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GROUP PRESENTATION OF THE SCHUR-MULTIPLIER DERIVED FROM A LOOP GROUP

By

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1. Introduction

In 1960s H. Matsumoto [2] considered the universal central extension and the Schur-multiplier of a Chevalley group which is derived from an arbitrary field F and an arbitrary Cartan matrix A of finite type. Then he showed that the corresponding Steinberg group (we denote it by $St(A,F)$) is its universal central extension and gave a presentation of its Schur-multiplier for almost every field. Now one sees this Schur-multiplier is an abelian group which is strongly connected with this root system.

In general, a Chevalley group $G(A,R)$ over a commutative ring R is constructed as a group using the functor represented by some Hopf algebra. And there are many results about the structure of the associated K_2 group.

In this paper we take Laurent polynomial rings $F[X,X^{-1}]$. A Chevalley group over a Laurent polynomial ring is sometimes called a loop group. Then we consider the structure of the K_2 group of a loop group and obtain the following theorem, where $\hat{K}_2$ will be given by generators and relations in section 3.1.2.

Theorem

Let A be a Cartan matrix of finite type. Then we have

$$K_2(A,F[X,X^{-1}]) \simeq \widehat{K_2}(A,F[X,X^{-1}]).$$

2. Preliminaries

In this section K is a field of characteristic 0. Let $X=(X_{ij})$ $(1\le i,j\le n)$ be an $n\times n$ symmetrizable generalized Cartan matrix. We denote a Kac-Moody Lie algebra over K, the standard Cartan subalgebra, the associated root system, the set of real roots obtained from X, by $\mathfrak{g}(X)$, $\mathfrak{h}$, Δ, Δ^{re} respectively. Using this notation we can decompose $\mathfrak{g}(X)$ as follows:

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$$\mathfrak{g}(X) = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha), \quad \text{where } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \ \forall h \in \mathfrak{h}^*\}.$$

We call this the root space decomposition with respect to $\mathfrak{h}$.

Now we introduce a nondegenerate, symmetric, invariant, bilinear form on $\mathfrak{g}(X)$ (cf. [12]). Using this we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ (here $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$). We can take $\Pi = \{\alpha_1, \dots \alpha_n\} \subset \mathfrak{h}^*$ and $\check{\Pi} = \{h_1, \dots h_n\} \subset \mathfrak{h}$ satisfying $2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i) = X_{ij}$, where $h_i := 2\alpha_i/(\alpha_i, \alpha_i)$. Then Π and $\check{\Pi}$ are called fundamental roots and fundamental coroots respectively.

Now we take $\sigma_{\alpha_i}(h) := h - (\alpha_i, h)h_i$ $(h \in \mathfrak{h}^*$ *or* $\mathfrak{h})$. Then $\sigma_{\alpha_i} \in Aut(\mathfrak{h}^*)$ *or* $Aut(\mathfrak{h})$. And the subgroup of $Aut(\mathfrak{h})$ or $Aut(\mathfrak{h}^*)$ generated by the σ_{α_i} $(1 \le i \le n)$ is called the Weyl group of the $\mathfrak{g}(X)$ (cf. [12]).

Next we take a Chevalley base $\{e_\alpha \mid \alpha \in \Delta^{re}\}$ of $\mathfrak{g}$ and fix an integrable representation (π, V) of $\mathfrak{g}(X)$ with

$$\pi : \mathfrak{g}(X) \to End(V).$$

We consider the group $G := \langle x_\alpha(t) \mid t \in K, \alpha \in \Delta^{re} \rangle \subset Aut(V)$, where $x_\alpha(t) := Exp(\pi(te_\alpha)) \in Aut(V)$. We call the group G a Kac-Moody group. In fact G is a central quotient of $G_{sc}(X, K)$ as in [1].

Theorem 2.1 (Universal Kac-Moody group) [1]. *Let F be an arbitrary field and let X be an $n \times n$ symmetrizable generalized Cartan matrix. Then the universal Kac-Moody group $G_{sc}(X, F)$ (cf. [1]) is isomorphic to the group generated by the symbols $x_\alpha(u)$ (for all $u \in F$) and charactarized by the following defining relations:*

(K1) $x_\alpha(u) \cdot x_\alpha(t) = x_\alpha(u + t)$,

(K2) $[x_\alpha(u), x_\beta(t)] = \prod_{i\alpha + j\beta \in Q_{\alpha,\beta}} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j} u^i t^j)$,

(K3) $w_\alpha(v) x_\beta(t) w_\alpha(-v) = x_{\sigma_\alpha \beta}(\eta_{\alpha,\beta} t v^{-\beta(h_\alpha)})$,

(K4) $h_\alpha(v) h_\alpha(w) = h_\alpha(vw)$

for all $u, t \in F$, $v, w \in F^*$ and $\alpha, \beta \in \Delta^{re}$, where $w_\alpha(v) = x_\alpha(v) x_{-\alpha}(-v^{-1}) x_\alpha(v)$, $h_\alpha(v) = w_\alpha(v) w_\alpha(-1)$.

Definition 2.1 (Steinberg group) [10] [7] [1]. *Under the same condition as in Theorem 2.1, a Steinberg group of type X over F is the group which is generated by the symbols $\hat{x}_\alpha(t)$ (for all $t \in F$) and charatarized by the conditions (K1)–(K3). Now we denote it by $St(X, F)$.*

In this paper the generators of a Kac-Moody group $G_{sc}(X, F)$ are denoted by $x_\alpha(u)$ (for all $u \in F^*$ and $\alpha \in \Delta^{re}$) and the generators of a Steinberg group $St(X, F)$ are denoted by $\hat{x}_\alpha(u)$ (for all $u \in F^*$ and $\alpha \in \Delta^{re}$).

Now $\eta_{\alpha,\beta} \in \{\pm 1\}$ is the number which satisfies $\exp(ade_\alpha)\exp(-ade_{-\alpha}) \cdot \exp(ade_\alpha)(e_\beta) = \eta_{\alpha,\beta} e_{\sigma_\alpha \beta}$. Then the following propositions hold (cf. [5]).

Proposition 2.1. *Let X be a symmetrizable generalized Cartan matrix and F an arbitrary field. Then the following formulas hold in $G(X,F)$ (cf. [4]) for all $u, v, t \in F^*$ and $\alpha, \beta \in \Delta^{re}$.*

1. $w_\alpha(v) x_\beta(t) w_\alpha(-v) = x_{\sigma_\alpha \beta}(t\eta_{\alpha,\beta} v^{-\beta(h_\alpha)})$.
2. $w_\alpha(v) w_\beta(t) w_\alpha(-v) = w_{\sigma_\alpha \beta}(t\eta_{\alpha,\beta} v^{-\beta(h_\alpha)})$.
3. $w_\alpha(v) h_\beta(t) w_\alpha(-v) = h_{\sigma_\alpha \beta}(t)$.
4. $h_\alpha(v) x_\beta(t) h_\alpha(v^{-1}) = x_\beta(tu^{\beta(h_\alpha)})$.
5. $h_\alpha(v) w_\beta(t) h_\alpha(v^{-1}) = w_\beta(tu^{\beta(h_\alpha)})$.
6. $h_\alpha(v) h_\beta(t) h_\alpha(v^{-1}) = h_\beta(t)$.

Here $w_\alpha(v) = x_\alpha(v) x_{-\alpha}(-v^{-1}) x_\alpha(v)$, $h_\alpha(v) = w_\alpha(v) w_\alpha(-1)$.

Proposition 2.2. *Let X be a symmetrizable generalized Cartan matrix and F be an arbitrary field. Then the following formulas hold in $St(X,F)$ for all $u, v, t \in F^*$ and $\alpha, \beta \in \Delta^{re}$.*

1. $\hat{w}_\alpha(v) \hat{x}_\beta(t) \hat{w}_\alpha(-v) = \hat{x}_{\sigma_\alpha \beta}(t\eta_{\alpha,\beta} v^{-\beta(h_\alpha)})$.
2. $\hat{w}_\alpha(v) \hat{w}_\beta(t) \hat{w}_\alpha(-v) = \hat{w}_{\sigma_\alpha \beta}(t\eta_{\alpha,\beta} v^{-\beta(h_\alpha)})$.
3. $\hat{w}_\alpha(v) \hat{h}_\beta(t) \hat{w}_\alpha(-v) = \hat{h}_{\sigma_\alpha \beta}(t\eta_{\alpha,\beta} u^{-\beta(h_\alpha)}) \hat{h}_{\sigma_\alpha \beta}(\eta_{\alpha,\beta} u^{\beta(h_\alpha)})$.
4. $\hat{h}_\alpha(v) \hat{x}_\beta(t) \hat{h}_\alpha(v^{-1}) = \hat{x}_\beta(tu^{\beta(h_\alpha)})$.
5. $\hat{h}_\alpha(v) \hat{w}_\beta(t) \hat{h}_\alpha(v^{-1}) = \hat{w}_\beta(tu^{\beta(h_\alpha)})$.
6. $\hat{h}_\alpha(v) \hat{h}_\beta(t) \hat{h}_\alpha(v^{-1}) = \hat{h}_\beta(tu^{\beta(h_\alpha)}) \hat{h}_\beta^{-1}(u^{\beta(h_\alpha)})$.

Proposition 2.3. *Notation is as above. Then the following formulas hold for all $\alpha, \beta \in \Delta$.*

1. $\eta_{\alpha,\beta} \eta_{\alpha,\sigma_\alpha \beta} = (-1)^{\beta(h_\alpha)}$.
2. $\eta_{\alpha,\alpha} = -1$.
3. $\eta_{\alpha,-\alpha} = -1$.

Definition 2.2 [11]. *Let X be a symmetrizable generalized Cartan matrix and F an arbitrary field. Now we can define a natural group homomorphism $\psi : St(X,F) \to G_{sc}(X,F)$ by $\Psi(\hat{x}_\alpha(u)) = x_\alpha(u)$ for all $\alpha \in \Delta^{re}$ and $u \in F^*$. Then the kernel of Ψ is denoted by $K_2(X,F)$. It is sometimes called the K_2 group of $G_{sc}(X,F)$.*

Theorem 2.2 [1] [4] [11]. *Let X be a symmetrizable generalized Cartan matrix and F an arbitrary field, and let Π be the set of fundamental roots obtained from X. Now we shall consider the following exact sequence*:

$$\{1\} \to K_2(X,F) \to St(X,F) \to G_{sc}(X,F) \to \{1\} \quad (exact).$$

Then the following results hold.

1: $K_2(X,F) = \langle \hat{h}_{\alpha_i}(u)\hat{h}_{\alpha_i}(v)\hat{h}_{\alpha_i}^{-1}(uv) \mid \alpha_i \in \Pi \;\; u,v \in F^* \rangle$.

2: $K_2(X,F)$ *is an abelian group and if* F *is an infinite field, then* $St(X,F)$ *is a universal centaral extension of* $G(X,F)$.

3: $K_2(X,F)$ *is isomorphic to the group which is generated by the symbols* $C_{\alpha_i}(u,v)$ *for all* $\alpha_i \in \Pi$ *and* $u,v \in F^*$, *and charactarized by the following relations* (M1)–(M8). *Usually we say that* $K_2(X,F)$ *has a Matsumoto-type presentation*:

(M1) $C_{\alpha_i}(u,v)C_{\alpha_i}(uv,w) = C_{\alpha_i}(u,vw)C_{\alpha_i}(v,w)$,

(M2) $C_{\alpha_i}(u,v) = C_{\alpha_i}(v,u^{-1})$,

(M3) $C_{\alpha_i}(u,1) = C_{\alpha_i}(1,u) = 1$,

(M4) $C_{\alpha_i}(u,v) = C_{\alpha_i}(u,-uv)$,

(M5) $C_{\alpha_i}(u,v) = C_{\alpha_i}(u,(1-u)v)$ *with* $(1-u) \in F^*$,

(M6) $C_{\alpha_i}(u, v^{\alpha_i(h_j)}) = C_{\alpha_j}(u^{\alpha_j(h_i)}, v)$ *denoting it by* $C_{\alpha_i\alpha_j}(u,v)$,

(M7) $C_{\alpha_i\alpha_j}(uv,w) = C_{\alpha_i\alpha_j}(u,w)C_{\alpha_i\alpha_j}(v,w)$,

(M8) $C_{\alpha_i\alpha_j}(u,vw) = C_{\alpha_i\alpha_j}(u,v)C_{\alpha_i\alpha_j}(u,w)$ *for all* $u,v,w \in F^*$ *and* $\alpha_i, \alpha_j \in \Pi$.

Here we can recognize that $C_{\alpha_i}(u,v)$ *corresponds to* $\hat{h}_{\alpha_i}(u)\hat{h}_{\alpha_i}(v)\hat{h}_{\alpha_i}(uv)^{-1}$.

Furthermore $\hat{h}_\alpha(u)\hat{h}_\alpha(v)\hat{h}_\alpha(uv)^{-1}$ is in $K_2(A,F)$, for any real root α. We denote it by $C_\alpha(u,v)$.

As above, the group structure of $K_2(X,F)$ is well known in case of an arbitrary field F. Now it is natural to study the group structure of K_2 group when we take rings instead of fields. And there are many results about this quastion. We introduce two of them.

In fact if X is a Cartan matrix of finite type, then we can obtain a certain group functor $G(X,\;) = Alg_{\mathbf{Z}}(H_{\mathbf{Z}},\;)$, using a Hopf algebra $H_{\mathbf{Z}}$, corresponding to our finite dimentional Kac-Moody group here. Then, the group $G(X,R)$ for a commutative ring R is called a Chevalley group (cf. [1]).

Definition 2.3 (Steinberg Groups over Rings). *Let* R *be a commutative ring and let* X *be a Cartan matrix of finite type. Let* Δ *be the root system obtained from* X. *Then we consider the group generated by the symbols* $\hat{x}_\alpha(t)$ *for all* $t \in R$ *and* $\alpha \in \Delta$, *and charactarized by the relation* (K1)–(K3) (*see Theorem* 2.1). *We call it a Steinberg group and denote it by* $St(X,R)$.

Now we can define a natural group homomorphism $\psi : St(X,R) \to G_{sc}(X,R)$ by $\Psi(\hat{x}_\alpha(u)) = x_\alpha(u)$, and we denote $Ker(\psi)$ by $K_2(X,R)$ (cf. [11] [1] [10]). Then there is a natural question asking whether or not $K_2(X,R)$ has a Matsumoto-Type presentation.

Theorem 2.3 [7]. *Let X be a Cartan matrix of finite type, and let R be a local ring whose residue field is infinite, and Π the set of fundamental roots obtained from X. Then we have following.*

1: $K_2(X,R) = \langle \hat{h}_{\alpha_i}(u)\hat{h}_{\alpha_i}(v)\hat{h}_{\alpha_i}^{-1}(uv) \mid \alpha_i \in \Pi \ \ u,v \in F^* \rangle$.

2: *$K_2(X,R)$ is generated by the symbols $C_{\alpha_i}(u,v)$ for all $u,v \in F^*$ and $\alpha_i \in \Pi$, and has a Matsumoto-type presentation.*

Theorem 2.4 [13]. *Let p be a prime number which is neither 2 nor 3, then $K_2(A_n, \boldsymbol{Z}[1/p])$ does not have a Matsumoto-type presentation for all $1 \leq n$.*

3. Mainresults

In this chapter we suppose that F is an arbitrary field and A is a Cartan matrix of finite type, and A^{aff} is the affine Cartan matrix obtained from A whose tier number is 1. (For the definition of the tier number of A, see [12].)

Now we consider the K_2 group obtained from a simply connected loop group $G_{sc}(A, F[X,X^{-1}])$, this is a universal Chevalley group generated by a Laurent polynomial ring (cf. [9]). Then we have

$$1 \to K_2(A, F[X,X^{-1}]) \to St(A, F[X,X^{-1}]) \to G_{sc}(A, F[X,X^{-1}]) \to 1 \quad \text{(exact)}.$$

In the above exact sequence, we want to determine a group presentation of $K_2(A, F[X,X^{-1}])$.

It is known that $G_{sc}(A, F[X,X^{-1}])$ is generated by the symbols $x_{\alpha_i}(uX^m)$ for all $u \in F^*$, $m \in \boldsymbol{Z}$, *and* $\alpha_i \in \Pi$, where Π is the set of fundamental roots obtained from A, and charactarized by the relations (K1)–(K4) as in Theorem 2.1 (cf. [9] [3]).

3.1. The Case of A_1

In this section $\{\alpha\}$ is the set of a fundamental root in the root system of A_1. And $\{\alpha_0, \alpha_1\}$ is the set of fundamental roots in the root system of A_1^{aff}.

It is known that $G_{sc}(A_1, F[X,X^{-1}]) = SL(2, F[X,X^{-1}])$ (cf. [13]). Now we give a presentation of $K_2(A_1, F[X,X^{-1}])$ which satisfies the following exact sequence:

$$1 \to K_2(A_1, F[X,X^{-1}]) \to St(A_1, F[X,X^{-1}]) \to SL(2, F[X,X^{-1}]) \to 1 \quad \text{(exact)}.$$

3.1.1. Exact Sequence of A_1 Type

Fig. 1[9]

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & 1 & & F^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_2(A_1^{aff}, F) & \longrightarrow & St(A_1^{aff}, F) & \longrightarrow & G_{sc}(A_1^{aff}, F) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \Phi & & \downarrow \Psi & & \\
1 & \longrightarrow & K_2(A_1, F[X, X^{-1}]) & \longrightarrow & St(A_1, F[X, X^{-1}]) & \longrightarrow & SL(2, F[X, X^{-1}]) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F^* & & 1 & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}
$$

The above diagram is commutative, and each sequence is exact. Here Φ and Ψ are group homomorphisms given by

$$\Phi : St(A_1^{aff}, F) \to St(A_1, F[X, X^{-1}])$$

$$\hat{x}_{n\alpha_0+(n+1)\alpha_1}(t) \mapsto \hat{x}_{\alpha}(tX^n)$$

$$\hat{x}_{n\alpha_0+(n-1)\alpha_1}(t) \mapsto \hat{x}_{-\alpha}(tX^n)$$

$$\hat{w}_{n\alpha_0+(n+1)\alpha_1}(t) \mapsto \hat{w}_{\alpha}(tX^n)$$

$$\hat{w}_{n\alpha_0+(n-1)\alpha_1}(t) \mapsto \hat{w}_{-\alpha}(tX^n)$$

and

$$\Psi : G_{sc}(A_1^{aff}, F) \to SL(2, F[X, X^{-1}])$$

$$x_{n\alpha_0+(n+1)\alpha_1}(t) \mapsto \begin{pmatrix} 1 & tX^n \\ 0 & 1 \end{pmatrix}$$

$$x_{n\alpha_0+(n-1)\alpha_1}(t) \mapsto \begin{pmatrix} 1 & 0 \\ tX^n & 1 \end{pmatrix}$$

$$w_{n\alpha_0+(n+1)\alpha_1}(t) \mapsto \begin{pmatrix} 0 & tX^n \\ -t^{-1}X^{-n} & 0 \end{pmatrix}$$

$$w_{n\alpha_0+(n-1)\alpha_1}(t) \mapsto \begin{pmatrix} 0 & -tX^{-n} \\ tX^n & 0 \end{pmatrix}$$

$$h_{n\alpha_0+(n+1)\alpha_1}(t) \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

$$h_{n\alpha_0+(n-1)\alpha_1}(t) \mapsto \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

for all $t \in F^*$ and $n \in \boldsymbol{Z}$, respectively (cf. [9]).

Since the group presentations of $St(A_1^{aff}, F)$ and $St(A_1, F[X, X^{-1}])$ are well known, it is easy to see that Φ is an isomorphism. And also it is easy to show that Ψ is well-defined. Here we note that $St(A_1, F[X, X^{-1}])$ has a Bruhat decomposition (cf. [4] [9]).

Proposition 3.1 [4] [2]. *Notation is as above. Then $K_2(A_1^{aff}, F)$ is the group generated by the symols $C_\alpha(u,v)$ for all $u, v \in F^*$ and $\alpha \in \{\alpha_0, \alpha_1\}$, where $C_\alpha(u,v) = \hat{h}(u)\hat{h}(v)\hat{h}^{-1}(uv)$, and charactarized by the relations* (L1)–(L7):

(L1) $C_\alpha(u,v)C_\alpha(uv,w) = C_\alpha(u,vw)C_\alpha(v,w)$,

(L2) $C_\alpha(u,1) = C_\alpha(1,v) = 1$,

(L3) $C_\alpha(u,v) = C_\alpha(v^{-1},u)$,

(L4) $C_\alpha(u,-uv) = C_\alpha(u,v)$,

(L5) $C_\alpha(u,v) = C_\alpha(u,(1-u)v)$ (*if* $1-u \in F^*$),

(L6) $C_{\alpha_0}(u,v^{-2}) = C_{\alpha_1}(u^{-2},v)$ (*denoting it by* $C_{\alpha_0\alpha_1}(u,v)$),

(L7) $C_{\alpha_0\alpha_1}(u,v)$ *is bimultiplicative*

for all $u, v \in F^*$ *and* $\alpha \in \{\alpha_0, \alpha_1\}$.

Proposition 3.2 [6]. *Notation is as above. Then $SL(2, F[X, X^{-1}])$ has a Bruhat decomposition.*

Proposition 3.3 [9]. *Notation is as above. Then we have $Ker\,\Psi = \{h_{\alpha_0}(t)h_{\alpha_1}(t) \mid t \in F^*\}$.*

Proof. Since both $SL(2, F[X, X^{-1}])$ and $G(A_1^{aff}, F)$ have Bruhat decompositions, and since Ψ preserves the Bruhat decomposition, we can see $Ker\,\Psi \subseteq \langle h_{\alpha_i}(t_i) \mid 0 \le i \le 1 \rangle$. Hence each element of $Ker\,\Psi$ can be written as $h_{\alpha_0}(t)h_{\alpha_1}(t)$ for some $t \in F$. Therefore we obtain the derived result. □

3.1.2. Construction of Isomorphism

Definition 3.1. *Now we recognize* $\{u,v\}_\alpha$ *as a symbol for all* $u,v \in F^*$. *We define the group* $\hat{K}_2(A_1, F[X,X^{-1}])$, *whose generators are* $\{u,v\}_\alpha$ *for all* $u,v \in F^*$ *and which is charactarized by the following relations* (M′1)–(M′5):

(M′1) $\{u,v\}_\alpha\{uv,w\}_\alpha = \{u,vw\}_\alpha\{v,w\}_\alpha$,

(M′2) $\{u,1\}_\alpha = \{1,u\}_\alpha = 1$,

(M′3) $\{u,v\}_\alpha = \{v^{-1},u\}_\alpha$,

(M′4) $\{u,-uv\}_\alpha = \{u,v\}_\alpha$,

(M′5) $\{u,(1-u)v\}_\alpha = \{u,v\}_\alpha$ (*if* $(1-u) \in F[X,X^{-1}]^*$)

for all $u,v,w \in F^*$.

Proposition 3.4. *Let* $\eta : \hat{K}_2(A_1, F[X,X^{-1}]) \to K_2(A_1, F[X,X^{-1}])$ *be a homomorphism with* $\eta(\{u,v\}_\alpha) = \hat{h}_\alpha(u)\hat{h}_\alpha(v)\hat{h}_\alpha(uv)^{-1}$, *for all* $u,v \in F^*$. *Then the* η *is a group homomorphism.*

Proof. We only have to check (M′1) to (M′5) (cf. [2]). □

By the commutative diagram in Fig 1, we can see $K_2(A_1, F[X,X^{-1}]) \subset St(A_1, F[X,X^{-1}])$ and $K_2(A_1^{aff}, F) \subset St(A_1^{aff}, F)$. Hence we can conclude that

$$\begin{aligned} K_2(A_1, F[X,X^{-1}]) &= \Phi(\langle Ker\,\Psi, K_2(A_1^{aff}, F)\rangle) \\ &= \langle \hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1}, K_2(A_1^{aff}, F) \mid t \in F^*\rangle. \end{aligned}$$

We restrict Φ to $\langle \hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1}, K_2(A_1^{aff}, F) \mid t \in F^*\rangle$. Then we have

$$\Phi(C_{\alpha_0}(u,v)) = C_\alpha(u,-X)C_\alpha(u,-vX),$$

$$\Phi(C_{\alpha_1}(u,v)) = C_\alpha(u,v),$$

$$\Phi(\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1}) = C_\alpha(t,-X)^{-1}C_\alpha(t,-1),$$

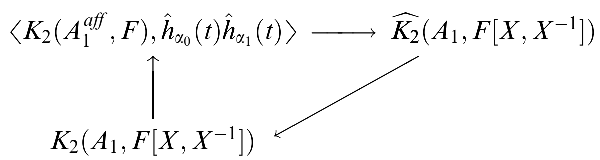

Figure A

where $\hat{h}_{-\alpha}(t) = \hat{h}_{\alpha}(t^{-2})\hat{h}_{\alpha}(t)C_{\alpha}(t,-1)^{-1}$ and $C_{-\alpha}(u,v) = C_{\alpha}(u,-v)C_{\alpha}(u,-1)^{-1}$ for all $u,v,t \in F^*$ (see Proposition 3.8, 3.9).

Now if the following correspondance ξ gives a group homomorphism, we can see that Fig A is a commutative diagram. From that we can conclude that $K_2(A, F[X, X^{-1}])$ has a Matsumoto-type presentation.

$$\xi : \langle K_2(A_1^{aff}, F), \hat{h}_{\alpha_0}(t)\hat{h}_{\alpha_1}(t) \mid t \in F^* \rangle \to \hat{K}_2(A_1, F[X, X^{-1}])$$

$$C_{\alpha_0}(u,v) \mapsto \{u,-X\}_{\alpha}^{-1}\{u,-vX\}_{\alpha}$$

$$C_{\alpha_1}(u,v) \mapsto \{u,v\}_{\alpha}$$

$$\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1} \mapsto \{t,-X\}_{\alpha}^{-1}\{t,-1\}_{\alpha} \quad \text{for all } u,v,t \in F^*. \tag{1}$$

3.1.3. Well-definedness of ξ

Central Extension. In this subsection, we make use of the theory of centaral extensions to analyse abelian groups which have a Matsumoto-type presentations (cf. [13]). Let R be a commutative ring. Let L be an abelian group generated by the symbols $\langle u,v \rangle$ for all $u,v \in R^*$, and charactarized by the relations (M′1)–(M′4) (as in Definition 3.1). Now we take the symbols $C(r)$ for all $r \in R^*$, and consider the set $H := \{C(r)\langle u,v \rangle \mid r \in R^*, u,v \in F^*\}$. We define a multiplication in H with the following defining relations:

$$C(r_1)C(r_2) = C(r_1 r_2)\langle r_1, r_2 \rangle,$$

$$\langle u,v \rangle C(r) = C(r)\langle u,v \rangle \quad \text{for all } r_1, r_2 \in R^* \text{ and } u,v \in F^*.$$

Lemma 3.1. *H has a group structure.*

Proof. To see the associativity of our multiplication in H is easy. The unit of H is $C(1)$. And the inverse element of $C(r)\langle u,v \rangle$ is $C(r^{-1})\langle r, r^{-1} \rangle^{-1}\langle u,v \rangle$ for all $r,u,v \in R^*$. Hence we obtain the derived result. □

Now we obtain the following exact sequence:

$$1 \to L \to H \to R^* \to 1 \quad \text{(exact)}.$$

And H is a central extension of R^* by L.

Lemma 3.2. *Let $[\,,\,] : H \times H \to L$ be the form defined by $[x,y] := xyx^{-1}y^{-1}$ for all $x,y \in H$. Then it is bimultiplicative. Furthermore if $b,c \in L$, then $[x,y] = [xb, yc]$. Usually we say that $[\,,\,]$ has a mod L stability.*

Proof. We have

$$[x,z][y,z] = xzx^{-1}z^{-1}yzy^{-1}z^{-1} \quad \text{(since } x \text{ and } zx^{-1}z^{-1} \text{ are commutative)}$$
$$= zx^{-1}z^{-1}xyzy^{-1}z^{-1} \quad \text{(since } zx^{-1}z^{-1}x \text{ and } yzy^{-1}z^{-1} \text{ are commutative)}$$
$$= yzy^{-1}x^{-1}z^{-1}x \quad \text{(since } yzy^{-1}x^{-1}z^{-1} \text{ and } x \text{ are commutative)}$$
$$= xyzy^{-1}x^{-1}z^{-1} = [xy,z].$$

Therefore we obtain the desired result. □

Lemma 3.3. *For all* $p, q \in R^*$, *we have* $[C(p), C(q)] = \langle p^2, q \rangle$.

Proof. We have

$$[C(p), C(q)] = C(p)C(q)C(p)^{-1}C(q)^{-1}$$
$$= C(pq)\langle p,q\rangle C(p^{-1})\langle p,p^{-1}\rangle^{-1}C(q^{-1})\langle q,q^{-1}\rangle^{-1}$$
$$= C(pq)C(p^{-1})C(q^{-1})\langle p,q\rangle\langle p,p^{-1}\rangle^{-1}\langle q,q^{-1}\rangle^{-1}$$
$$= \langle pq,p^{-1}\rangle\langle q,q^{-1}\rangle\langle p,q\rangle\langle p,p^{-1}\rangle^{-1}\langle q,q^{-1}\rangle^{-1}$$
$$= \langle p,-1\rangle^{-1}\langle p,q\rangle\langle p,-q\rangle.$$

Now we can see $\langle p,p\rangle\langle p^2,q\rangle = \langle p,pq\rangle\langle p,q\rangle$.
Hence $\langle p^2,q\rangle = \langle p,-1\rangle^{-1}\langle p,-q\rangle\langle p,q\rangle$.
Therefore we obtain the desired result. □

Lemma 3.4. *For all* $p, q, r \in R^*$, *we have* $\langle p^2 q, r\rangle = \langle p^2, r\rangle\langle q, r\rangle$.

Proof. Since

$$\langle p^2,q\rangle\langle p^2,r\rangle = \langle p^2,qr\rangle$$
$$\langle p^2,q\rangle\langle p^2q,r\rangle = \langle p^2,qr\rangle\langle q,r\rangle,$$

we obtain $\langle p^2q,r\rangle = \langle p^2,r\rangle\langle q,r\rangle$. □

Lemma 3.5. *For all* $p, q \in R^*$, *we have* $\langle p^2, q\rangle = \langle p, q^2\rangle$.

Proof. We see the left hand side $= [C(p), C(q)]$, and the right hand side $= \langle q^2, p^{-1}\rangle = [C(q), C(p^{-1})] = [C(q), C(p)^{-1}]$ for all $p, q \in R^*$, since $[\,,]$ has a *mod L* stability.

We put $C(p) := X$ and $C(q) := Y$. Then $XYX^{-1}Y^{-1} = YXY^{-1}X^{-1}$, so we have $[X, Y] = [Y, X^{-1}]$. $\square$

Proposition 3.5. *In $\hat{K}_2(A_1, F[X, X^{-1}])$ (resp. in $K_2(A_1, F[X, X^{-1}])$), we have $\{u, v^2\}_\alpha = \{u^2, v\}_\alpha$ (resp. $C_\alpha(u, v^2) = C_\alpha(u^2, v)$) and $\{u^2 v, w\}_\alpha = \{u^2, w\}_\alpha \{v, w\}_\alpha$ (resp. $C_\alpha(uv^2, w) = C_\alpha(u, w) C_\alpha(v^2, w)$) for all $u, v \in R^*$.*

Proof. From Lemma 3.4 and Lemma 3.5, this is easily shown. $\square$

The result of Proposition 3.5 is stated in [10] [14] without proof, and the proof of the proposition seems to be not trivial, so we give its proof here.

About $K_2(A_1, F[X, X^{-1}])$

Lemma 3.6. *In $St(A_1^{aff}, F)$ we have*

$$\hat{h}_{\alpha_0}(s)\hat{h}_{-\alpha_1}(s)^{-1}\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1}$$
$$= \hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}(st)^{-1} C_{\alpha_0}(t, s) C_{\alpha_1}(t, -s)^{-1} C_{\alpha_1}(t, -1) \quad \text{for all } s, t \in F^*.$$

Proof. Note $[\hat{h}_{\alpha_0}(t), \hat{h}_{-\alpha_1}(s)] = C_{\alpha_0}(t, s^2)$ and $\hat{h}_{\alpha_i}(s)\hat{h}_{\alpha_i}(s^{-1}) = C_{\alpha_i}(s, -1)$ (cf. [4]).

We have $[\hat{h}_{\alpha_0}(t)^{-1}, \hat{h}_{-\alpha_1}(s)^{-1}] = [\hat{h}_{\alpha_0}(t^{-1}), \hat{h}_{-\alpha_1}(s^{-1})] = C_{\alpha_0}(t^{-1}, s^{-2})$. Then

$$\begin{aligned}
\text{the left hand side} &= \hat{h}_{\alpha_0}(s)\hat{h}_{-\alpha_1}(s)^{-1}\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1} \\
&= \hat{h}_{\alpha_0}(s)\hat{h}_{\alpha_0}(t)\hat{h}_{\alpha_0}(t)^{-1}\hat{h}_{-\alpha_1}(s)^{-1}\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(s)\hat{h}_{-\alpha_1}(s)^{-1}\hat{h}_{-\alpha_1}(t)^{-1} \\
&= \hat{h}_{\alpha_0}(s)\hat{h}_{\alpha_0}(t)[\hat{h}_{\alpha_0}(t)^{-1}, \hat{h}_{-\alpha_1}(s)^{-1}]\hat{h}_{-\alpha_1}(s)^{-1}\hat{h}_{-\alpha_1}(t)^{-1} \\
&= \hat{h}_{\alpha_0}(st) C_{\alpha_0}(s, t) C_{\alpha_0}(t^{-1}, s^{-2})\hat{h}_{-\alpha_1}(st)^{-1} C_{-\alpha_1}(t, s)^{-1} \\
&= \hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}(st)^{-1} C_{\alpha_0}(s, t) C_{\alpha_0}(t^{-1}, s^{-2}) C_{\alpha_1}(t, -1) C_{\alpha_1}(t, -s)^{-1} \\
&= \hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}(st)^{-1} C_{\alpha_0}(t^{-1}, s) C_{\alpha_0}(t^{-1}, s^{-2}) C_{\alpha_1}(t, -1) C_{\alpha_1}(t, -s)^{-1} \\
&= \hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}(st)^{-1} C_{\alpha_0}(t, s) C_{\alpha_1}(t, -1) C_{\alpha_1}(t, -s)^{-1} \\
&= \text{the right hand side. Therefore we obtain the desired result.}
\end{aligned}$$

$\square$

From Lemma 3.6, the subgroup $\langle K_2(A_1^{aff}, F), \hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1} \mid t \in F^* \rangle$ is a central extension of $\{\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}(t)^{-1} \mid t \in F^*\}$ by the $K_2(A_1^{aff}, F)$. So we can obtain the group presentation of $\langle K_2(A_1^{aff}, F), \hat{h}_{\alpha_0}(t)\hat{h}_{\alpha_1}(t) \mid t \in F^* \rangle$ from the group presentation of $K_2(A_1^{aff}, F)$ and the following ♣:

$$♣ \quad \hat{h}_{\alpha_0}(s)\hat{h}_{-\alpha_1}^{-1}(s)\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}^{-1}(t)$$
$$= \hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}^{-1}(st)C_{\alpha_0}(t,s)C_{\alpha_1}(t,-s)^{-1}C_{\alpha_1}(t,-1) \quad \text{for all } s,t \in F^*. \quad (2)$$

Determine the Presentation of $K_2(A_1, F[X, X^{-1}])$

If we prove that $\xi|_{K_2(A_1^{aff}, F)}$ (as in (1)) is well-defined and ξ preserves ♣ (as in (2)), we can conclude that ξ is well-defined.

Lemma 3.7. *Notation is as above. Then $\xi|_{K_2(A_1^{aff}, F)}$ is well-defined.*

Proof. It is sufficient to confirm that $\xi|_{K_2(A_1^{aff}, F)}$ preserves the relations (M1)–(M7) (as in Theorem 2.2).

We remark that $C_{\alpha_1}(u,v)$ for all $u,v \in F^*$ satisfies the relations (M1)–(M5) and our $\xi|_{K_2(A_1^{aff}, F)}$ preserves the relation (M1)–(M5) in the case of $C_{\alpha_1}(u,v)$ for all $u,v \in F^*$. Hence we consider the case of (M1)–(M5) for $C_{\alpha_0}(u,v)$ and the case of (M6)–(M7) for both $C_{\alpha_1}(u,v)$ and $C_{\alpha_0}(u,v)$ for all $u,v \in F^*$.

First we prove that our $\xi|_{K_2(A_1^{aff}, F)}$ preserve the relation (M1)–(M5) in the case of $C_{\alpha_0}(u,v)$ for all $u,v \in F^*$.

(M1):
We have

$$\begin{aligned}
&\xi(C_{\alpha_0}(u,v))\xi(C_{\alpha_0}(uv,w)) \\
&\quad = \xi(C_{\alpha_0}(u,vw))\xi(C_{\alpha_0}(v,w)) \\
&\quad \Leftrightarrow \{u,-X\}_\alpha^{-1}\{u,-vX\}_\alpha\{uv,-X\}_\alpha^{-1}\{uv,-wX\}_\alpha \\
&\quad = \{u,-X\}_\alpha^{-1}\{u,-vwX\}_\alpha\{v,-X\}_\alpha^{-1}\{v,-wX\}_\alpha \\
&\qquad (\text{using } \{u,v\}_\alpha\{uv,-X\}_\alpha = \{u,-vX\}_\alpha\{v,-X\}_\alpha) \\
&\quad \Leftrightarrow \{u,-X\}_\alpha^{-1}\{u,v\}_\alpha\{v,-X\}_\alpha^{-1}\{uv,-wX\}_\alpha \\
&\quad = \{u,-X\}_\alpha^{-1}\{u,-vwX\}_\alpha\{v,-X\}_\alpha^{-1}\{v,-wX\}_\alpha \\
&\quad \Leftrightarrow \{u,v\}_\alpha\{uv,-wX\}_\alpha = \{u,-vwX\}_\alpha\{v,-wX\}_\alpha \quad \text{for all } u,v,w \in F^*.
\end{aligned}$$

Hence we have shown that our $\xi|_{K_2(A_1^{aff},F)}$ preserves (M1) in the case of $C_{\alpha_0}(u,v)$ for all $u,v \in F^*$. It is easily shown that our $\xi|_{K_2(A_1^{aff},F)}$ preserves the relations (M1)–(M5).

Next we show that our $\xi|_{K_2(A_1^{aff},F)}$ preserves the relation (M6), (M7) for both $C_{\alpha_0}(u,v)$ and $C_{\alpha_1}(u,v)$ for all $u,v \in F^*$.

(M6):
We have

$$\begin{aligned}\xi(C_{\alpha_0}(u, v^{\alpha_0(h_1)})) &= \xi(C_{\alpha_1}(u^{\alpha_1(h_0)}, v)) \\ &\Leftrightarrow \xi(C_{\alpha_0}(u, v^{-2})) = \xi(C_{\alpha_1}(u^{-2}, v)) \\ &\Leftrightarrow \{u, -X\}_\alpha \{u, -v^{-2}X\}_\alpha = \{u^{-2}, v\}_\alpha \\ &\Leftrightarrow \{u, v^{-2}\}_\alpha = \{u^{-2}, v\}_\alpha.\end{aligned}$$

Hence our correspondence preserves (M6). Finally we discuss (M7).

(M7):
We have

$$\xi(C_{\alpha_0\alpha_1}(u, vw)) = \xi(C_{\alpha_0\alpha_1}(u, v)C_{\alpha_0\alpha_1}(u, w)) \Leftrightarrow \{u^{-2}, vw\}_\alpha = \{u^{-2}, v\}_\alpha \{u^{-2}, w\}_\alpha.$$

Hence our correnpondence preserves (M7). Therefore we obtain the desired result. □

Proposition 3.6. *Notation is as above. Then ξ preserves ♣.*

Proof. We apply our ξ to the left hand side and right hand side of the above equation ♣. Now we have

$$\begin{aligned}&\xi(\hat{h}_{\alpha_0}(s)\hat{h}_{-\alpha_1}^{-1}(s)\hat{h}_{\alpha_0}(t)\hat{h}_{-\alpha_1}^{-1}(t)) \\ &\quad = \xi(\hat{h}_{\alpha_0}(st)\hat{h}_{-\alpha_1}^{-1}(st)C_{\alpha_0}(t,s)C_{\alpha_1}(t,-s)^{-1}C_{\alpha_1}(t,-1)) \\ &\quad \Leftrightarrow \{s,-X\}_\alpha^{-1}\{s,-1\}_\alpha\{t,-X\}_\alpha^{-1}\{t,-1\}_\alpha \\ &\quad = \{st,-X\}_\alpha^{-1}\{st,-1\}_\alpha\{t,-X\}_\alpha^{-1}\{t,-sX\}_\alpha\{t,-s\}_\alpha^{-1}\{t,-1\}_\alpha \\ &\quad \Leftrightarrow \{s,-X\}_\alpha^{-1}\{s,-1\}_\alpha = \{st,-X\}_\alpha^{-1}\{st,-1\}_\alpha\{t,-sX\}_\alpha\{t,-s\}_\alpha^{-1} \\ &\qquad\quad (\text{using } \{t,s\}_\alpha\{ts,-X\}_\alpha = \{t,-sX\}_\alpha\{s,-X\}_\alpha) \\ &\quad \Leftrightarrow \{s,-X\}_\alpha^{-1}\{s,-1\}_\alpha = \{t,s\}_\alpha\{s,-X\}_\alpha^{-1}\{st,-1\}_\alpha\{t,-s\}_\alpha^{-1} \\ &\quad \Leftrightarrow \{t,-s\}_\alpha\{s,-1\}_\alpha = \{t,s\}_\alpha\{st,-1\}_\alpha \quad \text{for all } s,t \in F^*.\end{aligned}$$

Hence our ξ preserves ♣. □

Hence the well-definedness of ξ has be shown, and we obtain the following theorem.

Theorem 3.1. *Notation is as above. Then we have* $K_2(A_1, F[X, X^{-1}]) \simeq \hat{K}_2(A_1, F[X, X^{-1}])$.

3.2. General Case

3.2.1. Exact Sequence of General Case

In this section A is any Cartan matrix of finite type ($A \neq A_1$) and A^{aff} is the affine Cartan matrix obtained from A whose tier number is 1. (For the definition of the tier number of A, see [12].) Let $\{\alpha_1, \dots \alpha_n\}$ be the set of fundamental roots in the root system Δ of A and denote it by Π, and $\{\alpha_0, \alpha_1, \dots \alpha_n\}$ is the set of fudamental roots in the root system Δ_{aff} of A^{aff} and denote it by Π_{aff}. The set of real roots in A^{aff} is denoted by Δ_{aff}^{re}. Furthermore W is the Weyl group obtained from A, and W_{aff} is the Weyl group obtained from A^{aff}. We choose $\delta \in \Delta_{aff}$ as a fundamental null root of Δ^{aff}, and θ is the highest root of Δ, and we put $h_\theta := 2\theta/(\theta, \theta)$.

Fig. 2[9]

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & 1 & & F^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_2(A^{aff}, F) & \longrightarrow & St(A^{aff}, F) & \longrightarrow & G_{sc}(A^{aff}, F) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Psi} & & \\
1 & \longrightarrow & K_2(A, F[X, X^{-1}]) & \longrightarrow & St(A, F[X, X^{-1}]) & \longrightarrow & G_{sc}(A, F[X, X^{-1}]) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F^* & & 1 & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}
$$

The above diagram is commutative and each sequence is exact. Here Φ and Ψ are group homomorphisms given by

$$\begin{aligned}\Phi : St(A_1^{aff}, F) &\to St(A, F[X, X^{-1}])\\ \hat{x}_{n\delta+\alpha}(t) &\mapsto \hat{x}_\alpha(tX^n)\\ \hat{w}_{n\delta+\alpha}(t) &\mapsto \hat{w}_\alpha(tX^n)\\ \hat{h}_{n\delta+\alpha}(t) &\mapsto \hat{h}_\alpha(tX^n)h_\alpha^{-1}(X^n)\\ \hat{h}_{\alpha_0}(t) &\mapsto \hat{h}_{-\theta}(tX)h_{-\theta}^{-1}(X)\\ \hat{h}_{\alpha_i}(t) &\mapsto \hat{h}_{\alpha_i}(t)\end{aligned}$$

and

$$\begin{aligned}\Psi : G_{sc}(A^{aff}, F) &\to G_{sc}(A, F[X, X^{-1}])\\ x_{n\delta+\alpha}(t) &\mapsto x_\alpha(tX^n)\\ w_{n\delta+\alpha}(t) &\mapsto w_\alpha(tX^n)\\ h_{n\delta+\alpha}(t) &\mapsto h_\alpha(t)\end{aligned}$$

for all $t \in F^*$, $n \in \boldsymbol{Z}$ and $\alpha \in \Delta$, respectively.

Proposition 3.7 [9]. *Notation is as above. Then we have*

$$Ker\ \Psi = \{h_{\alpha_0}(t_0)h_{\alpha_1}(t_1)\cdots h_{\alpha_l}(t_l) \mid h_\theta(t_0) = h_{\alpha_1}(t_1)\cdots h_{\alpha_l}(t_l) \in G(A, F[X, X^{-1}])\}.$$

Proof. Since $G(A^{aff}, F)$ has a standard Bruhat decomposition, we see

$$Ker\ \Psi \subset \langle h_{\alpha_i}(t) \mid i = 0, 1, \dots n \ \ t \in F^* \rangle = \{h_{\alpha_0}(t_0)h_{\alpha_1}(t_1)\cdots h_{\alpha_l}(t_l) \mid t_i \in F^*\}.$$

Applying Ψ to $h_{\alpha_0}(t_0)\cdots h_{\alpha_l}(t_l)$, we have $h_\theta(t_0) = h_{\alpha_1}(t_1)\cdots h_{\alpha_l}(t_l)$. □

Proposition 3.8 [4] [2]. *Let* α *be in* $\{\alpha_0, \alpha_1, \dots \alpha_n\}$, *and* u, v, w *in* F^*. *Then* $K_2(A^{aff}, F)$ *is generated by* $C_{\alpha_i}(u, v)$ *for all* $u, v \in F^*$ *and* $\alpha_i \in \Pi_{aff}$, *where* $C_{\alpha_i}(u, v) = \hat{h}_{\alpha_i}(u)\hat{h}_{\alpha_i}(v)\hat{h}_{\alpha_i}(uv)^{-1}$, *and charactarized by the following relations* (L1)–(L7):

(L1) $C_\alpha(u, v)C_\alpha(uv, w) = C_\alpha(u, vw)C_\alpha(v, w)$,

(L2) $C_\alpha(u, 1) = C_\alpha(1, v) = 1$,

(L3) $C_\alpha(u, v) = C_\alpha(v^{-1}, u)$,

(L4) $C_\alpha(u, -uv) = C_\alpha(u, v)$,

(L5) $C_\alpha(u, v) = C_\alpha(u, (1 - u)v)$ (*if* $1 - u \in F^*$), (L6) $C_{\alpha_i}(u, v^{\alpha_i(h_j)}) = C_{\alpha_j}(u^{\alpha_j(h_i)}, v)$ *denoting it by* $C_{\alpha_i\alpha_j}(u, v)$,

(L7) $C_{\alpha_i\alpha_j}(u, v)$ *is bimultiplicative*

for all $u, v, w \in F^*$ *and* $\alpha_i, \alpha_j \in \Pi_{aff}$.

Now we can recognize that $C_\alpha(u, v)$ is the element corresponding to $\hat{h}_\alpha(u)\hat{h}_\alpha(v)\hat{h}_\alpha(uv)^{-1}$ for all $\alpha \in \Delta^{re}$.

3.2.2. Action of Weyl Group

Proposition 3.9. *In* $St(A^{aff}, F)$, *we have* $\hat{h}_{\sigma_\alpha\beta}(t) = \hat{h}_\alpha(t^{-\alpha(h_\beta)})\hat{h}_\beta(t) \cdot C_\beta(t, \eta_{\alpha,\sigma_\alpha(\beta)})^{-1}$ *for all* $t \in F^*$ *and* $\alpha, \beta \in \Delta^{re}$.

Proof. Note that $\hat{h}_{\sigma_\alpha\beta}(t)w_\alpha(1)\hat{h}_{\sigma_\alpha\beta}(t)^{-1} = \hat{h}_{\sigma_\alpha\beta}(t)\hat{w}_\alpha(1)\hat{h}_{\sigma_\alpha\beta}(t)^{-1}\hat{w}_\alpha(-1)\hat{w}_\alpha(1)$. Then we have

$$\begin{aligned}\hat{w}_\alpha(t^{-\alpha(h_\beta)}) &= \hat{h}_{\sigma_\alpha\beta}(t)w_\alpha(1)\hat{h}_{\sigma_\alpha\beta}(t)^{-1}\hat{h}_{\sigma_\alpha\beta}(t)\hat{w}_\alpha(1)\hat{h}_{\sigma_\alpha\beta}(t)^{-1}\hat{w}_\alpha(-1)\hat{w}_\alpha(1)\\ &= \hat{h}_{\sigma_\alpha\beta}(t)C_\beta(t, \eta_{\alpha,\sigma_\alpha(\beta)})\hat{h}_\beta(t)^{-1}\hat{w}_\alpha(1).\end{aligned}$$

Hence we obtain the desired result. □

Proposition 3.10. *In* $St(A^{aff}, F)$, *we have* $C_{\sigma_\alpha\beta}(u, v) = C_\beta(u, v\eta)C_\beta(u, \eta)^{-1}$, *where* $\eta = \eta_{\alpha,\sigma_\alpha\beta}$, *for all* $u, v \in F^*$ *and* $\alpha, \beta \in \Delta^{re}$.

(3)

Proof. We see

$$\begin{aligned}C_{\sigma_\alpha\beta}(u, v) &= \hat{h}_{\sigma_\alpha\beta}(u)\hat{h}_{\sigma_\alpha\beta}(v)\hat{h}_{\sigma_\alpha\beta}(uv)^{-1}\\ &= \hat{h}_\alpha(u^{-\alpha(h_\beta)}\hat{h}_\beta(u)\hat{h}_\alpha(v^{-\alpha(h_\beta)})\hat{h}_\beta(v)\hat{h}_\beta(uv)^{-1}\hat{h}_\alpha(\{uv\}^{-\alpha(h_\beta)})^{-1}\\ &\quad \times C_\beta(u, \eta)^{-1}C_\beta(v, \eta)^{-1}C_\beta(uv, \eta)\\ &\quad \text{(using the formula } [\hat{h}_\alpha(u), \hat{h}_\beta(v)] = C_{\alpha\beta}(u, v) \text{ (cf. [4]))}\\ &= \hat{h}_\alpha(u^{-\alpha(h_\beta)}\hat{h}_\alpha(v^{-\alpha(h_\beta)}\hat{h}_\beta(u)\hat{h}_\beta(v)\hat{h}_\beta(uv)^{-1}\hat{h}_\alpha(\{uv\}^{-\alpha(h_\beta)})\\ &\quad \times C_\beta(u, \eta)^{-1}C_\beta(v, \eta)^{-1}C_\beta(uv, \eta)C_{\beta\alpha}(u, v^{-\alpha(h_\beta)})\\ &= C_\alpha(u^{-\alpha(h_\beta)}, v^{-\alpha(h_\beta)})C_\beta(u, v)C_\beta(u, \eta)^{-1}C_\beta(v, \eta)^{-1}C_\beta(uv, \eta)C_{\beta\alpha}(u, v^{-\alpha(h_\beta)})\\ &\quad \text{(using } C_\alpha(u^{-\alpha(h_\beta)}, v^{-\alpha(h_\beta)}) = C_{\beta\alpha}(u^{-1}, v^{-\alpha(h_\beta)}))\end{aligned}$$

$$= C_{\beta\alpha}(u^{-1}, v^{-\alpha(h_\beta)})$$

$$= C_\beta(u, v)C_\beta(u, \eta)^{-1}C_\beta(v, \eta)^{-1}C_\beta(uv, \eta)$$

$$= C_\beta(u, v\eta)C_\beta(u, \eta)^{-1}.$$

Therefore we obtain the desired result. □

Proposition 3.11. *Let* $\alpha \in \Delta_{aff}^{re}$. *Then there exist* $\eta \in \{\pm 1\}$ *and* $\alpha_i \in \Pi$ *such that* $C_\alpha(u, v) = C_{\alpha_i}(u, v\eta)C_{\alpha_i}(u, \eta)^{-1}$ *for all* $u, v \in F^*$.

Proof. We take $\alpha = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}(\alpha_i) \in W_{aff}(\{\alpha_0 \cdots \alpha_n\}) = \{w(\beta) \mid w \in W_{aff}, \beta \in \Pi_{aff}\}$.

Now put

$$\alpha = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}(\alpha_i)$$

$$\alpha(1) = \sigma_{i_2}\cdots\sigma_{i_r}(\alpha_i)$$

$$\alpha(2) = \sigma_{i_3}\cdots\sigma_{i_r}(\alpha_i)$$

$$\cdots.$$

Then using (3) above, we have

$$C_\alpha(u, v) = C_{\alpha(1)}(u, v\eta_1)C_{\alpha(1)}(u, \eta_1)^{-1}$$

$$= C_{\alpha(2)}(u, v\eta_1\eta_2)C_{\alpha(2)}(u, \eta_2)^{-1}C_{\alpha(2)}(u, \eta_2)C_{\alpha(2)}(u, \eta_1\eta_2)^{-1}$$

$$= C_{\alpha(2)}(u, v\eta_1\eta_2)C_{\alpha(2)}(u, \eta_1\eta_2)^{-1}\cdots$$

for some $\eta_1, \eta_2 \cdots \in \{\pm 1\}$.

Therefore we obtain the desired result. □

Definition 3.2. *We define* $\hat{K}_2(A, F[X, X^{-1}])$ *as a group whose generators are the symbols* $\hat{C}_{\alpha_i}(u, v)$ *for all* $u, v \in F[X, X^{-1}]$ *and* $\alpha_i \in \Pi$ *and charactarized by relations* (M1)–(M7):

(M1) $\hat{C}_{\alpha_i}(u, v)\hat{C}_{\alpha_i}(uv, w) = \hat{C}_{\alpha_i}(u, vw)\hat{C}_{\alpha_i}(v, w)$,
(M2) $\hat{C}_{\alpha_i}(u, 1) = \hat{C}_{\alpha_i}(1, v) = 1$,
(M3) $\hat{C}_{\alpha_i}(u, v) = \hat{C}_{\alpha_i}(v^{-1}, u)$,
(M4) $\hat{C}_{\alpha_i}(u, v) = \hat{C}_{\alpha_i}(u, -uv)$,
(M5) $\hat{C}_{\alpha_i}(u, v) = \hat{C}_{\alpha_i}(u, (1-u)v)$ (*if* $1 - u \in F^*$),

(M6) $\hat{C}_{\alpha_i}(u, v^{\alpha_i(h_{\alpha_j})}) = \hat{C}_{\alpha_j}(u^{\alpha_j(h_{\alpha_i})}, v)$ *denoting it by* $\hat{C}_{\alpha_i\alpha_j}(u, v)$,

(M7) $\hat{C}_{\alpha_i\alpha_j}(u, v)$ *is bimultiplicative*

for all $u, v, w \in F^*$ *and* $\alpha_i, \alpha_j \in \Pi$. *Note we can conclude that* $C_{\alpha_i}(u, v^2) = C_{\alpha_i}(u^2, v)$ *and* $C_{\alpha_i}(uv^2, w) = C_{\alpha_i}(u, w)C_{\alpha_i}(v^2, w)$ *(as in Proposition* 3.5*)*.

Proposition 3.12. *We can define the action of the Weyl group on* $\widehat{K_2}(A, F[X, X^{-1}])$ *as follows*:

$$\sigma_i(\hat{C}_\beta(u, v)) := \hat{C}_\beta(u, v\eta_{\alpha_i, \sigma_{\alpha_i}\beta})\hat{C}_\beta(u, \eta_{\alpha_i, \sigma_{\alpha_i}\beta})^{-1} \quad \textit{for all } u, v \in F^*, \sigma_i \in W \textit{ and } \beta \in \Pi.$$

Proof. Let β be a fundamental root. The statement is proved if we show:

$$\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}(\beta) = \beta \Rightarrow \hat{C}_{\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}(\beta)}(u, v) = \hat{C}_\beta(u, v).$$

Now we see

$$\begin{aligned}
&\hat{C}_{\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}(\beta)}(u, v) \\
&\quad = \hat{C}_{\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}(u, v\eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta})\hat{C}_{\sigma_{\gamma_2}, \sigma_{\gamma_r}\beta}(u, \eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}, \sigma_{\gamma_r}\beta})^{-1} \\
&\quad = \hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}(u, v\eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta})\hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}(\beta)}(u, \eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta})^{-1} \\
&\quad = \hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}(u, \eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta})^{-1}\hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}(\beta)}(u, \eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}) \\
&\quad = \hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}(u, v\eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}) \\
&\qquad \times \hat{C}_{\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}(\beta)}(u, \eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta})^{-1} \\
&\qquad \text{(continuing in this way } \cdots) \\
&\quad = \hat{C}_\beta(u, v\eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\cdots\eta_{\gamma_r, \sigma_{\gamma_r}\beta}) \\
&\qquad \times \hat{C}_\beta(u, \eta_{\gamma_1, \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2, \sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\cdots\eta_{\gamma_r, \sigma_{\gamma_r}\beta})^{-1}.
\end{aligned}$$

Note that $\eta_{\alpha,\beta} \in \{\pm 1\}$ satisfies

$$Exp(ade_\alpha)\ Exp(-ade_{-\alpha})\ Exp(ade_\alpha)e_\beta = \eta_{\alpha,\beta}e_{\sigma_\alpha\beta}.$$

If we put $w_\alpha(1) := Exp(ade_\alpha)\ Exp(-ade_{-\alpha})\ Exp(ade_\alpha)$, then we have

$$w_{\gamma_1}(1)e_{\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta} = \eta_{\gamma_1\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}e_{\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}$$

$$w_{\gamma_2}(1)e_{\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta} = \eta_{\gamma_2\sigma_{\gamma_2}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}e_{\sigma_{\gamma_3}\sigma_{\gamma_4}\cdots\sigma_{\gamma_r}\beta}$$

(continuing in this way $\cdots$)

$$w_{\gamma_r}(1)e_{\sigma_r\beta} = \eta_{\gamma_r,\sigma_r\beta}e_\beta.$$

Then we obtain

$$w_{\gamma_r}(1)w_{\gamma_2}(1)\cdots w_{\gamma_1}(1)e_{\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}$$
$$= \eta_{\gamma_1,\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2,\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\cdots\eta_{\gamma_r,\sigma_{\gamma_r}\beta}e_\beta, \quad \text{where}$$
$$\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta = \beta \quad \text{implies} \quad \sigma_{\gamma_r}\sigma_{\gamma_{r-1}}\cdots\sigma_{\gamma_1}\beta = \beta.$$

From the general theory of Kac-Moody groups, we can write $w_{\gamma_r}(1)\cdots w_{\gamma_2}(1)w_{\gamma_1}(1) = h_{\alpha_{i_1}}(-1)h_{\alpha_{i_2}}(-1)\cdots h_{\alpha_{im}}(-1)$ for some $\alpha_{i_1},\dots\alpha_{i_r}\in\Pi$. Since $h_\alpha(u)e_\beta = u^{\beta(h_\alpha)}e_\beta$, we can write

$$w_{\gamma_r}(1)w_{\gamma_2}(1)\cdots w_{\gamma_1}(1)e_\beta = h_{\alpha_{i_1}}(-1)h_{\alpha_{i_2}}(-1)\cdots h_{\alpha_{i_m}}(-1)e_\beta$$
$$= (-1)^{\beta(h_{\alpha_{i_1}})+\beta(h_{\alpha_{i_2}})+\cdots+\beta(h_{\alpha_{i_m}})}e_\beta.$$

Hence we get

$$\eta_{\gamma_1,\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2,\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\cdots\eta_{\gamma_r,\sigma_{\gamma_r}\beta} = (-1)^{\beta(h_{\alpha_{i_1}})+\beta(h_{\alpha_{i_2}})+\cdots+\beta(h_{\alpha_{i_m}})}.$$

Claim: Let α, β be fundamental roots. Then we have

$$\hat{C}_\beta(u,(-1)^{\beta(h_\alpha)}v) = \hat{C}_\beta(u,v)\hat{C}_\beta(u,(-1)^{\beta(h_\alpha)}).$$

(Proof of Claim)
In case of $\beta(h_\alpha)\in 2\boldsymbol{Z}$, there is nothing to show.
In case of $\beta(h_\alpha)\in 2\boldsymbol{Z}+1$, we have

$$\hat{C}_\beta(u,(-1)^{\beta(h_\alpha)}v) = \hat{C}_\beta(u,(-v)^{\beta(h_\alpha)}v^{1-\beta(h_\alpha)}) = \hat{C}_\beta(u,(-v)^{\beta(h_\alpha)})\hat{C}_\beta(u,v^{1-\beta(h_\alpha)})$$
$$= \hat{C}_\beta(u,(-1)^{\beta(h_\alpha)})\hat{C}_\beta(u,v^{\beta(h_\alpha)})\hat{C}_\beta(u,v^{1-\beta(h_\alpha)})$$
$$= \hat{C}_\beta(u,v)\hat{C}_\beta(u,(-1)^{\beta(h_\alpha)}). \qquad \square$$

From the above claim, we get

$$\hat{C}_\beta(u,v\eta_{\gamma_1,\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2,\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\cdots\eta_{\gamma_r,\sigma_{\gamma_r}\beta})$$
$$\hat{C}_\beta(u,\eta_{\gamma_1,\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_2,\sigma_{\gamma_2}\sigma_{\gamma_3}\cdots\sigma_{\gamma_r}\beta}\eta_{\gamma_r,\sigma_{\gamma_r}\beta})^{-1} = \hat{C}_\beta(u,v).$$

Therefore we obtain the desired result. $\square$

Definition 3.3. *For any* $\alpha \in \Delta$, *there exist* $w \in W$ *and* $i \in \{1, \dots n\}$ *such that* $\alpha = w(\alpha_i)$, *and for all* $u, v \in F[X, X^{-1}]$, *we define the element* $\hat{C}_\alpha(u, v)$ *as follows*:

$$\hat{C}_\alpha(u, v) := w(\hat{C}_{\alpha_i}(u, v)).$$

By Proposition 3.12, *the above is well-defined.*

3.2.3. A Subgroup of $St(A^{aff}, F)$ Which is Isomorphic to $K_2(A, F[X, X^{-1}])$

Viewing the commutative diagram in Fig 2, we have $St(A, F[X, X^{-1}]) \supset K_2(A, F[X, X^{-1}]) \simeq \langle K_2(A^{aff}, F), Ker\, \Psi \rangle$.

Proposition 3.13. *If* $h_{\alpha_0}(t_0) h_{\alpha_1}(t_1) \cdots h_{\alpha_l}(t_l) \in Ker\, \Psi$ *for some* $t_i \in F^*$, *then we have*

$$\hat{h}_{\alpha_0}(t_0) \hat{h}_{\alpha_1}(t_1) \cdots \hat{h}_{\alpha_l}(t_l) \equiv \hat{h}_{\alpha_0}(t_0) \hat{h}_\theta(t_0) \equiv \hat{h}_{\alpha_0}(t_0) \hat{h}_{-\theta}(t_0)^{-1} \quad mod\ K_2(A^{aff}, F).$$

Proof. We remark that if $\alpha \in \Delta^{re}$, then $\hat{h}_\alpha(t) \hat{h}_\alpha(t^{-1}) \in K_2(A^{aff}, F)$. By Proposition 3.9 we see $\hat{h}_{-\theta}(t) \equiv \hat{h}_\theta(t^{-2}) \hat{h}_\theta(t) \equiv \hat{h}_\theta(t^{-1}) \equiv \hat{h}_\theta(t)^{-1}\ mod\ K_2(A^{aff}, F)$.

From the fact $h_\theta(t_0) = h_{\alpha_1}(t_1) \cdots h_{\alpha_l}(t_l) \in G_{sc}(A, F[X, X^{-1}])$ (as in Proposition 3.7), it is easy to see the desired result. □

Hence we have

$$\langle K_2(A^{aff}, F), Ker\, \Psi \rangle = \langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t) \hat{h}_\theta(t) \mid t \in F^* \rangle = K_2(A, F[X, X^{-1}]).$$

In Fig 2, due to Φ, the correspondance f between the subgroup $\langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t) \hat{h}_\theta(t) \rangle$ and the subgroup $K_2(A, F[X, X^{-1}])$ is given as follows:

$$\begin{aligned} f : \langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t) \hat{h}_\theta(t) \rangle &\to K_2(A, F[X, X^{-1}]) \\ C_{\alpha_0}(u, v) &\mapsto C_{-\theta}(u, X)^{-1} C_{-\theta}(u, vX) \\ C_{\alpha_i}(u, v) &\mapsto C_{\alpha_i}(u, v) \quad (i \neq 0) \\ \hat{h}_{\alpha_0}(t) \hat{h}_{-\theta}^{-1}(t) &\mapsto C_{-\theta}(t, X)^{-1} \quad \text{for all } u, v, t \in F^*. \end{aligned} \tag{4}$$

From Definition 3.3, we can realize $\hat{C}_{-\theta}(u, v)$ with $u, v \in F^*$ in $\hat{K}_2(A, F[X, X^{-1}])$.

3.2.4. About $\langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t) \hat{h}_\theta(t) \rangle$

Lemma 3.8. *Notation is as above. Then we have* $\hat{h}_{\alpha_0}(s) \hat{h}_{-\theta}(s)^{-1} \hat{h}_{\alpha_0}(t) \hat{h}_{-\theta}(t)^{-1} = C_{\alpha_0}(s, t^{-1}) C_{-\theta}(t, s)^{-1} \hat{h}_{\alpha_0}(st) \hat{h}_{-\theta}(st)^{-1}$.

Proof. We see:

$$\begin{aligned}\hat{h}_{\alpha_0}(s)\hat{h}_{-\theta}(s)^{-1}h_{\alpha_0}(t)\hat{h}_{-\theta}(t)^{-1} &= \hat{h}_{\alpha_0}(s)\hat{h}_{\alpha_0}(t)\hat{h}_{-\theta}(s)^{-1}\hat{h}_{-\theta}(t)^{-1}C_{\alpha_0-\theta}(t,s)\\ &= C_{\alpha_0}(s,t)C_{-\theta}(t,s)^{-1}C_{\alpha_0-\theta}(t,s)\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}(st)^{-1}\\ &= C_{\alpha_0}(s,t)C_{-\theta}(t,s)^{-1}C_{\alpha_0}(t,s^2)\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}(st)^{-1}\\ &= C_{\alpha_0}(s,t^{-1})C_{-\theta}(t,s)^{-1}\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}(st)^{-1}.\end{aligned}$$

Hence we obtain the desired result. □

From Lemma 3.8, the subgroup $\langle K_2(A^{aff},F),\hat{h}_{\alpha_0}(t)\hat{h}_{\theta}(t) \mid t\in F^*\rangle$ is charactalized by the following two conditions:

(1) The Generators and relations of $K_2(A^{aff},F)$,

(2) $\hat{h}_{\alpha_0}(s)\hat{h}_{-\theta}^{-1}(s)\hat{h}_{\alpha_0}(t)\hat{h}_{-\theta}^{-1}(t) = C_{\alpha_0}(s,t^{-1})C_{-\theta}(t,s)^{-1}\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}^{-1}(st)$ for all $s,t\in F^*$.

Now we define g as follows:

$$\begin{aligned}g : \hat{K}_2(A,F[X,X^{-1}]) &\to K_2(A,F[X,X^{-1}])\\ \hat{C}_{\alpha_i}(u,v) &\mapsto C_{\alpha_i}(u,v)\quad (i\neq 0).\end{aligned}$$

Then g is well-defined (cf. [2] [7] [10]).

By the construction of $\hat{C}_{\alpha}(u,v)$ (see Proposition 3.10 and Definition 3.3) for each real root α, we have

$$g(\hat{C}_{\alpha}(u,v)) = \hat{h}_{\alpha}(u)\hat{h}_{\alpha}(v)\hat{h}_{\alpha}(uv)^{-1}.$$

By (4), we define H as follows:

$$\begin{aligned}H : \langle K_2(A^{aff},F),h_{\alpha_0}(t)h_{\theta}(t)\rangle &\to \hat{K}_2(A,F[X,X^{-1}])\\ C_{\alpha_0}(u,v) &\mapsto \hat{C}_{-\theta}(u,X)^{-1}\hat{C}_{-\theta}(u,vX)\\ C_{\alpha_i}(u,v) &\mapsto \hat{C}_{\alpha_i}(u,v)\\ \hat{h}_{\alpha_0}(t)\hat{h}_{-\theta}^{-1}(t) &\mapsto \hat{C}_{-\theta}(t,X)^{-1}\quad \text{for all } u,v,t\in F^*.\end{aligned}$$

If H is well-defined, then Fig B is commutative. Hence g is bijective, therefore we get $\hat{K}_2(A,F[X,X^{-1}]) \simeq K_2(A,F[X,X^{-1}])$. Thus our purpose is to prove that H is well-defined.

$$\begin{array}{ccc} \langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t)\hat{h}_{\theta}(t)\rangle & \longrightarrow & \widehat{K_2}(A, F[X, X^{-1}]) \\ \uparrow & \swarrow & \\ K_2(A, F[X, X^{-1}]) & & \end{array}$$

Figure B

3.2.5. Determine the Presentation of $K_2(A, F[X, X^{-1}])$

We can see that $\langle K_2(A^{aff}, F), \hat{h}_{\alpha_0}(t)\hat{h}_{\theta}(t) \mid t \in F^* \rangle$ is a central extension of F^* by $K_2(A^{aff}, F)$.

To prove our H is well-defined, it is sufficient that we show the following ♣ and ♠.

$$\clubsuit \ H \text{ is well-defined when we restrict it to } K_2(A^{aff}, F). \tag{5}$$

$$\spadesuit \ H(\hat{h}_{\alpha_0}(s)\hat{h}_{-\theta}(s)^{-1} H(\hat{h}_{\alpha_0}(t)\hat{h}_{-\theta}(t)^{-1}$$
$$= H(C_{\alpha_0}(t,s)C_{-\theta}(t,s)^{-1}\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}(st)^{-1} \quad \text{for all } s, t \in F^*. \tag{6}$$

About $\clubsuit \cdots (5)$

For $C_{\alpha_0}(u,v)$ and $C_{\alpha_i}(u,v)$ $(i \neq 0)$ for all $u, v \in F^*$, it is easy to show the consitions (M1)–(M5). And if $i, j \neq 0$, Then (M6) and (M7) are preserved by H.

Now we see

$$C_{\alpha_0\alpha_i}(u,v) = C_{\alpha_0}(u, v^{\alpha_0(h_i)}) = C_{\alpha_i}(u^{\alpha_i(h_{\alpha_0})}, v) \quad \text{and}$$
$$C_{\alpha_i\alpha_0}(u,v) = C_{\alpha_i}(u, v^{\alpha_i(h_{\alpha_0})}) = C_{\alpha_0}(u^{\alpha_0(h_i)}, v).$$

Hence it is sufficient that we show the following three statements ♭1, ♭2 and ♮.

$$H(C_{\alpha_0}(u, v^{\alpha_0(h_i)})) = H(C_{\alpha_i}(u^{\alpha_i(h_{\alpha_0})}, v)). \ \cdots \flat 1 \tag{7}$$
$$H(C_{\alpha_i}(u, v^{\alpha_i(h_{\alpha_0})})) = H(C_{\alpha_0}(u^{\alpha_0(h_i)}, v)). \ \cdots \flat 2 \tag{8}$$
$$C_{\alpha_i\alpha_0}(u,v) \text{ is bimultiplicative in } u, v \quad \text{for all } u, v \in F^*. \ \cdots \natural \tag{9}$$

Now we note $\alpha_0(h_i) = -\theta(h_i)$ and $\alpha_i(h_{\alpha_0}) = -\alpha_i(h_\theta)$.

About $\flat 1 \cdots (7)$ and $\flat 2 \cdots (8)$

We only prove $\flat 1 \cdots (7)$ because the proof of $\flat 2 \cdots (8)$ is similar. If $i = 0$, there is nothing to show, so we suppose that $i \neq 0$. Now it is sufficient to prove the following formula:

$$\hat{C}_{-\theta}(u, X)^{-1}\hat{C}_{-\theta}(u, v^{-\theta(h_i)}X) = \hat{C}_{\alpha_i}(u^{-\alpha_i(h_\theta)}, v) \quad \text{for all } u, v \in F^*, \ i \neq 0.$$

To prove this formula we use the so-called Extended Dynkin Diagrams in Fig C.

By Proposition 3.12 and by the proof of Proposition 3.11, we let $\alpha_k \in \Pi$ be some long root. We have the following.

$$\hat{C}_{-\theta}(u, X)^{-1}\hat{C}_{-\theta}(u, v^{-\theta(h_i)}X) = \hat{C}_{\alpha_k}(u, X\eta)^{-1}\hat{C}_{\alpha_k}(u, v^{-\theta(h_i)}X\eta) \quad \text{for } \eta \in \{\pm 1\}.$$

Therefore it is sufficient that we prove the following formula.

$$\begin{aligned}&\hat{C}_{\alpha_k}(u, X\eta)^{-1}\hat{C}_{\alpha_k}(u, v^{-\theta(h_i)}X\eta)\\ &\quad = \hat{C}_{\alpha_i}(u^{-\alpha_i(h_\theta)}, v) \quad \text{for all } u, v \in F^*,\ i \neq 0 \text{ and } \eta = \pm 1.\end{aligned} \tag{10}$$

In the case of $(\alpha_i, \theta) = 0$, our result is trivial, Hence we consider the case of $(\alpha_i, \theta) \neq 0$.

Lemma 3.9. *If $\alpha_i(h_j) = \alpha_j(h_i) = -1$ for some $\alpha_i, \alpha_j \in \Pi$ (or Π_{aff}), then $\hat{C}_{\alpha_i}(u, v) = \hat{C}_{\alpha_j}(u, v)$ for all $u, v \in F^*$ and it is bimultiplicative in u, v. Furthermore for any $\alpha \in W(\alpha_i)$, we have $\hat{C}_{\alpha}(u, v) = \hat{C}_{\alpha_i}(u, v)$, and it is bimultiplicative in u, v.*

Proof. The first statement is trivial by the fact $\hat{C}_{\alpha_i\alpha_j}(u, v) = \hat{C}_{\alpha_i}(u, v^{-1}) = \hat{C}_{\alpha_j}(u^{-1}, v)$. Also obviously we have that it is bimultiplicative in u, v.

Now if $\hat{C}_{\beta}(u, v)$ is bimultiplicative in u, v, then

$$\hat{C}_{\sigma_\alpha\beta}(u, v) = \hat{C}_{\beta}(u, v\eta)\hat{C}_{\beta}(u, \eta)^{-1} = \hat{C}_{\beta}(u, v).$$

Hence we have that $\hat{C}_{\sigma_\alpha\beta}(u, v)$ is also bimultiplicative. The statement of the second part is also true. Hence we obtain the desired result. □

We prove (10) case by case.

(1) The case of $A^{(1)}_{n\geq 2}$, $D^{(1)}_n$, $E^{(1)}_{6,7,8}$.

If $(\alpha_i, \theta) \neq 0$ then the left hand of (10) can be written as follows:

$$\begin{aligned}\hat{C}_{\alpha_k}(u, X\eta)^{-1}\hat{C}_{\alpha_k}(u, v^{-\theta(h_i)}X\eta) &= \hat{C}_{\alpha_k}(u, X\eta)^{-1}\hat{C}_{\alpha_k}(u, v^{-1}X\eta)\\ &= \hat{C}_{\alpha_k}(u, v^{-1}) \quad \text{for all } u, v \in F^* \text{ (by Lemma 3.9).}\end{aligned}$$

And the right hand is $\hat{C}_{\alpha_i}(u^{-\alpha_i(h_\theta)}, v) = \hat{C}_{\alpha_i}(u^{-1}, v)$ for all $u, v \in F^*$.

Hence by Lemma 3.9, in the case of $A^{(1)}_{n\geq 2}$, $D^{(1)}_n$, $E^{(1)}_{6,7,8}$, the statement ♭1 holds.

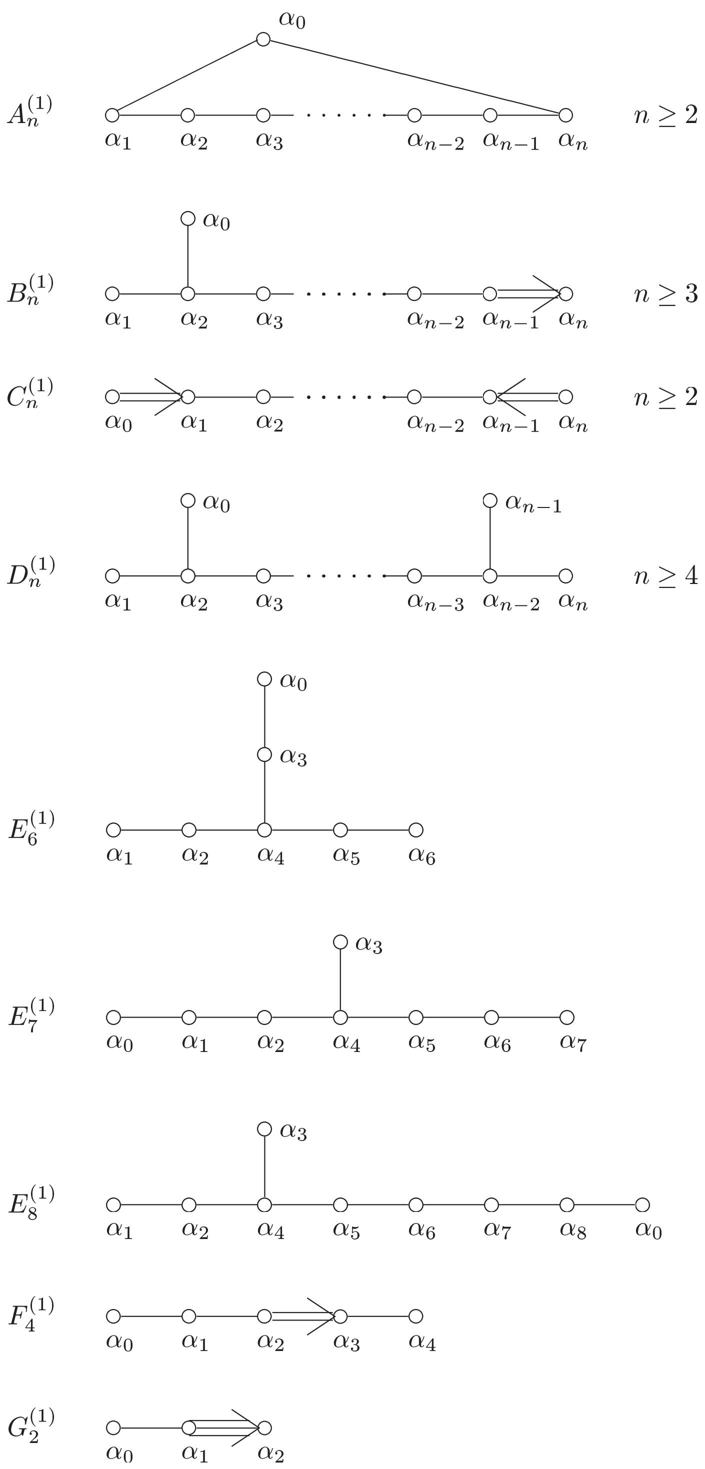

Figure C Extended Dynkin Diagram

(2) The case of $B_n^{(1)}$.

By Extended Dynkin diagrams in Fig. C we should only consider the case of $i=2$. Then we have $-\theta(h_{\alpha_2})=-1$ and $-\alpha_2(h_\theta)=-1$.

Now we suppose α_k is a long root. Then by the Extended Dynkin Diagram of $B_n^{(1)}$ and by the left hand side of (10), we have the following:

$$\hat{C}_{\alpha_k}(u,X\eta)^{-1}\hat{C}_{\alpha_k}(u,v^{-1}X\eta)=\hat{C}_{\alpha_k}(u,v^{-1}).$$

Then the right hand side is $\hat{C}_{\alpha_2}(u^{-\alpha_2(h_\theta)},v)=\hat{C}_{\alpha_2}(u^{-1},v)$ for all $u,v\in F^*$.

Because of α_2 being a long root, α_2 and α_k are transitive by some element of the Weyl group. Hence in the case of $B_n^{(1)}$, the statement $\flat 1(\cdots(7))$ holds.

(3) The case of $C_n^{(1)}$.

We should only consider the case of $i=1$. Then $-\theta(h_1)=-2$ and $-\alpha_1(h_\theta)=-1$.

Let α_k be a fundamental long root. Then we have $\alpha_k=\alpha_n$. Then the left hand side of (10) can be computed as follows:

$$\begin{aligned}\hat{C}_{\alpha_k}(u,X\eta)^{-1}\hat{C}_{\alpha_k}(u,v^{-2}X\eta)&=\hat{C}_{\alpha_n}(u,X\eta)^{-1}\hat{C}_{\alpha_n}(u,v^{-2}X\eta)\\&=\hat{C}_{\alpha_n}(u,v^{-2})\quad\text{for all } u,v\in F^* \text{ and } \eta\in\{\pm1\}.\end{aligned}$$

Hence it is sufficient that we show $\hat{C}_{\alpha_1}(u^{-\alpha_1(h_\theta)},v)=\hat{C}_{\alpha_1}(u^{-1},v)=\hat{C}_{\alpha_n}(u,v^{-2})$.

We note $\hat{C}_{\alpha_n\alpha_{n-1}}(u,v)=\hat{C}_{\alpha_n}(u,v^{-2})=\hat{C}_{\alpha_{n-1}}(u^{-1},v)$.

Also we see that α_1 and α_{n-1} are transitive by some element of the Weyl group. Therefore we have

$$\hat{C}_{\alpha_1}(u^{-1},v)=\hat{C}_{\alpha_n}(u,v^{-2})\quad\text{for all } u,v\in F^*,\ \eta\in\{\pm1\}.$$

Hence in the case of $C_n^{(1)}$, the statement $\flat 1(\cdots(7))$ holds.

(4) The case of $F_4^{(1)}$.

We should only consider the case of $i=1$. Then we have $-\theta(h_1)=-1$ and $-\alpha_1(h_\theta)=-1$. Let $\alpha_k\in\Pi$ be a long root. Then by the Extended Dynkin Diagram of $F_4^{(1)}$ and Lemma 3.9 , the left hand side of (10) can be calculated as follows:

$$\hat{C}_{\alpha_k}(u,X\eta)^{-1}\hat{C}_{\alpha_k}(u,v^{-1}X\eta)=\hat{C}_{\alpha_k}(u,v^{-1})=\hat{C}_{\alpha_1}(u,v^{-1}).$$

Also we see that the right hand side becomes $\hat{C}_{\alpha_1}(u^{-\alpha_1(h_\theta)},v)=\hat{C}_{\alpha_1}(u^{-1},v)=\hat{C}_{\alpha_1}(u,v^{-1})=\hat{C}_{\alpha_k}(u,v^{-1})$ for all $u,v\in F^*$. Hence in the case of the $F_4^{(1)}$, the statement $\flat 1$ holds.

(5) The case of $G_2^{(1)}$.

We should only consider the case of $i=1$. Then $-\theta(h_1)=-1$ and $-\alpha_1(h_\theta)=-1$.

Lemma 3.10. *In the case of the* $G_2^{(1)}$ *(see Fig C),* α_1 *is a fundamental long root, and* $\hat{C}_{\alpha_1}(u,v)$ *is bimultiplicative in* u, v *for all* $u, v \in F^*$.

Proof. We note $\hat{C}_{\alpha_1\alpha_2}(u,v) = \hat{C}_{\alpha_1}(u,v^3)$. Then we obtain

$$\hat{C}_{\alpha_1}(u,v) = \hat{C}_{\alpha_1\alpha_2}(u,v)\hat{C}_{\alpha_1}(u,v^2)^{-1}.$$

Also we see $\hat{C}_{\alpha_1\alpha_2}(u,v)$ and $\hat{C}_{\alpha_1}(u,v^2)^{-1}$ are bimultiplicative in u, v.

Therefore $\hat{C}_{\alpha_1}(u,v)$ is bimultiplicative in u, v. Hence we obtain the desired result. □

Now α_1 is the only element which is long and belongs to the set of fundamental roots. Therefore it is sufficient that we show the following:

$$\hat{C}_{\alpha_1}(u,X\eta)^{-1}\hat{C}_{\alpha_1}(u,v^{-1}X\eta) = \hat{C}_{\alpha_1}(u,v^{-1}) \quad \text{for all } u,v \in F^* \text{ and } \eta \in \{\pm 1\},$$

which is trivial by the Lemma 3.10.

Hence we obtain that the statement $\flat 1(\cdots(7))$ is true.

About $\natural(\cdots(9))$.

To prove $\natural$, it is sufficient that $\hat{C}_{\alpha_i}(u^{-\alpha_i(h_\theta)},v)$ is bimultiplicative in u, v for any i.

In the case of $(\alpha_i,\theta) = 0$, it is trivial. Therefore we consider the case of $(\alpha_i,\theta) \neq 0$. More explicitely, we have the following:

$$\begin{aligned}
&A_n^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_1}(u^{-1},v)\hat{C}_{\alpha_n}(u^{-1},v),\\
&B_n^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_2}(u^{-1},v),\\
&C_n^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_1}(u^{-1},v),\\
&D_n^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_2}(u^{-1},v),\\
&E_6^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_3}(u^{-1},v),\\
&E_7^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_1}(u^{-1},v),\\
&E_8^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_8}(u^{-1},v),\\
&F_4^1 \text{ type } \cdots\cdots \hat{C}_{\alpha_1}(u^{-1},v) \quad \text{for all } u,v \in F^*.
\end{aligned}$$

Using Extended Dynkin diagrams, these can be written as $\hat{C}_{\alpha_i\alpha_j}(u^{\pm 1},v)$ for some $\alpha_i, \alpha_j \in \Pi$.

Therefore we obtain that both statements $\flat 1(\cdots(7))$ and $\natural(\cdots(9))$ are true. Hence the statement $\clubsuit(\cdots(5))$ is true.

About $\spadesuit$.

We prove $H(\hat{h}_{\alpha_0}(s)\hat{h}_{-\theta}(s)^{-1}H(\hat{h}_{\alpha_0}(t)\hat{h}_{-\theta}(t)^{-1} = H(\hat{h}_{\alpha_0}(st)\hat{h}_{-\theta}(st)^{-1}C_{\alpha_0}(t,s)\cdot C_{-\theta}(t,s)^{-1}$ for all $s, t \in F^*$. We have the following:

$$\text{The left hand side} = \hat{C}_{-\theta}(s,X)^{-1}\hat{C}_{-\theta}(t,X)^{-1} \quad \text{for all } s,t \in F^*.$$

$$\begin{aligned}\text{The right hand side} &= \hat{C}_{-\theta}(st,X)^{-1}\hat{C}_{-\theta}(t,X)^{-1}\hat{C}_{-\theta}(t,sX)\hat{C}_{-\theta}(t,s)^{-1} \\ &= \hat{C}_{-\theta}(t,sX)^{-1}\hat{C}_{-\theta}(s,X)^{-1}\hat{C}_{-\theta}(t,X)^{-1}\hat{C}_{-\theta}(t,sX) \\ &= \hat{C}_{-\theta}(t,X)^{-1}\hat{C}_{-\theta}(s,X)^{-1} \quad \text{for all } s,t \in F^*.\end{aligned}$$

Hence we get the statement $\spadesuit$. Therefore we obtain the following theorem.

Theorem 3.2. *Notation is as above. Then we have* $\hat{K}_2(A, F[X, X^{-1}]) \simeq K_2(A, F[X, X^{-1}])$.

4. Applications

4.1. Motivations

First we note that the following theorem about $K_2(A, F[X, X^{-1}])$ is known.

Theorem 4.1 [3] [8]. *If* $A \neq C_n$ $(1 \leq n)$, *then* $K_2(A, F[X, X^{-1}]) \simeq K_2(F) \oplus F^*$. *If* $A = C_n$ $(1 \leq n)$, *then* $K_2(C_n, F[X, X^{-1}]) \simeq K_2Sp(F) \oplus P(F)$ *with the exact sequence*:

$$1 \to I^2(F) \to P(F) \to F^* \to 1,$$

where $I(F)$ *is the fundamental ideal of the Witt-ring* $W(F)$.

In the previous chapter we found the generators and the relations of $K_2(A, F[X, X^{-1}])$. In this chapter we will see how to split the elements in $K_2(A, F[X, X^{-1}])$ into the elemtnts in $K_2(F) \oplus F^*$ or the elements $K_2Sp(F) \oplus P(F)$. Using this, we will give the generators and the relations of $P(F)$ and $I^2(F)$.

4.2. Case of $A = C_n$ $(1 \leq n)$

Lemma 4.1. *Notation is as above. Then we have* $K_2(C_n, F[X, X^{-1}]) \simeq K_2(A_1, F[X, X^{-1}])$.

Proof. Note that $K_2(C_n, F[X, X^{-1}]) = \langle C_{\alpha_i}(uX^m, vX^n) \mid i = 1, \dots n \;\; m, n \in \boldsymbol{Z} \;\; u, v \in F^* \rangle$.

Considering the action of the Weyl group, we have

$$K_2(C_n, F[X, X^{-1}]) = \langle C_{\alpha_1}(uX^m, vX^n) \mid m, n \in \boldsymbol{Z} \;\; u, v \in F^* \rangle \simeq K_2(A_1, F[X, X^{-1}]).$$

□

By Lemma 4.1, we can consider $K_2(A_1, F[X, X^{-1}])$ instead of $K_2(C_n, F[X, X^{-1}])$.

Now we will split the generators $C_{\alpha_1}(uX^m, vX^n)$ for all $u, v \in F^*$ and $m, n \in \boldsymbol{Z}$ of $K_2(A_1, F[X, X^{-1}])$.

Since

$$C_{\alpha_1}(u, X^m)C_{\alpha_1}(uX^m, vX^n) = C_{\alpha_1}(u, vX^{m+n})C_{\alpha_1}(X^m, vX^n)$$

and

$$C_{\alpha_1}(u, v)C_{\alpha_1}(uv, X^{m+n}) = C_{\alpha_1}(u, vX^{m+n})C_{\alpha_1}(v, X^{m+n}),$$

we have

$$\begin{aligned}
&C_{\alpha_1}(uX^m, vX^n) \\
&\quad = C_{\alpha_1}(u, X^m)^{-1}C_{\alpha_1}(u, vX^{m+n})C_{\alpha_1}(X^m, vX^n) \\
&\quad = C_{\alpha_1}(u, v)C_{\alpha_1}(uv, X^{m+n})C_{\alpha_1}(v, X^{m+n})^{-1}C_{\alpha_1}(u, X^m)^{-1}C_{\alpha_1}(X^m, vX^n) \\
&\quad = C_{\alpha_1}(u, v)C_{\alpha_1}(u^{m+n}v^{m+n}, X)C_{\alpha_1}(v^{m+n}, X)^{-1}C_{\alpha_1}(u^m, X)^{-1}C_{\alpha_1}(X, v^mX^{mn}) \\
&\quad = C_{\alpha_1}(u, v)C_{\alpha_1}(u^{m+n}v^{m+n}, X)C_{\alpha_1}(v^{m+n}, X)^{-1}C_{\alpha_1}(u^m, X)^{-1}C_{\alpha_1}((-1)^{mn}v^{-m}, X) \\
&\qquad \text{for all } u, v, w \in F^* \text{ and } m, n \in \boldsymbol{Z}. \;\; \cdots\cdots\clubsuit \qquad (11)
\end{aligned}$$

Now we simplify the equation (11) case by case.

(1) The case of $(m, n) \equiv (0, 0)$ *mod* 2.
We have

$$\begin{aligned}
&\clubsuit(\cdots(11)) \\
&\quad = C_{\alpha_1}(u, v)C_{\alpha_1}(u^{m+n}v^{m+n}, X)C_{\alpha_1}(v^{m+n}, X)^{-1}C_{\alpha_1}(u^m, X)^{-1}C_{\alpha_1}(v^{-m}, X) \\
&\quad = C_{\alpha_1}(u, v)C_{\alpha_1}(u^nv^{-m}, X) \quad \text{for all } u, v \in F^*.
\end{aligned}$$

(2) The case of $(m, n) \equiv (1, 0)$ *mod* 2.
We have

♣$(\cdots(11))$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n}v^{m+n},X)C_{\alpha_1}(v^{m+n},X)^{-1}C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(v^{-m},X)$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n-1}v^{m+n-1},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v^{m+n-1},X)^{-1}$$

$$\times C_{\alpha_1}(v,X)^{-1}C_{\alpha_1}(u^{m-1},X)^{-1}C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(v^{-m},X)$$

(note that $m+n-1, m-1 \in 2\boldsymbol{Z}$)

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^n,X)C_{\alpha_1}(v^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(v,X)^{-1}$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^nv^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(v,X)^{-1} \quad \text{for all } u,v \in F^*.$$

(3) The case of $(m,n) \equiv (0,1)$ *mod* 2.
We have

♣$(\cdots(11))$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n}v^{m+n},X)C_{\alpha_1}(v^{m+n},X)^{-1}C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(v^{-m},X)$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n-1}v^{m+n-1},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v^{m+n-1},X)^{-1}$$

$$\times C_{\alpha_1}(v,X)^{-1}C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(v^{-m},X)$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n-1},X)C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(v^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{n-1},X)C_{\alpha_1}(v^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^n,X)C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(v^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^nv^{-m},X)C_{\alpha_1}(uv,X)C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(v,X)^{-1} \quad \text{for all } u,v \in F^*.$$

(4) The case of $(m,n) \equiv (1,1)$ *mod* 2.
We have

♣$(\cdots(11))$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n}v^{m+n},X)C_{\alpha_1}(v^{m+n},X)^{-1}C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(-v^{-m},X)$$

$$= C_{\alpha_1}(u,v)C_{\alpha_1}(u^{m+n},X)C_{\alpha_1}(u^m,X)^{-1}C_{\alpha_1}(-v^{-m},X)$$

(note that $C_{\alpha_1}(u^{m+n},X)C_{\alpha_1}(u^m,X)^{-1}$

$$= C_{\alpha_1}(u^{m+n-1},X)C_{\alpha_1}(u^{-1},X)^{-1}C_{\alpha_1}(u^m,X)^{-1}$$

$$= C_{\alpha_1}(u^{n-1}, X)C_{\alpha_1}(u^{-1}, X)^{-1})$$

$$= C_{\alpha_1}(u, v)C_{\alpha_1}(u^{n-1}, X)C_{\alpha_1}(-v^{-m}, X)C_{\alpha_1}(u^{-1}, X)^{-1}$$

$$= C_{\alpha_1}(u, v)C_{\alpha_1}(-u^{n-1}v^{-m}, X)C_{\alpha_1}(u^{-1}, X)^{-1}$$

$$= C_{\alpha_1}(u, v)C_{\alpha_1}(u^{n-1}v^{-m-1}, X)C_{\alpha_1}(-v, X)C_{\alpha_1}(u^{-1}, X)^{-1}$$

$$= C_{\alpha_1}(u, v)C_{\alpha_1}(-u^{n}v^{-m}, X)C_{\alpha_1}(-uv, X)^{-1}$$

$$\times C_{\alpha_1}(-v, X)C_{\alpha_1}(u^{-1}, X)^{-1} \quad \text{for all } u, v \in F^*.$$

Lemma 4.2. *Notation is as above. Then we have* $C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(-v, X) \cdot C_{\alpha_1}(u^{-1}, X)^{-1} = C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(u, X)^{-1}$ *for all* $u, v \in F^*$.

Proof. Since

$$C_{\alpha_1}(-v, X) = C_{\alpha_1}(v^{-1}, X)^{-1}C_{\alpha_1}(-1, X)AC_{\alpha_1}(u^{-1}, X)^{-1} = C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(u^2, X),$$

we have

$$C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(-v, X)C_{\alpha_1}(u^{-1}, X)^{-1}$$

$$= C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(v^{-1}, X)^{-1}C_{\alpha_1}(-1, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(u^2, X)$$

$$= C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(v^2, X)C_{\alpha_1}(-1, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(u^2, X)$$

$$= C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(-1, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(v, X)^{-1}$$

$$= C_{\alpha_1}(u^{-1}v^{-1}, X)C_{\alpha_1}(-1, X)^{-1}C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(-1, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(v, X)^{-1}$$

$$= C_{\alpha_1}(uv, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(v, X)^{-1}.$$

Hence we obtain the desired result. □

Using Lemma 4.2, we have (11)= $C_{\alpha_1}(u, v)C_{\alpha_1}(-u^n v^{-m}, X)$ for all $u, v \in F^*$, in the case of $(m, n) \equiv (1, 1)$ *mod* 2.

Hence we obtain the following proposition.

Proposition 4.1. *In the case of* $(m, n) \equiv (0, 1), (1, 0), (1, 1)$ *mod* 2, *we have*

$$C_{\alpha_1}(uX^m, vX^n)$$

$$= C_{\alpha_1}(u, v)C_{\alpha_1}((-1)^{mn}u^n v^{-m}, X)C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(u, X)^{-1}$$

for all $u, v \in F^*$.

In the case of $(m,n) \equiv (0,0)$ *mod* 2, *we have*

$$C_{\alpha_1}(uX^m, vX^n) = C_{\alpha_1}(u,v)C_{\alpha_1}(u^n v^{-m}, X) \quad \textit{for all } u,v \in F^*.$$

Now we put $\boldsymbol{S} := \langle C_{\alpha_1}(u,v) \mid u,v \in F^* \rangle$ and $\boldsymbol{M} := \langle C_{\alpha_1}(u,X) \mid u \in F^* \rangle$. We define two group homomorphisms $\Psi_{\boldsymbol{S}} : K_2(A_1, F[X,X^{-1}]) \to \boldsymbol{S}$ by $\Psi_{\boldsymbol{S}}(C_{\alpha_1}(uX^m, vX^n)) = C_{\alpha_1}(u,v)$ and $\Psi_{\boldsymbol{M}} : K_2(A_1, F[X,X^{-1}]) \to \boldsymbol{M}$ by $\Psi_{\boldsymbol{M}}(C_{\alpha_1}(uX^m, vX^n)) = C_{\alpha_1}((-1)^{mn}u^n v^{-m}, X)C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}C_{\alpha_1}(u,X)^{-1}$ in the case of $(m,n) \equiv (0,1),(1,0),(1,1)$ *mod* 2 and by $\Psi_{\boldsymbol{M}}(C_{\alpha_1}(uX^m, vX^n)) = C_{\alpha_1}(u^n v^{-m}, X)$ in the case of $(m,n) \equiv (0,0)$ *mod* 2.

Note that the group homomorphisms $\Psi_{\boldsymbol{S}}$ and $\Psi_{\boldsymbol{M}}$ above are well-defined.

Proposition 4.2. *Notation is as above. Let* $\Psi_{\boldsymbol{S}} \oplus \Psi_{\boldsymbol{M}} : K_2(A_1, F[X,X^{-1}]) \to \boldsymbol{S} \oplus \boldsymbol{M}$ *be a group homomorphism with* $\Psi_{\boldsymbol{S}} \oplus \Psi_{\boldsymbol{M}}(C_{\alpha_1}(uX^m, vX^n)) = (\Psi_{\boldsymbol{S}}(C_{\alpha_1}(uX^m, vX^n)), \Psi_{\boldsymbol{M}}(C_{\alpha_1}(uX^m, vX^n)))$ *for all* $u,v \in F^*$ *and* $m,n \in \boldsymbol{Z}$. *Then* $\Psi_{\boldsymbol{S}} \oplus \Psi_{\boldsymbol{M}}$ *is an isomorphism.*

Proof. Using Proposition 4.1, this is easily shown. □

Next we consider the group presentation of $\boldsymbol{S}$ and $\boldsymbol{M}$.

About $\boldsymbol{S}$

There is a natural one to one correspondance between $\boldsymbol{S}$ and $K_2(A_1, F)$, hence we have $\boldsymbol{S} \simeq K_2(A_1, F)$.

About $\boldsymbol{M}$

We put $C(u) := C_{\alpha_1}(u,X)$ for all $u \in F^*$ (these are the generators of $\boldsymbol{M}$).

We put $\langle u,v \rangle := C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}C_{\alpha_1}(u,X)^{-1}$ for all $u,v \in F^*$.

Lemma 4.3. *Notation is as above. Then for all* $u,v \in F^*$, $\langle u,v \rangle$ *satisfies the relation* (M1)–(M5).

Proof. First we prove that $\langle u,v \rangle$ satisfies (M1).

(M1): We have

$$\begin{aligned}
&\langle u,v \rangle \langle uv,w \rangle \\
&\quad = C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1}C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(uvw,X)C_{\alpha_1}(w,X)^{-1}C_{\alpha_1}(uv,X)^{-1} \\
&\quad = C_{\alpha_1}(uvw,X)C_{\alpha_1}(vw,X)^{-1}C_{\alpha_1}(u,X)^{-1}C_{\alpha_1}(vw,X)^{-1}C_{\alpha_1}(w,X)^{-1}C_{\alpha_1}(v,X)^{-1} \\
&\quad = \langle u,vw \rangle \langle v,w \rangle \quad \text{for all } u,v,w \in F^*.
\end{aligned}$$

Hence for all $u, v \in F^*$, $\langle u, v\rangle$ satisfies (M1).
In the case of (M2), there is nothing to prove.
We will show that $\langle u, v\rangle$ satisfies (M3)–(M5) are as follows.

(M3): We have

$$\begin{aligned}
\langle u, v\rangle = \langle v^{-1}, u\rangle \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(u, X)^{-1} = C_{\alpha_1}(v^{-1}u, X)C_{\alpha_1}(u, X)^{-1}C_{\alpha_1}(v^{-1}, X)^{-1} \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1} = C_{\alpha_1}(v^{-1}u, X)C_{\alpha_1}(v^{-1}, X)^{-1} \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v^{-1}u, X)^{-1} = C_{\alpha_1}(v^{-1}, X)^{-1}C_{\alpha_1}(v, X) \\
&\Leftrightarrow C_{\alpha_1}(v^2, X) = C_{\alpha_1}(v^2, X) \quad \text{for all } u, v \in F^*.
\end{aligned}$$

Hence for all $u, v \in F^*$, $\langle u, v\rangle$ satisfies (M3).

(M4): We have

$$\begin{aligned}
\langle u, v\rangle = \langle u, -uv\rangle \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(u, X)^{-1} \\
&= C_{\alpha_1}(-u^2v, X)C_{\alpha_1}(-uv, X)^{-1}C_{\alpha_1}(u, X)^{-1} \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1} = C_{\alpha_1}(-u^2v, X)C_{\alpha_1}(-uv, X)^{-1} \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(-uv, X) = C_{\alpha_1}(-u^2v, X)C_{\alpha_1}(v, X) \\
&\Leftrightarrow C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(u^{-1}v^{-1}, X)C_{\alpha_1}(-uv, X) \\
&= C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(-v^{-1}, X)C_{\alpha_1}(v, X) \\
&\Leftrightarrow C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(-1, X) = C_{\alpha_1}(u^2v^2, X)C_{\alpha_1}(-1, X) \quad \text{for all } u, v \in F^*.
\end{aligned}$$

Hence for all $u, v \in F^*$, $\langle u, v\rangle$ satisfies (M4).

(M5): We have

$$\begin{aligned}
\langle u, v\rangle = \langle u, (1-u)v\rangle \quad ((1-u) \in F^*) \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1}C_{\alpha_1}(u, X)^{-1} \\
&= C_{\alpha_1}(u(1-u)v, X)C_{\alpha_1}((1-u)v, X)^{-1}C_{\alpha_1}(u, X)^{-1} \\
&\Leftrightarrow C_{\alpha_1}(uv, X)C_{\alpha_1}(v, X)^{-1} = C_{\alpha_1}(u(1-u)v, X)C_{\alpha_1}((1-u)v, X)^{-1}
\end{aligned}$$

(since $C_{\alpha_1}(u, (1-u)v)C_{\alpha_1}(u(1-u)v, X)$

$$= C_{\alpha_1}(u,(1-u)vX)C_{\alpha_1}((1-u)v,X))$$

$$\Leftrightarrow C_{\alpha_1}(uv,X)C_{\alpha_1}(v,X)^{-1} = C_{\alpha_1}(u,(1-u)v)^{-1}C_{\alpha_1}(u,(1-u)vX)$$

$$(\text{since } C_{\alpha_1}(u,(1-u)v)^{-1}C_{\alpha_1}(u,(1-u)vX) = C_{\alpha_1}(u,v)^{-1}C_{\alpha_1}(u,vX))$$

$$\Leftrightarrow C_{\alpha_1}(u,v)^{-1}C_{\alpha_1}(uv,X)^{-1} = C_{\alpha_1}(u,vX)^{-1}C_{\alpha_1}(v,X)^{-1}.$$

Hence for all $u, v \in F^*$, $\langle u, v\rangle$ satisfies (M5).
Therefore we obtain the desired result. □

Now we see that $\boldsymbol{M}$ is generated by $C(u)$ for all $u \in F^*$. To obtain the relations among the generators $C(u)$ for all $u \in F^*$ in $\boldsymbol{M}$, using Proposition 4.1 and 4.2, we rewrite the relations (M1)–(M5) in $K_2(A_1, F[X, X^{-1}])$ to relations in $\boldsymbol{M}$. To rewrite the relations (M1)–(M5) in $K_2(A_1, F[X, X^{-1}])$ to relations in $\boldsymbol{M}$, if $(m,n) \equiv (0,1),(1,0),(1,1)\ mod\ 2$, we change the element $C_{\alpha_1}(uX^m, vX^n)$ to $C((-1)^{mn}u^n v^{-m})\langle u, v\rangle$ and if $(m,n) \equiv (0,0)\ mod\ 2$, we change the element $C_{\alpha_1}(uX^m, vX^n)$ to $C(u^n v^{-m})$ (see Proposition 4.1). Thus we obtain all relations among the generators $C(u)$ for all $u \in F^*$ in $\boldsymbol{M}$. The following lemma is trivial but useful.

Lemma 4.4. *Notation is as above. Then for all $u, v \in F$, we have $C(u^2 v) = C(u^2)C(v)$. This implies $\langle u, v^2\rangle = e$.*

Proof. In $K_2(A_1, F[X, X^{-1}])$, it is easy to see $C_{\alpha_1}(u^2 v, X) = C_{\alpha_1}(u^2, X) \cdot C_{\alpha_1}(v, X)$.
Then we apply $\Psi_{\boldsymbol{M}}$ to obtain $C(u^2 v) = C(u^2)C(v)$. □

First we rewrite the relation (M1) in $K_2(A_1, F[X, X^{-1}])$ to relations in $\boldsymbol{M}$.
(M1):

$$C_{\alpha_1}(uX^l, vX^m)C_{\alpha_1}(uvX^{l+m}, wX^n)$$

$$= C_{\alpha_1}(uX^l, vwX^{m+n})C_{\alpha_1}(vX^m, wX^n) \quad \text{for all } u, v, w \in F^* \text{ and } m, n \in \boldsymbol{Z}.$$

(1) The case of $(l,m,n) \equiv (0,0,0)\ mod\ 2$
We have

$$C(u^m v^{-l})C(u^n v^n w^{-l-m}) = C(u^{m+n} v^{-l} w^{-l})C(v^n w^{-m}) \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.4.

(2) The case of $(l,m,n) \equiv (0,0,1)$ *mod* 2
We have

$$C(u^m v^{-l})C(u^n v^n w^{-l-m})\langle uv, w\rangle$$

$$= C(u^{m+n}v^{-l}w^{-l})C(v^n w^{-m})\langle u, vw\rangle\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(u^n v^n)C(w^{-m})C(w^{-l})\langle uv, w\rangle$$

$$= C(u^m)C(u^n)C(v^{-l})C(w^{-l})C(v^n)C(w^{-m})\langle v, w\rangle\langle u, vw\rangle$$

$$\Leftrightarrow C(u^n v^n)C(u^n)^{-1}C(v^n)^{-1}\langle uv, w\rangle = \langle v, w\rangle\langle u, vw\rangle$$

(using Lemma 4.4)

$$\Leftrightarrow C(u^{n-1}v^{n-1})C(uv)C(u^{n-1})^{-1}C(u)^{-1}C(v^{n-1})^{-1}C(v)^{-1}\langle uv, w\rangle = \langle v, w\rangle\langle u, vw\rangle$$

$$\Leftrightarrow \langle u, v\rangle\langle uv, w\rangle = \langle u, vw\rangle\langle v, w\rangle \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.3

(3) The case of $(l,m,n) \equiv (0,1,0)$ *mod* 2
We have

$$C(u^m v^{-l})\langle u, v\rangle C(u^n v^n w^{-m-l})\langle uv, w\rangle$$

$$= C(u^{m+n}v^{-l}w^{-l})\langle u, vw\rangle C(v^n w^{-m})\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(u^n)C(v^n)C(w^{-m})C(w^{-l})\langle u, v\rangle\langle uv, w\rangle$$

$$= C(u^m)C(u^n)C(v^{-l})C(w^{-l})C(w^{-m})C(v^n)\langle v, w\rangle\langle u, vw\rangle \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.4 and Lemma 4.3.

(4) The case of $(l,m,n) \equiv (0,1,1)$ *mod* 2
We have

$$C(u^m v^{-l})\langle u, v\rangle C(-u^n v^n w^{-m-l})\langle uv, w\rangle$$

$$= C(u^{m+n}v^{-l}w^{-l})C(-v^n w^{-m})\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(-u^n v^n w^{-m-l})\langle u, v\rangle\langle uv, w\rangle$$

$$= C(u^{m+n})C(v^{-l})C(w^{-l})C(-v^n w^{-m})\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(-u^n v^n w^{-m})\langle u, v\rangle\langle uv, w\rangle = C(u^{m+n})C(-v^n w^{-m})\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(u^{n-1}v^{n-1}w^{-m-1})C(-uvw)\langle u, v\rangle\langle uv, w\rangle$$

$$= C(u^{m+n})C(v^{n-1}w^{-m-1})C(-vw)\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(u^{n-1})C(-uvw)\langle u, v\rangle\langle uv, w\rangle = C(u^{m+n})C(-vw)\langle v, w\rangle$$

$$\Leftrightarrow C(u^{m+n-1})C(-uvw)\langle u, v\rangle\langle uv, w\rangle = C(u^{m+n})C(-vw)\langle v, w\rangle$$

$$\Leftrightarrow C(u^{m+n})C(u^{-1})C(-uvw)\langle u, v\rangle\langle uv, w\rangle = C(u^{m+n})C(-vw)\langle v, w\rangle$$

$$\Leftrightarrow C(u^{-1})C(-uvw)\langle u, v\rangle\langle uv, w\rangle = C(-vw)\langle v, w\rangle$$

$$\Leftrightarrow C(u^{-1})C(-uvw) = C(-vw)C(uvw)^{-1}C(u)C(vw)$$

$$\Leftrightarrow C(uvw)C(-uvw) = C(-vw)C(vw)C(u^{-1})^{-1}C(u)$$

(note that $C(u^{-1})^{-1}C(u) = C(u^2)$)

$$\Leftrightarrow C(uvw)C(-uvw) = C(-vw)C(vw)C(u^2) \quad \text{for all } u, v, w \in F^*.$$

Hence we have $C(uvw)C(-uvw) = C(-vw)C(vw)C(u^2)$ for all $u, v, w \in F^*$.

(5) The case of $(l, m, n) \equiv (1, 0, 0) mod\ 2$
We have

$$C(u^m v^{-l})\langle u, v\rangle C(u^n v^n w^{-m-l})\langle uv, w\rangle$$

$$= C(u^{m+n}v^{-l}w^{-l})\langle u, vw\rangle C(v^n w^{-m})$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(u^n)C(v^n)C(w^{-m})C(w^{-l})\langle u, v\rangle\langle uv, w\rangle$$

$$= C(u^m)C(u^n)C(v^{-l}w^{-l})C(v^n)C(w^{-m})\langle u, vw\rangle$$

$$\Leftrightarrow C(v^{-l})C(w^{-l})\langle u, v\rangle\langle uv, w\rangle = C(v^{-l}w^{-l})\langle u, vw\rangle$$

$$\Leftrightarrow C(v^{-l-1})C(w^{-l-1})C(v)C(w)\langle u, v\rangle\langle uv, w\rangle$$

$$= C(v^{-l-1}w^{-l-1})C(vw)\langle u, vw\rangle \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.3.

(6) The case of $(l, m, n) \equiv (1, 0, 1)\ mod\ 2$
We have

$$C(u^m v^{-l})\langle u, v\rangle C(-u^n v^n w^{-m-l})\langle uv, w\rangle$$

$$= C(-u^{m+n}v^{-l}w^{-l})\langle u, vw\rangle C(v^n w^{-m})\langle v, w\rangle$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(u^{n-1}v^{n-1}w^{-m-l-1})C(-uvw)$$

$$= C(u^{m+n-1}v^{-l-1}w^{-l-1})C(-uvw)C(v^n)C(w^{-m})$$

$$\Leftrightarrow C(u^m)C(v^{-l})C(-uvw) = C(u^m v^{-l-n} w^m)C(-uvw)C(v^n)C(w^{-m})$$

$$\Leftrightarrow C(u^m)C(v^{-l}) = C(u^m)C(v^{-l-n})C(w^m)C(w^{-m})C(v^n) \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.3 and Lemma 4.4.

(7) The case of $(l, m, n) \equiv (1, 1, 0)$ *mod* 2
We have

$$C(-u^m v^{-l})\langle u, v\rangle C(u^n v^n w^{-m-l})$$

$$= C(-u^{m+n}v^{-l}w^{-l})\langle v, w\rangle\langle u, vw\rangle C(v^n w^{-m})$$

$$\Leftrightarrow C(u^{m-1}v^{-l-1})C(-uv)C(u^n)C(v^n)C(w^{-m-l})\langle u, v\rangle$$

$$= C(u^{m+n-1}v^{-l-1}w^{-l-1})C(-uvw)C(v^n)C(w^{-m})\langle v, w\rangle\langle u, vw\rangle$$

$$\Leftrightarrow C(-uv)C(u^n)C(v^n)C(w^{-m-l})\langle u, v\rangle$$

$$= C(u^n v^{-l-1})C(-uvw)C(v^n)C(w^{-m})\langle v, w\rangle\langle u, vw\rangle$$

$$\Leftrightarrow C(-uv)C(w^{-m-l})\langle u, v\rangle = C(w^{-l-1})C(-uvw)C(w^{-m})\langle v, w\rangle\langle u, vw\rangle$$

$$\Leftrightarrow C(uvw)^{-1}C(uv)C(w)C(-uv)C(w^{-m-l}) = C(w^{-l-1})C(-uvw)C(w^{-m})$$

$$\Leftrightarrow C(uvw)C(-uvw) = C(uv)C(w)C(-uv)C(w^{-m-l})C(w^{-l-1})C(w^{-m})$$

$$\Leftrightarrow C(uvw)C(-uvw) = C(uv)C(-uv)C(w)C(w^{-1})^{-1}$$

(using $C(w^{-1})^{-1}C(w) = C(w^2)$)

$$\Leftrightarrow C(uvw)C(-uvw) = C(uv)C(-uv)C(w^2) \quad \text{for all } u, v, w \in F^*.$$

It is derived from Lemma 4.3, Lemma 4.4 and the equation which we obtain in (4).

(8) The case of $(l, m, n) \equiv (1, 1, 1)$ *mod* 2
We have

$$C(-u^m v^{-l})\langle u, v\rangle C(u^n v^n w^{-m-l})\langle uv, w\rangle$$

$$= C(u^{m+n}v^{-l}w^{-l})\langle v, w\rangle\langle u, vw\rangle C(-v^n w^{-m})$$

$$\Leftrightarrow C(u^{m-1}v^{-l-1})C(-uv)C(u^n v^n)C(w^{-m-l})$$

$$= C(u^{m+n})C(v^{-l}w^{-l})C(v^{n-1}w^{-m-1})C(-vw)$$

$$\Leftrightarrow C(u^{m+n-1}v^{-l+n-1}w^{-m-l})C(-uv) = C(u^{m+n}v^{-l+n-1}w^{-m-l-1})C(-vw)$$

$$\Leftrightarrow C(u^{m+n}v^{-l+n}w^{-m-l})C(u^{-1}v^{-1})C(-uv)$$

$$= C(u^{m+n}v^{-l+n}w^{-m-l})C(v^{-1}w^{-1})C(-vw) \quad \text{for all } u, v, w \in F^*.$$

Using the fact that $C(u)C(-u^{-1}) = C(-1)$ for all $u \in F^*$, it is derived from Lemma 4.3 and Lemma 4.4.

It is easy to see the relation (M2) is equivalent to $C(1) = e$.

Next we rewrite the relations (M3)–(M5) in $K_2(A_1, F[X, X^{-1}])$ to relations in $\boldsymbol{M}$.

(M3):

$$C_{\alpha_1}(uX^m, vX^n) = C_{\alpha_1}(v^{-1}X^n, uX^m) \quad \text{for all } u, v \in F^* \text{ and } m, n \in \boldsymbol{Z}.$$

(1) The case of $(m, n) \equiv (0, 0)$ *mod* 2

We have

$$C(u^n v^{-m}) = C(v^{-m}u^n) \quad \text{for all } u, v \in F^*.$$

Nothing appears.

(2) The case of $(m, n) \equiv (1, 0)(1, 1)(0, 1)$ *mod* 2

We have

$$C((-1)^{mn}u^n v^{-m})\langle u, v\rangle = C((-1)^{mn}u^n v^{-m})\langle v^{-1}, u\rangle \quad \text{for all } u, v \in F^*.$$

It is derived from Lemma 4.3.

(M4):

$$C_{\alpha_1}(uX^m, vX^n) = C_{\alpha_1}(uX^m, -uvX^{m+n}) \quad \text{for all } u, v \in F^* \text{ and } m, n \in \boldsymbol{Z}.$$

(1) The case of $(m, n) \equiv (0, 0)$ *mod* 2

We have

$$C(u^n v^{-m}) = C(u^{m+n}u^{-m}v^{-m}) = C(u^n v^{-m}) \quad \text{for all } u, v \in F^*.$$

Nothing appears.

(2) The case of $(m, n) \equiv (1, 0)(1, 1)(0, 1)$ *mod* 2

We have

$$C((-1)^{mn}u^n v^{-m})\langle u, v\rangle$$

$$= C((-1)^{m(m+n)}u^{m+n}(-1)^m v^{-m}u^{-m})\langle u, -uv\rangle \quad \text{for all } u, v \in F^*.$$

It is derived from Lemma 4.3.

(M5):

$C_{\alpha_1}(u, vX^n) = C_{\alpha_1}(u, (1-u)vX^n)$ for all $u, v \in F^*, 1-u \in F^*$ and $n \in \boldsymbol{Z}$.

(1) The case of $n \equiv 0 \ mod\ 2$
We have

$$C(u^n) = C(u^n)$$

Nothing appears.

(2) The case of $n \equiv 1 \ mod\ 2$
We have

$$C(u^n)\langle u, v\rangle = C(u^n)\langle u, (1-u)v\rangle.$$

It is derived from Lemma 4.3.

From the above argument we conclude that $\boldsymbol{M}$ is generated by the symbols $C(u)$ for all $u \in F^*$ and charactarized by the following relation:

(1) $C(u^2v) = C(u^2)C(v)$ $C(1) = e$ for all $u, v \in F^*$.

(2) $C(uvw)C(-uvw) = C(u^2)C(vw)C(-vw)$ for all $u, v \in F^*$.

(3) We put $\langle u, v\rangle := C(uv)C(u)^{-1}C(v)^{-1}$ for all $u, v \in F^*$. Then $\langle u, v\rangle$ for all $u, v \in F^*$ satisfies the relation (M1)–(M5) and $\langle u, v^2\rangle = e$.

Lemma 4.5. *Notation is as above. Then* (2) *follows from* (1) *and* (3).

Proof. It is sufficient to confirm the following:

$$C(yx)C(-yx) = C(y^2)C(x)C(-x) \quad \text{for all } x, y \in F^*.$$

Indeed $e = d(x, 1) = d(x, -x^{-1}) = C(-1)C(x)^{-1}C(-x^{-1})^{-1}$, hence we have

$$C(x)C(-x^{-1}) = C(-1),$$

$$C(-x)C(x^{-1}) = C(-1), \quad \text{and} \quad C(-yx) = C(y^{-1}x^{-1})^{-1}C(-1).$$

Then we have

$$C(yx)C(-yx) = C(yx)C(-1)C(y^{-1}x^{-1})^{-1} = C(y^2x^2)C(-1) = C(y^2)C(x^2)C(-1)$$

$$= C(y^2)C(x^{-1})^{-1}C(x)C(-1) = C(y^2)C(x)C(-x).$$

Hence we have the desired result. □

Now we put $\boldsymbol{D} := \langle\langle u, v\rangle \mid u, v \in F^*\rangle \subset \boldsymbol{M}$, then by [3] [8], we have $\boldsymbol{M} = P(F)$ and $\boldsymbol{D} = I^2(F)$.

Proposition 4.3. *Notation is as above. Then* $\boldsymbol{M}$ *is generated by the symbols* $C(u)$ *for all* $u \in F^*$ *and charactarized by the following relation*:

(1) *$C(u^2v) = C(u^2)C(v)$ and $C(1) = e$ for all $u, v \in F^*$.*

(2) *We put $\langle u, v\rangle := C(uv)C(u)^{-1}C(v)^{-1}$ for all $u, v \in F^*$. Then $\langle u, v\rangle$ for all $u, v \in F^*$ satisfies the relation* (M1)–(M5) *and $\langle u, v^2\rangle = e$.*

Proposition 4.4. *$\boldsymbol{D}$ is generated by the symbols $\langle u, v\rangle$ and charactarized by the relations* (M1)–(M5) *and $\langle u, v^2\rangle = e$ for all $u, v \in F^*$.*

Now we obtain the group presentations of $\boldsymbol{M} = P(F)$ and $\boldsymbol{D} = I^2(F)$.

4.3. Case of $A \neq C_n$

Lemma 4.6. *Suppose $A \neq C_n$ Let $\alpha_j \in \Pi$ be a long root. Then we have*

$$K_2(A, F[X, X^{-1}]) = \langle C_{\alpha_j}(uX^p, vX^q) \mid u, v \in F^*, p, q \in \boldsymbol{Z}\rangle.$$

Furthermore for all $u, v, w \in F^*$ and $p, q, r \in \boldsymbol{Z}$, we have $C_{\alpha_j}(uX^p, vX^q) \cdot C_{\alpha_j}(uX^p, wX^r) = C_{\alpha_j}(uX^p, vwX^{q+r})$ and $C_{\alpha_j}(uX^p, vX^q)C_{\alpha_j}(wX^r, vX^q) = C_{\alpha_j}(uwX^{p+r}, vX^q)$.

Proof. We choose $\alpha_k, \alpha_l \in \Pi$ with $\alpha_k(h_l) = -1$, $\alpha_l(h_k) = -1$. Then we have $C_{\alpha_k}(u, v) = C_{\alpha_k}(u^{-1}, v^{-1})C_{\alpha_k}(u^{-1}, v^{\alpha_k(h_l)}) = C_{\alpha_k\alpha_l}(u^{-1}, v) = C_{\alpha_l}(u^{-\alpha_l(h_k)}, v) = C_{\alpha_l}(u, v)$. From this and seeing Dynkin-diagrams in Fig D, for some short root $\alpha_p \in \Pi$ and long root $\alpha_q \in \Pi$, we have

$$K_2(A, F[X, X^{-1}]) = \langle C_{\alpha_p}(u^m, v^n), C_{\alpha_q}(uX^m, vX^n) \mid u, v \in F^*, m, n \in \boldsymbol{Z}\rangle.$$

From the fact that every Dynkin-diagrams in Fig D is connected, we have

$$K_2(A, F[X, X^{-1}]) = \langle C_{\alpha_j}(uX^m, vX^n) \mid u, v \in F^*, m, n \in \boldsymbol{Z}\rangle.$$

The remaining result is easily obtained from the bimultiplicativity of C_j as is well-known.

Hence we obtain the desired result. □

Now we split the element $C_{\alpha_i}(uX^m, vX^n)$ for all $u, v \in F^*$, $m, n \in \boldsymbol{Z}$ and $\alpha_i \in \Pi$ long root as follows:

$$\begin{aligned} C_{\alpha_i}(uX^m, vX^n) &= C_{\alpha_i}(u, v)C_{\alpha_i}(u, X^n)C_{\alpha_i}(X^m, v)C_{\alpha_i}(X^m, X^n) \\ &= C_{\alpha_i}(u, v)C_{\alpha_i}(u^n, X)C_{\alpha_i}(v^{-m}, X)C_{\alpha_i}((-1)^{mn}, X) \\ &= C_{\alpha_i}(u, v)C_{\alpha_i}((-1)^{mn}u^n v^{-m}, v). \end{aligned}$$

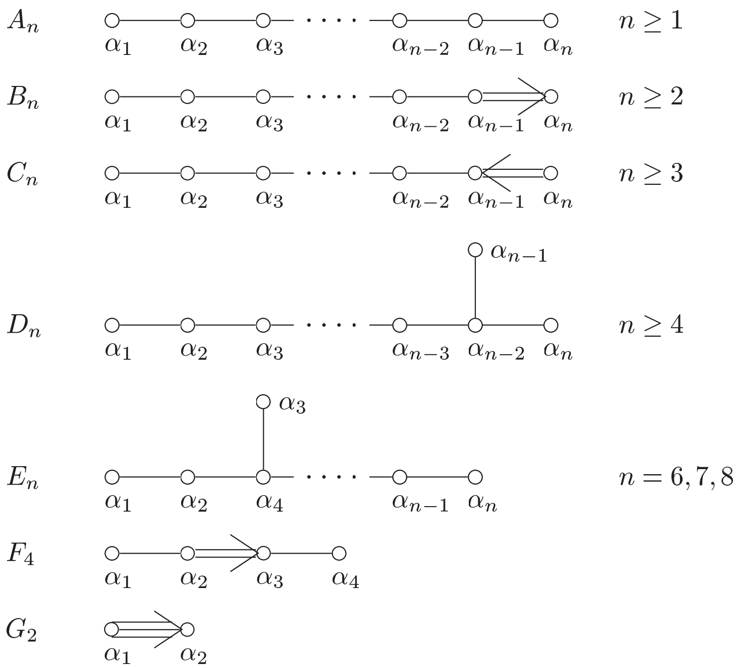

Figure D Dynkin-Diagram

Proposition 4.5. *The correspondance*

$$\Psi : K_2(A, F[X, X^{-1}]) \to K_2(A, F) \oplus F^*$$

$$C_{\alpha_i}(uX^m, vX^n) \mapsto C_{\alpha_i}(u, v) \oplus (-1)^{mn} u^n v^{-m}$$

for all $u, v \in F^*$, $m, n \in \boldsymbol{Z}$ *and* $\alpha_i \in \Pi$ (*long*) *gives a group isomorphism.*

Proof. It is easy to show the well-definedness of Ψ as a group homomorphism. Now we define Φ by

$$\Phi : K_2(A, F) \oplus F^* \to K_2(A, F[X, X^{-1}])$$

$$C_{\alpha_i}(u, v) \oplus t \mapsto C_{\alpha_i}(u, v) C_{\alpha_i}(t, X).$$

It is also easy to see the well-definedness of Φ as a group homomorphism. Then we see $\Phi \circ \Psi = Id$, $\Psi \circ \Phi = Id$. Hence we obtain the desired result. □

Here we see the following convention between Dynkin-diagrams and Cartan matrics.

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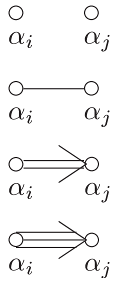

$a_{ij} = 0, a_{ji} = 0$

$a_{ij} = -1, a_{ji} = -1$

$a_{ij} = -1, a_{ji} = -2$

$a_{ij} = -1, a_{ji} = -3$

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