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COMMUTING STRUCTURE JACOBI OPERATORS FOR REAL HYPERSURFACES IN COMPLEX SPACE FORMS II

By

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Abstract. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper, we prove that if the structure Jacobi operator R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the Ricci tensor, then M is a Hopf hypersurface provided that the mean curvature of M is constant with respect to the structure vector field.

1. Introduction

Let $M_n(c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $4c \neq 0$, and let J be its complex structure. Complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbf{C}$ or a complex hyperbolic space $H_n\mathbf{C}$ for $c > 0$ or $c < 0$, respectively.

Let M be a connected submanifold of $M_n(c)$ with real codimension 1. We refer to this simply as a real hypersurface below.

For a local unit normal vector field N of M, we define the structure vector field ξ of M by $\xi = -JN$. The structure vector ξ is said to be principal if $A\xi = \alpha\xi$ is satisfied for some functuion α, where A is the shape operator of M.

A real hypersurface M is said to be a Hopf hypersurface if the structure vector ξ of M is principal.

Hopf hypersurfaces is realized as tubes over certain submanifolds in $P_n\mathbf{C}$, by using its focal map (see Cecil and Ryan [2]). By making use of those results and the mentioned work of Takagi ([16], [17]), Kimura [10] proved the local clas-

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sification theorem for Hopf hypersurfaces of $P_n\mathbf{C}$ whose all principal curvatures are constant. For the case $H_n\mathbf{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbf{C}$ or $H_k\mathbf{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbf{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [12] (resp. Montiel and Romero [11]) showed that a real hypersurface in $P_n\mathbf{C}$ (resp. $H_n\mathbf{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator A.

The Reeb vector field ξ plays an important role in the theory of real hypersurfaces in a complex space form $M_n(c)$. Related to the Reeb vector field ξ the Jacobi operator R_ξ defined by $R_\xi = R(\cdot, \xi)\xi$ for the curvature tensor R on a real hypersurface M in $M_n(c)$ is said to be a *structure Jacobi operator* on M. The structure Jacobi operator has a fundamental role in contact geometry. In [3], Cho and the first author started the study on real hypersurfaces in complex space form by using the operator R_ξ. In particular the structure Jacobi operator has been studied under the various commutative condition ([8], [15]). For example, Pérez *et al.* [15] called that real hypersurfaces M has commuting structure Jacobi operator if $R_\xi R_X = R_X R_\xi$ for any vector field X on M, and proved that there exist no real hypersurfaces in $M_n(c)$ with commuting structure Jacobi operator. On the other hand Ortega *et al.* [13] have proved that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field X on M. More generally, such a result has been extended by [14]. In this situation, if naturally leads us to be consider another condition weaker than parallelness. In the preceding work, we investigate the weaker condition $\phi\nabla_\xi\xi$-parallelness, that is, $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. Motivated the present authors proved following.

Theorem 1 (Ki and Kurihara [6]). *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$, $c \neq 0$ which satisfies $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following hypersurfaces*:

(I) *In cases that $M_n(c) = P_n\mathbf{C}$ with $\eta(A\xi) \neq 0$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ for some $k \in \{1, \dots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(II) *In cases $M_n(c) = H_n\mathbf{C}$,*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_{n-1}\mathbf{C}$;

(A_2) *a tube over a totally geodesic* $H_k\mathbf{C}$ *for some* $k \in \{1, \dots, n-2\}$.

In continuing work [7] the authors proved the following:

Theorem 2 (Ki and Kurihara [7]). *Let* M *be a real hypersurface in a non-flat complex space form* $M_n(c)$, $c \neq 0$ *which satisfies* $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. *If it satisfies* $R_\xi A = AR_\xi$. *Then* M *is Hopf hypersurfaces.*

Further, in the preceeding paper [7] we studied the structure Jacobi operator is $\phi\nabla_\xi\xi$-parallel under the condition that the structure Jacobi operator commute with the Ricci tensor. In this paper, we investigate such a real hypersurface in $M_n(c)$ under the condition with respect to the mean curvature. We prove that if the structure Jacobi operator R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with a Ricci tensor, then M is a Hopf hypersurface provided that mean curvature of M is constant with respect to the structure vector field.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, $c \neq 0$ with almost complex structure J, and N be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

where g denotes the Riemannian metric of M induced from that of $M_n(c)$ and A denotes the shape operator of M in direction N. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We call ξ the structure vector field (or the Reeb vector field) and its flow also denoted by the same latter ξ. The Reeb vector field ξ is said to be principal if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$.

A real hypersurface M is said to be a Hopf hypersurface if the Reeb vector field ξ is principal. It is known that the aggregate (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. From Kähler condition $\tilde{\nabla}J = 0$, and taking account of above equations, we see that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.2}$$

for any vector fields X and Y tangent to M.

Since we consider that the ambient space is of constant holomorphic sectional curvature $4c$, equations of the Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X, Y)Z = c(&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z) + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi) \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$ and $h = \operatorname{Tr} A$, and for a function f we denote by ∇f the gradient vector field of f.

From the Gauss equation (2.3), the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.5}$$

for any vector field X on M.

Now, we can put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W is a unit vector field orthogonal to ξ. In the sequel, we put $U = \nabla_\xi \xi$, then by (2.1) we see that

$$U = \mu\phi W \tag{2.7}$$

and hence U is orthogonal to W. So we have $g(U, U) = \mu^2$. Using (2.7), it is clear that

$$\phi U = -\mu W, \tag{2.8}$$

which shows that $g(U, U) = \beta - \alpha^2$. Thus it is seen that

$$\mu^2 = \beta - \alpha^2. \tag{2.9}$$

Making use of (2.1), (2.7) and (2.8), it is verified that

$$\mu g(\nabla_X W, \xi) = g(AU, X), \tag{2.10}$$

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.11}$$

because W is orthogonal to ξ.

Now, differentiating (2.8) covariantly and taking account of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi \nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.12}$$

which together with (2.4) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{2.13}$$

Applying (2.12) by ϕ and making use of (2.11), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.14}$$

which connected to (2.1), (2.9) and (2.13) gives

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.15}$$

Using (2.3), the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = R(X, \xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.16}$$

for any vector field X on M. Differentiating this covariantly along M, we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) &= g(\nabla_X(R_\xi Y) - R_\xi(\nabla_X Y), Z) \\ &= -c(\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)) + (X\alpha)g(AY, Z) \\ &\quad + \alpha g((\nabla_X A)Y, Z) - \eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\} \\ &\quad - \eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}. \end{aligned} \tag{2.17}$$

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M; A\xi - \alpha\xi \neq 0\}.$$

At each point of Ω, the Reeb vector field ξ is not principal. That is, ξ is not an eigenvector of the shape operator A of M if $\Omega \neq \emptyset$.

In what follows we assume that Ω is not an empty set in order to prove our main theorem by reductio ad absurdum, unless otherwise stated, all discussion concerns the set Ω.

3. Real hypersurfaces satisfying $R_\xi S = SR_\xi$

Let M be a real hypersurface in $M_n(c)$, $c \neq 0$ satisfying $R_\xi S = SR_\xi$, which means that the Ricci tensor S of type $(1,1)$ and the structure Jacobi operator R_ξ commute to each other. Then by (2.5) and (2.16) we have

$$(3.1)\quad \begin{aligned} g(R_\xi(Y), SX) - g(R_\xi(X), SY) &= g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) - g(A^2\xi, Y)g(hA\xi - c\xi, X) \\ &\quad + g(A^2\xi, X)g(hA\xi - c\xi, Y) - ch(g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)), \end{aligned}$$

which shows that

$$(3.2)\quad \alpha A^3\xi = (\alpha h - c)A^2\xi + (\gamma - \beta h + ch)A\xi + c(\beta - h\alpha)\xi.$$

Combining above two equations and using (2.7), we obtain

$$\mu(g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)) = \beta(\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)),$$

where a 1-form w is defined by $w(X) = g(W, X)$ for any vector field X. Putting $Y = A\xi$ in this, we find

$$(3.3)\quad \mu^2 g(A^2\xi, X) = \mu\gamma w(X) - \beta\alpha g(A\xi, X) + \beta^2\eta(X).$$

Thus, it follows that

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi,$$

which enables us to obtain

$$(3.4)\quad A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi,$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$ on Ω. From (2.6) and above equation we have

$$(3.5)\quad AW = \mu\xi + (\rho - \alpha)W$$

and hence

$$(3.6)\quad A^2W = \rho AW + (\beta - \rho\alpha)W.$$

Comparing (3.1) with (3.2), we find

$$(h - \rho)(\beta - \rho\alpha - c) = 0 \tag{3.7}$$

on Ω.

Now, differentiating (3.5) covariantly along Ω, we get

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X \xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{3.8}$$

By taking the inner product with W in this, since W is a unit vector field orthogonal to ξ, we obtain

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha. \tag{3.9}$$

Also applying this by ξ to (3.8) and making use of (2.10), we have

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{3.10}$$

which together with the Codazzi equation (2.4) gives

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu, \tag{3.11}$$

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - cU + \mu\nabla\mu. \tag{3.12}$$

From now on we put

$$\lambda = \rho - \alpha. \tag{3.13}$$

Putting $X = \xi$ in (3.9), and using (3.12) and (3.13), we get

$$W\mu = \xi\lambda. \tag{3.14}$$

Replacing X by ξ in (3.8) and taking account of (3.12), we obtain

$$\begin{aligned}(\rho - 2\alpha)AU - cU + \mu\nabla\mu + \mu(A\nabla_\xi W - \lambda\nabla_\xi W)\\ = \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi\lambda)W.\end{aligned} \tag{3.15}$$

Differentiating (2.8) covariantly and using (2.2), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Putting $X = \xi$ in this and using (2.16), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{3.16}$$

which enable to obtain

$$W\alpha = \xi\mu. \tag{3.17}$$

Substituting the last two equations into (3.15), we obtain

$$3A^2U - 2\rho AU + (\alpha\rho\beta - c)U + A\nabla\alpha + \frac{1}{2}\nabla\beta - \rho\nabla\alpha = 2\mu(W\alpha)\xi + \mu(\xi\rho)W + \lambda(W\alpha)W - (\lambda - \alpha)(\xi\alpha)\xi. \tag{3.18}$$

Differentiating (3.4) covariantly along Ω and using (2.1), we have

$$\begin{aligned} g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ = (X\rho)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) + X(\beta - \rho\alpha)\eta(Y) \\ + (\beta - \rho\alpha)g(\phi AX, Y). \end{aligned} \tag{3.19}$$

Taking the skew-symmetric part of this and using (2.4), we find

$$\begin{aligned} c(u(Y)\eta(X) - u(X)\eta(Y)) + 2c(\rho - \alpha)g(\phi Y, X) - g(A^2\phi AX, Y) + g(A^2\phi AY, X) \\ + 2\rho g(\phi AX, AY) - (\beta - \rho\alpha)(g(\phi AY, X) - g(\phi AX, Y)) \\ = g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Y\rho)g(A\xi, X) - (X\rho)g(A\xi, Y) \\ + Y(\beta - \rho\alpha)\eta(X) - X(\beta - \rho\alpha)\eta(Y), \end{aligned}$$

where we have defined a 1-form u by $u(X) = g(U, X)$ for any vector field X. Replacing X by μW in the last equation, and making use of (2.13), (3.5), (3.6), (3.10) and (3.11), we obtain

$$\begin{aligned} (3\alpha - 2\rho)A^2U + 2(\rho^2 + \beta - 2\rho\alpha + c)AU + (\rho - \alpha)(\beta - \rho\alpha - 2c)U \\ = \mu A\nabla\mu + (\rho\alpha - \beta)\nabla\alpha - \frac{1}{2}(\rho - \alpha)\nabla\beta + \mu^2\nabla\rho \\ - \mu(W\rho)A\xi - \mu W(\beta - \rho\alpha)\xi. \end{aligned} \tag{3.20}$$

Remark 3.1. If $\beta = \rho\alpha + c$, then $R_\xi A = AR_\xi$ on Ω.

In fact, from (2.16) we have

$$\begin{aligned} g(R_\xi Y, AX) - g(R_\xi X, AY) = g(A^2\xi, Y)g(A\xi, X) - g(A^2\xi, X)g(A\xi, Y) \\ + c(g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)). \end{aligned}$$

By the hypothesis and (3.4) we have $A^2\xi = \rho A\xi + c\xi$. Thus, we arrive at $R_\xi A = AR_\xi$.

4. Lemmas

We continue our arguments under the same hypothesis $R_\xi S = SR_\xi$ as in Section 3. Furthermore, suppose that $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. Then we have $\nabla_W R_\xi = 0$ on Ω because of (2.6), (2.7) and $\mu \neq 0$. Putting $X = W$ in (2.17) and using (2.2), we have

$$(4.1)\qquad -c(\eta(Z)g(\phi AW, Y) + \eta(Y)g(\phi AW, Z)) + (W\alpha)g(AY, Z) + \alpha g((\nabla_W A)Y, Z) - \eta(AZ)(g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)) - \eta(AY)(g((\nabla_W A)\xi, Z) + g(A\phi AW, Z)) = 0$$

because of $\nabla_W R_\xi = 0$. If we replace Y by ξ, and make use of (2.13) and (3.5), then we obtain

$$(4.2)\qquad \alpha A\phi AW + c\phi AW = 0.$$

Remark 4.1. $\alpha \neq 0$ on Ω.

If not, then we have $\alpha = 0$, and then we restrict our arguments on such a place. From (4.2) we have $\phi AW = 0$, which together with (3.5) yields $\rho = 0$ and hence (3.5) reformed as $AW = \mu\xi$. But, using (2.9) and (3.14), we get $W\beta = 0$. So by (2.9), equation (3.20) turns out to be

$$(4.3)\qquad 2(\beta + c)AU = \frac{1}{2}A\nabla\beta.$$

On the other hand, using $AW = \mu\xi$, we can write (4.1) as

$$\eta(AY)g((\nabla_W A)\xi, Z) + \eta(AZ)g((\nabla_W A)\xi, Y) = 0.$$

If we replace Y by W and take account of (3.10), then we obtain $(\nabla_W A)\xi = 0$. Thus (3.11) becomes $\mu\nabla\mu = 2cU$ and consequently $(1/2)\nabla\beta = 2cU$ and hence $\xi\beta = 0$. Accordingly (4.3) reformed as $\beta AU = 0$ and thus $AU = 0$. Using these facts, (3.18) is reduced to $(1/2)\nabla\beta = (\beta + c)U$. This contradicts the fact that $\nabla\beta = 4cU$. Therefore $\alpha \neq 0$ on Ω is proved.

If we make use of (4.2) and Remark 4.1, then (4.1) reformed as

$$\alpha(\nabla_W A)X = -(W\alpha)AX + g(A\xi, X)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi - \frac{c}{\alpha}\mu(w(X)\phi AW + g(\phi AW, X)W).$$

Using (3.5) and (3.10), we can write the last equation as

$$\alpha(\nabla_W A)X = -(W\alpha)AX - \frac{c}{\alpha}\lambda(w(X)U + u(X)W) + \frac{1}{\mu}\{(\rho - 2\alpha)AU - 2cU + \mu\nabla\mu\}g(A\xi, X) + \frac{1}{\mu}g((\rho - 2\alpha)AU - 2cU + \mu\nabla\mu, X)A\xi. \tag{4.4}$$

If we put $X = W$ in (3.20) and make use of (2.9), (3.8) and (3.12), then we obtain

$$\frac{1}{2}\nabla\beta - \alpha\nabla\rho = c\left(2 + \frac{\lambda}{\alpha}\right)U - \rho AU + (W\alpha)AW - (\xi\lambda)A\xi. \tag{4.5}$$

Taking the inner product W to this, and using (3.5) and (3.12), we find

$$\frac{1}{2}W\beta - \alpha(W\rho) = (\rho - \alpha)W\alpha - \mu(W\mu),$$

which together with (2.9) implies that

$$W\beta = \alpha(W\rho) + \rho(W\alpha). \tag{4.6}$$

From (2.9) we have $2\mu(W\mu) = W\beta - 2\alpha(W\alpha)$, which together with (3.14) and (4.6) yields

$$\alpha(W\lambda) = 2\mu(\xi\lambda) - \lambda(W\alpha). \tag{4.7}$$

According to the assumptions $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ and $R_\xi S = SR_\xi$, we have

$$h = \rho. \tag{4.8}$$

Indeed, if not, then by virtue of (3.7), we have $\beta = \rho\alpha + c$. Thus, (3.4) reformed as $A^2\xi = \rho A\xi + c\xi$. By Remark 3.1, we see that $R_\xi A = AR_\xi$. Owing to Theorem 2, we conclude that $\Omega = \varnothing$, a contradiction. Thus, $h = \rho$ is valid on everywhere Ω.

LEMMA 4.2. *If $\alpha AU = \tau U$ for some nonzero function τ on Ω, then*

$$(\alpha\lambda - \tau)\left\{\mu^2\nabla\alpha - (U\alpha)U + \frac{\alpha}{\tau}(\mu^2\nabla\tau - U\tau)U\right\} = \mu^2(W\alpha)(\alpha AW - \tau W).$$

PROOF. If we take the inner product U to (2.12), and use (2.4), (2.7) and the assumption, then we have

$$g(\nabla_X W, U) = \frac{1}{\mu} g((\nabla_U A)X, \xi) + \left(\frac{\tau}{\alpha} - 1\right) w(AX) - 2cw(X). \tag{4.9}$$

Putting $X = U$ in (2.14), we also obtain

$$\phi(\nabla_U A)\xi = \nabla_U U + \tau\left(\frac{\lambda}{\alpha} - 1\right) U. \tag{4.10}$$

Now, if we take the inner product U to (3.8), and make use of (2.4), (2.10) and the assumption, then we get

$$(\alpha\lambda - \tau) g((\nabla_W A)X, U) = \alpha g((\nabla_W A)X, U) + c\alpha\mu\eta(X) - \alpha\mu^2 w(AX). \tag{4.11}$$

By the way, replacing X by U in (4.4), we find

$$\alpha(\nabla_W A)U = (U\mu)A\xi - \frac{\tau}{\alpha}(W\alpha)U - \frac{c\lambda}{\alpha}\mu^2 W + \mu\left(\frac{\lambda\tau}{\alpha} - \tau - 2c\right)A\xi.$$

Combining last two equations, we see that

$$\begin{aligned}(\alpha\lambda - \tau)&g(\nabla_X W, U)\\ &= g\left(X, (U\mu)A\xi - \frac{c\lambda}{\alpha}\mu^2 W + \mu\left(\frac{\lambda\tau}{\alpha} - \tau - 2c\right)A\xi\right.\\ &\qquad \left. + c\alpha\mu\xi - \alpha\mu^2 AW - \frac{\tau}{\alpha}(W\alpha)U\right).\end{aligned} \tag{4.12}$$

Substituting (4.9) into (4.12), we find

$$\begin{aligned}(\alpha\lambda - \tau)&\left\{\frac{1}{\mu}(\nabla_U A)\xi + \left(\frac{\tau}{\alpha} - \alpha\right)AW - 2cW\right\}\\ &= (U\mu)A\xi - \frac{c\lambda}{\alpha}\mu^2 W + \mu\left(\frac{\lambda\tau}{\alpha} - \tau - 2c\right)A\xi - c\alpha\mu\xi - \alpha\mu^2 AW - \frac{\tau}{\alpha}(W\alpha)U.\end{aligned}$$

If we apply this by ϕ and take account of (3.5) and (4.10), then we get

$$\begin{aligned}(\alpha\lambda - \tau)&\left\{\nabla_U U + \left(\frac{2\lambda\tau}{\alpha} - \alpha\lambda - \tau - 2c\right)U\right\}\\ &= \frac{\tau}{\alpha}\mu^2(W\alpha)W + \mu^2\left\{\frac{\lambda(\tau - c)}{\alpha} - \tau - 2c - \alpha\lambda + \frac{U\mu}{\mu}\right\}U\end{aligned}$$

which shows that

$$(\alpha\lambda - \tau)\nabla_U U = \frac{\tau}{\alpha}\mu^2(W\alpha)W + \delta U, \tag{4.13}$$

where the function δ is given by

$$\delta = \mu(U\mu) + \mu^2\left\{\frac{\lambda(\tau - c)}{\alpha} - \tau - 2c - \alpha\lambda + \frac{U\mu}{\mu}\right\} - (\alpha\lambda - \tau)\left(\frac{2\lambda\tau}{\alpha} - \alpha\lambda - \tau - 2c\right). \tag{4.14}$$

Using (4.13) and the assumption, we verify that

$$(\alpha\lambda - \tau)(\alpha A\nabla_U U - \tau\nabla_U U) = \frac{\tau}{\alpha}\mu^2(W\alpha)(\alpha AW - \tau W). \tag{4.15}$$

On the other hand, differentiating $\alpha AU - \tau U = 0$ covariantly and using itself again, we find

$$\frac{\tau}{\alpha}(X\alpha)U + \alpha(\nabla_X A)U + \alpha A\nabla_X U - (X\tau)U - \tau\nabla_X U = 0.$$

If we take the inner product with Y to this, and make use of (2.4) and (2.8), then we have

$$\frac{\tau}{\alpha}(X\alpha)u(Y) + g(\alpha(\nabla_U A)X, Y) - c\alpha\mu(\eta(X)w(Y) + 2w(X)\eta(Y)) + \alpha g(A\nabla_X U, Y) - (X\tau)u(Y) - \tau g(\nabla_X U, Y) = 0.$$

Taking the skew-symmetric part with respect to X and Y, we get

$$\frac{\tau}{\alpha}((Y\alpha)u(X) - (X\alpha)u(Y)) + c\alpha\mu(\eta(Y)w(X) - \eta(X)w(Y)) + \alpha(g(A\nabla_Y U, X) - g(A\nabla_X U, Y)) + (X\tau)u(Y) - (Y\tau)u(X) - \tau du(Y, X) = 0 \tag{4.16}$$

where du the exterior derivate of 1-form u given by

$$du(X, Y) = Y(u(X)) - X(u(Y)) - u([X, Y]).$$

Putting $X = U$ in (4.16), we find

$$\frac{\tau}{\alpha}(\mu^2\nabla\alpha - (U\alpha)U) + (U\tau)U - \mu^2\nabla\tau = \alpha A\nabla_U U - \tau\nabla_U U,$$

which together with (4.15) yields

$$(\alpha\lambda - \tau)\left\{\mu^2\nabla\alpha - (U\alpha)U + \frac{\alpha}{\tau}((U\tau)U - \mu^2\nabla\tau)\right\} = \mu^2(W\alpha)(\alpha AW - \tau W).$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. *$\lambda \neq 0$ on Ω.*

Proof. If not, then we have $\lambda = 0$. It follows from (3.14) and (4.8) that $h = \rho = \alpha$. We restrict our arguments on such a place. By (3.5), we have $AW = \mu\xi$. So (4.5) are reduced to

$$\mu\nabla\mu = 2cU - \alpha AU + \mu(W\alpha)\xi, \tag{4.17}$$

where we have used (2.9) and (4.8). The equation (3.20) turns out to be

$$\alpha A^2U + 2(\mu^2 + c)AU = \mu A\nabla\mu - \mu(W\alpha)A\xi \tag{4.18}$$

with the aid of (3.14) and $W\alpha = 0$. Combining the last two equations, it follows that

$$\alpha A^2U + \mu^2AU = 0. \tag{4.19}$$

Differentiating $AW = \mu\xi$ covariantly along Ω, and taking account of (2.1) and (4.17), we obtain

$$\begin{aligned}(4.20)\qquad & \alpha\mu(\nabla_XA)W + \alpha\mu A\nabla_XW \\ &= \alpha(2cu(X) - \alpha g(AU, X) + \mu(W\alpha)\eta(X))\xi + \alpha\mu^2\phi AX.\end{aligned}$$

On the other hand, from the Codazzi equation (2.4), (2.7) and (4.17) we can write (4.4) as

$$\begin{aligned}\alpha\mu(\nabla_XA)W &= -\mu(W\alpha)AX - 2\alpha(g(A\xi, X)AU + g(AU, X)A\xi) \\ &\quad + c\alpha(\eta(X)U + 2u(X)\xi) + \mu(W\alpha)(g(A\xi, X)\xi + \eta(X)A\xi).\end{aligned}$$

Combining the last two equations, we get

$$\begin{aligned}(4.21)\qquad \alpha\mu A\nabla_XW &= \mu(W\alpha)\{AX - \alpha\eta(X)\xi - \mu(w(X)\xi + \eta(X)W)\} \\ &\quad + \alpha\mu^2\phi AX + 2\alpha(g(A\xi, X)AU + g(AU, X)A\xi) \\ &\quad - c\alpha\eta(X)U - \alpha^2g(AU, X)\xi.\end{aligned}$$

If we take the inner product U to this, then we obtain

$$\alpha\mu g(A\nabla_X W, U) = \mu(W\alpha)u(AX) + \alpha\mu^4\eta(X) + 2\alpha g(A\xi, X)u(AU) - c\alpha\mu^2\eta(X). \tag{4.22}$$

Taking the inner product AU to (4.21) and using (4.19), we have

$$-\mu^3 g(\nabla_X WAU) = -\frac{\mu^3}{\alpha}(W\alpha)g(AU, X) + \alpha\mu^2 g(AU, \phi AX) - 2\mu^2 u(AU)g(A\xi, X) - c\alpha u(AU)\eta(X) \tag{4.23}$$

Canceling $g(A\nabla_X W, U)$ from (4.22) and (4.23), we get

$$(\mu^6 - c\mu^4 - c\alpha u(AU))\eta(X) + \alpha\mu^2 g(AU, \phi AX) = 0.$$

Putting $X = \xi$, we have

$$(\mu^2 - c)(\mu^4 + \alpha u(AU)) = 0, \tag{4.24}$$

which enables us to obtain

$$\mu^4 + \alpha u(AU) = 0. \tag{4.25}$$

In fact, if not, then we gave the last equation $\mu^2 - c$. So μ is constant, which together with (3.17) and (4.17) gives $\alpha AU = 2cU$ on this subset. From this and (4.18) we verify that $\alpha A^2 U + 2(\mu^2 + c)AU = 0$, which implies that $\mu^2 + 2c = 0$, a contradiction. Therefore (4.25) is established. If we take the inner product U to (4.19) and make use of (4.25), then we obtain $\alpha^2 u(A^2 U) = \mu^6$. Comparing this with (4.25), we verify that $\|\alpha AU + \mu^2 U\| = 0$ and consequently

$$\alpha AU + \mu^2 U = 0. \tag{4.26}$$

So (4.17) turns out to be

$$\mu\nabla\mu = (\mu^2 + 2c)U + \mu(W\alpha)\xi. \tag{4.27}$$

Differentiating (4.26) covariantly along Ω and taking account of (4.26) and (4.27), we get

$$-\frac{\mu^2}{\alpha}(X\alpha)U + \alpha(\nabla_X A)U + \alpha A\nabla_X U + 2((\mu^2 + 2c)u(X) + \mu(W\alpha)\eta(X))U + \mu^2\nabla_X U = 0.$$

Taking the inner product Y to this and taking the skew-symmetric part, we have

$$
\begin{aligned}
-\frac{\mu^2}{\alpha}&((X\alpha)u(Y)-(Y\alpha)u(X))-c\alpha\mu(\eta(X)w(Y)-\eta(Y)w(X))\\
&+\alpha(g(\nabla_X U, AY)-g(\nabla_Y U, AX))+2\mu(W\alpha)(\eta(X)u(Y)-\eta(Y)u(X))\\
&+\mu^2(g(\nabla_X Y)-g(\nabla_Y U, X))=0.
\end{aligned}
$$

If we put $X=U$ in this and make use of (4.26), then we have

$$
(4.28)\qquad -\frac{\mu^2}{\alpha}((U\alpha)U-\mu^2\nabla\alpha)+\alpha A\nabla_U U+\mu^2\nabla_U U-2\mu^3(W\alpha)\xi=0.
$$

By the way, because of (4.26), it is satisfies the asuumption of Lemma 4.2. Thus since $\lambda=0$, $\tau=-\mu^2$ and $AW=\mu\xi$, it is seen that (4.15) reformed as

$$
\alpha A\nabla_U U+\mu^2\nabla_U U=-\frac{\mu^3}{\alpha}(W\alpha)(\alpha\xi+\mu W),
$$

which connected to (4.28) gives

$$
(4.29)\qquad \mu^2\nabla\alpha-(U\alpha)U=\mu(W\alpha)A\xi+2\alpha\mu(W\alpha)\xi.
$$

If we combine this to (2.14), then we obtain

$$
(4.30)\qquad \nabla_\xi U+\mu^2\xi=\left(\frac{3\mu^2}{\alpha}+\alpha\mu-\frac{U\alpha}{\mu}\right)W+\frac{W\alpha}{\mu}U.
$$

Taking the inner product X to (4.27) and differentiating this covariantly along Ω, we get

$$
\begin{aligned}
(Y\mu)(X\mu)+\mu Y(X\mu)=2((\mu^2+2c)u(Y)&+\mu(W\alpha)\eta(Y))u(X)\\
&+(\mu^2+2c)g(\nabla_Y U, X)+Y(\mu(W\alpha))\eta(X)\\
&+\mu(W\alpha)g(\phi AY, X)+\mu((\nabla_Y X)(\mu))
\end{aligned}
$$

where we have used (2.1) and (4.27). If we take the skew-symmetric part with respect to X and Y, then we obtain

$$
\begin{aligned}
(4.31)\qquad 2\mu(W\alpha)&(\eta(Y)u(X)-\eta(X)u(Y))\\
&+(\mu^2+2c)(g(\nabla_Y U, X)-g(\nabla_X U, Y))+Y(\mu(W\alpha))\eta(X)\\
&-X(\mu(W\alpha))\eta(Y)+\mu(W\alpha)g((\phi A+A\phi)Y, X)=0.
\end{aligned}
$$

Putting $Y = \xi$ in this and making use of (4.30), we get

$$X(\mu(W\alpha)) = \left(4\mu + \frac{2c}{\mu}\right)(W\alpha)u(X) + \xi(\mu(W\alpha))\eta(X) + (\mu^2 + 2c)\left(\frac{3\mu^2}{\alpha} + \alpha\mu - \frac{U\alpha}{\mu}\right)w(X), \tag{4.32}$$

which connected to (4.31) gives

$$(\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) + \mu(W\alpha)g((\phi A + A\phi)Y, X) + \frac{2}{\mu}(W\alpha)(\mu^2 + c)(u(X)\eta(Y) - u(Y)\eta(X)) + (\mu^2 + 2c)\left(\frac{3\mu^2}{\alpha} + \alpha\mu - \frac{U\alpha}{\mu}\right)(w(Y)\eta(X) - w(X)\eta(Y)) = 0.$$

Putting $\lambda = 0$ and $\tau = -\mu^2$ in (4.17), we get

$$\nabla_U U = -\frac{\mu^2}{\alpha}(W\alpha)W + \frac{\delta}{\mu^2}U.$$

If we put $Y = U$ in this, and make use of (4.27) and above equation, then we have

$$-\frac{2c\mu^2}{\alpha}(W\alpha)W + (\mu^2 + 2c)\left(\frac{\delta}{\mu^2} - \mu^2 - 2c\right)U - 4(\mu^2 + c)\mu(W\alpha)\xi = 0,$$

which tells us that $W\alpha = 0$. So (4.27) and (4.29) turns out to be

$$\mu\nabla\mu = (\mu^2 + 2c)U, \tag{4.33}$$

$$\mu^2\nabla\alpha = (U\alpha)U, \tag{4.34}$$

respectively, which implies that $\xi\alpha = 0$. Taking the inner product X to (4.33), differentiating this covariantly along Ω and taking the skew-symmetric part, we obtain

$$(\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0. \tag{4.35}$$

If we suppose that $\mu^2 + 2c \neq 0$, then we have $g(\nabla_Y U, X) - g(\nabla_X U, Y) = 0$. Replacing Y by ξ in this and using (4.30), we get

$$\frac{3\mu^2}{\alpha} - \frac{U\alpha}{\mu^2} = -\alpha. \tag{4.36}$$

On the other hand, putting $\rho = \alpha$, $\lambda = 0$, $W\alpha = 0$ and $\xi\alpha = 0$ in (3.18) and making use of (4.26), (4.33), (4.34) and above equation, we have $\mu^2 + c = 0$. Thus μ is some constant, which together with (4.33) that $\mu^2 + 2c = 0$, a contradiction. Therefore

$$\mu^2 + 2c = 0 \tag{4.37}$$

is established. By (4.34) we have

$$-2c\nabla\alpha = (U\alpha)U.$$

Taking the inner product X to this, differentiating this covariantly along Ω and taking the skew-symmetric part, we obtain

$$(Y(U\alpha))u(X) - (X(U\alpha))u(Y) + (U\alpha)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0. \tag{4.38}$$

If we put $X = U$, then since $W\alpha = 0$ and (4.37), we have

$$Y(U\alpha) = \frac{1}{\mu^2}\left(U(U\alpha) - \frac{\delta}{\mu^2}\right)u(Y).$$

Hence it follows from (4.38) that

$$(U\alpha)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0.$$

Using the same method as that used derive (4.37) from (4.35), we can deduce from this that (4.36). Thus putting $\rho = \alpha$, $\lambda = 0$, $W\alpha = 0$ and $\xi\alpha = 0$ in (3.18), and making use of (4.26) and (4.36), we have $c = 0$, a contradiction. Therefore $U\alpha = 0$, which means that α is some constant because of (4.34). Using (4.19) and (4.37), equation (3.18) reduced to $\alpha^2 - 4c = 0$. This contradics (4.37). Therefore Lemma 4.3 is proved. □

Remark 4.4. $\alpha\lambda + c \neq 0$ on Ω.

In fact, we assume that $\alpha\lambda + c = 0$, then from (4.14) and (4.15), we have $\alpha(U\mu) = 2c\lambda\mu$. Putting $X = \xi$ in (4.13), we get $\alpha(U\mu) = \mu(c\lambda + \alpha\mu^2)$. Compareing with the last two equations, we obtain $\alpha\mu^2 = c\lambda$, which connected to the fact that $\alpha\lambda + c = 0$ implies that $\lambda^2 + \mu^2 = 0$, a contradiction. Thus, $\alpha\lambda + c$ dose not vanish everywhere on Ω.

5. Real hypersurfaces satisfying $R_\xi S = SR_\xi$ and $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$

We will continue our discussions under the hypotheses as those stated in Section 4.

From (3.5) and (4.2) we have $\lambda(\alpha AU + cU) = 0$, which together with Lemma 4.3 implies that

$$\alpha AU + cU = 0. \tag{5.1}$$

Using (3.13), (4.8) and (5.1), we can write (4.4) as

$$\alpha(\nabla_W A)X = -(W\alpha)AX - \frac{c}{\alpha}\lambda(w(X)U + u(X)W) + g\left(\nabla\mu - \frac{ch}{\alpha\mu}U, X\right)A\xi + g(A\xi, X)\left(\nabla\mu - \frac{ch}{\alpha\mu}U\right). \tag{5.2}$$

Because of (4.8) and (5.1), we also have from (4.5)

$$\mu\nabla\mu - \alpha\nabla\lambda = \left(3c + \frac{2c}{\alpha}\lambda\right)U + (W\alpha)AW - (\xi\lambda)A\xi. \tag{5.3}$$

From (5.1) and Lemma 4.2, we obtain

$$(\alpha\lambda + c)(\mu^2\nabla\alpha - (U\alpha)U) = \mu^2(W\alpha)(\alpha AW + cW), \tag{5.4}$$

which tells us that

$$(\alpha\lambda + c)(\xi\alpha) = \mu\alpha(W\alpha). \tag{5.5}$$

Now, if we take account of (2.7), (2.9) and (5.1), then we can write (2.16) as

$$\alpha\nabla_\xi U = \mu(\alpha^2 + 3c)W - \alpha\mu^2\xi + \alpha\phi\nabla\alpha.$$

Applying (5.4) by ϕ, from Remark 4.4 we find

$$\mu\phi\nabla\alpha + (U\alpha)W = (W\alpha)U.$$

Combining the last two equations, we see that

$$\nabla_\xi U = \mu\left(\alpha + \frac{3c}{\alpha} - \frac{U\alpha}{\mu^2}\right)W - \mu^2\xi + \frac{W\alpha}{\mu}U, \tag{5.6}$$

which together with (5.1) implies that

$$A\nabla_\xi U + c\nabla_\xi U = \mu\left(\alpha + \frac{3c}{\alpha} - \frac{U\alpha}{\mu^2}\right)(\alpha AW + cW) - \mu^2(\alpha A\xi + c\xi). \tag{5.7}$$

Putting $Y = \xi$ in (4.16) with $\tau = -c$ and using (2.7), we get

$$\frac{c}{\alpha}(\xi\alpha)u(X) + c\alpha\mu w(X) - \alpha g(\nabla_X U, \alpha\xi + \mu W) + g(\alpha A\nabla_\xi U + c\nabla_\xi U, X) - cg(\nabla_X U, \xi) = 0,$$

which connected to (2.10) and (5.7) yields

$$\alpha g(\nabla_X W, U) = \frac{c}{\alpha\mu}(\xi\alpha)u(X) + \frac{U\alpha}{\mu^2} g(\alpha AW + cW, X) - 2(\alpha^2 + 2c)g(AW, X) - c\left(2\alpha + \frac{3c}{\alpha}\right)w(X) + \mu g(\alpha A\xi + c\xi, X). \tag{5.8}$$

If we combine this to (4.13) with $\tau = -c$, then we obtain

$$\frac{\alpha\lambda + c}{\mu^2}(U\alpha)(\alpha AW + cW) - \alpha(U\mu)A\xi = c\mu(\alpha^2\xi - \lambda\mu W) - c\mu(\alpha + \lambda)A\xi - \alpha^2\mu^2 AW + c(W\alpha)U + (\alpha\lambda + c)\left\{2(\alpha^2 + 2c)AW + c\left(2\alpha + \frac{3c}{\alpha}\right)W - \mu(\alpha A\xi + c\xi)\right\}.$$

Taking the inner product ξ and W to this, we have

$$\frac{\alpha\lambda + c}{\mu^2}(U\alpha) = \frac{\alpha}{\mu}(U\mu) - c\lambda - \alpha\mu^2 + (\alpha\lambda + c)\left(\alpha + \frac{3c}{\alpha}\right) \tag{5.9}$$

and

$$\frac{(\alpha\lambda + c)^2}{\mu^2}(U\alpha) = \alpha\mu(U\mu) - \mu^2(2c\lambda + c\alpha + \lambda\alpha^2) + (\alpha\lambda + c)\left\{2\lambda(\alpha^2 + 2c) + c\left(2\alpha + \frac{3c}{\alpha}\right) - \alpha\mu^2\right\},$$

respectively. From the last two equations it follows that

$$\{\alpha(U\mu) - \mu(\lambda\alpha^2 + 2c\lambda + c\alpha)\}(\alpha\lambda + c - \mu^2) = 0,$$

which implies that

$$\alpha(U\mu) = \mu(\lambda\alpha^2 + 2c\lambda + c\alpha). \tag{5.10}$$

In fact, if not, then we have $\alpha\lambda + c - \mu^2 = 0$, which together with (2.9) and (3.13) gives $\beta = \rho\alpha + c$ on this subset. Therefore, from Remark 3.1 we have $R_\xi A = AR_\xi$. By Theorem 2, we get $\Omega = \emptyset$. Thus, (5.10) is accomplished on Ω.

Substituting (5.10) into (5.9), we get

$$\alpha(\alpha\lambda + c)U\alpha = \mu^2(2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2). \tag{5.11}$$

Therefore (4.9) with $\tau = -c$ reformed as

$$(\alpha\lambda + c)\nabla\alpha = (W\alpha)(\alpha AW + cW) + \frac{2F}{\alpha}U, \tag{5.12}$$

where we have put $2F = 2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2$.

Because of (3.17), (5.5) and (5.12), it is seen that (3.16) reformed as

$$\mu\nabla_\xi W = 3AU - \alpha U + \frac{2F}{\alpha(\alpha\lambda + c)}U,$$

which connected to (5.1) gives

$$\mu\nabla_\xi W = \frac{1}{\alpha\lambda + c}(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2)U. \tag{5.13}$$

Because of (3.13), (3.17) and (5.13), we can write (3.15) as

$$\begin{aligned}\mu\nabla\mu &+ (\lambda - \alpha)AU - (\mu^2 + c)U \\ &= -\frac{1}{\alpha\lambda + c}(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2)(AU - \lambda U) + \mu(W\alpha)\xi + \mu(\xi\lambda)W\end{aligned}$$

which together with (5.1) gives

$$\mu\nabla\mu = \mu(W\alpha)\xi + \mu(\xi\lambda)W + \left(\alpha\lambda + \frac{2c\lambda}{\alpha} + c\right)U. \tag{5.14}$$

Using (4.7) and (5.14), we can write (5.3) as

$$\alpha\nabla\lambda = \alpha(\xi\lambda)\xi + \alpha(W\lambda)W + (\alpha\lambda - 2c)U. \tag{5.15}$$

Now, differentiating (5.1) covariantly and taking inner product to Y, we find

$$(X\alpha)u(AY) + \alpha g((\nabla_X A)U, Y) + \alpha g(A\nabla_X U, Y) + cg(\nabla_X U, Y) = 0. \tag{5.16}$$

Putting $X = W$ in this, we get

$$(W\alpha)g(AU, Y) + \alpha g((\nabla_W A)U, Y) + \alpha g(A\nabla_W U, Y) + cg(\nabla_W U, Y) = 0. \tag{5.17}$$

Taking the skew-symmetric part of (5.16) and using (2.4), we obtain

$$\begin{aligned}-\frac{c}{\alpha}&((X\alpha)u(Y) - (Y\alpha)u(X)) + c\alpha\mu(\eta(X)w(Y) - \eta(Y)w(X)) \\ &+ \alpha(g(A\nabla_X U, Y) - g(A\nabla_Y U, X)) + c(g(\nabla_X U, Y) - g(\nabla_Y U, X)) = 0.\end{aligned}$$

If we put $X = W$ in this and make use of (2.10), then we get

$$-\frac{c}{\alpha}(W\alpha)u(Y) - c\alpha\mu\eta(Y) + \alpha\mu^2 g(AW, Y) - (\alpha\lambda + c)g(\nabla_Y U, W) + g(\alpha A\nabla_W U + c\nabla_W U, Y) = 0. \tag{5.18}$$

On the other hand, if we replace X by U in (4.4), then we have

$$\alpha(\nabla_W A)U + (W\alpha)AU = -\frac{c}{\alpha}\lambda\mu^2 W + \left(-\frac{c}{\alpha}\mu(\lambda - \alpha) - 2c\mu + U\mu\right)A\xi,$$

which connected to (5.17) and (5.10) yields

$$\alpha A\nabla_W U + c\nabla_W U = -\lambda\alpha\mu A\xi - c\lambda\mu\xi.$$

Therefore (5.18) reformed as

$$(\alpha\lambda + c)g(U, \nabla_Y W) = \frac{c}{\alpha}(W\alpha)u(Y) + \mu(c\alpha + \lambda\alpha^2 - \alpha\mu^2 + c\lambda)\eta(Y). \tag{5.19}$$

Now, we put

$$x = \frac{c(W\alpha)}{\alpha(\alpha\lambda + c)}, \quad y = \frac{\mu(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2)}{\alpha\lambda + c}.$$

Then, (5.19) is written by

$$g(U, \nabla_Y W) = xu(Y) + y\eta(Y). \tag{5.20}$$

Differentiating this covariantly and taking the skew-symmetric part, we find

$$\begin{aligned} g(\nabla_X U, \nabla_Y W) - g(\nabla_Y U, \nabla_X W) + g(R(X, Y)W, U) \\ = (Xx)u(Y) - (Yx)u(X) + x(g(\nabla_X U, Y) - g(\nabla_Y U, X)) \\ + (Xy)\eta(Y) - (Yy)\eta(X) + y(g(\phi AX, Y) - g(\phi AY, X)). \end{aligned}$$

Gauss equation (2.3) becomes

$$\begin{aligned} g(R(X, Y)W, U) = 2c(w(Y)u(X) - w(X)u(Y) - \mu g(\phi X, Y)) \\ + \frac{c}{\alpha}(g(AW, Y)u(X) - g(AW, X)u(Y)). \end{aligned}$$

Combining the last two equations, we verify that

$$(5.21)\qquad g(\nabla_X U, \nabla_Y W) - g(\nabla_Y U, \nabla_X W) + 2c(w(Y)u(X) - w(X)u(Y) - \mu g(\phi X, Y)) + \frac{c}{\alpha}(g(AW, Y)u(X) - g(AW, X)u(Y))$$
$$= (Xx)u(Y) - (Yx)u(X) + x(g(\nabla_X U, Y) - g(\nabla_Y U, X)) + (Xy)\eta(Y) - (Yy)\eta(X) + y(g(\phi AX, Y) - g(\phi AY, X)).$$

On the other hand, from (4.10) and (4.13) with $\tau = -c$ we see

$$(5.22)\qquad \nabla_U U = -x\mu^2 W + \left(\alpha\lambda + \frac{2c\lambda}{\alpha} + c\right)U.$$

Differentiating (2.6) covariantly and using (2.1), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this, then by (3.14) we have

$$\mu(\nabla_W A)\xi + \lambda AU = \mu(W\alpha)\xi + \alpha\lambda U + \mu(\xi\lambda)W + \mu^2\nabla_W W,$$

which together with (3.17), (5.1) and (5.2) gives

$$(5.23)\qquad \nabla_W W = 0.$$

Differentiating (2.8) covariantly and using (2.2), we get

$$(5.24)\qquad \frac{c}{\alpha}u(X)\xi + \phi\nabla_X U = -(X\mu)W - \mu\nabla_X W.$$

Replacing X by W in this and using (2.7), (2.11), (3.5), (3.13) and (5.23), we find

$$(5.25)\qquad \nabla_W U = -\mu\lambda\xi + \frac{W\mu}{\mu}U.$$

From (5.14) we can write (5.24) as

$$\frac{c}{\alpha}u(X)\xi + \phi\nabla_X U = -\left\{(W\alpha)\eta(X) + (\xi\lambda)w(X) + \frac{1}{\mu}\left(\alpha\lambda + \frac{2c\lambda}{\alpha} + c\right)u(X)\right\}W - \mu\nabla_X W.$$

If we put $X = U$ in this, then from (5.22) we have

$$(5.26)\qquad \nabla_U W = -\frac{c}{\alpha}\mu\xi + xU.$$

Putting $Y = U$ in (5.21), from (2.10), (5.20), (5.22) and (5.26), we obtain

$$Xx = \frac{1}{\mu^2}\left\{Uy - y\left(\frac{2c\lambda}{\alpha} + \alpha\lambda + c + \mu^2\right)\right\}\eta(X) + \left\{-4c - x^2 + \frac{y}{\mu}\left(\frac{c}{\alpha} - \lambda\right)\right\}w(X) + \frac{Ux}{\mu^2}u(X). \tag{5.27}$$

Substituting this into (5.21), we find

$$\begin{aligned} g(\nabla_X U, \nabla_Y W) - g(\nabla_Y U, \nabla_X W) &= -2c\mu g(\phi X, Y) + \frac{c}{\alpha}(w(AY)u(X) - w(AX)u(Y)) \\ &+ \frac{1}{\mu^2}\left\{Uy - y\left(\frac{2c\lambda}{\alpha} + \alpha\lambda + c + \mu^2\right)\right\}(\eta(X)u(Y) - \eta(Y)u(X)) \\ &+ \left\{-2c - x^2 + \frac{y}{\mu}\left(\frac{c}{\alpha} - \lambda\right)\right\}(w(X)u(Y) - w(Y)u(X)) \\ &+ x(g(\nabla_X U, Y) - g(\nabla_Y U, X)) + (Xy)\eta(Y) - (Yy)\eta(X) \\ &+ y(g(\phi AX, Y) - g(\phi AY, X)). \end{aligned} \tag{5.28}$$

If we replace X by W in this, then from (2.10), (5.20), (5.23) and (5.25) we get

$$\mu(Wy) = y(W\mu) + x\mu(\mu\lambda - y). \tag{5.29}$$

6. Real hypersurfaces which satisfies the mean curvature is constant with respect to the structure vector field

We will continue our discussions under the hypotheses as those stated in Section 5.

Lemma 6.1. *If* $\xi h = 0$, *then* $\xi\alpha = \xi\lambda = W\alpha = 0$.

Proof. Since (3.13), (4.8) and $\xi h = 0$, we have

$$\xi\lambda = -\xi\alpha. \tag{6.1}$$

By (3.14) and (4.7) we find

$$W\mu = -\xi\alpha, \tag{6.2}$$

$$\alpha(W\lambda) = -2\mu(\xi\alpha) - \lambda(W\alpha). \tag{6.3}$$

Taking the inner product ξ to (5.12), we get

(6.4) $$(\alpha\lambda + c)(\xi\alpha) = \alpha\mu(W\alpha).$$

From (5.29), (6.2) and (6.4), we obtain

(6.5) $$\alpha^2\mu(Wy) = \{c(\lambda\mu - y) - \alpha^2 y\}(\xi\alpha).$$

Differentiating $(\alpha\lambda + c)y = \mu(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2)$ covariantly with respect the vector field W, from (6.2)–(6.5) we have

$$\begin{aligned}&(\alpha\lambda + c)\{c(\lambda\mu - y) - \alpha^2 y\}(\xi\alpha)\\ &\quad - \alpha^2\mu\left(2\mu y + \alpha\mu^2 - c\alpha - c\lambda - \alpha^2\lambda - \frac{2c\mu^2}{\alpha}\right)(\xi\alpha)\\ &\quad - \alpha\mu(\alpha\lambda + c)\left(\alpha\lambda + c - \mu^2 - \frac{c\lambda}{\alpha}\right)(\xi\alpha) = 0.\end{aligned}$$

Now, suppose that $\xi\alpha \neq 0$. Then above equation yields

$$\begin{aligned}&(\alpha\lambda + c)\{c\lambda\mu - (\alpha^2 + c)y\} - \alpha^2\mu\left(2\mu y + \alpha\mu^2 - c\alpha - c\lambda - \alpha^2\lambda - \frac{2c\mu^2}{\alpha}\right)\\ &\quad - \alpha\mu(\alpha\lambda + c)\left(\alpha\lambda + c - \mu^2 - \frac{c\lambda}{\alpha}\right) = 0,\end{aligned}$$

which implies that

$$\begin{aligned}-2\alpha^3\mu^4 &= (\alpha\lambda + c)^2(2c\lambda - 2c\alpha - \alpha^2\lambda) - c^2\lambda(\alpha\lambda + c)\\ &\quad + \alpha\mu^2\{(\alpha\lambda + c)^2 + 3c(\alpha\lambda + c) - 2\alpha(\alpha^2\lambda + c\alpha + c\lambda)\}.\end{aligned}$$

Differentiating this covariantly with respect the vector field ξ, from (6.1), (3.17) and (6.3) we get

$$\begin{aligned}-6\alpha^2\mu^4 &= (\alpha\lambda + c)(4c\lambda^2 - 8c\alpha\lambda - 2\alpha^2\lambda^2 - \alpha^3\lambda + c\alpha^2 + 5c^2) - c^2\lambda^2 + c^2\alpha\lambda\\ &\quad + \mu^2\{(\alpha\lambda + c)(3\alpha\lambda + 2\alpha^2 + 4c) - c\alpha\lambda - 3c\alpha^2 - 4\alpha^3\lambda + 2\alpha^4\}.\end{aligned}$$

Combining the last two equations, we obtain

$$\begin{aligned}&-11c^3 + c^2\alpha^2 - 10c^2\alpha\lambda + 2c\alpha^3\lambda + 2c\alpha^2\lambda^2 - c\alpha\lambda^3 + \alpha^4\lambda^2 + \alpha^3\lambda^3\\ &\quad = \mu^2(-8c^2 + 7c\alpha^2 - 3c\alpha\lambda + 4\alpha^3\lambda).\end{aligned}$$

Differentiating this covariantly with respect the vector field ξ, from (6.1), (3.17) and (6.3) we get

$$5c^3 - c^2\alpha^2 + 2c^2\alpha\lambda - 14c\alpha^3\lambda + 15c\alpha^2\lambda^2 - 2c\alpha\lambda^3 - 6\alpha^4\lambda^2 + 4\alpha^3\lambda^3$$
$$= \mu^2(-8c^2 + 24c\alpha^2 - 6c\alpha\lambda - 4\alpha^4 + 16\alpha^3\lambda).$$

Combining the last two equations, we obtain

$$128c^5 - 315c^4\alpha^2 + 177c^4\alpha\lambda + 77c^3\alpha^4 - 587c^3\alpha^3\lambda - 170c^3\alpha^2\lambda^2 - 8c^3\alpha\lambda^3$$
$$+ 10c^2\alpha^6 + 156c^2\alpha^5\lambda - 335c^2\alpha^4\lambda^2 - 17c^2\alpha^3\lambda^3$$
$$+ 146c\alpha^6\lambda^2 - 56c\alpha^5\lambda^3 - 2c\alpha^4\lambda^4 - 4\alpha^8\lambda^2 + 36\alpha^7\lambda^3 = 0.$$

Differentiating this covariantly nine times with respect the vector field ξ, from (6.1) we get $\xi\alpha = 0$, a contradiction. □

Thus, (5.12), (5.14) and (5.15) turns out

(6.6) $$(\alpha\lambda + c)\nabla\alpha = \frac{2F}{\alpha}U,$$

(6.7) $$\mu\nabla\mu = \left(\alpha\lambda + \frac{2c\lambda}{\alpha} + c\right)U,$$

(6.8) $$\alpha\nabla\lambda = (\alpha\lambda - 2c)U,$$

respectively. Since $x = 0$, equation (5.27) implies

$$Uy = y\left(\mu^2 + \alpha\lambda + \frac{2c\lambda}{\alpha} + c\right), \quad -4c\mu + y\left(\frac{c}{\alpha} - \lambda\right) = 0.$$

Combining the last two equations, it is verify that

$$(\alpha\lambda - c)y = -4c\mu,$$

which together with a definition of y yields

(6.9) $$-4c\alpha(\alpha\lambda + c) = (\alpha\lambda - c)(\alpha^2\lambda + c\lambda + c\alpha - \alpha\mu^2).$$

7. Proof of Main theorem

In this Section, we prove the following theorem.

THEOREM 7.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, such that $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ at the same time $R_\xi S = SR_\xi$. If the mean curvature of M is constant with respect to the structure vector field ξ, then M is a Hopf hypersurface.*

Proof. Taking the inner product X to (6.8) and differentiating this covariantly, we have

$$(Y\alpha)(X\lambda) + \alpha(Y(X\lambda)) = (\lambda(Y\alpha) + \alpha(Y\lambda))u(X) + (\alpha\lambda - 2c)(Y(u(X))).$$

Taking the skew-symmetric part of this, and using (6.6) and (6.8), we obtain

$$(\alpha\lambda - c)du(X, Y) = 0.$$

In the same way from (6.7) we see that

$$(\alpha^2\lambda + 2c\lambda + c\alpha)du(X, Y) = 0.$$

Now, let $\Omega_1 = \{p \in \Omega; (du)_p \neq 0\}$ and suppose that $\Omega_1 \neq \emptyset$. Using the last two equations, we verify that

$$\alpha\lambda = 2c, \quad \alpha^2\lambda + 2c\lambda + c\alpha = 0, \tag{7.1}$$

which shows that

$$3\alpha + 2\lambda = 0 \tag{7.2}$$

on Ω_1. Hence we have

$$3\alpha^2 + 4c = 0. \tag{7.3}$$

Therefore we see that α is some constant on Ω_1. So from (5.12) we have

$$2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2 = 0$$

on Ω_1, which connected to (7.1) gives

$$4\mu^2 + 9c = 0. \tag{7.4}$$

On the other hand, from (7.1)–(7.3) and Remark 4.1 we can write (6.9) as

$$\mu^2 - 12c = 0,$$

which contradicts (7.4). Hence $\Omega_1 = \emptyset$. So we have $du = 0$ and therefore $du(\xi, X) = 0$ for any vector field X. Namely,

$$g(\nabla_\xi U, X) + g(U, \nabla_X \xi) = 0.$$

Because of (2.11) and (2.15), it reformed as

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0,$$

which together with (2.7), (2.9), (3.5), (5.1) and Lemma 6.1 implies that

$$\alpha\nabla\alpha = (\alpha\lambda + \alpha^2 + 3c)U. \tag{7.5}$$

Comparing this with (5.16), we get

$$(\alpha\lambda + c)(\alpha\lambda + \alpha^2 + 3c) = 2\alpha^3\lambda + 4c\alpha\lambda + 2c\alpha^2 + 3c^2 - \alpha^2\mu^2,$$

which enable us to obtain

$$\mu^2 = \alpha\lambda - \lambda^2 + c. \tag{7.6}$$

Differentiation gives $2\mu\nabla\mu = (\alpha - 2\lambda)\nabla\lambda + \lambda\nabla\alpha$, which together with (6.7), (6.8) and (7.5) yields

$$2(\alpha^2\lambda + 2c\lambda + c\alpha) = (\alpha - 2\lambda)(\alpha\lambda - 2c) + \lambda(\alpha\lambda + \alpha^2 + 3c).$$

Accordingly we verify that

$$\alpha\lambda^2 + 4c\alpha - 3c\lambda = 0, \tag{7.7}$$

which shows that $(\lambda^2 + 4c)\nabla\alpha + (2\alpha\lambda - 3c)\nabla\lambda = 0$. This together with (6.8), (7.6) and (7.7) yields $3\lambda^2 + \alpha\lambda - 4\alpha^2 + 9c = 0$. Eliminating α to (7.7) and this, we obtain

$$\lambda^6 + 12c\lambda^4 + 32c^2\lambda^2 + 48c^3 = 0,$$

which shows that $\nabla\lambda = 0$ and hence from (6.9) we have $\alpha\lambda = 2c$. Thus, (7.7) implies that $\lambda = 4\alpha$ and therefore $\lambda^2 = 8c$. Consequently (7.6) becomes $\mu^2 + 5c = 0$, a contradiction. Hence we conclude that $\Omega = \varnothing$. Accordingly we see that M is Hopf hypersurfaces. This completes the proof of Theorem 7.1. □

Example. All examples of Takagi's list [16] and Berndt's list [1] satisfy the conditions of Theorem 7.1. In fact, the structure vector of these examples is principal and all principal curvatures are constant. Thus, we have the mean curvature is constant and $\phi\nabla_\xi\xi = 0$. Hence we obtain $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. Moreover from (2.5) and (2.16) it is easy to see that $R_\xi S = SR_\xi$.

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