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Quillen's plus construction and the D(2) problem

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Given a finite connected 3–complex with cohomological dimension 2, we show it may be constructed up to homotopy by applying the Quillen plus construction to the Cayley complex of a finite group presentation. This reduces the D(2) problem to a question about perfect normal subgroups.

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1 Introduction

Given a finite cell complex one may ask what the minimal dimension of a finite cell complex in its homotopy type is. If $n \neq 2$ and the cell complex has cohomological dimension n (with respect to all coefficient bundles), then the cell complex is in fact homotopy equivalent to a finite n–complex (a cell complex whose cells have dimension at most n). Although this has been known for around forty years (for $n > 2$ it is proved by Wall [13] and for $n = 1$ it follows from Swan [12] and Stallings [11]), it is an open question whether or not this holds when $n = 2$. This question is known as Wall's D(2) problem:

> Let X be a finite 3–complex with $H^3(X;\beta) = 0$ for all coefficient bundles β. Must X be homotopy equivalent to a finite 2–complex?

If X (as above) is not homotopy equivalent to a finite 2–complex, we say it is a counterexample which solves the D(2) problem.

For connected X with certain fundamental groups, it has shown been shown that X must be homotopy equivalent to a finite 2–complex (see for example Johnson [7], Edwards [4] and Mannan [9]). However no general method has been forthcoming.

Also, whilst potential candidates for counterexamples have been constructed (see Beyl and Waller [1] and Bridson and Tweedale [2]), no successful method has yet emerged for verifying that they are not homotopy equivalent to finite 2–complexes.

In Section 2 we apply the Quillen plus construction to connected 2–complexes, resulting in cohomologically 2–dimensional 3–complexes. These are therefore candidates for

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counterexamples which solve the D(2) problem. In Section 3 we show that in fact all finite connected cohomologically 2–dimensional 3–complexes arise this way, up to homotopy equivalence.

Finally, in Section 4 we use these results to reduce the D(2) problem to a question about perfect normal subgroups. This allows us to generalize existing approaches to the D(2) problem such as Johnson [8, Theorem I] and Harlander [6, Theorem 3.5].

Before moving on to the main argument we make a few notational points. All modules are right modules except where a left action is explicitly stated. The basepoint of a Cayley complex is always assumed to be its 0–cell.

If X is a connected cell complex with basepoint, we denote its universal cover by $\widetilde{X}$. Given two based loops $\gamma_1, \gamma_2 \in \pi_1(X)$ their product $\gamma_1\gamma_2$ is the composition whose initial segment is γ_2 and final segment is γ_1. With this convention, we have a natural right action of $\pi_1(X)$ on the cells of $\widetilde{X}$. Let $G = \pi_1(X)$. We can regard the associated chain complex of $\widetilde{X}$ as an algebraic complex of right modules over $\mathbb{Z}[G]$. We follow [8] in denoting this algebraic complex $C_*(X)$. Note that this differs from the convention in other texts. Thus in particular $C_*(X)$ and $C_*(\widetilde{X})$ have the same underlying sequence of abelian groups, but the former is a sequence of modules over $\mathbb{Z}[G]$ whilst the latter is a sequence of modules over $\mathbb{Z}[\pi_1(\widetilde{X})] = \mathbb{Z}$.

If Y is a subcomplex of X then $C_*(Y)$ is a sequence of right modules over $\pi_1(Y)$. Let $E = \pi_1(Y)$. The induced map $E \to G$ yields a left action of E on $\mathbb{Z}[G]$. Thus we have an algebraic complex $C_*(Y) \otimes_E \mathbb{Z}[G]$ over $\mathbb{Z}[G]$. The inclusion $Y \subset X$ induces a chain map $C_*(Y) \otimes_E \mathbb{Z}[G] \longrightarrow C_*(X)$. The complex $C_*(X, Y)$ is defined to be the relative chain complex associated to this chain map.

The basepoint allows us to interchange between coefficient bundles over X and right modules over $\mathbb{Z}[G]$. Thus for a right module N we have:

$$H^n(X; N) = H^n(C_*(X); N)$$

A left module over $\mathbb{Z}[G]$ may be regarded as a right module over $\mathbb{Z}[G]$, where right multiplication by a group element is defined to be left multiplication by its inverse. Hence a left module M may also be regarded as a coefficient bundle and we have:

$$H_n(X; M) = H_n(C_*(X); M), \quad H_n(X, Y; M) = H_n(C_*(X, Y); M)$$

Given a finitely generated Abelian group A we may regard it as a finitely generated module over $\mathbb{Z}$. Thus $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite dimensional vector space over $\mathbb{Q}$. The dimension of this vector space will be denoted $\mathrm{rk}_{\mathbb{Z}}(A)$.

Finally given a group G and elements $g, h \in G$, we follow the convention that $[g, h]$ denotes the element $ghg^{-1}h^{-1}$.

2 The plus construction applied to a Cayley complex

Let $\varepsilon = \langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle$ be a finite presentation for a group E. We say a normal subgroup of E is *finitely closed* when it is the normal closure in E of a finitely generated subgroup. Let $K \lhd E$ be finitely closed and perfect (so $K = [K, K]$). Let $\mathcal{K}_\varepsilon$ denote the Cayley complex associated to ε.

Theorem 2.1 (Quillen; see Rosenberg [10, Theorem 5.2.2]) *There is a 3–complex $\mathcal{K}_\varepsilon^+$, containing $\mathcal{K}_\varepsilon$ as a subcomplex, such that the inclusion $\mathcal{K}_\varepsilon \hookrightarrow \mathcal{K}_\varepsilon^+$ induces the quotient map $E \to E/K$ on fundamental groups and $H_*(\mathcal{K}_\varepsilon^+, \mathcal{K}_\varepsilon; M) = 0$ for all left modules M over $\mathbb{Z}[E/K]$. Further, given another such 3–complex X, there is a homotopy equivalence $\mathcal{K}_\varepsilon^+ \to X$ extending the identity map of the common subspace $\mathcal{K}_\varepsilon$.*

In fact we may construct $\mathcal{K}_\varepsilon^+$ explicitly, using the fact that K is finitely closed to ensure that we end up with a finite cell complex. Let $k_1, \ldots, k_r \in K$ generate a subgroup of E whose normal closure (in E) is K. As $K = [K, K]$, each k_i may be expressed as a product of commutators $k_i = \prod_{j=1}^{m_i} [a_{ij}, b_{ij}]$ with each $a_{ij}, b_{ij} \in K$. Then each a_{ij}, b_{ij} may be represented by words A_{ij}, B_{ij} in the g_l, $l = 1, \ldots, n$. For each $i = 1, \ldots, r$ attach a 2–cell E_i to $\mathcal{K}_\varepsilon$ whose boundary corresponds to the word $\prod_{j=1}^{m_i} [A_{ij}, B_{ij}]$. Denote the resulting chain complex $\mathcal{K}'_\varepsilon$.
The chain complex $C_*(\mathcal{K}_\varepsilon)$ may be written:

$$C_*(\mathcal{K}_\varepsilon)\colon C_2(\mathcal{K}_\varepsilon) \xrightarrow{\partial_2} C_1(\mathcal{K}_\varepsilon) \xrightarrow{\partial_1} C_0(\mathcal{K}_\varepsilon)$$

The boundary map ∂_2 applied to a 2–cell is the Fox free differential $\partial\colon F_{\{g_1, \ldots, g_n\}} \to C_1(\mathcal{K}_\varepsilon)$, applied to the word which the 2–cell bounds (see Johnson [8, Section 48] and Fox [5]). Let $\boldsymbol{e_i}$ denote the generator in $C_1(\mathcal{K}_\varepsilon)$ representing the generator g_i. The free Fox differential is then characterized by:

(i) $\partial g_i = \boldsymbol{e_i}$ for all $i = 1, \ldots, n$,

(ii) $\partial(AB) = \partial(A)B + \partial(B)$ for all words A, B.

Clearly the inclusion $\mathcal{K}_\varepsilon \hookrightarrow \mathcal{K}'_\varepsilon$ induces the quotient map $E \to E/K$ on fundamental groups. There is a right action of $\mathbb{Z}[E/K]$ on itself. Further there is a left action of E on $\mathbb{Z}[E/K]$.

Lemma 2.2 *As an algebraic complex of right $\mathbb{Z}[E/K]$ modules $C_*(\mathcal{K}'_\varepsilon)$ may be written:*

$$C_*(\mathcal{K}'_\varepsilon): \quad C_2(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[E/K] \oplus \mathbb{Z}[E/K]^r \xrightarrow{\partial_2 \oplus 0} C_1(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[E/K] \xrightarrow{\partial_1} C_0(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[E/K]$$

Proof The boundary of E_i is given by the free Fox differential ∂, applied to the word $\prod_{j=1}^{m_i}[A_{ij}, B_{ij}]$. However,

$$\partial \prod_{j=1}^{m_i}[A_{ij}, B_{ij}] = \sum_{j=1}^{m_i}[\partial A_{ij} + \partial B_{ij} - \partial A_{ij} - \partial B_{ij}] = 0$$

as each A_{ij}, B_{ij} represents an element of K and hence is trivial in $\pi_1(\mathcal{K}'_\varepsilon) = E/K$. □

Each E_i therefore generates an element of $H_2(\widetilde{\mathcal{K}'_\varepsilon};\mathbb{Z})$. By the Hurewicz isomorphism theorem we have isomorphisms $H_2(\widetilde{\mathcal{K}'_\varepsilon};\mathbb{Z}) \cong \pi_2(\widetilde{\mathcal{K}'_\varepsilon}) \cong \pi_2(\mathcal{K}'_\varepsilon)$ coming from the Hurewicz homomorphism and the covering map respectively. Let $\psi_i\colon S^2 \to \mathcal{K}'_\varepsilon$ represent the element of $\pi_2(\mathcal{K}'_\varepsilon)$ which corresponds to E_i under these isomorphisms.

For each $i \in 1, \dots, r$ we then attach a 3–cell B_i to $\mathcal{K}'_\varepsilon$ via the attaching map $\psi_i\colon \partial B_i \to \mathcal{K}'_\varepsilon$. Let $\mathcal{K}''_\varepsilon$ denote the resulting 3–complex. Then we have that $C_*(\mathcal{K}''_\varepsilon)$ is

$$C_*(\mathcal{K}''_\varepsilon): \quad \begin{aligned} &\mathbb{Z}[E/K]^r \xrightarrow{\partial_3} C_2(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K] \oplus \mathbb{Z}[E/K]^r \\ &\qquad \xrightarrow{\partial_2\oplus 0} C_1(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K] \xrightarrow{\partial_1} C_0(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K]\end{aligned}$$

where ∂_3 is inclusion of the second summand.

Hence we have:

Lemma 2.3 *$H_*(\mathcal{K}''_\varepsilon, \mathcal{K}_\varepsilon; M) = 0$ for all left modules M over $\mathbb{Z}[E/K]$.*

Proof We have the following relative complex:

$$C_*(\mathcal{K}''_\varepsilon, \mathcal{K}_\varepsilon): \quad \mathbb{Z}[E/K]^r \xrightarrow{\sim} \mathbb{Z}[E/K]^r \to 0 \to 0$$ □

Thus by Theorem 2.1 we may conclude that $\mathcal{K}''_\varepsilon$ has the homotopy type of $\mathcal{K}^+_\varepsilon$.

Lemma 2.4 *The complex $\mathcal{K}''_\varepsilon$ is cohomologically 2–dimensional.*

Proof The inclusion $\iota\colon \mathcal{K}_\varepsilon \hookrightarrow \mathcal{K}''_\varepsilon$ induces a chain homotopy equivalence:

$$C_*(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K] \to C_*(\mathcal{K}''_\varepsilon)$$ □

Corollary 2.5 *We may choose $\mathcal{K}^+_\varepsilon$ to be the cohomologically 2–dimensional finite 3–complex $\mathcal{K}''_\varepsilon$.*

3 Cohomologically 2–dimensional 3–complexes

Let X be a finite connected 3–complex with $H^3(X;\beta) = 0$ for all coefficient bundles β. In this section we will show that up to homotopy, X arises as the Quillen plus construction applied to a finite Cayley complex.

Let T be a maximal tree in the 1–skeleton of X. The quotient map $X \to X/T$ is a homotopy equivalence. Hence we may assume without loss of generality that X has one 0–cell. We take this to be the basepoint of X and any complexes obtained from X by adding or removing cells. Also we set $G = \pi_1(X)$ with respect to this basepoint.

Let $C_*(X)$ be denoted by

$$F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

where the F_i, $i = 0, 1, 2, 3$, are free modules over $\mathbb{Z}[G]$ and the ∂_i are linear maps over $\mathbb{Z}[G]$.

We have $H^3(X; F_3) = 0$ so in particular there exists ϕ such the following diagram commutes:

$$\begin{tikzcd} F_3 \arrow[r, "\partial_3"] \arrow[d, "1"] & F_2 \arrow[r, "\partial_2"] \arrow[dl, "\phi"] & F_1 \arrow[r, "\partial_1"] & F_0 \\ F_3 & & & \end{tikzcd}$$

Hence ∂_3 is the inclusion of the first summand $\partial_3\colon F_3 \hookrightarrow \partial_3(F_3) \oplus S = F_2$, where S is the kernel of ϕ. Let X' denote the wedge of X with one disk for each 3–cell in X. Then the inclusion of cell complexes $X \hookrightarrow X'$ is a homotopy equivalence and:

$$C_*(X'): \quad F_3 \xrightarrow{\partial_3'} F_2 \oplus F_3' \xrightarrow{\partial_2'} F_1 \oplus F_3' \xrightarrow{\partial_1'} F_0$$

Here $F_3' \cong F_3$ and the maps are defined as follows:

∂_1' restricts to ∂_1 on F_1 and restricts to 0 on F_3',

$\partial_2' = \begin{pmatrix} \partial_2 & 0 \\ 0 & 1 \end{pmatrix}$,

∂_3' is $\partial_3\colon F_3 \to F_2$ composed with the natural inclusion: $F_2 \hookrightarrow F_2 \oplus F_3'$.

Thus ∂_3' is the inclusion into the first summand $\partial_3'\colon F_3 \hookrightarrow \partial_3' F_3 \oplus S \oplus F_3'$.

Let m denote the number of 2–cells in X. The submodule $S \oplus F_3' \subset (\partial_3' F_3 \oplus S) \oplus F_3'$ is isomorphic to $S \oplus F_3 \cong F_2$ and hence has a basis $\boldsymbol{x_1}, \dots, \boldsymbol{x_m} \in F_2 \oplus F_3'$.

The cell complex X' has one 0–cell, so $F_0 \cong \mathbb{Z}[G]$. Let n denote the number of 1–cells in X'. Then each 1–cell corresponds to a generator g_i, $i \in [1, \dots, n]$ of G. Let $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ form the corresponding basis for $F_1 \oplus F_3'$.

Let r denote the number of 2–cells in X'. The attaching map for each 2–cell maps the boundary of a disk round a word in the g_i. For each 2–cell let R_j, $j \in [1, \dots, r]$ denote this word. Let $\{\boldsymbol{E_1}, \dots, \boldsymbol{E_r}\}$ form the corresponding basis for $F_2 \oplus F_3'$. Thus we have a presentation $G = \langle g_1, \dots, g_n \mid R_1, \dots, R_r \rangle$.

We may therefore express each $\boldsymbol{x_i}$ as a linear combination of the $\boldsymbol{E_j}$. Thus for some integers v_i and sequences $j_{i1}, \dots, j_{iv_i} \in \{1, \dots, r\}$ we have

$$\boldsymbol{x_i} = \sum_{l=1}^{v_i} \boldsymbol{E_{j_{il}}} \lambda_{il} \sigma_{il}$$

with each $\lambda_{il} \in G$ and $\sigma_{il} \in \{1, -1\}$. For each $i \in [1, \dots, m]$, $l \in [1, \dots, v_i]$ let w_{il} be a word in the g_k, $k = 1, \dots, n$, representing λ_{il}. Now for each $i = 1, \dots, m$, let:

$$S_i = \prod_{l=1}^{v_i} w_{il}^{-1} R_{j_{il}}^{\sigma_{il}} w_{il}$$

For each $i \in \{1, \dots, m\}$, attach a 2–cell a_i to X' by mapping the boundary of the disk around the path in the 1–skeleton of X' corresponding to the word S_i. Let Z denote the resulting finite cell complex. Note that each word S_i corresponds to a trivial element of G, so the inclusion $X' \subset Z$ induces an isomorphism $\pi_1(X') \cong \pi_1(Z)$. Hence we may write $C_*(Z)$:

$$C_*(Z): \quad F_3 \xrightarrow{\partial_3''} (F_2 \oplus F_3') \oplus F_2' \xrightarrow{(\partial_2' \ \partial_2'')} (F_1 \oplus F_3') \xrightarrow{\partial_1'} F_0$$

where ∂_3'' is understood to be $\partial_3'\colon F_3 \to (F_2 \oplus F_3')$ composed with the natural inclusion $(F_2 \oplus F_3') \hookrightarrow (F_2 \oplus F_3') \oplus F_2'$.

For $i = 1, \dots, m$ let $\boldsymbol{A_i}$ be the basis element of F_2' corresponding to the 2–cell a_i. Recall the Fox free differential, ∂. We have:

$$\partial_2'' \boldsymbol{A_i} = \partial S_i = \sum_{l=1}^{v_i} \partial(w_{il}^{-1} R_{j_{il}}^{\sigma_{il}} w_{il}) = \sum_{l=1}^{v_i} \partial_2' \boldsymbol{E_{j_{il}}} \lambda_{il} \sigma_{il} = \partial_2' \boldsymbol{x_i}$$

Thus $\boldsymbol{A_i} - \boldsymbol{x_i}$ represents a class in $H_2(\widetilde{Z^{(2)}}; \mathbb{Z})$ which is isomorphic to $\pi_2(Z^{(2)})$ via the Hurewicz isomorphism composed with the map $\pi_2(\widetilde{Z^{(2)}}) \to \pi_2(Z^{(2)})$ induced by the covering map. Let $\psi_i\colon S^2 \to Z^{(2)}$ represent the corresponding element of $\pi_2(Z^{(2)})$.

Then for each $i = 1, \dots, m$ we may attach a 3–cell b_i to Z via the map ψ_i. We denote the resulting complex X''.

Lemma 3.1 *The inclusion* $\iota \colon X' \subset X''$ *is a homotopy equivalence.*

Proof Starting with X', for each i we attached a 2–cell a_i with contractible boundary in X', and then attached a 3–cell b_i with a_i as a free face. Thus X'' is obtained from X' through a series of cell expansions and the inclusion $X' \subset X''$ is a simple homotopy equivalence. □

Let Y denote the subcomplex of X'' consisting of the 1–skeleton, $X''^{(1)}$, together with the a_i, $i = 1, \dots, m$. Let ε denote the group presentation $\langle g_1, \dots, g_n \mid S_1, \dots, S_m \rangle$ and let E denote the underlying group. By construction we have $Y = \mathcal{K}_\varepsilon$.

Let $k_1, \dots, k_r \in E$ denote the elements represented by the words $R_1, \dots, R_r$. Let K denote the normal closure in E of $k_1, \dots, k_r$. By construction then, K is finitely closed and we have a short exact sequence of groups:

$$1 \to K \to E \to G \to 1$$

Lemma 3.2 K *is a perfect group.*

Proof Clearly $\mathbb{Z}[G]$ is a right module over itself and there is a left action of E on $\mathbb{Z}[G]$. The algebraic complex $C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G]$ is given by:

$$F_2' \xrightarrow{\partial_2''} (F_1 \oplus F_3') \xrightarrow{\partial_1'} F_0$$

Now consider $C_*(X')$:

$$F_3 \xrightarrow{\partial_3'} F_2 \oplus F_3' \xrightarrow{\partial_2'} F_1 \oplus F_3' \xrightarrow{\partial_1'} F_0$$

As $\widetilde{X}'$ is simply connected, we have $\ker(\partial_1') = \mathrm{Im}(\partial_2')$.

Recall that $F_2 \oplus F_3' = \partial_3'(F_3) \oplus S \oplus F_3'$ and that $S \oplus F_3'$ has basis $\boldsymbol{x_1}, \dots, \boldsymbol{x_m}$. Clearly ∂_2' restricts to 0 on $\partial_3'(F_3)$, so $\ker(\partial_1') = \mathrm{Im}(\partial_2')$ which is generated by the $\partial_2'(\boldsymbol{x_i})$.

Also recall that $\partial_2' \boldsymbol{x_i} = \partial_2'' \boldsymbol{A_i}$. Hence $\ker(\partial_1') = \mathrm{Im}(\partial_2'')$ and $H_1(C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G]) = 0$.

However by restricting coefficients $C_*(\mathcal{K}_\varepsilon)$ may be regarded as an algebraic complex of free modules over $\mathbb{Z}[K]$. Hence we have

$$K/[K,K] = H_1(K;\mathbb{Z}) = H_1(C_*(\mathcal{K}_\varepsilon) \otimes_K \mathbb{Z}) = H_1(C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G]) = 0$$

where $\mathbb{Z}$ is regarded as having a trivial left K–action. □

Lemma 3.3 *$X'' = \mathcal{K}_\varepsilon^+$ where $+$ is taken with respect to K.*

Proof We may identify $\mathcal{K}_\varepsilon$ with the subcomplex $Y \subset X''$. The inclusion $\ell\colon \mathcal{K}_\varepsilon \hookrightarrow X''$ then induces the quotient map $E \to E/K$ on fundamental groups. By Theorem 2.1 it is sufficient to show that $H_*(X'', Y; M) = 0$ for all left coefficient modules M.

Let $\ell_*\colon C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G] \to C_*(X'')$ be the chain map induced by the inclusion $\ell\colon \mathcal{K}_\varepsilon \hookrightarrow X''$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & F_2' & \xrightarrow{\partial_2''} & (F_1 \oplus F_3') & \xrightarrow{\partial_1'} & F_0 \\
 & & \downarrow{\scriptstyle \ell_2} & & \downarrow{\scriptstyle \ell_1} & & \downarrow{\scriptstyle \ell_0} \\
F_3 \oplus F_2'' & \xrightarrow{(\partial_3''\ \partial_3''')} & (F_2 \oplus F_3') \oplus F_2' & \xrightarrow{(\partial_2'\ \partial_2'')} & (F_1 \oplus F_3') & \xrightarrow{\partial_1'} & F_0
\end{array}$$

where F_2'' has a basis $\boldsymbol{D_1}, \dots, \boldsymbol{D_m}$ corresponding to the 3–cells $b_1, \dots, b_m$, so for $i = 1, \dots, m$ we have $\partial_3'''(D_i) = \boldsymbol{A_i} - \boldsymbol{x_i}$. Here ℓ_0 and ℓ_1 are the identity maps and ℓ_2 is the inclusion of the second summand.

We have that $(F_2 \oplus F_3') = \partial_3'' F_3 \oplus (S \oplus F_3')$. Hence we have $(F_2 \oplus F_3') \oplus F_2' = \partial_3'' F_3 \oplus (S \oplus F_3') \oplus F_2'$.

The submodule $(S \oplus F_3')$ has basis $\boldsymbol{x_1}, \dots, \boldsymbol{x_m}$. The submodule F_2' has basis $\boldsymbol{A_1}, \dots, \boldsymbol{A_m}$. Also $\partial_3''' F_2''$ has basis $\boldsymbol{A_1} - \boldsymbol{x_1}, \dots, \boldsymbol{A_m} - \boldsymbol{x_m}$. Hence we have the following equality of submodules: $(S \oplus F_3') \oplus F_2' = \partial_3''' F_2'' \oplus F_2'$.

Thus: $$(F_2 \oplus F_3') \oplus F_2' = \partial_3'' F_3 \oplus \partial_3''' F_2'' \oplus F_2'$$

The relative chain complex $C_*(X'', Y)$ is therefore given by

$$F_3 \oplus F_2'' \xrightarrow{\sim} \partial_3'' F_3 \oplus \partial_3''' F_2'' \longrightarrow 0 \longrightarrow 0$$

and $H_*(X'', Y; M) = 0$ for all left coefficient modules M as required. □

As $X \sim X''$, we have proved the following theorem:

Theorem 3.4 *Let X be a finite connected 3–complex with $H^3(X; \beta) = 0$ for all coefficient bundles β. Then X has the homotopy type of $\mathcal{K}_\varepsilon^+$ for some finite presentation ε of a group E, where $+$ is taken with respect to some perfect finitely closed normal subgroup $K \lhd E$.*

4 Implications for the D(2) problem

The D(2) problem asks if every finite cohomologically 2–dimensional 3–complex must be homotopy equivalent to a finite 2–complex. Clearly a counterexample must have a connected component which is also cohomologically 2–dimensional but not homotopy equivalent to a finite 2–complex. By Theorem 3.4 this component must have the homotopy type of $\mathcal{K}_\varepsilon^+$ for some finite presentation ε of a group E, where $+$ is taken with respect to some perfect finitely closed normal subgroup $K \lhd E$.

Conversely, by Corollary 2.5, given any finite presentation ε of a group E together with some perfect finitely closed normal subgroup $K \lhd E$ we have a cohomologically 2–dimensional finite 3–complex, $\mathcal{K}_\varepsilon^+$. It follows that the D(2) problem is equivalent to:

> Given a finite presentation ϵ for a group E, and a finitely closed perfect normal subgroup $K \lhd E$, must $\mathcal{K}_\varepsilon^+$ be homotopy equivalent to a finite 2–complex?

Suppose that we have a homotopy equivalence $\mathcal{K}_\varepsilon^+ \sim Y$ for some finite 2–complex Y. Let T be a maximal tree in the 1–skeleton of Y. The quotient map $Y \to Y/T$ is a homotopy equivalence so $Y \sim \mathcal{K}_\mathcal{G}$ for some finite presentation $\mathcal{G}$ of $\pi_1(Y) = \pi_1(\mathcal{K}_\varepsilon^+) = E/K$.

Hence the affirmative answer to the D(2) problem would be equivalent to:

> For all finitely presented groups E and all perfect finitely closed normal subgroups $K \lhd E$ and all finite presentations ε of E, there exists a finite presentation $\mathcal{G}$ of E/K and a homotopy equivalence $\mathcal{K}_\varepsilon^+ \sim \mathcal{K}_\mathcal{G}$ inducing the identity $1\colon E/K \to E/K$ on fundamental groups.

Lemma 4.1 *The following are equivalent:*

(i) *There exists a homotopy equivalence $\mathcal{K}_\varepsilon^+ \sim \mathcal{K}_\mathcal{G}$ inducing the identity $1\colon E/K \to E/K$ on fundamental groups.*

(ii) *There exists a chain homotopy equivalence $C_*(\mathcal{K}_\varepsilon^+) \sim C_*(\mathcal{K}_\mathcal{G})$ over $\mathbb{Z}[E/K]$.*

Proof (i) $\Rightarrow$ (ii) is immediate. Conversely, from (ii) we have a chain homotopy equivalence between the algebraic complexes associated to a finite cohomologically 2–dimensional 3–complex and a finite 2–complex (with respect to an isomorphism of fundamental groups). To show that (ii) $\Rightarrow$ (i) we must construct a homotopy equivalence between the spaces, inducing the same isomorphism on fundamental groups. For finite fundamental groups this is done in [8, Proof of Theorem 59.4]. The same argument holds for all finitely presented fundamental groups [8, Appendix B, Proof of Weak Realization Theorem]. □

From the proof of Lemma 2.4, $C_*(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K]\sim C_*(\mathcal{K}_\varepsilon^+)$. Hence we have:

Theorem 4.2 *The following two statements are equivalent:*

(i) *Let X be a a finite 3–complex with $H^3(X;\beta)=0$ for all coefficient bundles β. Then X is homotopy equivalent to a finite 2–complex.*

(ii) *Let K be a perfect finitely closed normal subgroup of a finitely presented group E. For each finite presentation ε of E, there exists a finite presentation $\mathcal{G}$ of E/K, such that we have a chain homotopy equivalence over $\mathbb{Z}[E/K]$:*

$$C_*(\mathcal{K}_\varepsilon)\otimes_E \mathbb{Z}[E/K]\to C_*(\mathcal{K}_\mathcal{G})$$

Suppose we have a short exact sequence

$$1\to L\to F\to G\to 1$$

where G is a finitely presented group and F is a free group generated by elements $g_1,\dots,g_n$. Let $R_1,\dots,R_m$ be elements of L.

Definition 4.3 $\langle g_1,\dots,g_n \mid R_1,\dots,R_m\rangle$ is called a finite partial presentation for G when the normal closure $N_F(R_1,\dots,R_m)$ surjects onto $L/[L,L]$ under the quotient map $L\to L/[L,L]$.

Note that a finite partial presentation $\varepsilon=\langle g_1,\dots,g_n \mid R_1,\dots,R_m\rangle$ as above is an actual finite presentation of some group E, so it has a well defined Cayley complex $\mathcal{K}_\varepsilon$.

Let K denote the kernel of the homomorphism $E\to G$ sending each g_i to the corresponding element in G. If G is finitely presented then it is finitely presented on the generators in ε [3, Chapter 1, Proposition 17]. As K is the normal closure in E of the images of this finite set of relators we have that K is finitely closed.

Further K is perfect as every $k\in K$ may be lifted to an element of L which may be written in the form ab where $a\in[L,L]$ and $b\in N_F(R_1,\dots,R_m)$. Thus k is equal to the image of a in E, so $k\in[K,K]$. Thus a finite partial presentation ε of a finitely presented group G may be viewed as a presentation satisfying the hypothesis' of statement (ii) in Theorem 4.2.

Conversely, given ε as in statement (ii) of Theorem 4.2, we have that ε is a finite partial presentation of E/K (as $K=[K,K]$), and E/K is finitely presented (as K is finitely closed).

Thus statement (ii) is equivalent to:

(ii)' Given a finite partial presentation ε of a finitely presented group G, there exists a finite presentation $\mathcal{G}$ of G, such that we have a chain homotopy equivalence

$$C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G] \to C_*(\mathcal{K}_\mathcal{G})$$

where E is the group presented by ε and each $x \in E$ acts on $\mathbb{Z}[G]$ by left multiplication by its image in G.

One approach to the D(2) problem is to use Euler characteristic as an obstruction. That is, given a finite cohomologically 2–dimensional 3–complex X, if we can show that every finite 2–complex Y with $\pi_1(Y) = \pi_1(X)$ satisfies $\chi(X) < \chi(Y)$ then clearly X cannot be homotopy equivalent to any such Y. It has been shown that certain constructions involving presentations of a group would allow one to construct such a space [6, Theorem 3.5]. A candidate for such a space is given in [2]. In light of Corollary 2.5 and Theorem 3.4 we are able to generalize this approach.

The deficiency $\mathrm{Def}(\mathcal{G})$ of a finite presentation $\mathcal{G}$ is the number of generators minus the number of relators. We say a presentation of a group is minimal if it has the maximal possible deficiency. A finitely presented group G always has a minimal presentation, because an upper bound for the deficiency of a presentation is given by $\mathrm{rk}_\mathbb{Z}(G/[G,G])$. The deficiency $\mathrm{Def}(G)$ of a finitely presented group G is defined to be the deficiency of a minimal presentation.

Again let $K \lhd E$ be a perfect finitely closed normal subgroup. Then if ε is a finite presentation of E and $\mathcal{G}$ is a finite presentation for E/K we have:

$$\chi(\mathcal{K}_\varepsilon^+) = \chi(\mathcal{K}_\varepsilon) = 1 - \mathrm{Def}(\varepsilon), \quad \chi(\mathcal{K}_\mathcal{G}) = 1 - \mathrm{Def}(\mathcal{G})$$

Lemma 4.4 *If* $\mathrm{Def}(E) > \mathrm{Def}(E/K)$ *then given a minimal presentation* ε *of* E *we have that* $\chi(\mathcal{K}_\varepsilon^+) < \chi(\mathcal{K}_\mathcal{G})$ *for any finite presentation* $\mathcal{G}$ *of* E/K.

Proof $\chi(\mathcal{K}_\mathcal{G}) = 1 - \mathrm{Def}(\mathcal{G}) \geq 1 - \mathrm{Def}(E/K) > 1 - \mathrm{Def}(E) = 1 - \mathrm{Def}(\varepsilon) = \chi(\mathcal{K}_\varepsilon^+)$. □

Suppose we have a short exact sequence of groups

$$1 \to K \to E \to G \to 1$$

with E, G finitely presented. Then given a finite presentation for E, the images in G of the generators will generate G. We may present G on these generators with a finite set of relators [3, Chapter 1, Proposition 17]. Let $k_1, \ldots, k_r$ denote the elements of K represented by these relators. Then K is the normal closure in E of $k_1, \ldots, k_r$ and so

K is finitely closed in E. In particular $K/[K,K]$ is generated by the $k_1,\dots,k_r$ as a right module over $\mathbb{Z}[G]$ (where G acts on $K/[K,K]$ by conjugation). Let $\mathrm{rk}_G(K)$ denote the minimal number of elements required to generate $K/[K,K]$ over $\mathbb{Z}[G]$.

Theorem 4.5 *The following statements are equivalent:*

(i) *There exists a connected finite cohomologically 2–dimensional 3–complex X, such that for all finite connected 2–complexes Y with $\pi_1(Y)=\pi_1(X)$ we have $\chi(X)<\chi(Y)$.*

(ii) *There exists a short exact sequence of groups $1\to K\to E\to G\to 1$ with E, G finitely presented and:*

$$\mathrm{rk}_G(K)+\mathrm{Def}(G)<\mathrm{Def}(E)$$

Proof (i) $\Rightarrow$ (ii) By Theorem 3.4, X is homotopy equivalent to $\mathcal{K}_\varepsilon^+$ for some finite presentation ε of some group E and some perfect finitely closed normal subgroup K. Let $G=E/K$. We have a short exact sequence:

$$1\to K\to E\to G\to 1$$

As K is finitely closed, G is finitely presented. As K is perfect we have $\mathrm{rk}_G(K)=0$. Let $\mathcal{G}$ be some finite presentation of G. We have:

$$1-\mathrm{Def}(\varepsilon)=\chi(\mathcal{K}_\varepsilon^+)<\chi(\mathcal{K}_\mathcal{G})=1-\mathrm{Def}(\mathcal{G})$$

Thus $\mathrm{Def}(\mathcal{G})<\mathrm{Def}(\varepsilon)$. As $\mathcal{G}$ was chosen arbitrarily, we have $\mathrm{Def}(G)<\mathrm{Def}(\varepsilon)\le\mathrm{Def}(E)$. Hence $0+\mathrm{Def}(G)<\mathrm{Def}(E)$ as required.

(ii) $\Rightarrow$ (i) We start with the short exact sequence $1\to K\to E\to G\to 1$. Let $k_1,\dots,k_r\in K$ generate $K/[K,K]$ over $\mathbb{Z}[G]$, where $r=\mathrm{rk}_G(K)$. Let K' denote the normal closure in E of $k_1,\dots,k_r$. Then we have a short exact sequence:

$$1\to K/K'\to E/K'\to G\to 1$$

Then $K=K'[K,K]$ so K/K' is perfect. From the discussion preceding this theorem we know that K is finitely closed in E, so K/K' must be finitely closed in E/K'. Also E/K' may be presented by taking a minimal presentation of E and adding r relators (representing to $k_1,\dots,k_r$). Hence:

$$\mathrm{Def}(E/K')\ge\mathrm{Def}(E)-\mathrm{rk}_G(K)>\mathrm{Def}(G)$$

Take a minimal presentation ε of E/K' and let $X=\mathcal{K}_\varepsilon^+$, where $+$ is taken with respect to K/K'. Any finite connected 2–complex Y with $\pi_1(Y)=\pi_1(X)$ is homotopy equivalent to $\mathcal{K}_\mathcal{G}$ for some finite presentation $\mathcal{G}$ of G. Therefore by Lemma 4.4 we have $\chi(X)<\chi(Y)$ as required. □

We note that Michael Dyer proved (ii) $\Rightarrow$ (i) in the case where $H^3(G;\mathbb{Z}[G])=0$ and E is a free group whose generators are the generating set for some minimal presentation of G [6, Theorem 3.5].

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