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HIGSON COMPACTIFICATIONS OF WALLMAN TYPE

By

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Abstract. We provide a sufficient condition for a proper metric space in order that its Higson compactification may be of Wallman type.

1. Introduction

The notion of Higson compactification was introduced by N. Higson in analyzing Roe's index theorem for non-compact complete Riemannian manifolds (see [11, Chapter 5]) and it is one of the fundamental notions in coarse geometry (see also [12, Section 2.3]). Higson compactifications are defined for proper metric spaces by applying the Gelfand-Naimark theorem for the C^*-algebra consisting of slowly oscillating functions, or by embedding a Tychonoff cube with respect to the set of slowly oscillating functions (see Section 2). Here, a metric space (or its metric) is said to be *proper* if every closed bounded subspace is compact. Note that the Higson compactification of an unbounded proper metric space is never metrizable ([12, Exercise 2.49]).

A Wallman (or Wallman-Frink, Wallman-Shanin) compactification is a compactification defined by means of a closed base, called a Wallman base (see Section 2 for definition). A compactification is said to be of Wallman type if it is equivalent to some Wallman compactification. It is known that Čech-Stone compactifications, one-point compactifications and metrizable compactifications are of Wallman type (see [1]), while V. M. Ul'janov [13] proved that there exists a Hausdorff compactification of a Tychonoff (i.e. completely regular Hausdorff) space which is not of Wallman type.

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The main concern of this paper is the following question (see [2, p. 1692]).

Question 1.1. *Is every Higson compactification of Wallman type?*

C. Bandt [3, Theorem (1) in §10] proved that every compact Hausdorff space of weight $\le \omega_1$ is a Wallman compactification of each of its dense subspaces, where ω_1 is the first uncountable cardinal. Thus, under the continuum hypothesis, the answer to Question 1.1 is affirmative (see also [3, Theorem (5) in §10]).

The purpose of this paper is to give a partial answer to Question 1.1 providing a sufficient condition for a proper metric space in order that its Higson compactification may be of Wallman type (in ZFC without additional set-theoretic assumption). After reviewing basic properties of Higson compactifications and Wallman compactifications in Section 2, we introduce a condition (HW) for a metric space in Section 3. Our main result is that the Higson compactification of every proper metric space satisfying (HW) is of Wallman type (Theorem 3.2). It is also shown that condition (HW) is a coarse invariant and closed under taking finite Cartesian products. Examples of proper metric spaces with (HW) are given in Section 4. They include Euclidean spaces and trees of finite degree. We prove the main result in Section 6. For its proof we give a criterion of Wallman bases generating Higson compactifications in Section 5. Some questions are listed in Section 7.

2. Preliminaries

All spaces in this paper are assumed to be Tychonoff topological spaces. For a space X and $A \subset X$ the closure of A in X is denoted by $\mathrm{cl}_X A$. The letters $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}$ represent the real line, the set of integers and the set of positive integers, respectively. For $a, b \in \mathbf{R}$ let

$$[a, b] = \{x \in \mathbf{R} : a \le x \le b\} \quad \text{and} \quad [a, \infty) = \{x \in \mathbf{R} : x \ge a\}.$$

For a metric space (X, d), $x \in X$ and $R > 0$ let $B(x, R)$ denote the open R-ball centered at x. For $E \subset X$ and $R > 0$ let

$$B(E, R) = \bigcup_{x \in E} B(x, R).$$

For undefined notions we refer to [7] and [10].

We review Higson compactifications following [8, Section 1] (for another equivalent definition by means of the Gelfand-Naimark theorem, see [11, Section

5.1], [12, Section 2.3]). For a proper metric space (X, d) a bounded continuous function $f : X \to \mathbf{R}$ is said to be *slowly oscillating* (or a *Higson function*) if for every $\varepsilon > 0$ and every $R > 0$ there exists a bounded subset $B \subset X$ such that $|f(x_1) - f(x_2)| \le \varepsilon$ for every $x_1, x_2 \in X \backslash B$ with $d(x_1, x_2) \le R$.

The set of all slowly oscillating functions on X is denoted by $C_h(X)$. For $f \in C_h(X)$ let $\|f\| = \sup\{|f(x)| : x \in X\}$ and $I_f = [-\|f\|, \|f\|]$. Then $C_h(X)$ separates points and closed sets in X, and hence the map $e_X : X \to \prod_{f \in C_h(X)} I_f$ defined by $e_X(x) = (f(x))_{f \in C_h(X)}$, $x \in X$, is a topological embedding. The closure of $e_X(X)$ in the Tychonoff product $\prod_{f \in C_h(X)} I_f$ is called the *Higson compactification* of X and denoted by hX. We identify $e_X(x)$ with x for every $x \in X$.

For metric spaces (X, d_X) and (Y, d_Y) a map $f : X \to Y$ is called a *coarse equivalence* (see [9, Definition 1.4.4]) if

(1) there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} \rho_-(t) = \infty$ and

$$\rho_-(d_X(x, x')) \le d_Y(f(x), f(x')) \le \rho_+(d_X(x, x'))$$

for every $x, x' \in X$, and

(2) there exists $S > 0$ such that $B(f(X), S) = Y$.

Two metric spaces X and Y are *coarsely equivalent* if there exists a coarse equivalence $f : X \to Y$.

Remark 2.1. A proper metric space X is an open subset of its Higson compactification hX. Thus the remainder $hX \backslash X$, which is called the *Higson corona* of X, is compact. It is known that if two proper metric spaces X and Y are coarsely equivalent, then their Higson coronas $hX \backslash X$ and $hY \backslash Y$ are homeomorphic [12, Corollary 2.42] (note that the notion of coarse equivalence defined above is equivalent to that in [12, Definition 2.21] for the bounded coarse structure [12, Example 2.5]).

Next we review Wallman compactifications (see [10, Section 4.4]).

Definition 2.2. A family $\mathscr{L}$ of subsets of a space X is said to be a *Wallman base* on X if it satisfies the following conditions:

(i) $\mathscr{L}$ is a ring, that is, $A \cup B \in \mathscr{L}$ and $A \cap B \in \mathscr{L}$ for every $A, B \in \mathscr{L}$,

(ii) $\varnothing, X \in \mathscr{L}$,

(iii) $\mathscr{L}$ is a closed base for X, that is, $\mathscr{L}$ is a family of closed subsets of X such that for every closed subset F of X and for every $x \in X \backslash F$ there is $A \in \mathscr{L}$ such that $F \subset A \not\ni x$,

(iv) if $A \in \mathscr{L}$ and $x \in X \backslash A$, then there exists $B \in \mathscr{L}$ such that $x \in B$ and $A \cap B = \emptyset$, and

(v) if $A, B \in \mathscr{L}$ and $A \cap B = \emptyset$, then there exist $C, D \in \mathscr{L}$ such that $A \cap C = \emptyset = B \cap D$ and $C \cup D = X$.

Let $\mathscr{L}$ be a Wallman base on a space X and $w_{\mathscr{L}}X$ the set of all $\mathscr{L}$-ultrafilters on X. For $A \in \mathscr{L}$ and $x \in X$ let

$$S(A) = \{p \in w_{\mathscr{L}}X : A \in p\} \quad \text{and} \quad p_x = \{B \in \mathscr{L} : x \in B\}.$$

Then the following fact holds (for proof see [10, Section 4.4]).

Fact 2.3. (1) *$\{S(A) : A \in \mathscr{L}\}$ is a closed base for a topology on $w_{\mathscr{L}}X$. (We assume that $w_{\mathscr{L}}X$ has the topology induced by the base.)*

(2) *$w_{\mathscr{L}}X$ is compact.*

(3) *The map $e_X : X \to w_{\mathscr{L}}X$ defined by $e_X(x) = p_x$ is a topological embedding such that $e_X(X)$ is dense in $w_{\mathscr{L}}X$.*

(4) *$\mathrm{cl}_{w_{\mathscr{L}}X}(e_X(A)) = S(A)$ for every $A \in \mathscr{L}$.*

(5) *$\mathrm{cl}_{w_{\mathscr{L}}X}\, A \cap \mathrm{cl}_{w_{\mathscr{L}}X}\, B = \mathrm{cl}_{w_{\mathscr{L}}X}(A \cap B)$ for every $A, B \in \mathscr{L}$.*

The compactification $w_{\mathscr{L}}X$ is called the *Wallman compactification* of X with respect to $\mathscr{L}$. We identify $e_X(x)$ with x for every $x \in X$. Two compactifications c_1X and c_2X of a space X are said to be *equivalent* if there exists a homeomorphism $f : c_1X \to c_2X$ such that $f\restriction_X = \mathrm{id}_X$. A compactification γX of a space X is said to be of *Wallman type* if γX is equivalent to $w_{\mathscr{L}}X$ for some Wallman base $\mathscr{L}$ on X.

3. A sufficient condition

For a metric space (X, d) and $A, B \subset X$ let

$$\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\} \quad \text{and}$$
$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

where we let $\sup \emptyset = 0$, $\inf \emptyset = \infty$ and $\sup A = \infty$ when A is unbounded. For a family $\mathscr{F}$ of subsets of X let

$$\operatorname{mesh} \mathscr{F} = \sup\{\operatorname{diam} F : F \in \mathscr{F}\}$$

and $\mathscr{F}$ is said to be

- *uniformly bounded* if mesh $\mathscr{F} < \infty$, and
- *r-disjoint*, for $r > 0$, if $d(E, F) \geq r$ for every distinct $E, F \in \mathscr{F}$.

We say that a family $\mathscr{F}$ of subsets of a metric space X is *boundedly finite* if for every bounded subset B of X the set $\{F \in \mathscr{F} : B \cap F \neq \varnothing\}$ is finite. Clearly, every boundedly finite family of subsets of a metric space is locally finite, and every locally finite family of subsets of a proper metric space is boundedly finite.

Definition 3.1. For $R > 0$ a family $\mathscr{F}$ of subsets of a metric space (X, d) is said to have $HW(R)$ if

$$\forall \mathscr{F}' \subset \mathscr{F}\left(\bigcap \mathscr{F}' = \varnothing \Rightarrow \exists F, F' \in \mathscr{F}'(d(F, F') \geq R)\right).$$

A metric space X is said to satisfy (HW) if for every $R > 0$ there exists a boundedly finite uniformly bounded cover $\mathscr{F}$ of X with $\mathrm{HW}(R)$.

Here we state the main result of this paper.

Theorem 3.2. *The Higson compactification of every proper metric space satisfying* (HW) *is of Wallman type.*

Proof of Theorem 3.2 will be given in Section 6. The following proposition shows that (HW) is a coarse invariant.

Proposition 3.3. *Let* (X, d_X) *and* (Y, d_Y) *be coarsely equivalent metric spaces. If* X *satisfies* (HW)*, then so does* Y.

Proof. Let $f : X \to Y$ be a coarse equivalence, and take non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ and $S > 0$ so that $\lim_{t \to \infty} \rho_-(t) = \infty$,

$$\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))$$

for every $x, x' \in X$, and $Y = B(f(X), S)$.

Assume that X satisfies (HW). To show that Y has (HW), let $R > 0$. Taking $r > 0$ with $\rho_-(r) > R + 2S$ and a boundedly finite uniformly bounded cover $\mathscr{F}_X$ of X satisfying $\mathrm{HW}(r)$, let

$$\mathscr{F}_Y = \{B(f(F), S) : F \in \mathscr{F}_X\}.$$

Then $\mathscr{F}_Y$ is boundedly finite since $f^{-1}(B(A, S))$ is bounded for every bounded subset A of Y; $\mathscr{F}_Y$ is uniformly bounded since diam $F \leq \rho_+(\text{mesh } \mathscr{F}_X) + 2S$ for every $F \in \mathscr{F}_Y$; and $\mathscr{F}_Y$ covers X since $\mathscr{F}_X$ covers X and $Y = B(f(X), S)$. To show that $\mathcal{F}_Y$ has HW(R), let $\mathcal{F}' \subset \mathcal{F}_X$ and assume $\bigcap_{F \in \mathcal{F}'} B(f(F), S) = \emptyset$. Then $\bigcap \mathcal{F}' = \emptyset$ and, since $\mathcal{F}_X$ has HW(r), there exist $F, F' \in \mathcal{F}'$ such that $d(F, F') \geq r$. Then we have

$$d(B(f(F), S), B(f(F'), S)) \geq \rho_-(r) - 2S \geq R.$$

Hence $\mathcal{F}_Y$ has HW(R), and Y satisfies (HW). □

Next we show that (HW) is finitely multiplicative. For two metric spaces (X, d_X) and (Y, d_Y) the Cartesian product $X \times Y$ is assumed to have the ℓ_2-metric, that is, the metric d_2 defined by

$$d_2((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2},$$

for every $(x, y), (x', y') \in X \times Y$.

Proposition 3.4. *Let X and Y be metric spaces with (HW). Then, $X \times Y$ satisfies (HW).*

Proof. Let $R > 0$. Take boundedly finite uniformly bounded covers $\mathcal{F}_X$ and $\mathcal{F}_Y$ of X and Y, respectively, with HW(R). Then the family $\{F_X \times F_Y : F_X \in \mathcal{F}_X, F_Y \in \mathcal{F}_Y\}$ is a required cover of $X \times Y$. □

4. Examples

In this section, we give some examples of proper metric spaces satisfying (HW). By Theorem 3.2 the Higson compactifications of such proper metric spaces are of Wallman type.

Example 4.1. The real line $\mathbf{R}$ with the usual metric satisfies (HW). Indeed, for each $R > 0$ the family $\{[jR, (j+1)R] : j \in \mathbf{Z}\}$ is a boundedly finite uniformly bounded cover of $\mathbf{R}$ with HW(R).

By Proposition 3.4 and Example 4.1 we have the following.

Corollary 4.2. *Every Euclidean space $\mathbf{R}^n$ with the usual metric satisfies (HW). In particular, $h\mathbf{R}^n$ is of Wallman type.*

Recall that the asymptotic dimension of a metric space (X, d) is said to be at most n (denoted by asdim $X \leq n$) provided for every $r > 0$ there exist $n + 1$

uniformly bounded r-disjoint families $\mathscr{U}_0, \ldots, \mathscr{U}_n$ of subsets of X such that $\bigcup_{i=0}^n \mathscr{U}_i$ covers X. For information on asymptotic dimension see [4], [5], [9, Chapter 2], [12, Chapter 9].

Proposition 4.3. *Every proper metric space X of* $\operatorname{asdim} X \leq 1$ *satisfies* (HW).

Proof. Let X be a proper metric space of $\operatorname{asdim} X \leq 1$ and $R > 0$. Take two uniformly bounded $3R$-disjoint families $\mathscr{U}_0$ and $\mathscr{U}_1$ of subsets of X such that $\mathscr{U}_0 \cup \mathscr{U}_1$ covers X. Let $U \in \mathscr{U}_0$. For each $U' \in \mathscr{U}_1$ with $d(U', U) < R$ take $x_{U'} \in U'$ such that $d(x_{U'}, U) < R$ and let

$$V_U = U \cup \{x_{U'} : U' \in \mathscr{U}_1 \text{ and } d(U', U) < R\}.$$

Let $\mathscr{V}_0 = \{V_U : U \in \mathscr{U}_0\}$ and $\mathscr{F} = \mathscr{V}_0 \cup \mathscr{U}_1$. Then $\mathscr{V}_0$ is R-disjoint since $\mathscr{U}_0$ is $3R$-disjoint and $V_U \subset B(U, R)$ for every $U \in \mathscr{U}_0$. As $\mathscr{V}_0$ and $\mathscr{U}_1$ are uniformly bounded, so is $\mathscr{F}$. Since $\mathscr{V}_0$ and $\mathscr{U}_1$ are R-disjoint and X is a proper metric space, $\mathscr{F}$ is boundedly finite.

It remains to show that $\mathscr{F}$ satisfies $\mathrm{HW}(R)$. Let $\mathscr{F}' \subset \mathscr{F}$ and assume $\bigcap \mathscr{F}' = \varnothing$. Since $\mathscr{V}_0$ and $\mathscr{U}_1$ are R-disjoint, it suffices to consider the case that $\mathscr{F}' = \{F, F'\}$, $F \in \mathscr{V}_0$ and $F' \in \mathscr{U}_1$. Then $F = V_U$ for some $U \in \mathscr{U}_0$. Since $F' \in \mathscr{U}_1$ and $V_U \cap F' = \varnothing$, we have $d(U, F') \geq R$. Let $U' \in \mathscr{U}_1$ with $d(U', U) < R$. Then $U' \neq F'$ since $V_U \cap F' = \varnothing$. Thus, since $\mathscr{U}_1$ is R-disjoint, we have $d(U', F') \geq R$, which implies that $d(x_{U'}, F') \geq R$. Hence $d(F, F') = d(V_U, F') \geq R$. Therefore $\mathscr{F}$ satisfies $\mathrm{HW}(R)$. □

Example 4.4. Every graph G is assumed to have the path-metric d with edge length 1, that is, every edge is assume to have length 1 and, for two points $x, y \in G$, $d(x, y)$ is the length of a shortest path between x and y. A graph is said to be of *finite degree* if every its vertex is contained in only finitely many edges. A graph is of finite degree if and only if it is a proper metric space. A *tree* is a connected graph without cycle. According to [12, Proposition 9.8], $\operatorname{asdim} T \leq 1$ for every tree T. Thus every tree T of finite degree satisfies (HW) by Proposition 4.3, and hence hT is of Wallman type.

Example 4.5. The countable direct sum $\bigoplus_{k=1}^{\infty} \mathbf{Z}$ of integers is defied as the subset

$$\{(x_k) \in \mathbf{Z}^{\mathbf{N}} : x_k = 0 \text{ for all but finitely many } k\}$$

of the product $\mathbf{Z}^{\mathbf{N}}$. We assume that it has the metric defined by

$$d((x_k),(y_k)) = \sum_{k=1}^{\infty} k|x_k - y_k|, \quad (x_k),(y_k) \in \bigoplus_{k=1}^{\infty} \mathbf{Z},$$

which is proper. Note that the asymptotic dimension of $\bigoplus_{k=1}^{\infty} \mathbf{Z}$ is infinite (see [9, Example 2.6.1]).

The metric space $\bigoplus_{k=1}^{\infty} \mathbf{Z}$ satisfies (HW). Indeed, for each $R > 0$ let $\mathscr{I} = \{[jR,(j+1)R] \cap \mathbf{Z} : j \in \mathbf{Z}\}$ and take $i_R \in \mathbf{N}$ with $i_R > R$. Let

$$\mathscr{F} = \left\{ \prod_{k=1}^{i_R} I_k \times \prod_{k=i_R+1}^{\infty} \{n_k\} : I_1, \dots, I_{i_R} \in \mathscr{I}, (n_k) \in \bigoplus_{k=1}^{\infty} \mathbf{Z} \right\}.$$

Then $\mathscr{F}$ is a bounded finite uniformly bounded cover of $\bigoplus_{k=1}^{\infty} \mathbf{Z}$ with $\mathrm{HW}(R)$.

5. A criterion of Wallman bases generating Higson compactifications

In this section we give a criterion concerning Wallman bases for a proof of Theorem 3.2.

LEMMA 5.1. *Let X be a space and $\mathscr{L}$ a Wallman base on X. Then for every pair E, F of closed subsets of X, $\mathrm{cl}_{w_{\mathscr{L}}X} E \cap \mathrm{cl}_{w_{\mathscr{L}}X} F = \emptyset$ if and only if E and F are separated by disjoint elements of $\mathscr{L}$, that is, there exist $A, B \in \mathscr{L}$ such that $E \subset A$, $F \subset B$ and $A \cap B = \emptyset$.*

PROOF. Let E and F be closed subsets of X. For the "if" part, suppose that there exist $A, B \in \mathscr{L}$ such that $E \subset A$, $F \subset B$ and $A \cap B = \emptyset$. Then by (5) of Fact 2.3 we have

$$\mathrm{cl}_{w_{\mathscr{L}}X} E \cap \mathrm{cl}_{w_{\mathscr{L}}X} F \subset \mathrm{cl}_{w_{\mathscr{L}}X} A \cap \mathrm{cl}_{w_{\mathscr{L}}X} B = \mathrm{cl}_{w_{\mathscr{L}}X}(A \cap B) = \emptyset.$$

To show the "only if" part, suppose that $\mathrm{cl}_{w_{\mathscr{L}}X} E \cap \mathrm{cl}_{w_{\mathscr{L}}X} F = \emptyset$. By (1) of Fact 2.3, for every $p \in w_{\mathscr{L}}X$ we may take $A_p \in \mathscr{L}$ so that $p \notin S(A_p)$ and either $\mathrm{cl}_{w_{\mathscr{L}}X} E \subset S(A_p)$ or $\mathrm{cl}_{w_{\mathscr{L}}X} F \subset S(A_p)$. Since $w_{\mathscr{L}}X$ is compact and $\bigcap\{S(A_p) : p \in w_{\mathscr{L}}X\} = \emptyset$, there exists a finite $\mathscr{F} \subset \{A_p : p \in w_{\mathscr{L}}X\}$ such that $\bigcap\{S(L) : L \in \mathscr{F}\} = \emptyset$. Let

$$A = \bigcap\{L \in \mathscr{F} : \mathrm{cl}_{w_{\mathscr{L}}X} E \subset S(L)\} \quad \text{and}$$

$$B = \bigcap\{L \in \mathscr{F} : \mathrm{cl}_{w_{\mathscr{L}}X} F \subset S(L)\}.$$

Then $A, B \in \mathscr{L}$ since $\mathscr{L}$ is a ring, and $A \cap B = \bigcap\{S(L) : L \in \mathscr{F}\} = \varnothing$. The facts that $E \subset A$ and $F \subset B$ follow from the fact that $\mathscr{F}$ is a closed family. □

For a metric space (X, d) and $E, F \subset X$ the pair $\{E, F\}$ is said to *diverge* ([6, Definition 2.1]) if $B(E, R) \cap B(F, R)$ is bounded for every $R > 0$. The following lemma follows from [6, Proposition 2.3].

Lemma 5.2. *For every disjoint closed subsets E and F of a proper metric space X, $\mathrm{cl}_{hX}\, E \cap \mathrm{cl}_{hX}\, F = \varnothing$ if and only if $\{E, F\}$ diverges.*

For proof of the next lemma see [7, Theorem 3.5.5].

Lemma 5.3. *Compactifications c_0X and c_1X of a space X are equivalent if and only if for every pair E, F of closed subsets of X we have*

$$\mathrm{cl}_{c_0X}\, E \cap \mathrm{cl}_{c_0X}\, F = \varnothing \quad \textit{if and only if} \quad \mathrm{cl}_{c_1X}\, E \cap \mathrm{cl}_{c_1X}\, F = \varnothing.$$

Then we have the following.

Theorem 5.4. *Let X be a proper metric space and $\mathscr{L}$ a Wallman base. Then the Wallman compactification $w_{\mathscr{L}}X$ is equivalent to the Higson compactification hX if and only if $\mathscr{L}$ satisfies the following conditions*:

(1) *If A and B are disjoint elements of $\mathscr{L}$, then $\{A, B\}$ diverges.*
(2) *For every disjoint closed subsets E and F of X, if $\{E, F\}$ diverges, then they are separated by disjoint elements of $\mathscr{L}$.*

Proof. To show the "if" part, suppose that $\mathscr{L}$ satisfies (1) and (2). Let E and F be closed subsets of X. According to Lemma 5.3, it suffices to show that $\mathrm{cl}_{w_{\mathscr{L}}X}\, E \cap \mathrm{cl}_{w_{\mathscr{L}}X}\, F = \varnothing$ if and only if $\mathrm{cl}_{hX}\, E \cap \mathrm{cl}_{hX}\, F = \varnothing$.

Suppose that $\mathrm{cl}_{hX}\, E \cap \mathrm{cl}_{hX}\, F = \varnothing$. Then by Lemma 5.2 the pair $\{E, F\}$ diverges. This and (2) imply E and F are separated by disjoint elements of $\mathscr{L}$. Thus by Lemma 5.1 we have $\mathrm{cl}_{w_{\mathscr{L}}X}\, E \cap \mathrm{cl}_{w_{\mathscr{L}}X}\, F = \varnothing$.

Conversely, suppose that $\mathrm{cl}_{w_{\mathscr{L}}X}\, E \cap \mathrm{cl}_{w_{\mathscr{L}}X}\, F = \varnothing$. Then by Lemma 5.1 there exist $A, B \in \mathscr{L}$ such that $E \subset A$, $F \subset B$ and $A \cap B = \varnothing$. By (1) the pair $\{A, B\}$ diverges. Thus, $E \subset A$ and $F \subset B$ imply that $\{E, F\}$ diverges, and hence $\mathrm{cl}_{hX}\, E \cap \mathrm{cl}_{hX}\, F = \varnothing$ by Lemma 5.2.

For the "only if" part suppose that $w_{\mathscr{L}}X$ is equivalent to hX. Item (2) follows from Lemmas 5.2, 5.3 and 5.1. To show (1), let $A, B \in \mathscr{L}$ with $A \cap B =$

$\varnothing$. Then by (5) of Fact 2.3 we have

$$\mathrm{cl}_{w_{\mathscr{S}}X}\, A \cap \mathrm{cl}_{w_{\mathscr{S}}X}\, B = \mathrm{cl}_{w_{\mathscr{S}}X}(A \cap B) = \varnothing,$$

and hence $\{A, B\}$ diverges by Lemmas 5.3 and 5.2. $\square$

6. Proof of Theorem 3.2

Throughout this section, let (X, d) be a proper metric space with (HW). For $A \subset X$ and a family $\mathscr{F}$ of subsets of X let

$$\mathrm{St}(A, \mathscr{F}) = \bigcup \{F \in \mathscr{F} : A \cap F \neq \varnothing\}.$$

With ω, we represent the set of non-negative integers. Let $\mathscr{P}(A)$ denote the power set of a set A.

Lemma 6.1. *There exists a sequence $\{\mathscr{F}_i\}_{i\in\omega}$ of families $\mathscr{F}_i$ of subsets of X and a strictly increasing function $s : \omega \to \omega$ such that for each $i \in \omega$*

(1) *$\mathscr{F}_i$ is a boundedly finite uniformly bounded closed cover of X,*

(2) *if $i \geq 1$ and $F \in \mathscr{F}_i$, then $F = \bigcup \{F' \in \mathscr{F}_{i-1} : F' \subset F\}$,*

(3) $\mathrm{mesh}\, \mathscr{F}_i < s(i)$, *and*

(4) *$\mathscr{F}_i$ has $HW(i)$.*

Proof. Since X is a metric space, we can take a locally finite closed cover $\mathscr{F}_0$ of X which refines the open cover $\{B(x, 1) : x \in X\}$. Then $\mathscr{F}_0$ is boundedly finite since X is proper. Let $s(0) = 3$. Then $\mathscr{F}_0$ satisfies (1)–(4). Let $i \geq 1$ and assume that $\mathscr{F}_{i-1}$ and $s(i-1)$ have been defined. Since X has (HW), we can take a boundedly finite uniformly bounded cover $\mathscr{F}$ with $\mathrm{HW}(i + 2s(i-1))$. Let

$$\mathscr{F}_i = \{\mathrm{St}(F, \mathscr{F}_{i-1}) : F \in \mathscr{F}\}, \quad \text{and} \quad s(i) = \mathrm{mesh}\, \mathscr{F} + 2s(i-1) + 1.$$

Then $\mathscr{F}_i$ satisfies (1)–(4) and the resulting function $s : \omega \to \omega$ is strictly increasing. $\square$

We fix a sequence $\{\mathscr{F}_i\}_{i\in\omega}$ and a function $s : \omega \to \omega$ as in Lemma 6.1. For every $i \in \omega$ let $\mathscr{F}_i'$ be the family of all finite intersections of elements of $\mathscr{F}_i$. Take $K_0 \in \mathscr{F}_0' \setminus \{\varnothing\}$ and let

$$K_{n+1} = \mathrm{St}(B(K_n, s(n+1)), \mathscr{F}_{n+1}')$$

for $n \in \omega$. For each $n \in \omega$ and $i \leq n$ let

$$\mathscr{F}_i^n = \{F \in \mathscr{F}_i' : F \subset K_n\}.$$

LEMMA 6.2. *For every $n \in \omega$ and $i \leq n$*

(1) *$\mathscr{F}_i'$ is a boundedly finite uniformly bounded closed cover of X,*
(2) *$F = \bigcup\{F' \in \mathscr{F}_i' : F' \subset F\}$ for every $F \in \mathscr{F}_{i+1}'$,*
(3) *if $F, F' \in \mathscr{F}_i'$ and $F \cap F' = \varnothing$, then $d(F, F') \geq i$,*
(4) *K_n is a compact subset of X,*
(5) *$B(K_n, s(n+1)) \subset K_{n+1}$,*
(6) *$\bigcup \mathscr{F}_i^n = K_n$,*
(7) mesh $\mathscr{F}_i^n < s(i)$, *and*
(8) *$F \cap F' \in \mathscr{F}_i^n$ for every pair $F, F' \in \mathscr{F}_i'$ with $F \cap F' \subset K_n$.*

In particular,

(9) *if C is a compact subset of X, then there exists $n \in \omega$ such that $C \subset K_m$ for every $m \geq n$.*

PROOF. Items (1), (2) and (7) follow from (1), (2) and (3) of Lemma 6.1, respectively.

To show (3), let $F, F' \in \mathscr{F}_i'$ with $F \cap F' = \varnothing$. We may assume $F \neq \varnothing \neq F'$. Then $F = \bigcap \mathscr{F}$ and $F' = \bigcap \mathscr{F}'$ for some finite subsets $\mathscr{F}, \mathscr{F}' \subset \mathscr{F}_i$. Since $F \neq \varnothing \neq F'$ and $F \cap F' = \varnothing$, by (4) of Lemma 6.1, there exist $E \in \mathscr{F}$ and $E' \in \mathscr{F}'$ such that $d(E, E') \geq i$. This, $F \subset E$ and $F' \subset E'$ imply $d(F, F') \geq i$.

Item (4) follows from (1) and the fact that X is a proper metric space.

Item (5) follows from the definition of K_{n+1} and the fact that $\mathscr{F}_{n+1}'$ covers X.

Item (6) follows from the fact that each element of $\mathscr{F}_n'$ can be represented as a union of members of $\mathscr{F}_i'$ by (2).

Item (8) is immediate from the definitions of $\mathscr{F}_i'$ and $\mathscr{F}_i^n$.

Item (9) follows from (5) and the fact that s is strictly increasing. □

Let $\omega^{\uparrow\omega}$ be the set of all functions $f : \omega \to \omega$ satisfying $\lim_{n\to\infty} f(n) = \infty$ and set

$$\Phi = \{f \in \omega^{\uparrow\omega} : f(n) \leq \min\{n, f(n+1)\} \text{ for every } n \in \omega\}.$$

For $f \in \Phi$ let

$$\mathscr{S}_f = \left\{ \bigcup_{n\in\omega} \bigcup \mathscr{I}_n : (\mathscr{I}_n) \in \prod_{n\in\omega} \mathscr{P}(\mathscr{F}_{f(n)}^n) \right\}.$$

Finally let

$$\mathscr{L} = \left\{ C \cup S : C \text{ is a compact subset of } X \text{ and } S \in \bigcup_{f \in \Phi} \mathscr{S}_f \right\}.$$

LEMMA 6.3. *The family $\mathscr{L}$ is a Wallman base on X.*

PROOF. First we check the following properties on $\mathscr{S}_f$, $f \in \Phi$.

CLAIM 6.3.1. *Let $f, g \in \Phi$. Then*
(1) *every $S \in \mathscr{S}_f$ is closed in X,*
(2) *$d(S \backslash K_n, S' \backslash K_n) \geq \min\{f(i) : i > n\}$ for every $n \in \omega$ and every $S, S' \in \mathscr{S}_f$ with $S \cap S' = \varnothing$,*
(3) *$\mathscr{S}_g \subset \mathscr{S}_f$ when $f \leq g$, and*
(4) *there is $h \in \Phi$ such that $\mathscr{S}_f \cup \mathscr{S}_g \subset \mathscr{S}_h$.*

PROOF OF CLAIM 6.3.1. (1). By (2) of Lemma 6.2 every $S \in \mathscr{S}_f$ can be represented as a union of members of $\mathscr{F}_0'$, and the union is closed in X by (1) in Lemma 6.2.

(2). Let $n \in \omega$ and $S, S' \in \mathscr{S}_f$ with $S \cap S' = \varnothing$. Take $x \in S \backslash K_n$ and $x' \in S' \backslash K_n$ arbitrarily. Then there are $i, i' \in \omega$, $F \in \mathscr{F}_{f(i)}^i$ and $F' \in \mathscr{F}_{f(i')}^{i'}$ such that $x \in F \subset S$ and $x' \in F' \subset S'$. Since $x, x' \notin K_n$, we have that $\min\{i, i'\} > n$ by (6) of Lemma 6.2. Without loss of generality, we may assume $f(i) \leq f(i')$. By (2) in Lemma 6.2 there is $E' \in \mathscr{F}_{f(i)}'$ such that $x' \in E' \subset F'$. This, the facts that $F \in \mathscr{F}_{f(i)}'$ and $F \cap E' \subset F \cap F' \subset S \cap S' = \varnothing$, and (3) of Lemma 6.2 imply that $d(x, x') \geq d(F, E') \geq f(i)$, which shows the conclusion.

(3). Assume $f \leq g$ and let $S \in \mathscr{S}_g$. Then $S = \bigcup_{n \in \omega} \bigcup \mathscr{J}_n^g$ for some $(\mathscr{J}_n^g) \in \prod_{n \in \omega} \mathscr{P}(\mathscr{F}_{g(n)}^n)$. Let $n \in \omega$. Then $\mathscr{J}_n^g \subset \mathscr{F}_{g(n)}^n \subset \mathscr{F}_{g(n)}'$ and $\bigcup \mathscr{J}_n^g \subset K_n$. By (2) in Lemma 6.2 each $F \in \mathscr{F}_{g(n)}'$ can be represented as a union of members of $\mathscr{F}_{f(n)}'$. Thus we can take $\mathscr{J}_n^f \subset \mathscr{F}_{f(n)}^n$ so that $\bigcup \mathscr{J}_n^g = \bigcup \mathscr{J}_n^f$. Then $S = \bigcup_{n \in \omega} \bigcup \mathscr{J}_n^g = \bigcup_{n \in \omega} \bigcup \mathscr{J}_n^f \in \mathscr{S}_f$. Therefore $\mathscr{S}_g \subset \mathscr{S}_f$.

(4). Define $h : \omega \to \omega$ by $h(n) = \min\{f(n), g(n)\}$, $n \in \omega$. Then $h \in \Phi$, $h \leq f$ and $h \leq g$. Thus we have $\mathscr{S}_f \cup \mathscr{S}_g \subset \mathscr{S}_h$ by (3). □

Now we show that $\mathscr{L}$ satisfies (i)–(v) in Definition 2.2.

(i). Let $A_0, A_1 \in \mathscr{L}$. By (4) of Claim 6.3.1 we may take $f \in \Phi$, compact subsets C_0 and C_1 of X and $(\mathscr{J}_n^0), (\mathscr{J}_n^1) \in \prod_{n \in \omega} \mathscr{P}(\mathscr{F}_{f(n)}^n)$ so that $A_k = C_k \cup \bigcup_{n \in \omega} \bigcup \mathscr{J}_n^k$, $k < 2$. Since $C_0 \cup C_1$ is compact and $\bigcup_{n \in \omega} \bigcup(\mathscr{J}_n^0 \cup \mathscr{J}_n^1) \in \mathscr{S}_f$, we have

$$A_0 \cup A_1 = (C_0 \cup C_1) \cup \bigcup_{n\in\omega} \bigcup (\mathscr{I}_n^0 \cup \mathscr{I}_n^1) \in \mathscr{L}.$$

To show $A_0 \cap A_1 \in \mathscr{L}$, take $m \in \omega$ so that $C_0 \cup C_1 \subset K_m$ (applying (9) of Lemma 6.2). Let $\mathscr{I}_n = \varnothing$ when $n \leq m$ and

$$\mathscr{I}_n = \left\{ F \in \mathscr{F}_{f(n)}^n : \exists F_0 \in \bigcup_{n\in\omega} \mathscr{I}_n^0 \exists F_1 \in \bigcup_{n\in\omega} \mathscr{I}_n^1 (F \subset F_0 \cap F_1) \right\}$$

when $n > m$. Also let $C = A_0 \cap A_1 \cap K_m$ and $S = \bigcup_{n\in\omega} \bigcup \mathscr{I}_n$. Then $(\mathscr{I}_n) \in \prod_{n\in\omega} \mathscr{P}(\mathscr{F}_{f(n)}^n)$, and hence $S \in \mathscr{S}_f$. This and the compactness of C imply $C \cup S \in \mathscr{L}$. It is easy to see that $A_0 \cap A_1 \supset C \cup S$. To show that $A_0 \cap A_1 \subset C \cup S$, let $x \in A_0 \cap A_1$. If $x \in K_m$, then $x \in C \subset C \cup S$. Assume that $x \notin K_m$ and let $m_0 = \min\{n \in \omega : x \in K_n\}$. Then $m_0 > m$ and $x \in K_{m_0} \backslash K_{m_0-1}$. Since $x \in (\bigcup_{n\in\omega} \bigcup \mathscr{I}_n^0) \cap (\bigcup_{n\in\omega} \bigcup \mathscr{I}_n^1)$, there are $n_0, n_1 \in \omega$, $F_0 \in \mathscr{I}_{n_0}^0$ and $F_1 \in \mathscr{I}_{n_1}^1$ such that $x \in F_0 \cap F_1$. From the facts that $f \in \Phi$ and $x \notin K_{m_0-1}$, and by (6) of Lemma 6.2, we have $f(m_0) \leq m_0 \leq \min\{n_0, n_1\}$. Hence by (2) of Lemma 6.2 there exist $F_0', F_1' \in \mathscr{F}_{f(m_0)}'$ such that $x \in F_0' \subset F_0$ and $x \in F_1' \subset F_1$. On the other hand, since $x \in K_{m_0}$ and $f(m_0) \leq m_0$, by (6) of Lemma 6.2, there exists $F' \in \mathscr{F}_{f(m_0)}^{m_0}$ such that $x \in F'$. Then $F_0' \cap F_1' \cap F' \in \mathscr{F}_{f(m_0)}^{m_0}$ by (8) of Lemma 6.2. This and the fact that $F_0' \cap F_1' \cap F' \subset F_0 \cap F_1$ imply that $F_0' \cap F_1' \cap F' \in \mathscr{I}_{m_0}$. Since $x \in F_0' \cap F_1' \cap F'$, we have $x \in \bigcup_{n\in\omega} \bigcup \mathscr{I}_n = S \subset C \cup S$. Thus $A_0 \cap A_1 \subset C \cup S$. Hence $A_0 \cap A_1 = C \cup S \in \mathscr{L}$. Therefore $\mathscr{L}$ is a ring.

(ii). Note that the identity map $\mathrm{id}_\omega : \omega \to \omega$ belongs to Φ and $\mathscr{F}_{\mathrm{id}_\omega(n)}^n = \mathscr{F}_n^n$ for every $n \in \omega$. Define $(\mathscr{I}_n), (\mathscr{I}_n') \in \prod_{n\in\omega} \mathscr{P}(\mathscr{F}_n^n)$ by $\mathscr{I}_n = \varnothing$ and $\mathscr{I}_n' = \mathscr{F}_n^n$ for each $n \in \omega$. Then $\varnothing = \bigcup_{n\in\omega} \bigcup \mathscr{I}_n \in \mathscr{S}_{\mathrm{id}_\omega}$. By (9) and (6) of Lemma 6.2 we have $X = \bigcup_{n\in\omega} K_n = \bigcup_{n\in\omega} \bigcup \mathscr{I}_n' \in \mathscr{S}_{\mathrm{id}_\omega}$.

(iii). By (1) of Claim 6.3.1 the family $\mathscr{L}$ consists of closed subsets of X. Let E be a closed subset of X and $x \in X \backslash E$. Applying (9) of Lemma 6.2, we may take $m \in \omega \backslash \{0\}$ such that $x \in K_{m-1}$. Define $f : \omega \to \omega$ by $f(n) = \max\{0, n-1\}$ for $n \in \omega$, and $(\mathscr{I}_n) \in \prod_{n\in\omega} \mathscr{P}(\mathscr{F}_{f(n)}^n)$ by

$$\mathscr{I}_n = \begin{cases} \varnothing & \text{if } n \leq m, \\ \{F \in \mathscr{F}_{f(n)}^n : F \backslash K_{n-1} \neq \varnothing\} & \text{if } n > m. \end{cases}$$

Let $A = (E \cap K_m) \cup \bigcup_{n\in\omega} \bigcup \mathscr{I}_n$. Since $E \cap K_m$ is compact and $f \in \Phi$, we have that $A \in \mathscr{L}$.

To see that $E \subset A$, let $z \in E$. If $z \in K_m$, then $z \in A$ obviously. Assume that $z \notin K_m$ and let $n = \min\{i \in \omega : z \in K_i\}$. Then $n > m$ and $z \in K_n \backslash K_{n-1}$. Since

$z \in K_n = \bigcup \mathscr{F}^n_{f(n)}$ by (6) of Lemma 6.2, we can take $F \in \mathscr{F}^n_{f(n)}$ such that $z \in F$. Since $n > m$ and $z \in F \setminus K_{n-1}$, we have $F \in \mathscr{J}_n$, and hence $z \in F \subset \bigcup_{n\in\omega} \bigcup \mathscr{J}_n \subset A$. Therefore $E \subset A$.

It remains to show that $x \notin A$. Since $x \notin E$, it suffices to show $x \notin \bigcup_{n\in\omega} \bigcup \mathscr{J}_n$. For every $n > m$ we have $K_{m-1} \cap \bigcup \mathscr{J}_n = \emptyset$ from the facts that $K_{m-1} \subset K_{n-2}$, $B(K_{n-2}, s(n-1)) \subset K_{n-1}$ (because of (5) of Lemma 6.2), mesh $\mathscr{F}^n_{f(n)} < s(n-1)$ (by (7) of Lemma 6.2) and $F \setminus K_{n-1} \neq \emptyset$ for every $F \in \mathscr{J}_n$ (directly from the definition of $\mathscr{J}_n$). This and the fact that $\mathscr{J}_n = \emptyset$ for every $n \leq m+1$ imply $K_{m-1} \cap \bigcup_{n\in\omega} \bigcup \mathscr{J}_n = \emptyset$. Thus, as $x \in K_{m-1}$, we obtain that $x \notin \bigcup_{n\in\omega} \bigcup \mathscr{J}_n$. Therefore $\mathscr{L}$ is a closed base for X.

(iv). This follows from the fact that X is a proper metric space and every compact subset of X belongs to $\mathscr{L}$.

(v). Let $A_0, A_1 \in \mathscr{L}$ with $A_0 \cap A_1 = \emptyset$. By (4) in Claim 6.3.1 we may take $f \in \Phi$, compact subsets C_0 and C_1 of X and sets $S_0, S_1 \in \mathscr{S}_f$ so that $A_k = S_k \cup C_k$ for each $k < 2$. By (9) of Lemma 6.2 and the fact that $f \in \omega^{\uparrow\omega}$, there is $m \in \omega$ satisfying $C_0 \cup C_1 \subset K_{m-1}$ and $f(m) > 4$. By applying the normality of K_m, take compact subsets D_0, D_1 of K_m such that $D_k \cap A_{1-k} = \emptyset$ for $k < 2$ and $D_0 \cup D_1 = K_m$. Define $g : \omega \to \omega$ by $g(n) = 0$ when $n \leq m$ and

$$g(n) = \min\{n-1, \max\{j \in \omega : s(j) < \min\{f(i) : i > n-2\}/2\}\}$$

when $n > m$. Then $g \in \Phi$ since $f \in \omega^{\uparrow\omega}$ and s is increasing. Note that $2s(g(n)) < \min\{f(i) : i > n-2\}$ whenever $n > m$. For each $k < 2$ let

$$\mathscr{J}^k_n = \begin{cases} \emptyset & \text{if } n \leq m, \\ \{F \in \mathscr{F}^n_{g(n)} : F \cap A_{1-k} = \emptyset\} & \text{if } n > m, \end{cases}$$

and $B_k = D_k \cup \bigcup_{n\in\omega} \bigcup \mathscr{J}^k_n$. It is clear that $B_0, B_1 \in \mathscr{L}$ and $B_k \cap A_{1-k} = \emptyset$ for $k < 2$.

To show that $B_0 \cup B_1 = X$, let $x \in X$ and $n = \min\{i \in \omega : x \in K_i\}$. If $n \leq m$, then $x \in K_m = D_0 \cup D_1$, which implies $x \in B_0 \cup B_1$. Assume that $n > m$. Then $n-1 \in \omega$ and $x \in K_n \setminus K_{n-1}$ by the choice of n. Since $A_0 \cap A_1 = \emptyset$ and $C_0 \cup C_1 \subset K_{m-1} \subset K_{n-2}$, we have that

$$d(A_0 \setminus K_{n-2}, A_1 \setminus K_{n-2}) \geq \min\{f(i) : i > n-2\} > 2s(g(n))$$

because of (2) in Claim 6.3.1. Thus $d(x, A_0 \setminus K_{n-2}) > s(g(n))$ or $d(x, A_1 \setminus K_{n-2}) > s(g(n))$. Without loss of generality, we may assume that $d(x, A_0 \setminus K_{n-2}) > s(g(n))$. Since $K_n = \bigcup \mathscr{F}^n_{g(n)}$ (by (6) of Lemma 6.2) and $x \in K_n$, we may take $F_1 \in \mathscr{F}^n_{g(n)}$ such that $x \in F_1$. By (7) of Lemma 6.2 we have $\operatorname{diam} F_1 < s(g(n)) < d(x, A_0 \setminus K_{n-2})$. Hence $F_1 \cap (A_0 \setminus K_{n-2}) = \emptyset$. Furthermore, since $x \notin K_{n-1}$, we have

$x \notin B(K_{n-2}, s(n-1))$ by (5) in Lemma 6.2. This, $x \in F_1$ and $\operatorname{diam} F_1 < s(g(n)) \le s(n-1)$ yield $F_1 \cap K_{n-2} = \emptyset$. Hence $F_1 \cap A_0 = \emptyset$, which implies that $F_1 \in \mathscr{I}_n^1$, and then $x \in B_1$. Therefore $x \in B_0 \cup B_1$, and we have $B_0 \cup B_1 = X$. □

Lemma 6.4. *The Wallman compactification $w_{\mathscr{L}}X$ with respect to $\mathscr{L}$ is equivalent to the Higson compactification hX.*

Proof. It suffices to show that $\mathscr{L}$ satisfies (1) and (2) in Theorem 5.4.

For (1) in Theorem 5.4 let $A_0, A_1 \in \mathscr{L}$ with $A_0 \cap A_1 = \emptyset$. By (4) of Claim 6.3.1 we may take $f \in \Phi$, compact subsets C_0 and C_1 of X and $S_0, S_1 \in \mathscr{S}_f$ so that $A_k = C_k \cup S_k$, $k < 2$. To see that $\{A_0, A_1\}$ diverges, fix $R > 0$ arbitrary. By (9) in Lemma 6.2 and the fact that $f \in \omega^{\uparrow\omega}$, there exists $m \in \omega$ such that $C_0 \cup C_1 \subset K_m$ and $f(n) > 2R$ for every $n \in \omega$ with $n > m$. By (2) of Claim 6.3.1 we have

$$d(A_0 \backslash K_m, A_1 \backslash K_m) \ge \min\{f(i) : i > m\} > 2R,$$

and hence $B(A_0, R) \cap B(A_1, R) \subset B(K_m, R)$, which shows that $\{A_0, A_1\}$ diverges.

For (2) in Theorem 5.4 let E_0 and E_1 be disjoint closed subsets of X such that $\{E_0, E_1\}$ diverges. Take $m \in \omega$ such that $B(E_0, 2) \cap B(E_1, 2) \subset K_{m-1}$ and define $f : \omega \to \omega$ by

$$f(n) = \begin{cases} 0 & \text{if } n \le m, \\ \max\left\{i \le n-1 : s(i) \le \frac{d(E_0 \backslash K_{n-2}, E_1 \backslash K_{n-2})}{2}\right\} & \text{if } n > m, \end{cases}$$

where we let $\max \emptyset = 0$.

Since $\{E_0, E_1\}$ diverges, we have $f \in \omega^{\uparrow\omega}$. For every $n > m$ we also have $d(E_0 \backslash K_{n-2}, E_1 \backslash K_{n-2}) \ge 2s(f(n))$, $f(n) \le n-1$ and $f(n) \le f(n+1)$. In particular, $f \in \Phi$.

For $k < 2$ define $(\mathscr{I}_n^k) \in \prod_{n \in \omega} \mathscr{P}(\mathscr{F}_{f(n)}^n)$ by letting

$$\mathscr{I}_n^k = \begin{cases} \emptyset & \text{if } n \le m, \\ \{F \in \mathscr{F}_{f(n)}^n : F \cap (E_k \backslash K_{n-1}) \ne \emptyset\} & \text{if } n > m \end{cases}$$

and let $A_k = (E_k \cap K_m) \cup \bigcup_{n \in \omega} \bigcup \mathscr{I}_n^k$. Then $A_k \in \mathscr{L}$.

Claim 6.4.1. *$E_k \subset A_k$ for each $k < 2$.*

Proof of Claim 6.4.1. Let $k < 2$ and $x \in E_k$. We may assume $x \notin K_m$. Let $n = \min\{i \in \omega : x \in K_i\}$. Then $n > m$ and $x \in K_n \backslash K_{n-1}$. Since $x \in K_n = \bigcup \mathscr{F}_{f(n)}^n$ by

(6) of Lemma 6.2, there exists $F \in \mathscr{F}^n_{f(n)}$ such that $x \in F$. Then $x \in F \cap (E_k \setminus K_{n-1})$, which implies $F \in \mathscr{J}^k_n$, and hence $x \in \bigcup \mathscr{J}^k_n \subset A_k$. □

Claim 6.4.2. *$K_{n-2} \cap \bigcup \mathscr{J}^k_n = \emptyset$ for every $n > m$ and $k < 2$.*

Proof of Claim 6.4.2. Let $k < 2$ and $n > m$. By (5) and (7) in Lemma 6.2 we have that $B(K_{n-2}, s(n-1)) \subset K_{n-1}$ and $\operatorname{mesh} \mathscr{F}^n_{f(n)} < s(f(n)) \leq s(n-1)$. Thus, as $F \setminus K_{n-1} \neq \emptyset$ for every $F \in \mathscr{J}^k_n$, the claim follows. □

Claim 6.4.3. *$E_{1-k} \cap \bigcup_{n \in \omega} \bigcup \mathscr{J}^k_n = \emptyset$ for each $k < 2$.*

Proof of Claim 6.4.3. Let $k < 2$. Take $n \in \omega$ and $F \in \mathscr{J}^k_n$ arbitrarily. Then $n > m$ since $\mathscr{J}^k_n \neq \emptyset$. From the facts that $\emptyset \neq F \cap (E_k \setminus K_{n-1}) \subset F \cap (E_k \setminus K_{n-2})$ and $d(E_0 \setminus K_{n-2}, E_1 \setminus K_{n-2}) \geq 2s(f(n)) > \operatorname{diam} F$, we have $F \cap (E_{1-k} \setminus K_{n-2}) = \emptyset$. This and Claim 6.4.2 imply that $E_{1-k} \cap F = \emptyset$. Therefore $E_{1-k} \cap \bigcup_{n \in \omega} \bigcup \mathscr{J}^k_n = \emptyset$. □

Claim 6.4.4. *$(\bigcup_{n \in \omega} \bigcup \mathscr{J}^0_n) \cap (\bigcup_{n \in \omega} \bigcup \mathscr{J}^1_n) = \emptyset$.*

Proof of Claim 6.4.4. Let $n_0, n_1 \in \omega$, $F_0 \in \mathscr{J}^0_{n_0}$ and $F_1 \in \mathscr{J}^1_{n_1}$ arbitrarily. Then $m < \min\{n_0, n_1\}$. It suffices to show that $F_0 \cap F_1 = \emptyset$. Without loss of generality, we may assume $n_0 \leq n_1$.

If $n_0 \leq n_1 - 2$, then $F_0 \subset K_{n_0} \subset K_{n_1-2}$ since $F_0 \in \mathscr{J}^0_{n_0} \subset \mathscr{F}^{n_0}_{f(n_0)}$ and by (6) of Lemma 6.2. Furthermore $K_{n_1-2} \cap \bigcup \mathscr{J}^1_{n_1} = \emptyset$ by Claim 6.4.2. Thus $F_0 \cap F_1 = \emptyset$.

Assume that $n_1 - 1 \leq n_0 \leq n_1$. Then we have

$$F_0 \cap (E_0 \setminus K_{n_1-2}) \neq \emptyset \neq F_1 \cap (E_1 \setminus K_{n_1-2}).$$

This and the facts that $d(E_0 \setminus K_{n_1-2}, E_1 \setminus K_{n_1-2}) \geq 2s(f(n_1))$, $\operatorname{diam} F_0 < s(f(n_0)) \leq s(f(n_1))$ and $\operatorname{diam} F_1 < s(f(n_1))$ imply $F_0 \cap F_1 = \emptyset$. □

By Claims 6.4.3 and 6.4.4 we have $A_0 \cap A_1 = \emptyset$. Hence E_0 and E_1 are separated by disjoint elements A_0 and A_1 of $\mathscr{L}$.

Therefore hX is a Wallman type compactification by Theorem 5.4. □

7. Questions

If the following question is affirmative, then so is Question 1.1 by Theorem 3.2.

Question 7.1. *Does every proper metric space satisfy* (HW)?

We do not even know the answer to the following questions weaker than Question 7.1.

Question 7.2. *Let* X *be a proper metric space such that* hX *is of Wallman type. Does* X *satisfy* (HW)?

Concerning Propositions 3.4 and 4.3, we may also ask the following.

Question 7.3. *Let* X *be a metric space satisfying* (HW) *and* Y *a metric subspace of* X. *Does* Y *satisfy* (HW)?

Question 7.4. *Let* X *be a metric space and* A *and* B *metric subspaces of* X *such that* $A \cup B = X$ *and both* A *and* B *satisfy* (HW). *Does* X *satisfy* (HW)?

Question 7.5. *Does every metric space of finite asymptotic dimension satisfy* (HW)?

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