we will also denote it by  $V_{r,g}$  when this needs to be emphasized. Let  $\mathcal{L} = \{0, 2, 4, \dots, r-3\}$  be the set of even labels. The basis elements of  $V$  correspond to the admissible  $\mathcal{L}$ -labelings of trivalent spines of a handlebody  $H$  with  $\partial H = \Sigma$ . The dimension of  $V$  is given by the Verlinde formula [3] as

$$(3.1) \quad d = \frac{1}{2^g} \left(\frac{r}{2}\right)^{g-1} \sum_{j=1}^{r-1} \left(\sin \frac{\pi j}{r}\right)^{2-2g}$$

Moreover, for our choice of  $A$ , there is an inner product  $\langle \cdot, \cdot \rangle$  induced from the splitting  $\#_g S^1 \times S^2 = H \cup (-H)$ , and these basis elements are orthogonal with respect to  $\langle \cdot, \cdot \rangle$ . See [16] for details.

Underlying this inner product, and indeed the entire WRT TQFT theory, is a skein element  $\omega$  which has the key property that if two links related by Kirby handleslides are colored by  $\omega$ , then the two resulting skeins have the same Kauffman bracket evaluation. More precisely, we have chosen  $\omega$  so that the Kauffman bracket of a 0-framed unlink in  $S^3$  colored by  $\omega$  is the number  $\mu^{-1}$ , where  $\mu^2 = -(A^2 - A^{-2})^2/r$  and so  $\mu = \frac{2}{\sqrt{r}} \sin \frac{\pi}{r}$ . We denote the Kauffman bracket of a  $(-1)$ -framed unlink colored by  $\omega$  by  $\kappa^3$ . Formulas for  $\kappa^3$  appear in, for example, [3], but here all that is needed is that  $\kappa^3$  is a  $4r^{\text{th}}$  root of unity.

We now describe the (projective) action of the mapping class group  $\Gamma$  of  $\Sigma$  on  $V$ . Let  $\mathcal{D}_{ns}$  denote the set of Dehn twists along non-separating curves on  $\Sigma$ . These generate  $\Gamma$ , and to start, we define an action of the free group  $\text{Free}(\mathcal{D}_{ns})$  on  $V$ . To a word  $w$  in  $\text{Free}(\mathcal{D}_{ns})$ , we associate a link  $L(w)$  whose components have framing  $\pm 1$  and that lives in  $\Sigma \times I$ , as in the proof of the Dehn–Lickorish theorem. Color the components of  $L(w)$  by the skein element  $\omega$  to obtain  $s(w)$ . Consider the transformation on  $V$  which adds to a collar neighborhood of  $H$  the skein  $s(w)$ . Because  $\omega$  behaves well under handleslides, this gives a unitary transformation on  $V$ . In other words, we have a homomorphism  $\rho: \text{Free}(\mathcal{D}_{ns}) \rightarrow \text{U}(V)$ . This construction is the “skein-theoretic version of the original geometric action on Dehn twists” from [18]. While the representation of  $\text{Free}(\mathcal{D}_{ns})$  on  $V$  does not descend to  $\Gamma$  itself, it does give a projective representation of  $\Gamma$ , that is, there is a homomorphism  $\bar{\rho}: \Gamma \rightarrow \text{PU}(V)$  with

the property that

$$(3.2) \quad \begin{array}{ccc} \text{Free}(\mathcal{D}_{ns}) & \xrightarrow{\rho} & \text{U}(V) \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\bar{\rho}} & \text{PU}(V) \end{array}$$

commutes.

The WRT 3–manifold invariant may be described easily using this language of unitary representations. For any word  $w$  in  $\text{Free}(\mathcal{D}_{ns})$ , let  $M_w$  be the closed 3–manifold obtained from the corresponding element of  $\Gamma$ . The *vacuum vector*  $v_\emptyset \in V$  is the element corresponding to the empty link. Up to a phase factor involving  $\kappa^3$ , the action of  $\rho(w)$  on  $v_\emptyset$  determines  $Z(M_w)$ . The phase factor is in some sense a correction factor for the framing, and depends on the exponent sum  $e(w)$  of the word  $w$  and on the signature  $\sigma_b(w)$  of the linking matrix of  $L(w)$ . Precisely, the WRT invariant is given by

$$(3.3) \quad Z(M_w) = \kappa^{-3(\sigma_b(w)+e(w))} \cdot \mu \cdot \langle \rho(w)v_\emptyset, v_\emptyset \rangle$$

For example, since the vacuum vector has  $|v_\emptyset|^2 = \mu^{-g}$ , we have  $Z(S^3) = \mu$  and  $Z(S^1 \times S^2) = 1$ ; in general  $Z(\#_g S^1 \times S^2) = \mu^{1-g}$ . Note that  $Z(M)$  here differs from  $I_p(M)$  appearing in [18] by a factor of  $\mu$ .

It will be convenient to rewrite  $Z(M_w)$  using matrix notation. Take the normalized vacuum vector  $\frac{v_\emptyset}{|v_\emptyset|}$  as the first basis element in an orthonormal basis for  $V$ , and denote the  $(1, 1)$ –entry of  $V(w)$  by  $\rho(w)_{(1,1)}$ . Then,

$$(3.4) \quad Z(M_w) = \kappa^{-3(\sigma_b(w)+e(w))} \cdot \mu^{1-g} \cdot \rho(w)_{(1,1)},$$

and since  $\kappa$  is a root of unity and  $\mu$  positive, we have

$$(3.5) \quad |Z(M_w)| = \mu^{1-g} |\rho(w)_{(1,1)}|.$$

In particular,  $|Z(M_w)| \leq \mu^{1-g}$ , leading to the key observation of Garoufalidis [9] and Turaev [24]:

**3.6 Theorem** *Suppose  $M$  is a closed orientable 3–manifold with Heegaard genus  $g(M)$ . Then  $|Z(M)| \leq \mu^{1-g(M)}$ .*

### 3.7 The image of $\rho$

We now turn to the properties of  $\rho$  that will be needed later. The first proposition asserts that the image of  $\rho$  is contained in  $\text{SU}(V)$ , or in some exceptional cases, a slightly larger group.

**3.8 Proposition** *Let  $r \geq 5$  be odd. The image of the homomorphism*

$$\det \circ \rho: \text{Free}(\mathcal{D}_{ns}) \rightarrow \mathbb{C}^\times$$

*is 1 for all  $g \geq 2$ , except when both  $g = 2$  and  $r \equiv 0 \pmod{5}$ . In the latter case, or when  $g = 1$ , the image is contained in the subgroup generated by  $A^4$ .*

**Proof** When  $g \geq 3$ , we need to show that  $\det \circ \rho: \text{Free}(\mathcal{D}_{ns}) \rightarrow \mathbb{C}^\times$  is the trivial homomorphism. Following Masbaum and Roberts [18], there is a central extension  $\tilde{\Gamma}_1$  of  $\Gamma$  to which  $\rho$  descends, and in this proof we consider  $\tilde{\Gamma}_1$  as the domain of  $\rho$ . As discussed in the proof of [18, Theorem 3.10], the abelianization  $\tilde{\Gamma}_1^{ab} = H_1(\tilde{\Gamma}_1; \mathbb{Z})$  is 0 when  $g \geq 3$ . Thus the homomorphism  $\det \circ \rho: \tilde{\Gamma}_1 \rightarrow \mathbb{C}^\times$  must be trivial as claimed, since it factors through the trivial group.

So now we focus on  $g = 1, 2$ , where  $\tilde{\Gamma}_1^{ab} = H_1(\tilde{\Gamma}_1; \mathbb{Z}) = \mathbb{Z}$ . By [18, Theorem 3.8 and Remark 3.9(ii)], all the Dehn twists in  $\mathcal{D}_{ns}$  become conjugate in  $\tilde{\Gamma}_1$ . Thus the abelianization  $H_1(\tilde{\Gamma}_1; \mathbb{Z})$  is generated by any single non-trivial element  $t_\alpha$  of  $\mathcal{D}_{ns}$ . We proceed by calculating the determinant of  $\rho(t_\alpha)$  explicitly, and find  $\rho(t_\alpha) = A^e$  where

$$e = \begin{cases} \frac{1}{6}(r-3)(r-1)(r+1) & \text{when } g = 1, \\ \frac{r}{5} \binom{r-3}{2} \binom{r-1}{2} \binom{r+1}{2} \binom{r+3}{2} & \text{when both } g = 2 \text{ and } r \equiv 0 \pmod{5}, \\ 0 & \text{in all other cases.} \end{cases}$$

In the first two cases, it is easy to see that  $4|e$ , and so the image of  $\det \circ \rho$  is contained in  $\langle A^4 \rangle \leq \mathbb{C}^\times$ .

First, consider the case  $g = 2$ . We view the genus 2 handlebody  $H$  as a neighborhood of the theta graph, with edges labeled by  $a, b, c \in \{0, 2, 4, \dots, r-3\}$ . The basis elements of  $V$  correspond to those labelings that are admissible, namely those satisfying

$$a + b + c \leq 2(r - 2), \quad b + c \geq a, \quad a + c \geq b, \quad a + b \geq c.$$

Consider a simple closed curve  $\alpha \subset \Sigma$  which bounds a disc in  $H$  which is dual to the edge labeled by  $a$ . Then each basis element  $(a, b, c)$  of  $V$  is an eigenvector for  $\rho(t_\alpha)$ , with eigenvalue  $A^{a^2+2a}$ . Now  $\det \rho(t_\alpha)$  is the product of these eigenvalues, and hence is  $A^e$  for  $e = \sum_{(a,b,c) \text{ admissible}} a^2 + 2a$ .

For a fixed label  $a$ , it can be shown that the number of admissible triples containing  $a$  is exactly  $(a + 1)\binom{r-1-a}{2}$ . Thus we can apply standard formulas to show that

$$\begin{aligned} e &= \sum_{\substack{(a,b,c) \\ \text{admissible}}} a^2 + 2a \\ &= \sum_{\substack{\text{even } a=0 \\ r-3}} (a^2 + 2a)(a + 1)\binom{r-1-a}{2} \\ &= \sum_{k=0}^{\frac{r-3}{2}} -8k^4 + (4r - 16)k^3 + (6r - 10)k^2 + (2r - 21)k \\ &= \frac{r}{5}\binom{r-3}{2}\binom{r-1}{2}\binom{r+1}{2}\binom{r+3}{2}. \end{aligned}$$

To conclude, we claim  $4r$  divides  $e$  and hence  $\det \rho(t_\alpha) = A^e = 1$  since  $A$  is a  $4r^{\text{th}}$  root of unity. As  $5 \nmid r$ , then  $5$  must divide one of the consecutive integers  $\frac{r-3}{2}, \frac{r-1}{2}, \frac{r+1}{2}, \frac{r+3}{2}$  and hence  $r|e$ . It is also clear that  $4$  divides  $a^2 + 2a$  since  $a$  is even, and hence  $4|e$ . Thus  $4r|e$  as needed.

When  $g = 1$ , we view the solid torus  $H$  as a neighborhood of a circle, with the Dehn twist curve  $\alpha$  being a meridian linking the circle once. Thus the eigenvalues of  $\rho(t_\alpha)$  are  $A^{a^2+2a}$  all with multiplicity one. Then  $e = \sum_{\text{even } a=0}^{r-2} a^2 + 2a = \frac{1}{6}(r-3)(r-1)(r+1)$ , as claimed. □

The next result is a simple corollary of the results of Freedman, Larsen, and Wang [8; 14] that the image of  $\bar{\rho}$  is dense in  $\text{PU}(V)$  when  $r$  is an odd prime.

**3.9 Proposition** *Let  $G \leq \text{U}(V)$  be the closure of the image of  $\rho$ . If  $g \geq 2$  and  $r \geq 5$  is prime, the group  $G$  contains  $\text{SU}(V)$ . Moreover  $G = \text{SU}(V)$  except if both  $g = 2$  and  $r = 5$ .*

**Proof** First suppose that  $(g, r) \neq (2, 5)$ , so that  $G \subset \text{SU}(V)$  by Proposition 3.8. Since  $\rho$  is a homomorphism,  $G$  is a compact Lie subgroup of  $\text{SU}(V)$  and hence  $\dim G \leq \dim \text{SU}(V)$ . On the other hand, the results of Freedman, Larsen, and Wang [14, Theorem 3] show that the map  $\bar{\rho}$  has dense image. Thus the commutativity of (3.2) implies that  $G$  maps onto  $\text{PU}(V)$ , and so  $\dim G \geq \dim \text{PU}(V)$ . As  $\dim \text{SU}(V) = \dim \text{PU}(V)$  and  $\text{SU}(V)$  is connected, we must have  $G = \text{SU}(V)$  as claimed.

If  $(g, r) = (2, 5)$ , then we saw in the proof of Proposition 3.8 that the image of  $\det \circ \rho$  is contained in the finite subgroup of  $\mathbb{C}^*$  generated by  $A^4$ . Let  $G_0$  be the connected

component of the identity, which must be contained in  $\ker(\det \circ \rho)$  since the image of  $\det \circ \rho$  is finite. Now as before, we must have  $\dim G \geq \dim \text{PU}(V)$ . Since  $G_0 \leq \text{SU}(V)$  and  $\dim G_0 = \dim G$  this forces  $G_0 = \text{SU}(V)$  as needed.  $\square$

**3.10 Remark** Note that Proposition 3.9 remains true if we restrict the domain of  $\rho$  to any subgroup of  $\text{Free}(\mathcal{D}_{ns})$  whose image generates  $\Gamma$ ; this is because the proof only uses that the image of  $\bar{\rho}$  is dense  $\text{PU}(V)$ .

We will also need to understand the dimension of  $V = V_{r,g}$  as a function of  $r$  and  $g$  (recall here that  $\mu$  depends on  $r$ ).

**3.11 Lemma** Fix an odd integer level  $r \geq 5$ . Let  $d_g$  be the dimension of the WRT representation space  $V_{r,g}$ . Then

$$\lim_{g \rightarrow \infty} \frac{d_g}{\mu^{2-2g}} = 1.$$

As  $\mu < 1$ , this implies  $d_g \rightarrow \infty$  as  $g \rightarrow \infty$ .

**Proof** Using (3.1) and the formula for  $\mu$  we have:

$$\frac{d_g}{\mu^{2-2g}} = \frac{1}{2} \sum_{j=1}^{r-1} \left( \frac{\sin(\pi j/r)}{\sin(\pi/r)} \right)^{2-2g} = \frac{1}{2} \sum_{j=1}^{r-1} a_j^{2g-2} \quad \text{where} \quad a_j = \frac{\sin(\pi/r)}{\sin(\pi j/r)}$$

Now when  $j = 1$  or  $r - 1$ , then  $\sin(\pi j/r) = \sin(\pi/r)$  and so  $a_j = 1$ . However, for any other  $j$ , we have  $\sin(\pi j/r) > \sin(\pi/r)$  and so  $0 \leq a_j < 1$ . Thus

$$(3.12) \quad \frac{d_g}{\mu^{2-2g}} = \frac{1}{2} \left( 2 + \sum_{j=2}^{r-2} a_j^{2g-2} \right)$$

where all  $a_j$  left are  $< 1$  and hence  $a_j^{2g-2} \rightarrow 0$  as  $g \rightarrow \infty$ . As  $r$  is fixed, we see that  $\lim_{g \rightarrow \infty} d_g / \mu^{2-2g} = 1$  as claimed. Indeed, it converges exponentially fast in  $g$ .  $\square$

## 4 Distribution of $|Z(M)|$

In this section, we calculate the distribution of  $|Z(M)|$  for random 3-manifolds. To state our results, we first define some distributions. The *standard complex Gaussian* is the probability distribution on  $\mathbb{C}$  where the real and imaginary components of  $z$  are independently distributed Gaussians of mean 0 and variance  $1/2$ ; equivalently the distribution function is  $\frac{1}{\pi} e^{-|z|^2}$  (see, for example, Novak [19, page 5]) and hence the

expectation of  $|z|^2$  is 1. The probability distribution describing of the absolute value of a standard complex Gaussian is the Rayleigh distribution with mean  $\frac{\sqrt{\pi}}{2}$ ; explicitly the density function is  $2xe^{-x^2}$  and  $P\{R \geq x\} = e^{-x^2}$  (see, for example, Papoulis [20]). We refer to this distribution here as the *standard Rayleigh distribution*; however, some authors use this term for the Rayleigh distribution with mean  $\sqrt{\pi/2}$  which arises when the components of the original complex distribution have variance 1. Recall our main result is:

**1.1 Theorem** *Let  $Z$  be the  $SO(3)$  TQFT associated to a prime level  $r \geq 5$ . Then for a random 3-manifold  $M$ , the invariant  $|Z(M)|$  is distributed by the standard Rayleigh.*

Along the way we give a complete picture for each fixed Heegaard genus, using the following distribution. Let  $X_d$  be the random variable on  $\mathbb{C}$  which gives the distribution of the  $(1, 1)$ -entry of a unitary matrix of size  $d$  with respect to Haar measure; it is well-known (see, for example, Hiai and Petz [12, page 140]) that the density in polar coordinates is  $\frac{d-1}{\pi}(1-r^2)^{d-2} dA$  for  $0 \leq r \leq 1$ . Focusing on the absolute value  $|X_d|$ , we easily calculate

$$P\{|X_d| \geq x\} = (1-x^2)^{d-1} \quad \text{for } 0 \leq x \leq 1.$$

We first show

**4.1 Theorem** *Fix a genus  $g$  and prime level  $r \geq 5$ . Let  $d_g$  be the dimension of the Hilbert space  $V_{r,g}$  associated to the surface of genus  $g$ . Then the invariant  $|Z(M)|$  is distributed by  $\mu^{1-g}|X_{d_g}|$  for a random Heegaard splitting of genus  $g$ . In particular*

$$P\{|Z(M)| \geq x\} = (1-\mu^{2g-2}x^2)^{d_g-1} \quad \text{for } 0 \leq x \leq \mu^{1-g}.$$

**Proof** Consider a set  $T$  consisting of words  $t_1, \dots, t_n \in \text{Free}(\mathcal{D}_{ns})$  which together generate  $\Gamma$ . Let  $w$  be a random word in  $T$  of length  $\ell$ . Then by (3.5), the manifold  $M_w$  has  $|Z(M_w)| = \mu^{1-g}|\rho(w)_{(1,1)}|$ . We need to show that the distribution of  $|Z(M_w)|$  limits on  $\mu^{1-g}|X_{d_g}|$  as the length  $\ell \rightarrow \infty$ .

Let  $G$  be the closure of image of the subgroup  $\langle T \rangle$  under  $\rho$ . By Proposition 3.9 and Remark 3.10, we know that  $G$  contains  $SU(V)$ . Now for the random word  $w$ , note that  $\rho(w)$  is the product of the  $\rho(t_i)$ 's. Thus by the Ito-Kawada Theorem [13], the matrix  $\rho(w)$  becomes equidistributed on  $G$  with respect to Haar measure as  $\ell \rightarrow \infty$ . Thus if  $X$  denotes the distribution of the  $(1, 1)$ -entry of a random matrix in  $G$ , we have that  $|Z(M)|$  is distributed by  $\mu^{1-g}|X|$ .

Thus it remains to show  $X$  and  $X_d$  have the same distribution where  $d = d_g = \dim V$ . If  $G = U(d)$  this is obvious, so since  $SU(V) \leq G$  we must have  $\dim G =$

$\dim \text{SU}(V) = \dim \text{U}(V) - 1$ . Consider the homomorphism  $\phi: S^1 \times G \rightarrow \text{U}(d)$  given by  $\phi(\zeta, M) = \zeta M$ . Since  $S^1 \times G \rightarrow \text{U}(d)$  is a covering map, the Haar measure (thought of as a differential form) on  $\text{U}(d)$  pulls back to (a multiple of) the Haar measure on  $S^1 \times G$ . Now the pullback of the random variable  $|X_d|$  on  $\text{U}(d)$  to  $S^1 \times G$  is equal to  $|X|$ , and as the Haar measure pulls back we have that  $|X_d|$  and  $|X|$  have the same distribution. To see that the distributions themselves are the same, for any  $\zeta \in S^1$  consider left multiplication on  $G$  by the matrix  $A_\zeta \in G$  which is diagonal with entries  $(\zeta, \zeta^{-1}, 1, 1, \dots, 1)$ . Notice that the absolute value of the  $(1, 1)$ -entry remains unchanged, but the phase shifts by  $\zeta$ . Since Haar measure is left-invariant, this shows  $X$  to be rotationally symmetric, and hence the same as  $X_d$ .  $\square$

We now consider the limit as the genus  $g$  gets large.

**Proof of Theorem 1.1** We need to determine the limit of the random variables  $\mu^{1-g}|X_{d_g}|$  as  $g \rightarrow \infty$ . It is well known that  $\sqrt{d_g}X_{d_g}$  converges in distribution to the standard complex Gaussian (see, for example, Novak [19]), and by Lemma 3.11 we know  $\sqrt{d_g} \approx \mu^{1-g}$ . Combining these can give that  $\mu^{1-g}|X_{d_g}|$  converges to the standard Rayleigh distribution  $R$ , but instead we show this directly starting from Theorem 4.1. If we fix  $x \geq 0$ , then since  $\mu < 1$  we have

$$\begin{aligned} \lim_{g \rightarrow \infty} P\{|Z(M)| \geq x\} &= \lim_{g \rightarrow \infty} (1 - \mu^{2g-2}x^2)^{d_g-1} \\ &= \lim_{g \rightarrow \infty} \left(1 - \frac{x^2}{\mu^{2-2g}}\right)^{\mu^{2-2g}} \cdot \lim_{g \rightarrow \infty} \left(1 - \frac{x^2}{\mu^{2-2g}}\right)^{\frac{d_g-1}{\mu^{2-2g}}} \\ &= e^{-x^2} \cdot 1^1 \\ &= e^{-x^2} \\ &= P\{R \geq x\} \end{aligned}$$

as needed.  $\square$

As  $|Z(M)|$  is distributed by the standard Rayleigh distribution, it is very natural to postulate

**4.2 Conjecture** *Let  $r \geq 5$  be prime. For a random 3-manifold  $M$ , the  $\text{SO}(3)$  invariant  $Z(M)$  is distributed by a standard complex Gaussian.*

Looking at the proof of Theorem 1.1, in fact we showed that  $\mu^{1-g}\rho(w)_{(1,1)}$  is essentially distributed by a standard complex Gaussian when  $g$  is large. From (3.4), we see this quantity differs from  $Z(M)$  by  $\kappa^{-3(\sigma_b(w)+e(w))}$ , where  $\kappa$  is a certain  $4r^{\text{th}}$  root of unity. Since  $r$  is fixed, this tells us the image of  $Z(M)$  in  $\mathbb{C}/(z \mapsto \kappa^3 z)$  is

distributed by the push-forward of the standard complex Gaussian. The exponent sum  $e(w)$  is a homomorphism  $e: \text{Free}(\mathcal{D}_{ns}) \rightarrow \mathbb{Z}$  and is thus easy to deal with, but  $\sigma_b(w)$  is not a homomorphism. Indeed, the signature  $\sigma_b$  is not Markovian in the sense that  $\sigma_b(w \cdot t)$  does not depend just on  $\sigma_b(w)$  and  $t$ . While it seems almost certain that  $\sigma_b(w) + e(w)$  must be uncorrelated with  $\rho(w)_{(1,1)}$ , and hence  $Z(M)$  is the standard complex Gaussian, we were unable to show this. If [Conjecture 4.2](#) is true, it would give an alternate proof that the values of  $Z(M)$  are dense in  $\mathbb{C}$  (see Wong [\[28\]](#)).

**4.3 Remark** A natural approach to [Conjecture 4.2](#) is to work with one of the extended mapping class groups discussed by Masbaum and Roberts [\[18\]](#). For instance, we used one such group  $\tilde{\Gamma}_1$  above in the proof of [Proposition 3.8](#) since  $\tilde{\Gamma}_1$  has an honest representation on  $V$ , not just a projective one. Certainly, one can generate a random Heegaard splitting via a random walk in  $\tilde{\Gamma}_1$  instead of  $\Gamma$ , but one should think of the output as a 3-manifold  $M$  and a  $p_1$ -structure  $\xi$ . That is, such a random model really produces  $p_1$ -manifolds in the language of [\[18, Section 4\]](#). The  $\text{SO}(3)$  TQFT makes sense for such  $p_1$ -manifolds, and the plain  $Z(M)$  is simply that more general invariant of  $M$  and its canonical  $p_1$ -structure. Now the above arguments show that  $Z(M, \xi)$  is distribution by the standard complex Gaussian. However, when one attempts to use this to understand the original question, the key issue is how a randomly chosen  $\xi$  differs from the canonical one, which hinges on the signature issue discussed in the preceding paragraph. Thus, while this line of thinking is a nice way to frame the underlying issue, it does not seem to give purchase on [Conjecture 4.2](#).

Finally, we compute the probability that  $Z(M)$  gives a sharp genus bound for each genus  $g$  and level  $r$ .

**Proof of [Theorem 1.2](#)** Fix a genus  $g$ . By Maher [\[17\]](#), a random Heegaard splitting of genus  $g$  actually has Heegaard genus  $g$  with probability 1. Thus by [Theorem 3.6](#), the quantum invariant will give a sharp lower bound on Heegaard genus whenever  $|Z(M)| > \mu^{2-g}$ . By [Theorem 1.1](#), this happens with probability  $(1 - \mu^2)^{d_g - 1}$ , as claimed. □

## 5 Random surface bundles

A simple class of 3-manifolds are those which are surface bundles fibering over the circle. Such manifolds are built from some  $f \in \Gamma$  by considering the mapping torus

$$MT_f = \Sigma \times [0, 1] / (p, 1) \sim (f(p), 0)$$

For reasons of homology, a random 3-manifold in the sense of Section 2 fibers over the circle with probability 0 (see Dunfield and Thurston [7, Corollary 8.5]) and even among those manifolds with  $b_1 > 0$ , surface bundles appear to be rare (see Dunfield and Thurston [6]). Despite this, we show that for a natural model of random bundles the distribution of  $|Z(M)|$  is the same as that of random manifolds more generally. This can be taken as evidence for the naturality and robustness of this notion of random 3-manifolds.

Here, a random bundle with fiber a surface  $\Sigma$  of genus  $g$  is defined just as one expects from Section 2: after fixing generators  $T$  of the mapping class group  $\Gamma$  of  $\Sigma$ , one considers  $MT_f$  for  $f \in \Gamma$  a random word in  $T$  of length  $\ell$ , as  $\ell$  tends to infinity. If we then send  $g \rightarrow \infty$ , we find:

**5.1 Theorem** *Consider the  $SO(3)$  invariant for a prime level  $r \geq 5$ . Then  $|Z(M)|$  is distributed by the standard Rayleigh for  $M$  a random surface bundle.*

**Proof** Fix a genus  $g > 2$  for the fiber  $\Sigma$ . In the notation of Section 3, if  $w \in \text{Free}(\mathcal{D}_{ns})$  then we have (see, for example, Turaev [24, Section IV.7.2]):

$$|Z(MT_w)| = |\text{tr } \rho(w)|$$

Thus as in the proof of Theorem 4.1, the Ito–Kawada Theorem tells us that  $|Z(MT_w)|$  is distributed the same as  $|\text{tr } A|$  for  $A \in SU(V)$  chosen with respect to Haar measure. Again as in Theorem 4.1, this is the same as the distribution of  $|\text{tr } A|$  for  $A \in U(V)$ . Now as  $d \rightarrow \infty$ , the distribution of  $\text{tr } A$  for  $A \in U(d)$  converges to the standard complex Gaussian (see Diaconis and Shahshahani [5], see also Petz–Réffy [21] and Novak [19]). Thus since  $\dim V \rightarrow \infty$  as  $g \rightarrow \infty$  by Lemma 3.11, we have that  $|Z(M)|$  is distributed by the standard Rayleigh for a random surface bundle.  $\square$

A genus- $g$  surface bundle  $M$  has a natural Heegaard splitting of genus  $2g + 1$  obtained by tubing together two of the fibers. Combining Maher [17] with Souto [23, Theorem 4.2] shows that this is the minimal genus splitting with probability 1, and explicit examples where this is the case are easy to construct by taking the monodromy from the Torelli group. However, we claim that the  $SO(3)$  quantum invariants can never certify that  $M$  has Heegaard genus  $2g + 1$ . As in the proof of Theorem 5.1, we have  $|Z(M)| = |\text{tr } \rho(w)|$ , and since  $\rho(w)$  is unitary, it follows that  $|\text{tr } \rho(w)| \leq \text{tr}(\text{Id}) = d_g$ . From (3.12), we see that  $d_g \leq \mu^{2-2g} < \mu^{1-2g}$  for any  $g$  when  $r \geq 5$ . Thus  $|Z(M)| < \mu^{1-2g}$ , and so  $Z(M)$  fails to give a sharp bound on Heegaard genus. To summarize, this shows:

**5.2 Proposition** *There exist closed hyperbolic 3-manifolds of arbitrary large Heegaard genus such that no  $SO(3)$  quantum invariant gives a sharp genus bound.*

## References

- [1] **M Atiyah**, *The geometry and physics of knots*, Lezioni Lincee, Cambridge University Press, Cambridge (1990) [MR1078014](#)
- [2] **M Atiyah**, *On framings of 3-manifolds*, *Topology* 29 (1990) 1–7 [MR1046621](#)
- [3] **C Blanchet**, **N Habegger**, **G Masbaum**, **P Vogel**, *Topological quantum field theories derived from the Kauffman bracket*, *Topology* 34 (1995) 883–927 [MR1362791](#)
- [4] **M Boileau**, **H Zieschang**, *Heegaard genus of closed orientable Seifert 3-manifolds*, *Invent. Math.* 76 (1984) 455–468 [MR746538](#)
- [5] **P Diaconis**, **M Shahshahani**, *On the eigenvalues of random matrices*, *J. Appl. Probab.* 31A (1994) 49–62 [MR1274717](#)
- [6] **N M Dunfield**, **D P Thurston**, *A random tunnel number one 3-manifold does not fiber over the circle*, *Geom. Topol.* 10 (2006) 2431–2499 [MR2284062](#)
- [7] **N M Dunfield**, **W P Thurston**, *Finite covers of random 3-manifolds*, *Invent. Math.* 166 (2006) 457–521 [MR2257389](#)
- [8] **M H Freedman**, **M J Larsen**, **Z Wang**, *The two-eigenvalue problem and density of Jones representation of braid groups*, *Comm. Math. Phys.* 228 (2002) 177–199 [MR1911253](#)
- [9] **S Garoufalidis**, *Applications of quantum invariants in low-dimensional topology*, *Topology* 37 (1998) 219–224 [MR1480887](#)
- [10] **W Haken**, *Various aspects of the three-dimensional Poincaré problem*, from: “Topology of Manifolds (Proc Inst., Univ. of Georgia, Athens, Ga., 1969)”, Markham, Chicago, Ill. (1970) 140–152 [MR0273624](#)
- [11] **T Hall**, **S Schleimer**, *Hyperbolic genus 2 bundles with short monodromy* (2002) Available at <http://www.warwick.ac.uk/~masgar/Maths/twister.html>
- [12] **F Hiai**, **D Petz**, *The semicircle law, free random variables and entropy*, *Mathematical Surveys and Monographs* 77, American Mathematical Society, Providence, RI (2000) [MR1746976](#)
- [13] **Y Kawada**, **K Itô**, *On the probability distribution on a compact group I*, *Proc. Phys.-Math. Soc. Japan* (3) 22 (1940) 977–998 [MR0003462](#)
- [14] **M Larsen**, **Z Wang**, *Density of the  $SO(3)$  TQFT representation of mapping class groups*, *Comm. Math. Phys.* 260 (2005) 641–658 [MR2183960](#)
- [15] **W B R Lickorish**, *Three-manifolds and the Temperley–Lieb algebra*, *Math. Ann.* 290 (1991) 657–670 [MR1119944](#)
- [16] **W B R Lickorish**, *Quantum invariants of 3-manifolds*, from: “Handbook of geometric topology”, North-Holland, Amsterdam (2002) 707–734 [MR1886680](#)
- [17] **J Maher**, *Random Heegaard splittings*, *J. Topol.* (to appear) [arXiv:0809.4881](#)

- [18] **G Masbaum, J D Roberts**, *On central extensions of mapping class groups*, Math. Ann. 302 (1995) 131–150 [MR1329450](#)
- [19] **J Novak**, *Truncations of random unitary matrices and Young tableaux*, Electron. J. Combin. 14 (2007) 21 [MR2285825](#)
- [20] **A Papoulis**, *Probability, random variables, and stochastic processes*, McGraw-Hill Book Co., New York (1965) [MR0176501](#)
- [21] **D Petz, J Réffy**, *On asymptotics of large Haar distributed unitary matrices*, Period. Math. Hungar. 49 (2004) 103–117 [MR2092786](#)
- [22] **N Reshetikhin, V G Turaev**, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991) 547–597 [MR1091619](#)
- [23] **J Souto**, *Geometry, Heegaard splittings and rank of the fundamental group of hyperbolic 3-manifolds*, from: “Workshop on Heegaard Splittings”, Geom. Topol. Monogr. 12, Geom. Topol. Publ., Coventry (2007) 351–399 [MR2408255](#)
- [24] **V G Turaev**, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics 18, Walter de Gruyter & Co., Berlin (1994) [MR1292673](#)
- [25] **F Waldhausen**, *Some problems on 3-manifolds*, from: “Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2”, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I. (1978) 313–322 [MR520549](#)
- [26] **E Witten**, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989) 351–399 [MR990772](#)
- [27] **H Wong**, *Quantum invariants can provide sharp Heegaard genus bounds* [arXiv:0801.0738](#)
- [28] **H Wong**, *SO(3) quantum invariants are dense*, Math. Proc. Cambridge Philos. Soc. 148 (2010) 289–295 [MR2600143](#)

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