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SUBMETACOMPACTNESS AND WEAK SUBMETACOMPACTNESS IN COUNTABLE PRODUCTS, II

By

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Abstract. In this paper, we shall discuss submetacompactness and weak submetacompactness in countable products of Čech-scattered spaces and prove the following: (1) If $\{X_n : n \in \omega\}$ is a countable collection of submetacompact Čech-scattered spaces, then the product $\prod_{n\in\omega} X_n$ is submetacompact. (2) If Y is a hereditarily weakly submetacompact space and $\{X_n : n \in \omega\}$ is a countable collection of weakly submetacompact Čech-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is weakly submetacompact.

1 Introduction

A space X is said to be *subparacompact* (*metacompact*) if every open cover of X has a σ-locally finite closed (point finite open) refinement. A space X is said to be *submetacompact* (*weakly submetacompact*) if for every open cover $\mathscr{U}$ of X, there is a sequence $(\mathscr{V}_n : n \in \omega)$ of open refinements (an open refinement $\bigcup_{n\in\omega} \mathscr{V}_n$) of $\mathscr{U}$ such that for each $x \in X$, there is an $n \in \omega$ with $ord(x, \mathscr{V}_n) < \omega$ $(1 \le ord(x, \mathscr{V}_n) < \omega)$. For a collection $\mathscr{A}$ of subsets of X and $x \in X$, let $\mathscr{A}_x = \{A \in \mathscr{A} : x \in A\}$ and $ord(x, \mathscr{A}) = |\mathscr{A}_x|$. We call such a sequence $(\mathscr{V}_n : n \in \omega)$ of open refinements (an open refinement $\bigcup_{n\in\omega} \mathscr{V}_n$) of $\mathscr{U}$ a *θ-sequence* (*weak θ-refinement*) of $\mathscr{U}$. It is clear that a space X is weakly submetacompact if and only if for every open cover $\mathscr{U}$ of X, there is an open refinement $\bigcup_{n\in\omega} \mathscr{V}_n$ of $\mathscr{U}$ such that for each $x \in X$, there is an $n \in \omega$ with $ord(x, \mathscr{V}_n) = 1$. It is well known that (1) every paracompact space is subparacompact and metacompact, (2) every

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subparacompact (metacompact) space is submetacompact, and (3) every submetacompact space is weakly submetacompact. Smith [S, Corollary 3.6(2)] proved that every countably compact, weakly submetacompact space is compact.

Telgársky [Te] introduced the notion of C-scattered spaces and proved that the product of a paracompact (Lindelöf) C-scattered space and a paracompact (Lindelöf) space is paracompact (Lindelöf). Yajima [Y1], Gruenhage and Yajima [GY] proved similar results for subparacompact (metacompact, submetacompact, weakly submetacompact) spaces. Furthermore, the author [T1, T2, T3, T4] proved the following: (1) if Y is a perfect paracompact (hereditarily Lindelöf, perfect subparacompact, hereditarily weakly submetacompact) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf, subparacompact, weakly submetacompact) C-scattered spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is paracompact (Lindelöf, subparacompact, weakly submetacompact) and (2) if $\{X_n : n \in \omega\}$ is a countable collection of metacompact (submetacompact) C-scattered spaces, then the product $\prod_{n \in \omega} X_n$ is metacompact (submetacompact).

On the other hand, Hohti and Ziqiu [HZ] introduced the notion of Čech-scattered spaces, which is a generalization of C-scattered spaces and studied paracompactness (Lindelöf property) of countable products. Furthermore Aoki, Mori and the author [AMT], Higuchi and the author [HT] proved that (1) if Y is a perfect paracompact (hereditarily Lindelöf, perfect subparacompact) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf, subparacompact) Čech-scattered spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is paracompact (Lindelöf, subparacompact) and (2) if $\{X_n : n \in \omega\}$ is a countable collection of metacompact Čech-scattered spaces, then the product $\prod_{n \in \omega} X_n$ is metacompact.

It seems to be natural to consider submetacompactness and weak submetacompactness of countable products of Čech-scattered spaces. So, Higuchi and the author [HT] raised the following:

PROBLEM. (1) If $\{X_n : n \in \omega\}$ is a countable collection of submetacompact Čech-scattered spaces, then is the product $\prod_{n \in \omega} X_n$ submetacompact?

(2) If Y is a hereditarily weakly submetacompact space and $\{X_n : n \in \omega\}$ is a countable collection of weakly submetacompact Čech-scattered spaces, then is the product $Y \times \prod_{n \in \omega} X_n$ weakly submetacompact?

In this paper, we shall answer to these problems affirmatively.

All spaces are assumed to be Tychonoff spaces. Let ω denote the set of natural numbers. Let $|A|$ denote the cardinality of a set A. Undefined terminology can be found in Engelking [E].

2 Preliminaries

A space X is said to be *scattered* if every nonempty (closed) subset A has an isolated point in A and X is said to be *C-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a compact neighborhood in A. Then scattered spaces and locally compact spaces are C-scattered. A space X is said to be *Čech-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a Čech-complete neighborhood in A. Thus locally Čech-complete spaces and C-scattered spaces are Čech-scattered. It is well known that the space of irrationals $\mathbf{P} = \omega^\omega$ is not C-scattered. However, it is Čech-complete and hence, Čech-scattered.

Let X be a space. For a closed subset A of X, let

$$A^* = \{x \in A : x \text{ has no Čech-complete neighborhood in } A\}.$$

Let $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal α. Note that every $A^{(\alpha)}$ is a closed subset of X and X is Čech-scattered if and only if $X^{(\alpha)} = \varnothing$ for some ordinal α. Let X be a Čech-scattered space. If A is open or closed in X, then A is also Čech-scattered. Let A be a subset of X. Put

$$\lambda(X) = \inf\{\alpha : X^{(\alpha)} = \varnothing\} \quad \text{and}$$

$$\lambda(A) = \inf\{\alpha : A \cap X^{(\alpha)} = \varnothing\} \le \lambda(X).$$

It is clear that if A, B are subsets of X such that $A \subset B$, then $\lambda(A) \le \lambda(B)$. A subset A of X is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty Čech-complete subset of X and $A \cap X^{(\alpha(A)+1)} = \varnothing$. We denote $Top(A) = A \cap X^{(\alpha(A))}$. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $rank(x) = \alpha$. Then there is a neighborhood base $\mathscr{B}_x$ of x in X, consisting of open subsets of X, such that for each $B \in \mathscr{B}_x$, $\bar{B}$ is topped in X and $\alpha(\bar{B}) = rank(x)$.

It is clear that if X and Y are Čech-scattered spaces, then the product $X \times Y$ is Čech-scattered.

Lemma 2.1 (Engelking [E]). *A space X is Čech-complete if and only if there is a sequence $(\mathscr{A}_n)$ of open covers of X satisfying that if $\mathscr{F}$ is a collection of closed subsets of X, with the finite intersection property, such that for each $n \in \omega$, there are $F_n \in \mathscr{F}$ and $A_n \in \mathscr{A}_n$ with $F_n \subset A_n$, then the intersection $\bigcap \mathscr{F}$ is nonempty.*

In Lemma 2.1, we may assume that for each $n \in \omega$, $\mathscr{A}_{n+1}$ is a refinement of $\mathscr{A}_n$. The sequence $(\mathscr{A}_n)$ is said to be a *complete* sequence of open covers of X. The following fact was well known.

Fact. In Lemma 2.1,

(1) The intersection $\bigcap \mathscr{F}$ is countably compact. So, if X is weakly submetacompact, then $\bigcap \mathscr{F}$ is compact (Smith [S]).

(2) If $\mathscr{F} = \{F_n : n \in \omega\}$ is a decreasing sequence of nonempty closed subsets of X such that for each $n \in \omega$, there is an $A_n \in \mathscr{A}_n$ with $F_n \subset A_n$, then the nonempty countably compact closed subset $F = \bigcap_{n \in \omega} F_n$ satisfies the following: for every open neighborhood U of F, there is an $n \in \omega$ with $F_n \subset U$.

Lemma 2.2 (Gruenhage and Yajima [GY]). *There is a filter $\mathscr{F}$ on ω satisfying: for every submetacompact space X and every open cover $\mathscr{U}$ of X, there is a sequence $(\mathscr{V}_n)_{n \in \omega}$ of open refinements of $\mathscr{U}$ such that for each $x \in X$,*

$$\{n \in \omega : ord(x, \mathscr{V}_n) < \omega\} \in \mathscr{F}.$$

By Lemma 2.2, let $\mathscr{F}^{n+1}$ denote the filter on ω^{n+1} generated by sets of the form

$$\prod_{i \le n} F_i, \quad \text{where } F_i \in \mathscr{F} \text{ for each } i \le n.$$

The proof of the following lemma is routine and hence, we omit it.

Lemma 2.3. *If X is a weakly submetacompact Čech-scattered space and Y is a closed subset of X, then for every open cover $\mathscr{U}$ of Y, there is an open cover $\bigcup_{n \in \omega} \mathscr{V}_n$ of Y such that:*

(a) *for each $V \in \bigcup_{n \in \omega} \mathscr{V}_n$, $\overline{V}$ is topped and is contained in some member of $\mathscr{U}$,*

(b) *for each $y \in Y$, there is an $n \in \omega$ with $ord(y, \mathscr{V}_n) = 1$.*

Reduction 2.4. In considering submetacompactness and weak submetacompactness of countable products of Čech-scattered spaces, we may consider X^ω or $Y \times X^\omega$. Furthermore, we may assume that X has a single top point a, that is, $Top(X) = \{a\}$ (cf. Alster [A, Theorem]). For, let $\{X_n : n \in \omega\}$ be a countable collection of submetacompact (weakly submetacompact) Čech-scattered spaces. Take an $a \notin \bigcup_{n \in \omega} X_n$ and let

$$Y_m = \bigoplus_{n \in \omega} X_n \quad \text{for each } m \in \omega \quad \text{and}$$

$$X = \bigoplus_{m \in \omega} Y_m \cup \{a\}.$$

The topology of X is as follows: every X_n is open and closed in X and the neighborhood base at a is $\{U_m \cup \{a\} : m \in \omega\}$, where $U_m = \bigoplus_{k \ge m} Y_k$ for each $m \in \omega$. Then X is a submetacompact (weakly submetacompact) Čech-scattered space with $Top(X) = \{a\}$. Let Y be a space. Then $\prod_{n\in\omega} X_n(Y \times \prod_{n\in\omega} X_n)$ is a closed subset of X^ω $(Y \times X^\omega)$ and hence, if X^ω $(Y \times X^\omega)$ is submetacompact (weakly submetacompact), then $\prod_{n\in\omega} X_n(Y \times \prod_{n\in\omega} X_n)$ is also submetacompact (weakly submetacompact).

Let X be a Čech-scattered space and Y be a space. A subset of the form $B = \tilde{B} \times \prod_{i\le n} B_i$ in $Y \times X^n$, $n \in \omega$, is said to be *rectangle*. A subset of the form $B = \tilde{B} \times \prod_{i\in\omega} B_i$ in $Y \times X^\omega$ is said to be *basic open* if $\tilde{B}$ is an open subset of Y and there is an $n \in \omega$ such that B_i is an open subset of X for each $i < n$ and $B_i = X$ for each $i \ge n$. Let

$$n(B) = \inf\{i \in \omega : B_j = X \text{ for each } j \ge i\}.$$

We call $n(B)$ the *length* of B. Let $n \in \omega$. If $A = \prod_{i\le n} A_i$ $(\prod_{i\in\omega} A_i)$ is a subset of X^{n+1} (X^ω) such that for each $i \le n$ $(i \in \omega)$, A_i is topped, then we denote $Top(A) = \prod_{i\le n} Top(A_i)$ $(\prod_{i\in\omega} Top(A_i))$.

Let Φ_n be an index set for each $n \in \omega$ and $\Phi = \bigcup_{n\in\omega} \Phi_n$. If $\phi = (\tau_0, \tau_1, \dots, \tau_n, \tau_{n+1}) \in \Phi_{n+1}$ is constructed by $\mu = (\tau_0, \tau_1, \dots, \tau_n) \in \Phi_n$ for $n \in \omega$, then we denote $\phi_- = \mu$ and $\phi = \mu \oplus \tau_{n+1}$. If $\phi \in \Phi_0$, let $\phi_- = \emptyset$.

3 Submetacompactness

Let X be a space. If $\mathscr{U}$, $\mathscr{V}$ are collections of subsets of X, let $\mathscr{U} \wedge \mathscr{V} = \{U \cap V : U \in \mathscr{U} \text{ and } V \in \mathscr{V}\}$. $\mathscr{V}$ is said to be a *partial refinement* of $\mathscr{U}$ if for each $V \in \mathscr{V}$, there is a $U \in \mathscr{U}$ such that $V \subset U$. It is well known that X is submetacompact if and only if for every open cover $\mathscr{U}$ of X, which is closed under finite unions, there is a θ-sequence of open refinements of $\mathscr{U}$.

By the Reduction 2.4, in order to prove Problem (1) in the Introduction, it suffices to prove the following.

Theorem 3.1. *If X is a submetacompact Čech-scattered space with $Top(X) = \{a\}$, then the product X^ω is submetacompact.*

Proof. Let $\mathscr{B}$ be the base of X^ω, consisting of all basic open subsets of X^ω. Let $\mathscr{U}$ be an open cover of X^ω, which is closed under finite unions and $\mathscr{O} = \{B \in \mathscr{B} : \bar{B} \subset U \text{ for some } U \in \mathscr{U}\}$.

For each $B = \prod_{i\in\omega} B_i \in \mathscr{B}$, let $\mathscr{N}(B) = \{i < n(B) : \overline{B_i} \text{ is topped in } X\}$. Define $(B, (\mathscr{A}(B)_{i,m})) \in \mathscr{C}$ if $B = \prod_{i\in\omega} B_i \in \mathscr{B}$ and for each $i \in \mathscr{N}(B)$, $(\mathscr{A}(B)_{i,m})$ is a complete sequence of open (in $Top(\overline{B_i})$) covers of $Top(\overline{B_i})$. For each $i \le n(B)$, we shall construct an open (in $\overline{B_i}$) cover $\mathscr{B}(i,B)$ of $\overline{B_i}$ such that for each $B' \in \mathscr{B}(i,B)$, $\overline{B'}$ is topped, as follows. Let $i \le n(B)$.

Case 1. $i \in \mathscr{N}(B)$.

Since $i \in \mathscr{N}(B)$, the complete sequence $(\mathscr{A}(B)_{i,m})$ of open (in $Top(\overline{B_i})$) covers of $Top(\overline{B_i})$ is given. For each $A \in \mathscr{A}(B)_{i,0}$, take an open subset A' of $\overline{B_i}$ such that $A' \cap Top(\overline{B_i}) = A$. For each $x \in \overline{B_i} - Top(\overline{B_i})$, take an open neighborhood $B(x)$ of x in $\overline{B_i}$ such that $\overline{B(x)}$ is topped with $\alpha(\overline{B(x)}) = rank(x)$ and $\overline{B(x)} \cap Top(\overline{B_i}) = \emptyset$. Let $\mathscr{B}(i,B) = \{A' : A \in \mathscr{A}(B)_{i,0}\} \cup \{B(x) : x \in \overline{B_i} - Top(\overline{B_i})\}$.

Case 2. $i < n(B)$, $i \notin \mathscr{N}(B)$ and $\lambda(\overline{B_i}) = \gamma + 1$ for some ordinal γ.

Since $\lambda(\overline{B_i}) = \gamma + 1$, $Top(\overline{B_i})$ is a nonempty locally Čech-complete subspace of X. For each $x \in Top(\overline{B_i})$, there is an open neighborhood $B(x)$ of x in $\overline{B_i}$ such that $\overline{B(x)} \cap Top(\overline{B_i})$ is Čech-complete. For each $x \in \overline{B_i} - Top(\overline{B_i})$, take an open neighborhood $B(x)$ of x in $\overline{B_i}$ such that $\overline{B(x)}$ is topped with $\alpha(\overline{B(x)}) = rank(x)$ and $\overline{B(x)} \cap Top(\overline{B_i}) = \emptyset$. Let $\mathscr{B}(i,B) = \{B(x) : x \in \overline{B_i}\}$.

Case 3. $i < n(B)$, $i \notin \mathscr{N}(B)$ and $\lambda(\overline{B_i})$ is limit.

For each $x \in \overline{B_i}$, take an open neighborhood $B(x)$ of x in $\overline{B_i}$ such that $\overline{B(x)}$ is topped with $\alpha(\overline{B(x)}) = rank(x)$. Let $\mathscr{B}(i,B) = \{B(x) : x \in \overline{B_i}\}$.

Case 4. $i = n(B)$.

Since $Top(X) = \{a\}$, take a proper open neighborhood $B(a)$ of a in X, and for each $x \in X - \{a\}$, take an open neighborhood $B(x)$ of x in X such that $a \notin \overline{B(x)}$, $\overline{B(x)}$ is topped in X and $\alpha(\overline{B(x)}) = rank(x)$. Let $\mathscr{B}(i,B) = \{B(x) : x \in \overline{B_i}\}$.

For $i \le n(B)$ and $B' \in \mathscr{B}(i,B)$, $\lambda(\overline{B'}) \le \lambda(\overline{B_i})$. Furthermore, $\lambda(\overline{B'}) = \lambda(\overline{B_i})$ if and only if $\lambda(\overline{B_i}) = \gamma + 1$ for some ordinal γ and $Top(\overline{B'}) \subset \overline{B_i} \cap X^{(\gamma)}$. Furthermore, if $i \in \mathscr{N}(B)$, then $\lambda(\overline{B'}) = \lambda(\overline{B_i})$ if and only if $Top(\overline{B'}) \subset Top(\overline{B_i})$ and hence, $Top(\overline{B'}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$.

Since X is submetacompact, there is a θ-sequence $(\mathscr{V}^j_{B,i})$ of open (in X) partial refinements of $\mathscr{B}(i,B)$, $\mathscr{V}^j_{B,i} = \{V_\xi : \xi \in \Xi^j_{B,i}\}$, $j \in \omega$, such that for each $j \in \omega$, $B_i = \bigcup \mathscr{V}^j_{B,i}$ and for each $x \in B_i$, $\{j \in \omega : ord(x, \mathscr{V}^j_{B,i}) < \omega\} \in \mathscr{F}$, where $\mathscr{F}$ is the filter on ω described in Lemma 2.2. For each $j \in \omega$ and $\xi \in \Xi^j_{B,i}$, take $A(\xi) \in \mathscr{A}(B)_{i,0}$ or $x(\xi) \in \overline{B_i}$ such that $V_\xi \subset A(\xi)'$ or $V_\xi \subset B(x(\xi))$. Then $\lambda(\overline{V_\xi}) = \lambda(\overline{B_i})$ if and only if $\overline{V_\xi}$ is topped and $\lambda(\overline{B_i}) = \alpha(\overline{V_\xi}) + 1$.

For each $\eta = (j_0, j_1, \dots, j_{n(B)}) \in \omega^{n(B)+1}$, put $\Xi_{B,\eta} = \prod_{i \le n(B)} \Xi^{j_i}_{B,i}$. For each $\xi = (\xi(i)) \in \Xi_{B,\eta}$, let $V(\xi) = \prod_{i \le n(B)} V_{\xi(i)} \times X \times \cdots \in \mathscr{B}$ and $\mathscr{V}_\eta(B) = \{V(\xi) : \xi \in \Xi_{B,\eta}\}$. Then every $\mathscr{V}_\eta(B)$ is an open cover of B. For each $\xi = (\xi(i)) \in \Xi_{B,\eta}$, let $K(\xi) = \prod_{i \in \mathscr{N}(V(\xi))} Top(\overline{V_{\xi(i)}}) \times \prod_{i \le n(B), i \notin \mathscr{N}(V(\xi))} V_{\xi(i)} \times \{a\} \times \cdots = \prod_{i \in \omega} K_{\xi,i}$ and $\mathscr{K}(B,\eta) = \{K(\xi) : \xi \in \Xi_{B,\eta}\}$.

We consider the following condition (*) for $K(\xi)$.

(*) There are open subsets $O, O' \in \mathscr{B}$ with $n(O) = n(O')$ and $U \in \mathscr{U}$ such that $K(\xi) \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U$.

Then $O, O' \in \mathscr{O}$.

If $K(\xi)$ satisfies (*), define

$$n(\xi) = \inf\{n(O) : K(\xi) \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U \text{ with } n(O) = n(O'), O, O' \in \mathscr{O} \text{ and } U \in \mathscr{U}\}.$$

Put

$$r(\xi) = \max\{n(B), n(\xi)\}.$$

There are $O(\xi) = \prod_{i \in \omega} O_{\xi,i}$, $O'(\xi) = \prod_{i \in \omega} O'_{\xi,i} \in \mathscr{O}$, $U(\xi) \in \mathscr{U}$ such that:

(3) $K(\xi) \subset O(\xi) \subset \overline{O(\xi)} \subset O'(\xi) \subset \overline{O'(\xi)} \subset U(\xi)$,

(4) $n(\xi) = n(O(\xi)) = n(O'(\xi))$.

Let $\mathscr{P}(B) = \{P : P \subset \{0, 1, \ldots, n(B)\}\}$ and $P \in \mathscr{P}(B)$. Define

$$G(\xi) = \prod_{i \in \omega} G_{\xi,i} \quad \text{and} \quad B(\xi, P) = \prod_{i \in \omega} B_{\xi,P,i}$$

as follows:

(5) (a) Suppose $r(\xi) = n(B)$. For each $i \le n(B)$, let $G_{\xi,i} = V_{\xi(i)} \cap O'_{\xi,i}$ and for each $i > n(B)$, let $G_{\xi,i} = X$.

(b) Suppose $r(\xi) = n(\xi) > n(B)$. For each $i \in \omega$, let $G_{\xi,i} = \varnothing$.

(c) In either case, for each $i \le n(B)$, if $i \in P$, let $B_{\xi,P,i} = V_{\xi(i)} - \overline{O_{\xi,i}}$ and if $i \notin P$, let $B_{\xi,P,i} = V_{\xi(i)} \cap O'_{\xi,i}$. For each $i > n(B)$, let $B_{\xi,P,i} = X$.

Clearly, if $r(\xi) = n(B)$, then $B(\xi, \varnothing) = G(\xi)$. Notice that for each $i \in \omega$, $B_{\xi,P,i} \subset B_i$ and if $B(\xi, P) \neq \varnothing$, then $n(B(\xi, P)) = n(B) + 1$. Let $i \le n(B)$. If $i \in P$ and $i \notin \mathscr{N}(V(\xi))$, then $B_{\xi,P,i} = \varnothing$. Let

$$\mathscr{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathscr{P}(B) - \{\varnothing\}, B(\xi, P) \neq \varnothing\} \quad \text{if } r(\xi) = n(B),$$

$$\mathscr{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathscr{P}(B), B(\xi, P) \neq \varnothing\} \quad \text{if } r(\xi) = n(\xi) > n(B).$$

We have that if $P \in \mathscr{P}(B)$, $B(\xi, P) \in \mathscr{B}_{\eta,\xi}(B)$ and $r(\xi) = n(B)$, then there is an $i < n(\xi)$ with $i \in P$.

If $K(\xi)$ does not satisfy the condition (*), define $G(\xi)$, $B(\xi, P)$ and $\mathscr{B}_{\eta,\xi}(B)$ as follows: Let $G(\xi) = \varnothing$. Take a $P \in \mathscr{P}(B)$. If $P = \varnothing$, let $B(\xi, P) = V(\xi)$. If $P \neq \varnothing$, let $B(\xi, P) = \varnothing$. Put $\mathscr{B}_{\eta,\xi}(B) = \{V(\xi)\}$.

Then, in each case, we have $V(\xi) = G(\xi) \cup (\bigcup \mathscr{B}_{\eta,\xi}(B))$. The proof of the following claim is similar to that of Claim 2 in Tanaka [T4].

Claim. *Let* $i \le n(B)$, $\xi = (\xi(i)) \in \Xi_{B,\eta}$, $K(\xi) = \prod_{i\in\omega} K_{\xi,i}$, $P \in \mathscr{P}(B)$ *and* $B(\xi,P) = \prod_{i\in\omega} B_{\xi,P,i} \in \mathscr{B}_{\xi,\eta}(B)$.

(a) *If* $i \in P$, *then* $K(\xi)$ *satisfies* (*), $i \in \mathscr{N}(V(\xi))$ *and* $\lambda(\overline{B_{\xi,P,i}}) < \lambda(\overline{B}_i)$.

(b) *Let* $i \notin P$. *Then* $\lambda(\overline{B_{\xi,P,i}}) \le \lambda(\overline{B}_i)$. *Furthermore,* $\lambda(\overline{B_{\xi,P,i}}) = \lambda(\overline{B}_i)$ *if and only if* $\overline{B_{\xi,P,i}}$ *is topped* $(i \in \mathscr{N}(B(\xi,P)))$ *and* $\lambda(\overline{B}_i) = \alpha(\overline{B_{\xi,P,i}}) + 1$. *Hence, if* $i \in \mathscr{N}(B)$, *then* $\lambda(\overline{B_{\xi,P,i}}) = \lambda(\overline{B}_i)$ *if and only if* $Top(\overline{B_{\xi,P,i}}) \subset Top(\overline{B}_i)$ *and hence,* $Top(\overline{B_{\xi,P,i}}) \subset A$ *for some* $A \in \mathscr{A}(B_{i,0})$.

Let $B(\xi,P) \in \mathscr{B}_{\eta,\xi}(B)$, where $B(\xi,P) = \prod_{i\in\omega} B_{\xi,P,i}$. Let $i \in \mathscr{N}(B(\xi,P))$. If $i \in \mathscr{N}(B)$ with $\lambda(\overline{B_{\xi,P,i}}) = \lambda(\overline{B}_i)$, then $Top(\overline{B_{\xi,P,i}}) \subset Top(\overline{B}_i)$. Let $\mathscr{A}(B(\xi,P))_{i,m} = \{A \cap Top(\overline{B_{\xi,P,i}}) : A \in \mathscr{A}(B)_{i,m+1}\}$ for each $m \in \omega$. If i does not satisfy the above condition, take a complete sequence $(\mathscr{A}(B(\xi,P))_{i,m})$ of open (in $Top(\overline{B_{\xi,P,i}})$) covers of $Top(\overline{B_{\xi,P,i}})$. In each case, we have $(B(\xi,P), (\mathscr{A}(B(\xi,P))_{i,m})) \in \mathscr{C}$. Let

$$\mathscr{G}_\eta(B) = \{G(\xi) : \xi \in \Xi_{B,\eta} \text{ and } G(\xi) \ne \varnothing\} \quad \text{and}$$

$$\mathscr{B}_\eta(B) = \bigcup\{\mathscr{B}_{\eta,\xi}(B) : \xi \in \Xi_{B,\eta}\}.$$

Then

(6) (a) for each $G \in \mathscr{G}_\eta(B)$, $\overline{G}$ is contained in some member of $\mathscr{U}$,

(b) $\mathscr{G}_\eta(B) \cup \mathscr{B}_\eta(B)$ is a cover of B,

(c) the length of element of $\mathscr{B}_\eta(B)$ is $n(B)+1$,

for each $B' = B(\xi,P) = \prod_{i\in\omega} B'_i \in \mathscr{B}_{\eta,\xi}(B)$, $\xi \in \Xi_\eta$, $P \in \mathscr{P}(B)$,

(d) if $K(\xi)$ satisfies (*) and $r(\xi) = n(B)$, then there is an $i < n(\xi)$ with $i \in P$,

(e) if $i \in P$, then $\lambda(\overline{B'_i}) < \lambda(\overline{B}_i)$,

(f) if $i \notin P$, then $\lambda(\overline{B'_i}) \le \lambda(\overline{B}_i)$. Furthermore, $\lambda(\overline{B'_i}) = \lambda(\overline{B}_i)$ if and only if $\overline{B'_i}$ is topped $(i \in \mathscr{N}(B'_i))$ and $\lambda(\overline{B}_i) = \alpha(\overline{B'_i}) + 1$. Hence, if $i \in \mathscr{N}(B)$, then $\lambda(\overline{B'_i}) = \lambda(\overline{B}_i)$ if and only if $Top(\overline{B'_i}) \subset Top(\overline{B}_i)$ and hence, $Top(\overline{B'_i}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$,

(g) $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ and furthermore, if $i \in \mathscr{N}(B) \cap \mathscr{N}(B')$ with $Top(\overline{B'_i}) \subset Top(\overline{B}_i)$, then $\mathscr{A}(B')_{i,m} = \{A \cap Top(\overline{B'_i}) : A \in \mathscr{A}(B)_{i,m+1}\}$ for each $m \in \omega$.

For the filter $\mathscr{F}^{n(B)+1}$, we have

(7) For each $x \in B$, $\{\eta \in \omega^{n(B)+1} : ord(x, \mathscr{V}_\eta(B)) < \omega\} \in \mathscr{F}^{n(B)+1}$.

To show this, take an $x = (x_i) \in B$. For each $i \le n(B)$, let $F_i = \{j \in \omega : ord(x_i, \mathscr{V}^j_{B,i}) < \omega\} \in \mathscr{F}$ and $F = \prod_{i \le n(B)} F_i \in \mathscr{F}^{n(B)+1}$. Then, for each $\eta \in F$, $ord(x, \mathscr{V}_\eta(B)) < \omega$. So, $\{\eta \in \omega^{n(B)+1} : ord(x, \mathscr{V}_\eta(B)) < \omega\} \in \mathscr{F}^{n(B)+1}$.

By (7), we obtain

(8) For each $x \in B$, $\{\eta \in \omega^{n(B)+1} : ord(x, \mathscr{G}_\eta(B) \cup \mathscr{B}_\eta(B)) < \omega\} \in \mathscr{F}^{n(B)+1}$.

Put $\Phi_n = \prod_{i \le n} \omega^{i+1}$ for each $n \in \omega$ and $\Phi = \bigcup\{\Phi_n : n \in \omega\}$. Let $B(-1) = X^\omega$. Then $n(B(-1)) = 0$. Since $Top(X) = \{a\}$, let $\mathscr{A}(B(-1))_{0,m} = \{a\}$ for each $m \in \omega$. Then $(B(-1), (\mathscr{A}(B(-1))_{0,m})) \in \mathscr{C}$. For each $k \in \Phi_0 = \omega$, let $\mathscr{G}_k = \mathscr{G}_k(B(-1))$ and $\mathscr{B}_k = \mathscr{B}_k(B(-1))$ and for each $B \in \mathscr{B}_k$, define a complete sequence $(\mathscr{A}(B_{i,m}))$, satisfying (6)(g).

Assume that for $n \in \omega$ and $\mu \in \Phi_n$, we have already obtained $\mathscr{G}_\mu$, $\mathscr{B}_\mu$ of elements of $\mathscr{B}$ as before. Let $\tau \in \Phi_{n+1}$ and $\tau = \mu \oplus \eta$, where $\mu = \tau_- \in \Phi_n$ and $\eta \in \omega^{n+2}$. Define $\mathscr{G}_\tau = \bigcup\{\mathscr{G}_\eta(B) : B \in \mathscr{B}_\mu\}$ and $\mathscr{B}_\tau = \bigcup\{\mathscr{B}_\eta(B) : B \in \mathscr{B}_\mu\}$. For $B \in \mathscr{B}_\mu$, $B' \in \mathscr{B}_\eta(B)$, $\eta \in \omega^{n+2}$, by the same method, define a complete sequence $(\mathscr{A}(B')_{i,m})$ such that $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$. Inductively, we have

(9) For $\tau \in \Phi_{n+1}$ and $\mu = \tau_- \in \Phi_n$, $\eta \in \omega^{n+2}$, $n \in \omega$ with $\tau = \mu \oplus \eta$,
  (a) $\mathscr{G}_\tau \subset \mathscr{B}$ and for each $G \in \mathscr{G}_\tau$, $\overline{G}$ is contained in some member of $\mathscr{U}$,
  (b) $\mathscr{B}_\tau \subset \mathscr{B}$,
  and for each $B \in \mathscr{B}_\mu$,
  (c) $B = \bigcup \mathscr{G}_\eta(B) \cup (\bigcup \mathscr{B}_\eta(B))$,
  (d) the length of element of $\mathscr{B}_\eta(B)$ is $n+2$,
  for $B' = B(\xi, P) = \prod_{i \in \omega} B'_i \in \mathscr{B}_{\eta,\xi}(B)$, $\xi \in \Xi_\eta$ and $P \in \mathscr{P}(B)$,
  (e) if $K(\xi)$ satisfies (*) and $r(\xi) = n(B)$, then there is an $i < n(\xi)$ with $i \in P$,
  (f) if $i \in P$, then $\lambda(\overline{B'_i}) < \lambda(\overline{B_i})$,
  (g) if $i \notin P$, then $\lambda(\overline{B'_i}) \le \lambda(\overline{B_i})$. Furthermore, $\lambda(\overline{B'_i}) = \lambda(\overline{B_i})$ if and only if $\overline{B'_i}$ is topped $(i \in \mathscr{N}(B'))$ and $\lambda(\overline{B_i}) = \alpha(\overline{B'_i}) + 1$. Hence, if $i \in \mathscr{N}(B)$, then $\lambda(\overline{B'_i}) = \lambda(\overline{B_i})$ if and only if $Top(\overline{B'_i}) \subset Top(\overline{B_i})$ and hence, $Top(\overline{B'_i}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$.
  (h) $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ and furthermore, if $i \in \mathscr{N}(B) \cap \mathscr{N}(B')$ with $Top(\overline{B'_i}) \subset Top(\overline{B_i})$, then $\mathscr{A}(B')_{i,m} = \{A \cap Top(\overline{B'_i}) : A \in \mathscr{A}(B)_{i,m+1}\}$ for each $m \in \omega$.

(10) For $\mu \in \Phi_n$, $B \in \mathscr{B}_\mu$ and $x \in B$, $\{\eta \in \omega^{n+2} : ord(x, \mathscr{G}_\eta(B) \cup \mathscr{B}_\eta(B)) < \omega\} \in \mathscr{F}^{n+2}$.

It suffices to prove that $\{\mathscr{G}_\tau \cup (\mathscr{B}_\tau \wedge \mathscr{O}) : \tau \in \Phi\}$ is a θ-sequence of open refinements of $\mathscr{O}$. By (9) (a), (c), every $\mathscr{G}_\tau \cup (\mathscr{B}_\tau \wedge \mathscr{O})$ is an open refinement of $\mathcal{O}$. Take an $x = (x_i) \in B(-1)$. By (10), take a $\tau(0) = \eta(0) \in \omega$ such that $ord(x, \mathcal{G}_{\tau(0)} \cup \mathcal{B}_{\tau(0)}) < \omega$. Then $\tau(0) \in \Phi_0$. If $\mathcal{B}_{\tau(0)_x} = \emptyset$, then we are done. So, assume tha $\mathcal{B}_{\tau(0)_x} \neq \emptyset$. By (9) (d), every nonempty element of $\mathcal{B}_{\tau(0)}$ has the length 1. By (10) again, we can take an $\eta(1) \in \omega^2$ such that

$$\eta(1) \in \bigcap \{\{\eta \in \omega^2 : ord(x, \mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)) < \omega\} : x \in B \in \mathcal{B}_{\tau(0)}\} \in \mathcal{F}^2.$$

Let $\tau(1) = (\eta(0), \eta(1)) \in \Phi_1$. Then we have $ord(x, \mathcal{G}_{\tau(1)} \cup \mathcal{B}_{\tau(1)}) < \omega$. Assume that $\mathcal{B}_{\tau(1)_x} \neq \emptyset$ and also that we can continue this method infinitely. For each $t \in \omega$, choose a $\tau(t) = (\eta(0), \eta(1), \ldots, \eta(t)) \in \Phi_t$ such that

$$ord(x, \mathcal{G}_{\tau(t)} \cup \mathcal{B}_{\tau(t)}) < \omega \quad \text{and} \quad \mathcal{B}_{\tau(t)_x} \neq \emptyset.$$

Since $\mathcal{B}_{\tau(t)_x} \neq \emptyset$ and finite for each $t \in \omega$, it follows from König's lemma (cf. Kunen [K]) that there are sequences $\{\eta(t) : t \in \omega\}$, $\{\tau(t) : t \in \omega\}$, $\{\xi(t) : t \in \omega\}$, $\{\mathcal{N}(t) : t \in \omega\}$, $\{K(t) : t \in \omega\}$, $\{P(t) : t \in \omega\}$, $\{B(t) = B(\xi(t), P(t)) : t \in \omega\}$, $\{A(i,t) : i, t \in \omega\}$ (if possible), satisfying: for each $t \in \omega$,

(11) (a) $\eta(t) \in \omega^{t+1}$ and $\tau(t) = (\eta(0), \eta(1), \ldots, \eta(t))$,
(b) $(B(t), (\mathcal{A}(B(t))_{i,m})) \in \mathcal{C}$ and $x \in B(t) = \prod_{i \in \omega} B(t)_i \in \mathcal{B}_{\eta(t)}(B(t-1))$,
(c) $n(B(t)) = t + 1$,
(d) $\mathcal{N}(t) = \mathcal{N}(B(t))$,
(e) $\xi(t) \in \Xi_{B(t-1), \eta(t)}$,
(f) $K(t) = K(\xi(t)) = \prod_{i \in \omega} K(t)_i \in \mathcal{K}(B(t-1), \eta(t))$,
(g) $P(t) \in \mathcal{P}(\{0, 1, \ldots, t\})$,
(h) if $K(t)$ satisfies the condition (*) and $r(\xi(t)) = t$, then there is an $i < n(\xi(t))$ with $i \in P(t)$,
(i) if $i \in P(t)$, then $\lambda(\overline{B(t)_i}) < \lambda(\overline{B(t-1)_i})$,
(j) if $i \notin P(t)$, then $\lambda(\overline{B(t+1)_i}) \leq \lambda(\overline{B(t)_i})$. Furthermore, $\lambda(\overline{B(t+1)_i}) = \lambda(\overline{B(t)_i})$ if and only if $\overline{B(t+1)_i}$ is topped $(i \in \mathcal{N}(B(t+1)))$ and $\lambda(\overline{B_i}) = \alpha(\overline{B(t+1)_i}) + 1$. Hence, if $i \in \mathcal{N}(B(t))$, then $\lambda(\overline{B(t+1)_i}) = \lambda(\overline{B(t)_i})$ if and only if $Top(\overline{B(t+1)_i}) \subset Top(\overline{B(t)_i})$ and hence, $Top(\overline{B(t+1)_i}) \subset A$ for some $A \in \mathcal{A}(B(t)_{i,0})$.
(k) $(B(t+1), (\mathcal{A}(B(t+1))_{i,m})) \in \mathcal{C}$ and furthermore, if $i \in \mathcal{N}(t) \cap \mathcal{N}(t+1)$ with $Top(\overline{B(t+1)_i}) \subset Top(\overline{B(t)_i})$, then $\mathcal{A}(B(t+1))_{i,m} = \{A \cap Top(\overline{B(t+1)_i}) : A \in \mathcal{A}(B(t))_{i,m+1}\}$ for each $m \in \omega$.

Let $i \in \omega$. By (11)(c), let $\tilde{t} \geq 1$ such that $n(B(\tilde{t})) > i$. By (11)(i), if $i \in P(t)$ for $t \geq \tilde{t}$, $\lambda(\overline{B(t)_i}) < \lambda(\overline{B(t-1)_i})$. Since there does not exist an infinite decreasing sequence of ordinals, there is a $t_i \in \omega$ with $t_i \geq \tilde{t}$ such that for each $t \geq t_i$, $i \notin P(t)$. By (11) (j), (k), there is a k_i such that $k_i \geq t_i$ and for each $t \geq k_i$, $t \in \mathcal{N}(t)$ and

$K(t+1)_i = Top(\overline{B(t+1)_i}) \subset A(i,t) \subset K(t)_i = Top(\overline{B(t)_i})$. Let $K = \prod_{i \in \omega} K(t)_i$. Then it follows from Fact (1) in Section 2 that K is a nonempty compact subset of X^{ω}. Since $\mathscr{U}$ is an open cover of X^{ω}, which is closed under finite unions, there are $O = \prod_{i \in \omega} O_i$, $O' = \prod_{i \in \omega} O'_i \in \mathscr{O}$, $U \in \mathscr{U}$ such that $K \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U$ and $n(K) = n(O) = n(O')$, where $n(K)$ is defined as that of $n(\xi)$. By Fact (2) and (11) (c), take an $s \geq 1$ such that:

(12) (a) $n(O) \leq n(B(s-1))$,
(b) for each $i < n(O)$, $k_i \leq s$ and $K(s)_i \subset O_i$.

Then $K(s) \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U$ and hence, $K(s)$ satisfies the conditon (*). Since $n(\xi_s) \leq n(O)$, $r(\xi_s) = s$. By (11)(h), there is an $i < n(\xi_s)$ with $i \in P(s)$, which is a contradiction.

4 Weak Submetacompactness

In order to give an affirmative answer to the Problem (2) in the Introduction, it suffices to prove the following.

Theorem 4.1. *If Y is a hereditarily weakly submetacompact space and X is a weakly submetacompact Čech-scattered space with $Top(X) = \{a\}$, then the product $Y \times X^{\omega}$ is weakly submetacompact.*

Proof. Let $\mathscr{B}$ be the base of $Y \times X^{\omega}$, consisting of all basic open subsets of $Y \times X^{\omega}$ and $\mathscr{U}$ be an open cover of $Y \times X^{\omega}$, which is closed under finite unions. It suffices to prove that there is an open weak θ-refinement of $\mathscr{U}$.

Define $(B, (\mathscr{A}(B)_{i,m})) \in \mathscr{C}$ if $B = \tilde{B} \times \prod_{i \in \omega} B_i \in \mathscr{B}$ such that for each $i < n(B)$, $\overline{B_i}$ is topped and $(\mathscr{A}(B)_{i,m})$ is a complete sequence of open (in $Top(\overline{B_i})$) covers of $Top(\overline{B_i})$.

Take a $(B, (\mathscr{A}(B)_{i,m})) \in \mathscr{C}$ and $B = \tilde{B} \times \prod_{i \in \omega} B_i$. For $i < n(B)$ and $A \in \mathscr{A}(B)_{i,0}$, take an open subset A' in $\overline{B_i}$ as before. Then $\{A' : A \in \mathscr{A}(B)_{i,0}\} \cup \{\overline{B_i} - Top(\overline{B_i})\}$ is an open (in $\overline{B_i}$) cover of $\overline{B_i}$. Since X is weakly submetacompact, by Lemma 2.3, there is a collection $\mathscr{H}(B)_i = \bigcup_{s \in \omega} \mathscr{H}(B)_{i,s}$ of open subsets of B_i such that:

(1) (a) $\mathscr{H}(B)_i$ partial refines $\{A' : A \in \mathscr{A}(B)_{i,0}\} \cup \{\overline{B_i} - Top(\overline{B_i})\}$,
(b) for each element H of $\mathscr{H}(B)_i$, $\overline{H}$ is topped,
(c) for each $x \in B_i$, there is an $s \in \omega$ with $ord(x, \mathscr{H}(B)_{i,s}) = 1$.

Since $Top(X) = \{a\}$, take an open cover $\mathscr{H}_{n(B)} = \bigcup_{s \in \omega} \mathscr{H}(B)_{n(B),s}$ of X such that:

(2) (a) $\mathscr{H}(B)_{n(B),0} = \{B\}$, where B is a proper open subset of X with $a \in B$,
(b) for $s \geq 1$, $H \in \mathscr{H}(B)_{n(B),s}$, $a \notin \overline{H}$,
(c) for each element H of $\mathscr{H}(B)_{n(B)}$, $\overline{H}$ is topped,
(d) for each $x \in X$, there is an $s \in \omega$ with $ord(x, \mathscr{H}(B)_{n(B),s}) = 1$.

Let $\mathscr{H}(B) = \prod_{i \leq n(B)} \mathscr{H}(B)_i$. For each $x \in \prod_{i \leq n(B)} B_i$, there is an $(s_0, s_1, \dots, s_{n(B)}) \in \omega^{n(B)+1}$ with $ord(x, \prod_{i \leq n(B)} \mathscr{H}(B)_{i,s_i}) = 1$. Since $|\omega^{n(B)+1}| = \omega$, we denote $\mathscr{H}(B) = \bigcup_{s \in \omega} \mathscr{H}(B)_s$ such that for each $x \in \prod_{i \leq n(B)} B_i$, there is an $s \in \omega$ with $ord(x, \mathscr{H}(B)_s) = 1$. Take an $H = \prod_{i \leq n(B)} H_n \in \mathscr{H}(B)$ with $Top(\overline{H}) \cap Top(\overline{\prod_{i \leq n(B)} B_i}) = Top(\prod_{i \leq n(B)} \overline{H_i}) \cap Top(\prod_{i \leq n(B)} \overline{B_i}) \neq \emptyset$. Then for each $i \leq n(B)$, $Top(\overline{H_i}) \cap Top(\overline{B_i}) \neq \emptyset$ and hence, $Top(\overline{H_i}) \subset Top(\overline{B_i})$. Let $\hat{H} = H \times X \times \cdots = \prod_{i \in \omega} \hat{H}_i$. Then $n(\hat{H}) = n(B) + 1$ and $Top(\hat{H}) = Top(H) \times \{a\} \times \cdots$. For each $y \in \tilde{B}$, let $H_y = \{y\} \times Top(\hat{H})$. Define the condition (**) for H_y as follows:

$$H_y \text{ satisfies (**)} \Leftrightarrow \text{there are } O, O' \in \mathscr{B} \text{ with } n(O) = n(O') \text{ and } U \in \mathscr{U}$$
$$\text{such that } H_y \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U.$$

Let

$$n(H_y) = \min\{n(O) : O, O' \in \mathscr{B} \text{ with } n(O) = n(O') \text{ such that}$$
$$H_y \subset O \subset \bar{O} \subset O' \subset \overline{O'} \subset U \text{ for some } U \in \mathscr{U}\}.$$

We say that H *satisfies* (**) if there is a $y \in \tilde{B}$ such that H_y satisfies (**). If H_y satisfies (**), take basic open subsets $O(H_y) = \widetilde{O(H_y)} \times \prod_{i \in \omega} O(H_y)_i$, $O'(H_y) = \widetilde{O'(H_y)} \times \prod_{i \in \omega} O'(H_y)_i$ in $Y \times X^{\omega}$ and $U(H_y) \in \mathscr{U}$ such that

(3) (a) $H_y \subset O(H_y) \subset \overline{O(H_y)} \subset O'(H_y) \subset \overline{O'(H_y)} \subset U(H_y)$,
(b) $n(H_y) = n(O(H_y)) = n(O'(H_y))$.

Define

$$r(H_y) = \max\{n(B), n(H_y)\}.$$

Assume that H satisfies (**). For each $k \in \omega$, let $W(H)_k = \{y \in \tilde{B} : n(H_y) \leq k\}$. Then $W(H)_k = \bigcup\{\widetilde{O(H_y)} \cap \tilde{B} : n(H_y) \leq k\}$ and hence, $W(H)_k$ is an open subspace of Y. Since Y is hereditarily weakly submetacompact, there is a collection $\mathscr{V}(H)_k = \bigcup_{j \in \omega} \mathscr{V}(H)_{k,j}$ of open subsets of $W(H)_k$ (and hence, open subsets in Y) such that:

(4) (a) $\mathscr{V}(H)_k$ partial refines $\{\widetilde{O(H_y)} \cap \tilde{B} : n(H_y) \leq k\}$,
(b) for each $x \in W(H)_k$, there is an $j \in \omega$ with $ord(x, \mathscr{V}(H)_{k,j}) = 1$.

For each $V \in \mathscr{V}(H)_k$, take a $y(V) \in W(H)_k$ such that $V \subset O(\widetilde{H_{y(V)}}) \cap \tilde{B}$. Define a basic open subset $G(V)$ as follows:

$$G(V) = V \times \prod\nolimits_{i \le n(B)} (\hat{H}_i \cap O'(H_{y(V)})_i) \times X \times \cdots \quad \text{if } r(H_{y(V)}) = n(B) \quad \text{and}$$

$$G(V) = V \times \prod\nolimits_{i < n(H_{y(V)})} (\hat{H}_i \cap O'(H_{y(V)})_i) \times X \times \cdots$$

$$\text{if } r(H_{y(V)}) = r(H_{y(V)}) > n(B).$$

We denote $G(V) = V \times \prod_{i \in \omega} G(V)_i$. For each $i \in \omega$, $\overline{G(V)_i}$ is topped and $V \times Top(\hat{H}) \subset \overline{G(V)} \subset U(H_{y(V)})$. We obtain the following collection $\mathscr{B}(V) = \bigcup_{t \in \omega} \mathscr{B}(V)_t \subset \mathscr{B}$ (cf. [AMT] or [HT]) such that:

(5) (a) for each $B' \in \mathscr{B}(V)$, $pr_Y(B') = V$, where $pr_Y : Y \times X^\omega \to Y$ is the projection of $Y \times X^\omega$ onto Y,
(b) $V \times \hat{H} - G(V) \subset \bigcup \mathscr{B}(V) \subset V \times \hat{H}$,
(c) for each $x \in V \times \hat{H} - G(V)$, there is a $t \in \omega$ with $ord(x, \mathscr{B}(V)_t) = 1$,
for each $B' = V \times \prod_{i \in \omega} B'_i \in \mathscr{B}(V)$,
(d) $n(B') = r(H_{y(V)}) > n(B)$,
(e) for each $i \in \omega$, $\alpha(\overline{B'_i}) \le \alpha(\overline{B_i})$,
(f) $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ such that for each $i \le n(B)$, if $\alpha(\overline{B'_i}) = \alpha(\overline{B_i})$, then $Top(\overline{B'_i}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$ and $\mathscr{A}(B')_{i,m} = \{A \cap Top(\overline{B'_i}) : A \in \mathscr{A}(B)_{i,m+1}\}$ for each $m \in \omega$,
(g) if $r(H_{y(V)}) = n(B)$, then there is an $i < n(H_{y(V)})$ such that $\alpha(\overline{B'_i}) < \alpha(\overline{B_i})$.

For $k, j, t \in \omega$, let $\mathscr{G}(H)_{k,j,t} = \{G(V) : V \in \mathscr{V}(H)_{k,j}\}$, $\mathscr{B}(H)_{k,j,t} = \bigcup \{\mathscr{B}(V)_t : V \in \mathscr{V}(H)_{k,j}\}$.

If H does not satisfy (**) or $Top(\overline{H}) \cap Top(\prod_{i \le n(B)} \overline{B_i}) = \emptyset$, for $k, j, t \in \omega$, let $\mathscr{G}(H)_{k,j,t} = \{\emptyset\}$, $\mathscr{B}(H)_{k,j,t} = \{B'\}$, where $B' = \tilde{B} \times \hat{H}$. Take a sequence $(\mathscr{A}(B')_{i,m})$ such that $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ as (5)(f).

For $s, k, j, t \in \omega$, let

$$\mathscr{G}(B)_{s,k,j,t} = \bigcup \{\mathscr{G}(H)_{k,j,t} : H \in \mathscr{H}(B)_s\} \quad \text{and}$$

$$\mathscr{B}(B)_{s,k,j,t} = \bigcup \{\mathscr{B}(H)_{k,j,t} : H \in \mathscr{H}(B)_s\}.$$

Then we have

(6) (a) for $s, k, j, t \in \omega$, $\mathscr{G}(B)_{s,k,j,t} \subset \mathscr{B}$ and for each $G \in \mathscr{G}(B)_{s,k,j,t}$, $\overline{G}$ is contained in some member of $\mathscr{U}$,
(b) for $s, k, j, t \in \omega$, $\mathscr{B}(B)_{s,k,j,t} \subset \mathscr{B}$,
(c) for each $x \in B$, there is a 4-tuple $(s, k, j, t) \in \omega^4$ such that

(c-1) $1 \le ord(x, \mathscr{G}(B)_{s,k,j,t} \cup \mathscr{B}(B)_{s,k,j,t})$,
(c-2) $ord(x, \mathscr{G}(B)_{s,k,j,t}) \le 1$,
(c-3) $ord(x, \mathscr{B}(B)_{s,k,j,t}) \le 1$,

for $B' = \widetilde{B}' \times \prod_{i \in \omega} B'_i \in \mathscr{B}(H)_{s,k,j,t}$, $s, k, j, t \in \omega$, $H = \prod_{i \le n(B)} H_i \in \mathscr{H}(B)_s$,

(d) $n(B') > n(B)$,
(e) for each $i \in \omega$, $\alpha(\overline{B'_i}) \le \alpha(\overline{B_i})$,
(f) $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ such that for each $i \le n(B)$, if $\alpha(\overline{B'_i}) = \alpha(\overline{B_i})$, then $Top(\overline{B'_i}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$ and $\mathscr{A}(B')_{i,m} = \{A \cap Top(\overline{B'_i}) : A \in \mathscr{A}(B)_{i,m+1}\}$ for each $m \in \omega$,
(g) if $B' = V \times \prod_{i \in \omega} B'_i$ and $r(H_{y(V)}) = n(B)$, then there is an $i < n(H_{y(V)})$ such that $\alpha(\overline{B'_i}) < \alpha(\overline{B_i})$.

Let $\Phi_0 = \{\emptyset\}$. For each $n \ge 1$, let $\Phi_n = (\omega^4)^n = (\omega^{4n})$ and $\Phi = \bigcup_{n \in \omega} \Phi_n$. Let $\mathscr{G}_0 = \{\emptyset\}$, $B(0) = Y \times X^\omega$, $\mathscr{B}_0 = \{B(0)\}$, $\mathscr{A}_{0,m} = \{\{a\}\}$ for $m \in \omega$ with $(B(0), (\mathscr{A}(B(0)_{0,m}))) \in \mathscr{C}$ and $Y(\emptyset) = \emptyset$. By the above construction, for each $n \ge 1$ and $\phi \in \Phi_n$, we obtain collections $\mathscr{G}_\phi$ and $\mathscr{B}_\phi$ of subsets of $Y \times X^\omega$ and a subset $Y(\phi)$ of Y, satisfying the following:

(7) for $\phi = \phi_- \oplus (s, k, j, t)$, $(s, k, j, t) \in \omega^4$,
  (a) $\mathscr{G}_\phi = \bigcup \{\mathscr{G}(B)_{s,k,j,t} : B \in \mathscr{B}_{\phi_-}\} \subset \mathscr{B}$ and for every element $G \in \mathscr{G}_\phi$, $\overline{G}$ is contained in some member of $\mathscr{U}$,
  (b) $\mathscr{B}_\phi = \bigcup \{\mathscr{B}(B)_{s,k,j,t} : B \in \mathscr{B}_{\phi_-}\} \subset \mathscr{B}$,

(8) for $x \in B$ and $B \in \mathscr{B}_{\phi_-}$, there is a $(s, k, j, t) \in \omega^4$ such that
  (a) $1 \le ord(x, \mathscr{G}(B)_{s,k,j,t} \cup \mathscr{B}(B)_{s,k,j,t})$,
  (b) $ord(x, \mathscr{G}(B)_{s,k,j,t}) \le 1$,
  (c) $ord(x, \mathscr{B}(B)_{s,k,j,t}) \le 1$,

  for $\phi = \phi_- \oplus (s, k, j, t)$, $(s, k, j, t) \in \omega^4$, $B = \tilde{B} \times \prod_{i \in \omega} B_i \in \mathscr{B}_{\phi_-}$, $B' = \widetilde{B}' \times \prod_{i \in \omega} B'_i \in \mathscr{B}(H)_{s,k,j,t}$, $H \in \mathscr{H}(B)_s$,

(9) $(B, (\mathscr{A}(B)_{i,m})) \in \mathscr{C}$,

(10) $n(B) < n(B')$,

(11) for $i \in \omega$, $\alpha(\overline{B'_i}) \le \alpha(\overline{B_i})$,

(12) $(B', (\mathscr{A}(B')_{i,m})) \in \mathscr{C}$ such that for each $i \le n(B)$, if $\alpha(\overline{B'_i}) = \alpha(\overline{B_i})$, then $Top(\overline{B'_i}) \subset A$ for some $A \in \mathscr{A}(B)_{i,0}$ and $\mathscr{A}(B')_{i,m} = \{A \cap Top(\overline{B'_i}) : A \in \mathscr{A}(B)_{i,m+1}\}$,

(13) let $Y(\phi) = \bigcup \{W(H)_k : B \in \mathscr{B}_{\phi_-}, H \in \mathscr{H}(B)_s$ and H satisfies (**)$\}$,

(14) if $B' = V \times \prod_{i \in \omega} B'_i \in \mathscr{B}(H)_{s,k,j,t}$, $H \in \mathscr{H}(B)_s$ and $r(H_{y(V)}) = n(B)$, then there is an $i < n(H_{y(V)})$ such that $\alpha(\overline{B'_i}) < \alpha(\overline{B_i})$.

Let $\mathscr{G} = \bigcup\{\mathscr{G}_\phi : \phi \in \Phi\}$. By (7)(a), $\mathscr{G}$ is a collection of basic open subsets of $Y \times X^\omega$ and for each $G \in \mathscr{G}$, $\bar{G}$ is contained in some member of $\mathscr{U}$. Take a point $(y, (x_u)) \in Y \times X^\omega$. We shall show that there is a $\phi \in \Phi$ such that $ord((y, (x_u)), \mathscr{G}_\phi) = 1$. Since $(y, (x_u)) \in B(0)$, by (8), there is a $\tau(1) = \phi(1) = (s(1), k(1), j(1), t(1)) \in \Phi_1 = \omega^4$ such that

$$1 \leq ord((y, (x_u)), \mathscr{G}_{\phi(1)} \cup \mathscr{B}_{\phi(1)}),$$

$$ord((y, (x_u)), \mathscr{G}_{\phi(1)}) \leq 1 \quad \text{and} \quad ord((y, (x_u)), \mathscr{B}_{\phi(1)}) \leq 1.$$

If $(y, (x_u)) \in \bigcup \mathscr{G}_{\phi(1)}$, then $ord((y, (x_u)), \mathscr{G}_{\phi(1)}) = 1$. So we have done. Assume that $(y, (x_u)) \notin \bigcup \mathscr{G}_{\phi(1)}$. Then $(y, (x_u)) \in \bigcup \mathscr{B}_{\phi(1)}$ and hence, $ord((y, (x_u)), \mathscr{B}_{\phi(1)}) = 1$. Take a unique $B(1) \in \mathscr{B}_{\phi(1)}$ such that $(y, (x_u)) \in B(1)$. By (8) again, there is a $\tau(2) = (s(2), k(2), j(2), t(2)) \in \omega^4$ such that

$$1 \leq ord((y, (x_u)), \mathscr{G}(B(1))_{\tau(2)} \cup \mathscr{B}(B(1))_{\tau(2)}),$$

$$ord((y, (x_u)), \mathscr{G}(B(1))_{\tau(2)}) \leq 1 \quad \text{and} \quad ord((y, (x_u)), \mathscr{B}(B(1))_{\tau(2)}) \leq 1.$$

Let $\phi(2) = (\tau(1), \tau(2)) \in \Phi_2$. Since $((y, (x_u)) \notin B$ for $B \in \mathscr{B}_{\phi(1)} - \{B(1)\}$, if $(y, (x_u)) \in \bigcup \mathscr{G}(B(1))_{\tau(2)}$, then $ord((y, (x_u)), \mathscr{G}_{\phi(2)}) = 1$. So, assume that $(y, (x_u)) \notin \bigcup \mathscr{G}(B(1))_{\tau(2)}$. Then $(y, (x_u)) \in \bigcup \mathscr{B}(B(1))_{\tau(2)}$ and hence, $ord((y, (x_u)), \mathscr{B}_{\phi(2)}) = 1$. Take a unique $B(2) \in \mathscr{B}_{\phi(2)}$ such that $(y, (x_u)) \in B(2)$. We continue this method by the same manner and assume that it is continued infinitely. Then there are sequences $\{\tau(n) = (s(n), k(n), j(n), t(n)) \in \omega^4 : n \geq 1\}$, $\{\phi(n) \in \Phi_n : n \geq 1\}$, $\{B(n) : n \in \omega\}$, $\{H(n) : n \geq 1\}$, $\{A(i, n) : i, n \in \omega\}$ (if possible), $\{y(n) : n \geq 1\}$ (if possible) such that: for each $n \geq 1$,

(15) (a) $\phi(n) = \phi(n-1) \oplus \tau(n) \in \Phi_n$, where $\phi(0) = \emptyset$ and $B_{\phi(0)} = B(0)$,
(b) $B(n) = \widetilde{B(n)} \times \prod_{i \in \omega} B_{n,i}$ and $\{B \in \mathscr{B}_{\phi(n)} : (y, (x_u)) \in B\} = \{B(n)\}$,
(c) $H(n) = \prod_{i \in \omega} H_{n,i} \subset \prod_{i \in \omega} B_{n-1,i}$,
(d) $n(B(n-1)) < n(B(n))$,
(e) for each $i \in \omega$, $\alpha(\overline{B_{n,i}}) \leq \alpha(\overline{B_{n-1,i}})$,
(f) $(B(n), (\mathscr{A}(B(n))_{i,m})) \in \mathscr{C}$ such that for each $i \leq n(B(n-1))$, if $\alpha(\overline{B_{n,i}}) = \alpha(\overline{B_{n-1,i}})$, then $Top(\overline{B_{n,i}}) \subset A(n) \in \mathscr{A}(B(n-1))_{i,0}$ and $\mathscr{A}(B_n)_{i,m} = \{A \cap Top(\overline{B_{n,i}}) : A \in \mathscr{A}(B(n-1))_{i,m+1}\}$,
(g) if $H(n)_y$ satisfies (**), then $y(n) \in Y(\phi(n))$, $n(H(n)_y) = n(H(n)_{y(n)})$ and furthermore, if $r(H(n)_y) = n(B(n-1))$, then there is an $i < n(H(n)_y)$ such that $\alpha(\overline{B_{n,i}}) < \alpha(\overline{B_{n-1,i}})$.

The rest of the proof is similar to that of Theorem 3.1. So we omit it.

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