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TRANSFERRED KINEMATIC FORMULAE IN TWO POINT HOMOGENEOUS SPACES

By

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Abstract. We give kinematic formulae for integral invariants of degree 2 for hypersurfaces in two point homogeneous spaces explicitly. The discussion here we use is a certain generalization of the transfer principle in integral geometry.

1. Introduction

Let M and N be submanifolds in a Riemannian homogeneous space G/K, one fixed and the other moving under the action of $g \in G$. Consider an "*integral invariant*" $I(M \cap gN)$ of the intersection submanifold $M \cap gN$. Then a formula which expresses the integral

$$\int_G I(M \cap gN)\, d\mu_G(g) \tag{1.1}$$

in terms of some geometric invariants of M and N, where $d\mu_G$ is the invariant measure of G, is called a kinematic formula. For example, in the case where M and N are submanifolds of a real space form G/K and $I(M \cap gN) = \mathrm{vol}(M \cap gN)$, then the evaluation of (1.1) leads to the Poincaré formula, which expresses it as a constant times of volumes of M and N (see [8] for reference). Chern [3] and Federer [4] obtained a remarkable kinematic formula as follows:

THEOREM 1.1. *Let $I(\boldsymbol{R}^n)$ denote the isometry group of n-dimensional Euclidean space $\boldsymbol{R}^n$. Assume that $0 \le 2l \le p + q - n$. Then there exist constants $c(p, q, n, i, l)$ determined by indicated parameters so that*

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$$\int_{I(\boldsymbol{R}^n)} \mu_{2l}(M \cap gN)\, d\mu_G(g) = \sum_{i=0}^{l} c(p,q,n,i,l)\mu_{2i}(M)\mu_{2(l-i)}(N)$$

holds for any compact submanifolds M and N in $\boldsymbol{R}^n$ of dimensions p and q, respectively.

Here the invariants μ_{2i} are those that appear in the Weyl tube formula. Definition and some fundamental properties of them will be explained in Section 2.

Later Howard [5] defined integral invariants of submanifolds in Riemannian homogeneous spaces from invariant polynomials on the space of second fundamental forms. He showed that kinematic formulae for these invariants can be expressed by invariants of M and N if G is unimodular and acts transitively on the sets of tangent spaces to each of M and N. Moreover, he showed the "*transfer principle*" in integral geometry. Roughly speaking, it guarantees that the same kinematic formulae hold in homogeneous spaces which have the same isotropy groups.

The linear isotropy action of a two point homogeneous space is transitive on the hypersphere in the tangent space at the origin. Therefore, from the transfer principle, kinematic formulae for hypersurfaces in two point homogeneous spaces can be expressed by invariants of two submanifolds. However, it is not obvious how to obtain explicit forms of such kinematic formulae. In his paper, Howard showed the following Poincaré formula by transferring from the case of real space forms.

Proposition 1.2 ([5] paragraph 3.12). *Let G/K be a two point homogeneous space of dimension n. Let M be a submanifold of dimension p and N a hypersurface in G/K. If M and N have finite volume then*

$$\int_G \mathrm{vol}(M \cap gN)\, d\mu_G(g) = \frac{\mathrm{vol}(K)\,\mathrm{vol}(S^{p-1})\,\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\,\mathrm{vol}(S^{n-1})}\,\mathrm{vol}(M)\,\mathrm{vol}(N)$$

holds.

A Poincaré formula is a kinematic formula for the volume functional, that is, an integral invariant of degree 0. Therefore our interest goes to the higher degree cases.

In the present paper, we shall study the kimematic formulae for hypersurfaces in two point homogeneous spaces. We generalize the transfer principle, and eventually obtain the following kinematic formulae.

Main Theorem. *Let M and N be real hypersurfaces in a two point homogeneous space G/K. Then the following kinematic formulae hold*:

$$\int_G I^{\mathscr{W}_2}(M \cap gN)\, d\mu_G(g)$$
$$= \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1, n-1, n)(I^{\mathscr{W}_2}(M)\, \mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathscr{W}_2}(N)),$$

$$\int_G I^{\mathscr{U}_{n-2}}(M \cap gN)\, d\mu_G(g)$$
$$= \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n-1, n-1, n)(I^{\mathscr{U}_{n-1}}(M)\, \mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathscr{U}_{n-1}}(N)).$$

Integral invariants $I^{\mathscr{W}_2}$ and $I^{\mathscr{U}_p}$ will be explained in the next section.

2. Preliminaries

We shall use this section to recall the general theory of the kinematic formulae in Riemannian homogeneous spaces due to Howard, which is necessary for our discussion. Refer to his paper [5] for details.

Let G be a Lie group and K a compact subgroup of G. We assume that G has a left invariant metric that is also right invariant under K, then G/K is a homogeneous space with an invariant metric. We denote by $T = T_o(G/K)$ the tangent space of G/K at the origin o. Let V be a linear subspace of T. A submanifold M of G/K is said to be of type V if and only if for each $x \in M$ there exists $g_x \in G$ such that $(g_x)_* V = T_x M$.

For a linear subspace V of T, we define a vector space $\mathrm{II}(V)$ to be

$$\mathrm{II}(V) = \{h \mid h : V \times V \to V^\perp;\ \text{symmetric bilinear}\},$$

where $V^\perp$ is the normal space of V in T. A second fundamental form of a submanifold of G/K which passes through o and has V as the tangent space at o is an element of $\mathrm{II}(V)$. Let $K(V)$ be the stabilizer of V in K, that is, $K(V) = \{k \in K \mid k_* V = V\}$. The group $K(V)$ acts on $\mathrm{II}(V)$ by the following manner:

$$(kh)(u, v) = k_*(h(k_*^{-1}u, k_*^{-1}v)) \quad (u, v \in V) \tag{2.1}$$

for $k \in K(V)$ and $h \in \mathrm{II}(V)$. Here we consider a polynomial $\mathscr{P}$ on $\mathrm{II}(V)$ which is invariant under $K(V)$, that is, $\mathscr{P}(kh) = \mathscr{P}(h)$ for all $k \in K(V)$ and $h \in \mathrm{II}(V)$. Let M be a submanifold of G/K of type V. For the second fundamental form h_x^M of M at $x \in M$, we define

$$\mathscr{P}(h_x^M) = \mathscr{P}(h_o^{g_x^{-1}M}).$$

Then we can define an integral invariant $I^{\mathscr{P}}(M)$ of M from a polynomial $\mathscr{P}$ by

$$I^{\mathscr{P}}(M) = \int_M \mathscr{P}(h_x^M)\, d\mu_M(x). \tag{2.2}$$

We also define a vector space $\mathrm{EII}(T)$ to be

$$\mathrm{EII}(T) = \{h \mid h : T \times T \to T;\ \text{symmetric bilinear}\}.$$

Since K also acts on $\mathrm{EII}(T)$ in the same way as in (2.1), we can define integral invariants of a submanifold from polynomials on $\mathrm{EII}(T)$ invariant under K in the same way as in (2.2).

With these notations, we can now state the kinematic formulae in Riemannian homogeneous spaces as follows:

THEOREM 2.1 ([5] paragraph 4.10). *Let G/K be a Riemannian homogeneous space and assume that G is unimodular. Let V and W be linear subspaces of T with* $\dim(V) + \dim(W) \geq \dim(T)$, *and $\mathscr{P}$ a homogeneous polynomial of degree l on* $\mathrm{EII}(T)$ *which is invariant under K, such that*

$$\int_K \sigma(V^{\perp}, k_* W^{\perp})^{1-l}\, d\mu_K(k) < \infty. \tag{2.3}$$

Then there exists a finite set of pairs $(\mathscr{Q}_\alpha, \mathscr{R}_\alpha)$ such that

(1) *each $\mathscr{Q}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(V)$ *invariant under* $K(V)$,

(2) *each $\mathscr{R}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(W)$ *invariant under* $K(W)$,

(3) $\deg \mathscr{Q}_\alpha + \deg \mathscr{R}_\alpha = l$ *for each* α,

(4) *for all compact submanifolds (possibly with boundaries) M of type V and N of type W in G/K the kinematic formula*

$$\int_G I^{\mathscr{P}}(M \cap gN)\, d\mu_G(g) = \sum_\alpha I^{\mathscr{Q}_\alpha}(M) I^{\mathscr{R}_\alpha}(N) \tag{2.4}$$

holds.

Here $\sigma(V, W)$ is the angle between linear subspaces V of dimension p and W of dimension q in an inner product space E. That is defined by

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|,$$

where $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ are orthonormal bases of V and W, respectively. In the condition (2.3), we required the integral to be convergent. If G/K is a real space form, then the condition (2.3) can be replaced by the manageable inequality $l \le \dim(V) + \dim(W) - \dim(T) + 1$.

In order to explain how Theorem 2.1 is obtained, we need some definitions and lemmas. For $0 < p < n$, $Gr_p(T)$ denotes the Grassmannian manifold of all p-dimensional subspaces in T. Then we set

$$\mathrm{II}_p(T) = \{(V,h) \mid V \in Gr_p(T),\ h \in \mathrm{II}(V)\}.$$

For $(V,h) \in \mathrm{II}_p(T)$ and a subspace W in T with $V + W = T$, we define $G_W(V,h) \in \mathrm{EII}(T)$ by

$$G_W(V,h)(u,v) = P_W^V(h(Pu, Pv)) \quad (u, v \in T).$$

Here P_W^V is the projection $T \to (V \cap W)^\perp \cap W$ with kernel V, and P is the orthogonal projection $T \to V \cap W$.

Assume that $p + q \ge n$. For $(V, h_1) \in \mathrm{II}_p(T)$, $(W, h_2) \in \mathrm{II}_q(T)$ and a polynomial $\mathscr{P}$ on $\mathrm{EII}(T)$ invariant under K, we define

$$\begin{aligned} I_K^{\mathscr{P}}&(V, h_1, W, h_2) \\ &= \int_K \mathscr{P}(G_{k_*^{-1}W}(V, h_1) + G_V(k_*^{-1}W, k^{-1}h_2))\sigma(V^\perp, k_*^{-1}W^\perp)\, d\mu_K(k) \end{aligned}$$

provided this integral converges.

Lemma 2.2 ([5] paragraph 6.5). *Under the hypothesis of Theorem* 2.1 *there exists a finite set of pairs* $(\mathscr{Q}_\alpha, \mathscr{R}_\alpha)$ *such that*

(1) *each* $\mathscr{Q}_\alpha$ *is a homogeneous polynomial on* $\mathrm{II}(V)$ *invariant under* $K(V)$,

(2) *each* $\mathscr{R}_\alpha$ *is a homogeneous polynomial on* $\mathrm{II}(W)$ *invariant under* $K(W)$,

(3) $\deg \mathscr{Q}_\alpha + \deg \mathscr{R}_\alpha = l$ *for each* α,

(4) *for all* $h_1 \in \mathrm{II}(V)$ *and* $h_2 \in \mathrm{II}(W)$

$$I_K^{\mathscr{P}}(V, h_1, W, h_2) = \sum_\alpha \mathscr{Q}_\alpha(h_1)\mathscr{R}_\alpha(h_2).$$

When M and N are submanifolds in G/K of type V and W, we define

$$I_K^{\mathscr{P}}(V, h_x^M, W, h_y^N) = I_K^{\mathscr{P}}(V, h_o^{g_x^{-1}M}, W, h_o^{g_y^{-1}N}).$$

Lemma 2.3 ([5] paragraph 7.2). *Under the hypothesis of Theorem* 2.1

$$\int_G I^{\mathscr{P}}(M \cap gN)\, d\mu_G(g) = \int_{M\times N} I_K^{\mathscr{P}}(V, h_x^M, W, h_y^N)\, d\mu_{M\times N}(x, y)$$

holds for any compact submanifolds M *of type* V *and* N *of type* W *in* G/K.

From these two lemmas we conclude Theorem 2.1.

REMARK 2.4. From these facts finally we arrive at the "*transfer principle*", that is a method of transferring kinematic formulae from one homogeneous space to any other homogeneous space with the same isotropy subgroup.

Now we give some concrete forms of invariant polynomials and kinematic formulae. Take an orthonormal basis $e_1, \dots, e_n$ of T such that $e_1, \dots, e_p$ is a basis of V and $e_{p+1}, \dots, e_n$ is a basis of $V^\perp$. Then components of $h \in \mathrm{II}(V)$ and $H \in \mathrm{EII}(T)$ are represented by

$$h_{ij}^k = \langle h(e_i, e_j), e_k \rangle \quad (1 \le i, j \le p, p+1 \le k \le n)$$

$$H_{ij}^k = \langle H(e_i, e_j), e_k \rangle \quad (1 \le i, j, k \le n)$$

The following polynomials $\mathscr{W}_{2l}$ are homogeneous polynomials on $\mathrm{II}(V)$ of degree $2l$ invariant under $O(V) \times O(V^\perp)$.

$$\mathscr{W}_{2l}(h) = \sum_{\substack{1 \le i_1, \dots, i_{2l} \le p \\ p+1 \le k_1, \dots, k_l \le n}} \det \begin{bmatrix} h_{i_1 i_1}^{k_1} & h_{i_1 i_2}^{k_1} & \cdots & h_{i_1 i_{2l}}^{k_1} \\ h_{i_2 i_1}^{k_1} & h_{i_2 i_2}^{k_1} & \cdots & h_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_{2l-1} i_1}^{k_l} & h_{i_{2l-1} i_2}^{k_l} & \cdots & h_{i_{2l-1} i_{2l}}^{k_l} \\ h_{i_{2l} i_1}^{k_l} & h_{i_{2l} i_2}^{k_l} & \cdots & h_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}.$$

We define homogeneous polynomials, also denoted by $\mathscr{W}_{2l}$, on $\mathrm{EII}(T)$ of degree $2l$ invariant under $O(T)$ by

$$\mathscr{W}_{2l}(H) = \sum_{\substack{1 \le i_1, \dots, i_{2l} \le n \\ 1 \le k_1, \dots, k_l \le n}} \det \begin{bmatrix} H_{i_1 i_1}^{k_1} & H_{i_1 i_2}^{k_1} & \cdots & H_{i_1 i_{2l}}^{k_1} \\ H_{i_2 i_1}^{k_1} & H_{i_2 i_2}^{k_1} & \cdots & H_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{i_{2l-1} i_1}^{k_l} & H_{i_{2l-1} i_2}^{k_l} & \cdots & H_{i_{2l-1} i_{2l}}^{k_l} \\ H_{i_{2l} i_1}^{k_l} & H_{i_{2l} i_2}^{k_l} & \cdots & H_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}.$$

In the both cases, $\mathscr{W}_0 = 1$ by definition. A second fundamental form $h \in \mathrm{II}(V)$ can be extended to $H \in \mathrm{EII}(T)$ by

$$H(u, v) = h(Pu, Pv) \quad (u, v \in T),$$

where $P : T \to V$ is the orthogonal projection. If $H \in \mathrm{EII}(T)$ is the extension of $h \in \mathrm{II}(V)$, then we have

$$\mathscr{W}_{2l}(h) = \mathscr{W}_{2l}(H).$$

Furthermore, these polynomials $\mathscr{W}_{2l}$ are characterized as the invariant polynomials which vanish on (extended) second fundamental forms with relative rank less than $2l$. For a submanifold M of G/K, we introduce the integral invariants $\mu_{2l}(M)$ defined by

$$\mu_{2l}(M) = I^{\mathscr{W}_{2l}}(M).$$

For these integral invariants μ_{2l}, the Chern-Federer kinematic formula (Theorem 1.1) holds. In fact, this formula holds in any real space forms by the transfer principle. The value of the constants $a(p,q,n,i,l)$ were computed by Chern [3] and Nijenhuis [7].

The space of homogeneous polynomials on $\mathrm{II}(V)$ of degree 2 invariant under $O(V) \times O(V^{\perp})$ is spanned by two polynomials

$$\mathscr{Q}_1(h) = \sum_{i,j,k} (h_{ij}^k)^2, \quad \mathscr{Q}_2(h) = \sum_k \left(\sum_i h_{ii}^k \right)^2,$$

where $1 \le i, j \le p,\ p+1 \le k \le n$. If $2 \le p \le n-1$, these two polynomials are independent. Geometrically, $\mathscr{Q}_1(h)$ is the square of the norm of the second fundamental form, and $\mathscr{Q}_2(h)$ is p^2 times the square of the mean curvature. However, it is convenient for us to take the basis

$$\mathscr{W}_2 = \mathscr{Q}_2 - \mathscr{Q}_1, \quad \mathscr{U}_p = p\mathscr{Q}_1 - \mathscr{Q}_2.$$

For these polynomials we have the following:

Proposition 2.5 ([5] paragraph 8.5). *Assume that $2 \le p+q-n$. Then there exist constants $a(p,q,n)$ and $b(p,q,n)$ so that*

$$\int_G I^{\mathscr{W}_2}(M \cap gN)\, d\mu_G(g) = a(p,q,n) I^{\mathscr{W}_2}(M) \operatorname{vol}(N) + a(q,p,n) \operatorname{vol}(M) I^{\mathscr{W}_2}(N)$$

$$\int_G I^{\mathscr{U}_{p+q-n}}(M \cap gN)\, d\mu_G(g) = b(p,q,n) I^{\mathscr{U}_p}(M) \operatorname{vol}(N) + b(q,p,n) \operatorname{vol}(M) I^{\mathscr{U}_q}(N)$$

holds for any compact submanifolds M and N of dimensions p and q in a real space form G/K.

The first one is entirely the Chern-Federer formula of degree 2. The constants $a(p,q,n)$ and $b(p,q,n)$ were determined in the previous paper [6]. The polynomial

$\mathscr{U}_p$ is characterized as the invariant polynomial which vanishes at an umbilic point. The integral invariant

$$I^{\mathscr{U}_p^{p/2}}(M) = \int_M (\mathscr{U}_p(h_x^M))^{p/2}\, d\mu_M(x)$$

is an conformal invariant, called the Willmore-Chen functional, of p-dimensional submanifold M (see [1], [2], [9]).

3. Two Point Homogeneous Spaces

A connected Riemannian manifold M is called a *two point homogeneous space* if, for any pairs of points $x_i, y_i \in M$ with distance $d(x_1, y_1) = d(x_2, y_2)$, there exists an isometry $g \in I(M)$ such that $gx_1 = x_2$ and $gy_1 = y_2$. On the other hand, a Riemannian manifold M is said to be *isotropic* at $x \in M$ if $I(M)_x = \{g \in I(M) \mid gx = x\}$ acts transitively on the unit hypersphere in T_xM, and M is *isotropic* if and only if it is isotropic at every point. It is well known that these two notions are equivalent. Furthermore, two point homogeneous spaces are completely classified; a two point homogeneous space is a Euclidean space $\boldsymbol{R}^n = I(\boldsymbol{R}^n)/O(n)$ or an irreducible symmetric space of rank 1: a sphere $S^n = O(n+1)/O(n)$, a real projective space $\boldsymbol{RP}^n = O(n+1)/O(1) \times O(n)$, a complex projective space $\boldsymbol{CP}^n = U(n+1)/U(1) \times U(n)$, a quaternionic projective space $\boldsymbol{HP}^n = Sp(n+1)/Sp(1) \times Sp(n)$, the Cayley projective plane $\mathbf{Cay}\,P^2 = F_4/Spin(9)$, and their non-compact duals.

LEMMA 3.1. *Let $M = G/K$ be a two point homogeneous space. Assume that G is the isometry group of M listed above. Then there is no homogeneous polynomial on* $\mathrm{II}(V)$ *(resp.* $\mathrm{EII}(T)$*) of odd degree invariant under $K(V)$ (resp. K).*

PROOF. Since $K(V)$ (resp. K) acts on $\mathrm{II}(V)$ (resp. $\mathrm{EII}(T)$) by (2.1), it is enough if we find an element $k \in K$ which acts on T as $-\mathrm{id}_T$. It is easy to find such $k \in K$ in the case of $K = O(n), O(1) \times O(n), U(1) \times U(n), Sp(1) \times Sp(n)$. It remains the case of $K = Spin(9)$. The spinor group $Spin(n)$ is defined as a subset of the Clifford algebra Cl_n, and it is well known that a Clifford algebra is isomorphic to a matrix algebra. In this case,

$$Spin(9) \subset Cl_9^{even} \cong Cl_8 \cong M(16, \boldsymbol{R})$$

where $M(16, \boldsymbol{R})$ denotes the algebra of 16×16 matrices over $\boldsymbol{R}$. This inclusion defines the spin representation of $Spin(9)$, that is equivalent to the linear isotropy representation of $K = Spin(9)$. Through these isomorphisms, $-1 \in Spin(9)$ corresponds to minus the identity matrix $-I \in M(16, \boldsymbol{R})$. Thus $-1 \in Spin(9)$ acts on T as $-\mathrm{id}_T$. This completes the proof. $\square$

4. Proof of the Main Theorem

Let G/K be a two point homogeneous space of dimension n. Then it is isotropic; K acts transitively on the hypersphere in T by the linear isotropy representation. For $v \in T$ we denote by $K(v)$ the stabilizer of v in K, and then $K/K(v)$ is homothetic to the unit sphere S^{n-1}. We note that if we put $W = v^{\perp}$ then $K(v) \subset K(W)$.

Let $\mathscr{P}$ be a polynomial on $\mathrm{EII}(T)$ invariant under the orthogonal group $O(T)$ acting on T. From the definition (2.5), we have

$$
\begin{aligned}
(4.1)\qquad & I_K^{\mathscr{P}}(V, h_1, W, h_2) \\
&= \int_K \mathscr{P}(G_{k_*^{-1}W}(V, h_1) + G_V(k_*^{-1}W, k^{-1}h_2))\sigma(V^{\perp}, k_*^{-1}W^{\perp})\, d\mu_K(k) \\
&= \int_K \mathscr{P}(G_{k_*W}(V, h_1) + G_V(k_*W, kh_2))\sigma(V^{\perp}, k_*W^{\perp})\, d\mu_K(k)
\end{aligned}
$$

for $(V, h_1) \in \mathrm{II}_p(T)$ and $(W, h_2) \in \mathrm{II}_{n-1}(T)$. The last equality holds since K is a compact Lie group.

We take a second fundamental form $h(r) \in \mathrm{II}(W)$ of a hypersurface which is tangent to W and umbilic at that point with principal curvature r. It is not a problem whether such a hypersurface exists. We are just considering an element of $\mathrm{II}(W)$ formally. If we take an orthonormal basis of T and regard $\mathrm{II}(W)$ as the space of $(n-1)$ by $(n-1)$ symmetric matrices then $h(r)$ is expressed as rI_{n-1}, where I_{n-1} is the identity matrix. Since $K(v)$ acts on $\mathrm{II}(W)$ by (2.1), it is clear that $h(r) \in \mathrm{II}(W)$ is fixed by the action of $K(v)$;

$$
(4.2)\qquad gh(r) = h(r) \quad ({}^{\forall}g \in K(v)).
$$

If $[k] = [k'] \in K/K(v)$, then there exists $g \in K(v)$ such that $k' = kg$. Therefore, when we apply $h_2 = h(r)$ in (4.1), from (4.2) we have

$$
\begin{aligned}
& \mathscr{P}(G_{k'_*W}(V, h_1) + G_V(k'_*W, k'h(r)))\sigma(V^{\perp}, k'_*W^{\perp}) \\
&\quad = \mathscr{P}(G_{kg_*W}(V, h_1) + G_V(kg_*W, kgh(r)))\sigma(V^{\perp}, kg_*W^{\perp}) \\
&\quad = \mathscr{P}(G_{k_*W}(V, h_1) + G_V(k_*W, kh(r)))\sigma(V^{\perp}, k_*W^{\perp}).
\end{aligned}
$$

This implies that, when we regard K as a principal fiber bundle on $K/K(v)$ with fiber $K(v)$, the integrand in (4.1) is constant on each fiber. Thus the integration on K is reduced to that on $K/K(v)$. Hence we have

$$
\begin{aligned}
(4.3)\qquad & I_K^{\mathscr{P}}(V,h_1,W,h(r)) \\
&= \mathrm{vol}(K(v))\int_{K/K(v)} \mathscr{P}(G_{[k]_*W}(V,h_1)+G_V([k]_*W,[k]h(r))) \\
&\quad \times \sigma(V^{\perp},[k]_*W^{\perp})\, d\mu_{K/K(v)}([k]) \\
&= \frac{\mathrm{vol}(K)}{\mathrm{vol}(S^{n-1})}\int_{K/K(v)} \mathscr{P}(G_{[k]_*W}(V,h_1)+G_V([k]_*W,[k]h(r))) \\
&\quad \times \sigma(V^{\perp},[k]_*W^{\perp})\, d\mu_{S^{n-1}}([k]) \\
&= \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} I_{SO(n)}^{\mathscr{P}}(V,h_1,W,h(r)).
\end{aligned}
$$

We have the second equality normalizing the invariant measure of $K/K(v)$ to that of unit sphere S^{n-1}. The last equality is obtained by the opposite procedure of reducing the integration on K to that on the sphere.

Now we restrict ourselves to the case $\dim V = \dim W = n-1$. In addition, we take an $O(T)$-invariant homogeneous polynomial $\mathscr{P} = \mathscr{W}_2$ on $\mathrm{EII}(T)$. Without loss of generality, we can assume $V = W$, since K acts transitively on $Gr_{n-1}(T)$. In Lemma 3.1 we showed that there is no homogeneous polynomial of odd degree invariant under $K(W)$. Therefore, from Lemma 2.2, there exists a homogeneous polynomial $\mathscr{Q}$ on $\mathrm{II}(W)$ of degree 2 invariant under $K(W)$ so that

$$I_K^{\mathscr{W}_2}(W,h_1,W,h_2) = \mathscr{Q}(h_1)+\mathscr{Q}(h_2).$$

In the case of $K = SO(n)$, this is entirely Proposition 2.5. Thus we have

$$I_{SO(n)}^{\mathscr{W}_2}(W,h_1,W,h_2) = a(n-1,n-1,n)(\mathscr{W}_2(h_1)+\mathscr{W}_2(h_2)).$$

From (4.3) we have

$$\mathscr{Q}(h_1)+\mathscr{Q}(h(r)) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1,n-1,n)(\mathscr{W}_2(h_1)+\mathscr{W}_2(h(r))).$$

Since $\mathscr{Q}$ is homogeneous polynomial of degree 2 and $h(r) = rI_{n-1}$

$$\mathscr{Q}(h_1)+r^2\mathscr{Q}(h(1)) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1,n-1,n)(\mathscr{W}_2(h_1)+r^2\mathscr{W}_2(h(1))).$$

Here r is arbitrary real number, thus coefficients of polynomials with respect to r agree in each degree. Thus we have

$$\mathscr{Q}(h_1) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1, n-1, n) \mathscr{W}_2(h_1).$$

The same discussion holds when we take an invariant polynomial $\mathscr{P} = \mathscr{U}_{n-2}$. Consequently we have the Main Theorem.

5. Problems

We shall conclude this paper by posing some related problems. In the Main Theorem we showed that the kinematic formulae for integral invariants of degree 2 can be obtained transferring from the case of real space forms. Then our interest is in the case of higher degree.

Problem 5.1. Can all kinematic formulae for hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$-invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

In Proposition 1.2, Howard showed the Poincaré formula for a real hypersurface N and *any* dimensional submanifold M in a two point homogeneous space. Therefore it is natural to pose the following problem:

Problem 5.2. Does the Main Theorem hold for a real hypersurface N and any dimensional submanifold M?

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