Generalised Swan modules and the D(2) problem

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We give a detailed proof that, for any natural number n, each algebraic two complex over $C_n \times C_\infty$ is realised up to congruence by a geometric complex arising from a presentation for the group.

57M20, 16D70, 55P15; 55Q05

1 Introduction

It is well known that Whitehead's theorem allows the study of homotopy types of two dimensional CW complexes to be phrased in terms of chain homotopy types of algebraic complexes, arising as the cellular chains on the universal cover. It is natural to ask whether the category of algebraic complexes fully represents the category of CW complexes, in particular whether every algebraic complex is realised geometrically. The case of two dimensional complexes is of special interest, partly due to the relationship between such complexes and group presentations and partly since, as was recently proved, it relates to the question as to when cohomology is a suitable indicator of dimension.

By an *algebraic 2–complex* over a group Γ we mean any exact sequence of (right) $\mathbb{Z}[\Gamma]$ modules of the form:

$$0 \longrightarrow M \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where each F_i is finitely generated free and $\mathbb{Z}$ denotes the trivial $\mathbb{Z}[\Gamma]$ module. The Realisation Problem asks if all such complexes are chain homotopy equivalent to a complex arising from a two dimensional CW complex, which may be assumed to be the Cayley complex of some presentation for Γ. If so, we say that the *realisation property* holds for Γ.

The D(2) Problem, as originally formulated by Wall in [18], asks if any three dimensional CW complex is necessarily homotopic to a complex of dimension two provided that, ranging over all possible coefficient systems, the complex has zero homology and cohomology in dimensions higher than two. The problem is parameterised by the fundamental group in the sense that, since homotopy equivalence induces an

Published: 24 February 2006 DOI: 10.2140/agt.2006.6.71

isomorphism on fundamental groups, one may prove or disprove the D(2) problem for CW complexes with a specified fundamental group. Accordingly, we say that *the D(2) property* holds for Γ if every three dimensional complex with fundamental group isomorphic to Γ, satisfying the hypothesis of the D(2) problem, is homotopy equivalent to a two dimensional complex. In [8], Johnson relates the D(2) problem to the realization problem as follows:

Theorem 1.1 (F E A Johnson) *Let Γ be a finitely presented group such that there is an algebraic 2–complex:*

$$0 \longrightarrow M \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with F_i, M finitely generated over $\mathbb{Z}[\Gamma]$ and each F_i free. Then the D(2) property holds for Γ if and only if the realization property holds for Γ.

We prove:

Theorem A *The D(2) Property holds for $C_n \times C_\infty$.*

In the case of Γ abelian and of rank one, the Bass–Murthy paper [1] shows that all stably free $\mathbb{Z}[\Gamma]$ modules are necessarily free. This result encouraged us to investigate the realization problem in this case and we remark that the Bass–Murthy result is essential to our work, although its role is subtle. Indeed, in all confirmed cases of the D(2) property, the proof is dependent on a proof that stably free modules are free. At present, the class of groups to which this applies is limited to finite abelian groups (Latiolais and Browning [11; 3]), free groups (Johnson [8]), the dihedral groups of order $4n+2$ (Johnson [7]) and the dihedral group of order 8 (Mannan [13]). In the appendix to this paper we give a proof that the D(2) property holds for the group $C_\infty \times C_\infty$.

2 Preliminaries

Suppose that $\mathcal{G} = \langle x_1, \dots x_g \mid W_1, \dots, W_r \rangle$ is a presentation for Γ, then the corresponding Cayley complex $X_{\mathcal{G}}$ is a 2–dimensional CW complex with fundamental group Γ. The chain complex of the universal cover $\widetilde{X}_{\mathcal{G}}$ gives rise to a complex of $\mathbb{Z}[\Gamma]$ modules thus:

$$C_*(\widetilde{X}_{\mathcal{G}}) = \left(0 \longrightarrow \pi_2(X_{\mathcal{G}}) \longrightarrow \mathbb{Z}[\Gamma]^r \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^g \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \right)$$

where:

(i) We have identified $\mathrm{Ker}(\partial_2) = H_2(\widetilde{X}_{\mathcal{G}})$ with $\pi_2(X_{\mathcal{G}})$ via the Hurewitz isomorphism.

(ii) Since the universal cover is simply connected, and by (i) above, the complex is an exact sequence, so $C_*(\widetilde{X}_{\mathcal{G}})$ is an algebraic 2–complex.

Note that ∂_2 and ∂_1 are completely determined by the relations and generators of $\mathcal{G}$ and may be given explicitly, following Fox [5]. In what follows, we shall say that an algebraic 2–complex E is *geometrically realized* if is there exists a presentation $\mathcal{G}$ for Γ and a chain homotopy equivalence $E \simeq C_*(\widetilde{X}_{\mathcal{G}})$. The realization property, as defined in the introduction, holds for Γ if and only if every algebraic 2–complex over Γ is geometrically realized.

The first reduction is due to Schanuel [16] and shows that, in the context of Theorem 1.1, M is determined up to stability by appearing in such a sequence – where stable equivalence of modules M and N is taken to be the existence of natural numbers a,b and an isomorphism $M \oplus \mathbb{Z}[\Gamma]^a \cong N \oplus \mathbb{Z}[\Gamma]^b$. The first task is to determine the class of modules which are stably isomorphic to M, and has historically presented the more difficult objective. The second determines each congruence class of algebraic two complexes with a given M, and asks which of these are chain homotopy equivalent to a complex arising from a group presentation. Two key reductions are then possible in some cases:

Proposition 2.1 (F E A Johnson) *Suppose that we are given Γ satisfying the hypothesis of Theorem 1.1 and such that $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$. Then:*

(i) *If each algebraic 2–complex of the form*

$$0 \longrightarrow M \longrightarrow \mathbb{Z}[\Gamma]^{a_1} \longrightarrow \mathbb{Z}[\Gamma]^{a_2} \longrightarrow \mathbb{Z}[\Gamma]^{a_3} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is geometrically realized, then each algebraic 2–complex of the form

$$0 \longrightarrow M \oplus \mathbb{Z}[\Gamma]^{b_0} \longrightarrow \mathbb{Z}[\Gamma]^{b_1} \longrightarrow \mathbb{Z}[\Gamma]^{b_2} \longrightarrow \mathbb{Z}[\Gamma]^{b_3} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is geometrically realized.

(ii) *There exists a ring structure on $\mathrm{Ext}^3(\mathbb{Z}, M)$ under which congruence classes of algebraic 2–complexes are units.*

We present a concise version of the proof, following Humphreys [6]. Suppose that a presentation $\mathcal{H}$ for Γ realizes the extension

$$C_*(\widetilde{X}_{\mathcal{H}}) = 0 \to M \xrightarrow{i} \mathbb{Z}[\Gamma]^{a_1} \xrightarrow{\delta_2} \mathbb{Z}[\Gamma]^{a_2} \xrightarrow{\delta_1} \mathbb{Z}[\Gamma]^{a_3} \to \mathbb{Z} \to 0,$$

then the addition of n trivial relations to $\mathcal{H}$ may be seen to realize the extension

$$f(C_*(\widetilde{X}_{\mathcal{H}})) = 0 \to M \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{[i,Id]} \mathbb{Z}[\Gamma]^{a_1+n} \xrightarrow{[\delta_2,0]} \mathbb{Z}[\Gamma]^{a_2} \xrightarrow{\delta_1} \mathbb{Z}[\Gamma]^{a_3} \to \mathbb{Z} \to 0.$$

If E is an arbitrary extension in $\mathrm{Ext}^3(\mathbb{Z}, M)$, then we may define $f(E)$ to be the extension in $\mathrm{Ext}^3(\mathbb{Z}, M \oplus \mathbb{Z}[\Gamma]^n)$ obtained similarly; f is an isomorphism $\mathrm{Ext}^3(\mathbb{Z}, M) \cong \mathrm{Ext}^3(\mathbb{Z}, M \oplus \mathbb{Z}[\Gamma]^n)$ and is a bijection on extensions which are chain homotopy equivalent to algebraic complexes. This completes the proof of (i).

Define the relation $\sim$ on the endomorphism ring $\mathrm{End}(M)$ by $f \sim h$ if and only if $f - h$ factors through $i\colon M \to \mathbb{Z}[\Gamma]^{a_1}$. Then $\sim$ is an equivalence relation. By standard homological algebra, there exists an additive group isomorphism

$$\mathrm{Ext}^3(\mathbb{Z}, M) \cong \mathrm{End}(M)/\sim$$

we show that the ring structure is preserved in the quotient, i.e. $ga \sim hb$ for any $a \sim b$ and $g \sim h$. For any such $a, b, g, h \in \mathrm{End}(M)$ it is trivial that $ga \sim gb$ (draw the diagram!). Since $(g - h) \sim 0$ there exists some factorisation $(g - h) = \varphi_1 \circ i$. Consider the homomorphism $i \circ b\colon M \to \mathbb{Z}[\Gamma]^{a_1}$; this represents a congruence class of $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]^{a_1})$ and since $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$, by additivity $i \circ b$ factors through $\mathbb{Z}[\Gamma]^{a_1}$ as some $\varphi_2 \circ i$ and $(g-h) \circ b = \varphi_1 \circ \varphi_2 \circ i$. Thus the product structure is well defined on the quotient.

Suppose that some $f \in \mathrm{End}(M)$ represents the congruence class of an algebraic 2–complex E'. Then E' may be extended to a projective resolution for $\mathbb{Z}$ which one may use to calculate the elements of $\mathrm{Ext}^3(\mathbb{Z}, M)$ as a quotient of $\mathrm{End}(M)$. Let h represent the congruence class of the given complex E under these terms, then $hf \sim Id$ and hence h is a left inverse for f. Since $\mathrm{Ext}^3(\mathbb{Z}, M)$ is finitely generated as an abelian group, h is also a right inverse. The result follows.

3 Generalised Swan modules over $C_n \times C_\infty$

Throughout, we shall assume that $G = C_n$ is the cyclic group on n elements and Γ is the product $G \times C_\infty$. Fix R to represent the ring and trivial $\mathbb{Z}[\Gamma]$ module $\mathbb{Z}[C_\infty]$, so that $\mathbb{Z}[\Gamma] = R[G]$. In this case, a standard representative of the stable class of the second homotopy module may be constructed from the presentation

$$\mathcal{G} = \langle x, t | x^n = 1, tx = xt \rangle.$$

Writing N for $\sum_{g\in C_n} g$, $\pi_2(X_{\mathcal{G}})$ may be identified with the submodule of $\mathbb{Z}[\Gamma]^2$ generated by the elements

$$\begin{pmatrix} 0 \\ x-1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} N \\ t-1 \end{pmatrix}.$$

with corresponding chain complex

$$C_*(\widetilde{X}_{\mathcal{G}}) = \left(0 \longrightarrow \pi_2(X_{\mathcal{G}}) \longrightarrow \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0\right)$$

$$\partial_2 = \begin{pmatrix} 1-t & N \\ x-1 & 0 \end{pmatrix} \qquad \partial_1 = (x-1 \quad t-1).$$

Since $\pi_2(X_{\mathcal{G}})$ is finitely generated, Γ satisfies the hypothesis of Theorem 1.1. To see that Γ satisfies the conditions of Proposition 2.1, note first that the augmentation ideal $I = \mathrm{Ker}(\varepsilon\colon \mathbb{Z}[\Gamma] \to \mathbb{Z})$ is free as an R module, generated by the elements $(t-1)$ and $\{(x^i-1)\colon 1 \le i \le n-1\}$. The class of such modules is *tame* in the sense of Johnson [8, Chapter 4], so that projective modules are relatively injective and $\mathrm{Ext}^2(I, \mathbb{Z}[\Gamma]) = 0$. The result follows by dimension shifting.

It shall prove easier to calculate the stable class of the dual module $\pi_2(X_{\mathcal{G}})^*$, and we justify doing so by [8, Proposition 28.1], which shows that cancellation holds for $\pi_2(X_{\mathcal{G}})^*$ if and only if it holds for $\pi_2(X_{\mathcal{G}})$.

Directly, one may verify that $\pi_2(X_{\mathcal{G}})^*$ may be identified with the submodule of $\mathbb{Z}[\Gamma]^2$ generated by the the elements:

$$\begin{pmatrix} 1-t \\ x-1 \end{pmatrix} \quad \begin{pmatrix} N \\ 0 \end{pmatrix}$$

R imbeds in $\pi_2(X_{\mathcal{G}})^*$ by identification with the module generated by the second element above, resulting in a short exact sequence of $\mathbb{Z}[\Gamma]$ modules:

$$0 \to R \to \pi_2(X_{\mathcal{G}})^* \to S \to 0$$

where $S = \mathbb{Z}[\Gamma]/\langle N \rangle$. Embed R into $\mathbb{Z}[\Gamma]$ by identifying R with the submodule generated by N, so that for each $k \in \mathbb{N}$ there is an exact sequence:

$$0 \longrightarrow R^k \longrightarrow \pi_2(X_{\mathcal{G}})^* \oplus \mathbb{Z}[\Gamma]^{k-1} \longrightarrow S^k \longrightarrow 0$$

giving a naïve model for modules potentially stably isomorphic to $\pi_2(X_{\mathcal{G}})^*$ as a subclass of the central modules occurring in extensions of R^k by S^k. Through a detailed study of such extensions we shall show that this model is appropriate and prove the required cancellation result.

Proposition 3.1 *Let $\mathbb{Z}_n$ denote the integers modulo n, and $\mathcal{M}_k(R_n)$ the ring of k by k matrices over the ring $\mathbb{Z}_n[C_\infty$. Then $\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(S^k, R^k) \cong \mathcal{M}_k(R_n)$.*

Proof The truth of this statement may be deduced from the 'change of rings formula' given in Cartan and Eilenberg [4] and from the cohomology of C_n. We give two explicit proofs, with the justification that the given constructions of congruence classes will prove useful in the sequel.

A projective resolution of S^k is given by:

$$\cdots \xrightarrow{(x-1)I_k} \mathbb{Z}[\Gamma]^k \xrightarrow{(N)I_k} \mathbb{Z}[\Gamma]^k \xrightarrow{\pi} S^k \longrightarrow 0$$

where S^k is identified with $\mathbb{Z}[\Gamma]^k/(N)\mathbb{Z}[\Gamma]^k$ and I_k denotes the identity matrix. Note that the kernel of π may be identified with R^k. Through the standard homological classification of Ext, Mac Lane [12], it is enough to determine the additive group of homomorphisms $f\colon R^k \to R^k$, modulo those which factor through the inclusion of R^k in $\mathbb{Z}[\Gamma]^k$. Since R may be considered as a ring, and f an R module homomorphism, we may represent f as multiplication by some $k \times k$ matrix F with entries in R. Since the image of R^k in $\mathbb{Z}[\Gamma]^k$ contains only elements divisible by N, if f factors through $\mathbb{Z}[\Gamma]^k$, then all of the entries in F must be divisible by n. Conversely if F is divisible by n, then f clearly factors through $\mathbb{Z}[\Gamma]^k$, i.e. F represents a unique congruence class modulo (n). For each matrix F in $\mathcal{M}_k(R_n)$, we may pick a lift for F in $\mathcal{M}_k(R)$ and define $E(F)$ to be the extension constructed through the pushout:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^k & \longrightarrow & \mathbb{Z}[\Gamma]^k & \xrightarrow{\pi} & S^k & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow & & \| & & \\
0 & \longrightarrow & R^k & \longrightarrow & M(F) & \longrightarrow & S^k & \longrightarrow & 0
\end{array}$$

where the bottom row is $E(F)$ and $M(F)$, the central module, is determined by F. The correspondence between $\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(S^k, R^k)$ and $\mathcal{M}_k(R_n)$ clearly commutes with the addition operations, and our first characterisation is complete.

We perform a second calculation following the methods of [8] regarding extensions within tame classes. Since R, S and $\mathbb{Z}[\Gamma]$ are free as R modules we may take a (relatively) injective coresolution of R^k to be:

$$0 \longrightarrow R^k \longrightarrow \mathbb{Z}[\Gamma]^k \xrightarrow{(x-1)I_k} \mathbb{Z}[\Gamma]^k \xrightarrow{(N)I_k} \cdots$$

We wish to determine the group of homomorphisms $f\colon S^k \to S^k$ modulo those which factor through π as $f = \pi \circ h$ for some $h\colon S^k \to \mathbb{Z}[\Gamma]^k$. Since $S = \mathbb{Z}[\Gamma]/(N)$ may be considered as a quotient ring of $\mathbb{Z}[\Gamma]$, each homomorphism $f\colon S^k \to S^k$ is a S module homomorphism and may be represented as multiplication by some $k \times k$ matrix F with entries in S. By observation of the coresolution above, f factors through π if and only if $(x-1)$ divides each entry of F. Note that $\hat{\varepsilon}$ induces a surjective augmentation homomorphism $\hat{\varepsilon}_S\colon S \to R_n$, which we extend to $\hat{\varepsilon}_S\colon \mathcal{M}_k(S) \to \mathcal{M}_k(R_n)$. Then the kernel of $\hat{\varepsilon}_S$ is precisely the set of matrices with $(x-1)$ a divisor of each entry. This establishes a one-to-one correspondence between $\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(S^k, R^k)$ and $\mathcal{M}_k(R_n)$ which, again, clearly commutes with addition. For each matrix F in $\mathcal{M}_k(R_n)$ we may pick a lift of F in $\mathcal{M}_k(S)$ and define $\overline{E}(F)$ to be the bottom row in the pullback:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^k & \longrightarrow & \mathbb{Z}[\Gamma]^k & \xrightarrow{\pi} & S^k & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow F & & \\
0 & \longrightarrow & R^k & \longrightarrow & \overline{M}(F) & \longrightarrow & S^k & \longrightarrow & 0
\end{array}$$

so that each extension of R^k by S^k is congruent to $\overline{E}(F)$ for some unique $F \in \mathcal{M}_k(R_N)$. □

To give concrete examples of the congruence classes and central modules, given any $A \in \mathcal{M}_k(R_n)$, we may pick a lift of A in $\mathcal{M}_k(\mathbb{Z}[\Gamma])$ and define $M(A)$ to be the submodule of $\mathbb{Z}[\Gamma]^{2k}$ generated by the columns of the matrix:

$$A' = \begin{pmatrix} A & (N)I_k \\ (x-1)I_k & 0 \end{pmatrix}$$

Then R^k embeds in $M(A)$ by identifying the i^{th} basis element of R^k with the $i+k^{th}$ column of A', and the image of R^k in $M(A)$ is the kernel of the surjective map $\pi_A\colon M(A) \to S^k$ given by sending the i^{th} column of A' (for $1 \le i \le k$) to the i^{th} generator of S^k. We represent the resulting exact sequence as:

$$E'(A) = \qquad 0 \longrightarrow R^k \xrightarrow{i_A} M(A) \xrightarrow{\pi_A} S^k \longrightarrow 0$$

and $E'(A) \equiv E(A) \equiv \overline{E}(A)$.

We wish to characterise the isomorphism classes of the modules $M(A)$. Since the action of N is zero on S and multiplication by n on R, and since S is torsion free:

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(R^k, S^k) = 0,$$

so that any isomorphism $\phi\colon M(A) \cong M(B)$ induces an isomorphism of extensions $E(A) \cong E(B)$, where the isomorphisms on each end may be represented as matrices C and D. We distinguish the matrices in $\mathcal{M}_k(R_n)$ which are images of such isomorphisms:

- $GR(k)$ denotes the image of $GL_k(R)$ in $GL_k(R_n)$ under the natural map.
- $GS(k)$ denotes the image of $GL_k(S)$ in $GL_k(R_n)$ under ε_S.

Explicitly, any isomorphism $\phi\colon M(A) \cong M(B)$ results in a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^k & \longrightarrow & M(A) & \longrightarrow & S^k & \longrightarrow & 0 \\ & & \downarrow C & & \downarrow \phi & & \downarrow D & & \\ 0 & \longrightarrow & R^k & \longrightarrow & M(B) & \longrightarrow & S^k & \longrightarrow & 0 \end{array}$$

and it is easily shown that the image of $C^{-1}AD$ in $\mathcal{M}_k(R_n)$ must be equal to B, so that $M(B) \cong M(C^{-1}AD)$. A diagram similar to the above gives, through the five lemma:

Theorem 3.2 *For arbitrary matrices $A, B \in \mathcal{M}_k(R_n)$, $M(A) \cong M(B)$ if and only if there exists $C \in GR(k), D \in GS(k)$ such that $CAD = B$.*

Having characterised the isomorphism classes of the modules $M(A)$, we wish to characterise their stable isomorphism classes. Clearly

$$M(A) \oplus \mathbb{Z}[\Gamma]^m \cong M(A \oplus I_m),$$

where for $B \in \mathcal{M}_m(R_n)$, the matrix $A \oplus B$ is taken to be the $(k+m) \times (k+m)$ matrix:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Corollary 3.3 *$M(A)$ is stably equivalent to $M(B)$ if and only if there exists $C \in GR(k+m), D \in GS(k+m)$ such that $C(A \oplus I_m)D = (B \oplus I_m)$.*

As is standard, we write $E_k(R_n)$ for the group of $k \times k$ matrices over R_n which are products of elementary matrices. Note that $E_k(R_n) \subseteq GS(k)$ and $E_k(R_n) \subseteq GR(k)$. The reader may wish to recall some of the basic results in algebraic K–theory, available for example in Milnor [14], from which we take our notation.

Proposition 3.4 *The determinant homomorphism* det: $GL_k(R_n)/E_k(R_n) \to R_n^+$ *is an isomorphism for all* $k \geq 1$*. Equivalently, every invertible matrix in* $\mathcal{M}_k(R_n)$ *with determinant one is a product of elementary matrices.*

Proof Let the prime decomposition of n be $\Pi_{i=1}^s p_i^{e_i}$. Then R_n is isomorphic to the product $\Pi_{i=1}^s R_{p_i^{e_i}}$ and $\mathcal{M}_k(R_n)$, R_n^+ decompose as products similarly, as does the determinant homomorphism. A fortiori, it is enough to prove the proposition in the case where $n = p^e$ is a power of a prime.

Suppose that $\det(E) = 1$ where $E \in \mathcal{M}_k(R_n)$, and let E_p denote the equivalence class of E in $\mathcal{M}_k(R_p)$. We shall show that we may reduce E by elementary row and column operations to the identity. Since $R_p = \mathbb{F}_p[t, t^{-1}]$ is a Euclidean domain there are elementary matrices $E_p^1, \dots E_p^j$ such that $E_p \cdot E_p^1 \cdot \dots E_p^j = Id_p$. Thus we may pick elementary matrices $E^1, \dots E^j \in E_k(R_n)$ such that:

$$E \cdot E^1 \dots E^j = \begin{pmatrix} 1 + c_{1,1}p & c_{1,2}p & \dots & c_{1,k}p \\ c_{1,2}p & 1 + c_{2,2}p & \dots & c_{2,k}p \\ \vdots & \vdots & & \vdots \\ c_{k,1}p & c_{k,2}p & \dots & 1 + c_{k,k}p \end{pmatrix} \qquad c_{i,j} \in R_n$$

By the binomial theorem each $(1 + c_{i,i}p)$ is a unit (take the $(p^e)^{th}$ power!) and thus the above matrix has diagonal entries which are all units. Thus the above matrix, and hence E, may be reduced by the action of column operations to a matrix of the form:

$$\begin{pmatrix} 1 + d_{1,1}p & 0 & \dots & 0 \\ 0 & 1 + d_{2,2}p & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 + d_{k,k}p \end{pmatrix} \qquad d_{i,i} \in R_n$$

By Whitehead's Lemma, $\mathrm{Diag}(1, \dots, 1, u, u^{-1}, 1, \dots, 1) \in E_k(R_n)$ and E may be further reduced to

$$E' = \mathrm{Diag}(u, 1, \dots, 1)$$

for some $u \in R_n^+$. Since all elementary matrices have determinant one, $\det(E') = \det(E) = 1$, i.e. $u = 1$ and $E' = I_k$. Hence E is a product of elementary matrices. □

Recall that for any Euclidean domain $\mathcal{R}$ and any $k \times k$ matrix A over $\mathcal{R}$, there are elementary matrices E_1 and E_2 over $\mathcal{R}$ such that

- $E_1 \cdot A \cdot E_2 = \mathrm{Diag}(a_1, a_2, \dots, a_k)$ is a diagonal matrix with $|a_{i+1}|$ a divisor of $|a_i|$.

- $\text{Diag}(a_1, a_2, \ldots, a_k)$ is unique up to multiplication by diagonal elementary matrices and is sometimes called the *Smith Normal Form* of A, denoted $\text{SNF}(A)$.

E_1 and E_2 may easily be constructed through a process derived from the Gauss algorithm for Euclidean domains.

Theorem 3.5 *For each non-zero $\alpha \in R_n$ and all $B \in \mathcal{M}_k(R_n)$, $M(\alpha)$ is stably equivalent to $M(B)$ and only if $M(B) \cong M(\alpha) \oplus \mathbb{Z}[\Gamma]^{k-1}$.*

Proof Suppose that $M(\alpha)$ is stably equivalent to $M(B)$ for some non-zero $\alpha \in R_n$ and some $B \in \mathcal{M}_k(R_n)$. Then by $\mathbb{Z}[C_\infty]$ rank considerations we may assume that there exists an $m \in \mathbb{N}$ and an isomorphism $M(B) \oplus \mathbb{Z}[\Gamma]^m \cong M(\alpha) \oplus \mathbb{Z}[\Gamma]^{k+m-1}$, so that by Theorem 3.2 there exist matrices $C \in GR(k+m)$, $D \in GS(k+m)$ such that

$$C(B \oplus I_m)D = (\alpha \oplus I_{m+k-1}).$$

Define $B_{new} = (\det(C) \oplus I_{k-1}) \cdot B \cdot (\det(D) \oplus I_{k-1})$. Since R and S are commutative rings, the determinant homomorphisms are well defined and

$$(\det(C) \oplus I_{k-1}) \in GR(k)$$
$$(\det(D) \oplus I_{k-1}) \in GS(k).$$

By Theorem 3.2 $M(B) \cong M(B_{new})$, so that we may assume that $B = B_{new}$ and $\det(B) = \alpha$. We shall show that B must be reducible by the action of elementary matrices to $(\alpha \oplus I_{k-1})$, which would imply the result. Again, this is essentially a statement about matrices over R_n and we may assume that $n = p^e$ is a power of a prime. As in the proof of the lemma, let B_p denote the congruence class of B mod p. Since R_p is a Euclidean domain, we may reduce B_p by row and column operations to a matrix of the form $\text{SNF}(B_p) = \text{Diag}(b_1, \ldots b_k)$ and moreover we we may assume that $\text{SNF}(B_p) = \text{Diag}(\alpha_p, 1, \ldots 1)$. Thus, over R_n with $n = p^e$, we may reduce B by the action of elementary matrices to a matrix of the form:

$$\begin{pmatrix} \alpha + c_{1,1}p & c_{1,2}p & \ldots & c_{1,k}p \\ c_{1,2}p & 1 + c_{2,2}p & \ldots & c_{2,k}p \\ \vdots & & & \vdots \\ c_{k,1}p & c_{k,2}p & \ldots & 1 + c_{k,k}p \end{pmatrix} \qquad c_{i,j} \in R_n$$

Again, each $1 + c_{i,i}p$ is a unit and we may reduce B to a matrix of the form $\text{Diag}(\alpha + dp, 1, \ldots, 1)$ for some $d \in R_n$. This completes the proof, since we have insisted that $\det(B) = \alpha$ and hence B may be reduced by the action of elementary matrices to $(\alpha \oplus I_{k-1})$. □

Theorem 3.5 shows that a limited form of cancellation holds within the class of modules $M(A)$, of which $\pi_2(X_{\mathcal{G}})^* = M(t-1)$ is a member. In order to extend this class we shall employ a technique for constructing projectives over fibre products due to Milnor [14]. A *Cartesian* square of rings is a commutative diagram of rings and ring homomorphisms:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{i_1} & \mathcal{R}_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathcal{R}_2 & \xrightarrow{j_2} & \mathcal{R}_0 \end{array}$$

such that

(i) $\mathcal{R}$ is the fibre product of $\mathcal{R}_1$ and $\mathcal{R}_2$ over $\mathcal{R}_0$. This means that $\mathcal{R}$ may be identified with the set of pairs $(r_1, r_2) \in \mathcal{R}_1 \times \mathcal{R}_2$ such that $j_1(r_1) = j_2(r_2)$.

(ii) At least one of j_1 or j_2 is surjective.

If P_i is a projective module over $\mathcal{R}_i$ then there is an induced $\mathcal{R}_0$ module $j_{i\#}(P_i)$ given by $P_i \otimes_{\mathcal{R}_i} \mathcal{R}_0$ and a canonical R_i–linear map $j_{i*}\colon P_i \to j_{i\#}(P_i)$ given by

$$j_{i*}(p) = 1 \otimes p.$$

Given projective modules P_1 over $\mathcal{R}_1$, P_2 over $\mathcal{R}_2$ and an $\mathcal{R}_0$ isomorphism

$$h\colon j_{1\#}(P_1) \cong j_{2\#}(P_2),$$

let $M(P_1, P_2, h)$ denote the subgroup of $P_1 \times P_1$ consisting of all pairs (p_1, p_2) such that $hj_{1*}(p_1) = j_{2*}(p_2)$. Then $M(P_1, P_2, h)$ has a natural $\mathcal{R}$ module structure.

Proposition 3.6 (J Milnor) *The module $M(P_1, P_2, h)$ is projective over $\mathcal{R}$, and every projective $\mathcal{R}$ module is isomorphic to some $M(P_1, P_2, h)$. Moreover, the modules P_1 and P_2 are naturally isomorphic to $i_{1\#}M(P_1, P_2, h)$ and $i_{2\#}M(P_1, P_2, h)$ respectively.*

We may now prove the following theorem.

Theorem 3.7 *If M is any module such that $M \oplus \mathbb{Z}[\Gamma]^m \cong M(B)$ for some $B \in \mathcal{M}_{k+m}(R_n)$, then $M \cong M(A)$ for some $A \in \mathcal{M}_k(R_n)$.*

We remark that it is here that we use the work of Bass–Murthy. In particular we shall use the details in section 9 of [1].

Proof It is sufficient to show that there exists an exact sequence

$$0 \to R^k \to M \to S^k \to 0$$

where as before $S = \mathbb{Z}[\Gamma]/(N)$. It is clear that, in the extension $E(B)$, R^{k+m} is identified precisely with the submodule of $M(B)$ on which the action of x is trivial. Let M_G denote the submodule of M on which x acts trivially, so that there is an exact sequence:

$$0 \to M_G \to M \to M/M_G \to 0.$$

Any isomorphism $M \oplus \mathbb{Z}[\Gamma]^m \cong M(B)$ induces isomorphisms $M_G \oplus R^m \cong R^{k+m}$, and $M/M_G \oplus S^m \cong S^{k+m}$. Thus M_G is stably free, and hence free, as an R module. Since the action of x on M_G is trivial, we deduce that $M_G \cong R^k$ as a $\mathbb{Z}[\Gamma]$ module. It remains to show that $M/M_G \cong S^k$. Note that M/M_G has an S module structure which is then stably free, and if we can show that M/M_G is free over S we may deduce the result.

We claim that all stably free S modules are free. Let $S_{\mathbb{Z}}$ denote $\mathbb{Z}[G]/(N)\mathbb{Z}[G]$, where as before $G = C_n$, so that there is a natural identification of S with $S_{\mathbb{Z}}[C_\infty$. By Bass–Murthy [1, 9.1] it is enough to prove that $S_{\mathbb{Z}}$ has finitely many non-projective maximal ideals. We shall deduce it from the fact that $\mathbb{Z}[G]$ has finitely many non-projective maximal ideals, (which [1] claims is "not difficult to verify", and we prove for the sake of completeness in an appendix).

Let $i_1\colon \mathbb{Z}[G] \to S_{\mathbb{Z}}$ be the natural map onto the quotient, $i_2\colon \mathbb{Z}[G] \to \mathbb{Z}$ be augmentation, $j_1\colon S_{\mathbb{Z}} \to \mathbb{Z}_n$ be defined as $j_1(\alpha + (N)) = i_2(\alpha) \mod n$ and $j_2\colon \mathbb{Z} \to \mathbb{Z}_n$ be the natural map. Then the following is a commutative diagram of rings and surjective ring homomorphisms:

$$\begin{array}{ccc} \mathbb{Z}[G] & \xrightarrow{i_1} & S_{\mathbb{Z}} \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \mathbb{Z} & \xrightarrow{j_2} & \mathbb{Z}_n \end{array}$$

and $\mathbb{Z}[G]$ may be identified with the fibre product of $S_{\mathbb{Z}}$ and $\mathbb{Z}$ over $\mathbb{Z}_n$. Let I be a maximal ideal of $S_{\mathbb{Z}}$, and let I' denote the (necessarily projective) maximal ideal of $\mathbb{Z}$ given by $I' = j_2^{-1}(j_1(I))$. Then (I, I') is a maximal ideal of $\mathbb{Z}[G]$ which is isomorphic to the module $M(I, I', Id)$ constructed as above. Furthermore, by Proposition 3.6, (I,I') is projective if and only if I is projective. Lastly, if J is another maximal ideal of $S_{\mathbb{Z}}$, then $(J, J') = (I, I')$ if and only if $J = I$ and hence there is an injective map from the maximal ideals of $S_{\mathbb{Z}}$ to the maximal ideals of $\mathbb{Z}[G]$ which preserves projectivity. □

Corollary 3.8 *Any module stably isomorphic to $\pi_2(X_{\mathcal{G}})$ is necessarily isomorphic to $\pi_2(X_{\mathcal{G}}) \oplus \mathbb{Z}[\Gamma]^m$ for some m, where $\pi_2(X_{\mathcal{G}})$ is the module defined in the beginning of this section.*

Proof Observe, from our description of the generators of $\pi_2(X_{\mathcal{G}})^*$, that there is an isomorphism $\pi_2(X_{\mathcal{G}})^* \cong M(t-1)$. The required result is then a clear consequence of Theorem 3.5 and Theorem 3.7. □

4 The D(2) problem for $C_n \times C_\infty$

We bring together the work of the previous sections:

Proposition 4.1 *In order to prove the D(2) Problem for $C_n \times C_\infty$ it is sufficient to realise geometrically all algebraic 2–complexes of the form:*

$$0 \longrightarrow \pi_2(X_{\mathcal{G}}) \longrightarrow \mathbb{Z}[\Gamma]^a \longrightarrow \mathbb{Z}[\Gamma]^b \longrightarrow \mathbb{Z}[\Gamma]^c \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Proof By Theorem 1.1 we may consider the D(2) problem to be equivalent to the realization problem for Γ. The result then follows from Proposition 2.1 and from Corollary 3.8. □

Lemma 4.2 *There is an isomorphism* $\mathrm{Ext}^3_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \pi_2(X_{\mathcal{G}})) \cong \mathbb{Z}_n$.

Proof One may calculate this result directly using the resolution:

$$\cdots \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_3} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where ∂_1, ∂_2 are as before and

$$\partial_3 = \begin{pmatrix} 0 & N \\ x-1 & t-1 \end{pmatrix}.$$

We simplify our proof by noticing that, immediately from the generators given for $\pi_2(X_{\mathcal{G}})$ in the start of the previous section, there is a clear isomorphism $\pi_2(X_{\mathcal{G}}) \cong I$, where I is the augmentation ideal and is generated as a submodule of $\mathbb{Z}[\Gamma]$ by the elements $(x-1)$ and $(t-1)$.

If $f\colon \mathbb{Z}[\Gamma]^2 \longrightarrow I$ is a homomorphism, then we may represent f as multiplication by some matrix

$$\begin{pmatrix} a & b \end{pmatrix} \qquad a, b \in I$$

f is a cocycle if and only if $a = \alpha_1(x-1)$, $b = (\alpha_1 + \alpha_2 N)(t-1)$ for some $\alpha_1 \in \mathbb{Z}[\Gamma], \alpha_2 \in R$; α_1 is well defined modulo (N) and α_2 is determined by α_1.

f is a coboundary if there exists some $a, b \in I$ such that

$$(a \quad b)\cdot\begin{pmatrix} 0 & N \\ x-1 & t-1 \end{pmatrix} = \begin{pmatrix}\alpha_1(x-1) & (\alpha_1+\alpha_2 N)(t-1)\end{pmatrix},$$

from which we deduce that f if a coboundary if and only if there exists a choice of $\alpha_1 \in I$. The reader may verify that the map $\varphi\colon \mathrm{Ext}^3_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \pi_2(X_{\mathcal{G}})) \to \mathbb{Z}_n$ given by:

$$\varphi\,(\alpha_1(x-1) \qquad (\alpha_1+\alpha_2 N)(t-1)) \quad = \quad \varepsilon(\alpha_1) \quad \bmod n$$

is a (ring) isomorphism. □

Proof of Theorem A We have already shown that it is sufficient to realize all extensions of the form

$$0 \longrightarrow \pi_2(X_{\mathcal{G}}) \longrightarrow \mathbb{Z}[\Gamma]^a \longrightarrow \mathbb{Z}[\Gamma]^b \longrightarrow \mathbb{Z}[\Gamma]^c \longrightarrow \mathbb{Z} \longrightarrow 0$$

and by Proposition 2.1, $\mathrm{Ext}^3(\mathbb{Z}, \pi_2(X_{\mathcal{G}}))$ has the structure of a ring under which algebraic 2–complexes are necessarily units. For each unit $w \in \mathbb{Z}_n^+$ we shall realize the class of w up to congruence, and hence up to chain homotopy equivalence.

An obvious change one may make to the standard presentation

$$\mathcal{G} = \langle\, x, t \mid x^n = 1, tx = xt \,\rangle$$

is to replace the generator x for C_n by the generator $y(v) = x^v$ where $1 \le v \le n-1$ is a natural number coprime to n. Denote each such presentation by

$$\mathcal{G}(v) = \langle\, y(v),\ t \mid y(v)^n = 1\ ,\ y(v)t = ty(v) \,\rangle$$

where $y(v) = x^v$.

We remark that the Cayley complex of $\mathcal{G}(v)$ is homotopy equivalent to the standard one, since as presentations they are identical. Indeed, one may view this complex as that arising from changing the isomorphism between fundamental group of the original complex determined by $\mathcal{G}$ and the abstract group determined by the presentation. The aforementioned homotopy equivalence does not induce the identity on the fundamental groups of the spaces, and so does not imply that the resulting algebraic complexes are chain homotopic, although the distinction is meaningless from a geometric viewpoint.

The corresponding chain complex of the universal cover of $X_{\mathcal{G}(v)}$ is then

$$0 \longrightarrow \mathrm{Ker}(\partial_2^v) \longrightarrow \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_2^v} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1^v} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\partial_2^v = \begin{pmatrix} 1-t & N \\ x^v-1 & 0 \end{pmatrix}$ and $\partial_1^v = (x^v-1 \quad t-1)$

so that $\mathrm{Ker}(\partial_2^v) = \mathrm{Ker}(\partial_2) = \pi_2(X_{\mathcal{G}})$. Let w denote the inverse of v mod n. Set τ to be the element $1 + x^v + \ldots x^{v(w-1)}$, then $(1-x^v)\tau = (1-x)$ and the following diagram commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \pi_2(X_{\mathcal{G}}) & \xrightarrow{j} & \mathbb{Z}[\Gamma]^2 & \xrightarrow{\partial_2} & \mathbb{Z}[\Gamma]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[\Gamma] & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow Id & & \downarrow Id & & \\
0 & \longrightarrow & \pi_2(X_{\mathcal{G}}) & \xrightarrow{j} & \mathbb{Z}[\Gamma]^2 & \xrightarrow{\partial_2^v} & \mathbb{Z}[\Gamma]^2 & \xrightarrow{\partial_1^v} & \mathbb{Z}[\Gamma] & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

where each f_i may be represented as multiplication on the left by:

$$f_3 = \tau \quad f_2 = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \quad f_1 = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

Finally, since $\varepsilon(\tau) = w$, $C_*(\widetilde{X}_{\mathcal{G}(v)})$ realizes geometrically the congruence class of $w \in \mathrm{Ext}^3(\mathbb{Z}, \pi_2(X_{\mathcal{G}}))$. This completes the proof. □

5 Swan modules and weak cancellation

The machinery developed to prove the cancellation result in section 3 may be applied to Swan modules over a finite group, where $R = \mathbb{Z}[C_\infty]$ is replaced by $\mathbb{Z}$.

Suppose that $G = \{g_i\}_{i=1}^n$ is a finite group of order n, then the trivial $\mathbb{Z}[G]$ module $\mathbb{Z}$ embeds into $\mathbb{Z}[G]$ by identification with the ideal generated by the element $N = \sum g_i$. There is a resulting exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]/(N) \longrightarrow 0.$$

Similarly to before, we shall write S for the $\mathbb{Z}[G]$ module and quotient ring $\mathbb{Z}[G]/(N)$. Note that S is the dual of the augmentation ideal, and may be identified with the submodule of $\mathbb{Z}[G]^n$ generated by the element:

$$\begin{pmatrix} g_1 - 1 \\ \vdots \\ g_n - 1 \end{pmatrix}$$

For each $k \in \mathbb{N}$ and $A \in \mathcal{M}_k(\mathbb{Z}_n)$, we may define $M(A)$ to be the submodule of $\mathbb{Z}[G]^{k(n+1)}$ generated by the columns of the matrix:

$$A' = \begin{pmatrix} A'' & N \cdot I_k \\ (g_1 - 1)I_k & 0 \\ \vdots & \vdots \\ (g_n - 1)I_k & 0 \end{pmatrix}$$

where A'' is a lift of A under the natural map $\mathbb{Z}[G] \to \mathbb{Z}_n$. Then we may identify $\mathbb{Z}^k$ with the submodule of $M(A)$ generated by the last k columns of A', and the quotient under this embedding is clearly isomorphic to S^k. Define $\varphi\colon \mathbb{Z}[G]^k \to M(A)$ by sending the i^{th} generator of $\mathbb{Z}[G]$ to the i^{th} column of A', so that the following commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}[G]^k & \longrightarrow & S^k & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & M(A) & \longrightarrow & S^k & \longrightarrow & 0 \end{array}$$

and we may define $E(A)$ to be the congruence class of the bottom row. Let $GR(k)$ denote the image of $GL_k(\mathbb{Z})$ in $GL_k(\mathbb{Z}_n)$, and $GS(k)$ the image of $GL_k(S)$. Note that since $GR(k)$ contains matrices of determinant ± 1 we have $GR(k) \subseteq GS(k)$. The reader may verify that the analysis of section 3 transfers immediately to such extensions and proves the following:

Proposition 5.1

(a) *Each extension of* $\mathrm{Ext}^1(S^k, \mathbb{Z}^k)$ *is congruent to* $E(A)$ *for some* $A \in \mathcal{M}_k(\mathbb{Z}_n)$ *and if* A_1, A_2 *are both lifts of* A*, then* $E(A_1) \equiv E(A_2)$.

(b) *For arbitrary matrices* $A, B \in \mathcal{M}_k(\mathbb{Z}_n)$, $M(A) \cong M(B)$ *if and only if there exists* $C \in GR(k), D \in GS(k)$ *such that* $CAD = B$.

(c) *For each* $A \in \mathcal{M}_k(\mathbb{Z}_n)$, $M(A)$ *is a free module if and only if* $A \in GS(k)$.

(d) *For each* $r \in \mathbb{Z}_n^+$, $M(r)$ *is a stably free non-free module if and only if* $(r \oplus I_k) \in GS(k+1)$ *for some* k *but* $r \notin GS(1)$.

Note that we may apply the Swan Jakobinski theorem in order to reduce (d) to the case where $k = 1$.

For each unit $r \in \mathbb{Z}_n$, the *Swan module* (N, r) is the submodule of $\mathbb{Z}[G]$ generated by the elements N and $\hat{r}$, where $\hat{r}$ is any element such that $\varepsilon(\hat{r}) = r \mod n$. Note that $(N, r) \cong M(r)$. Swan modules, as originally defined in [16], are projective and form a well studied subset of the projective class group of finite groups. We shall only use the fact that they represent the class of modules $M(r)$ for $r \in \mathbb{Z}_n$ which are projective.

Now let G denote the quaternion group of order $4n$ with $n \geq 6$, or in general any finite group of period four such that there are stably free modules $\mathbb{Z}[G]$ which are not free. We say that weak cancellation holds for G if all stably free Swan modules are free. Johnson has shown [9], that if weak cancellation holds for G, then one may construct modules which are stably equivalent to $\pi_2(X_{\mathcal{G}})$ for some presentation $\mathcal{G}$, but which are *not* isomorphic to $\pi_2(X_{\mathcal{G}}) \oplus \mathbb{Z}[G]^m$. We refer the interested reader to the recently published Beyl and Waller [2] for an explicit construction of such a module.

For G a 2–group, or for G of order $4p$ with p an odd prime, Swan has shown that weak cancellation holds [17, Theorem IV].

6 Appendix

Proposition 6.1 *For $G = C_n$, the integral group ring $\mathbb{Z}[G]$ has finitely many non-projective maximal ideals.*

Proof Let J be a maximal ideal of $\mathbb{Z}[G]$. Then $\mathbb{Z}[G]/J = \mathbb{F}$ is a field, which is necessarily finite with ground ring $\mathbb{Z}_p$ for some prime p. Define $J\colon \mathbb{Z}[G]$ to be the set of elements $r \in \mathbb{Z}$ such that $r\mathbb{Z}[G] \subseteq J$. Clearly, $J\colon \mathbb{Z}[G] = (p)$ and so by proposition 7.1 of [15] J is projective unless p divides the order of G.

Thus, if J is not projective, we may assume that $J \cap \mathbb{Z} = (p)$ for some p dividing n, where we consider $\mathbb{Z}$ to be a subring of $\mathbb{Z}[G]$. Again, since $\mathbb{Z}[G]/J = \mathbb{F}$ is a field, the generator x for C_n has some minimal polynomial $\omega(x)$ over $\mathbb{F}$. Then J is necessarily the ideal generated by p and $\omega(x)$. Since the degree of $\omega(x)$ is less than or equal to n, there are finitely many $\omega(x)$ such that the ideals $(p, \omega(x))$ are distinct. This completes the proof. □

Proposition 6.2 *The D(2) property holds for $C_\infty \times C_\infty$.*

Proof If Γ is a free abelian group, then every projective $\mathbb{Z}[\Gamma]$ module is free, see for example proposition 4.12 of [10]. In particular all stably free modules are free. We may take the the presentation $\mathcal{G} = \langle x, t | xt = tx \rangle$ for $C_\infty \times C_\infty = \Gamma$, leading to the algebraic complex:

$$0 \longrightarrow \mathbb{Z}[\Gamma] \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} 0$$

$$\partial_2 = \begin{pmatrix} 1-t \\ x-1 \end{pmatrix} \qquad \partial_1 = (x-1 \quad t-1).$$

Immediately we deduce that $\mathrm{Ext}^3(\mathbb{Z}, M) = 0$ for any $\mathbb{Z}[\Gamma]$–module M, and if there exists an exact sequence:

$$0 \longrightarrow M \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with each F_i free, then M is free. This may be seen to complete the proof by Proposition 2.1. □

Other than S^2 and $\mathbb{R}P^2$, which are dealt with in [8], the second homotopy module of any 2–manifold is necessarily zero, and hence if Γ is the fundamental group of *any* surface and all stably free $\mathbb{Z}[\Gamma]$ modules are free, a similar proof shows that the D(2) property holds for Γ.

Acknowledgements The author wishes to express his gratitude to his PhD supervisor, Professor F E A Johnson, who suggested the problem and who provided valuable insights regarding its eventual solution. The author also wishes to thank EPSRC for providing sponsorship.

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Received: 16 September 2005 Revised: 13 December 2005