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REMARKS ON THE CLIFFORD INDEX OF ALGEBRAIC CURVES

By

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Abstract. Let $\mathrm{Cliff}(X)$ be the Clifford index of a curve X and L be a base-point-free line bundle on X which satisfies $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \geq 3)$. We determine sufficient conditions for the $|L| = g_d^r$ being simple (*i.e.* birationally very ample).

1. Introduction

Let X be a smooth irreducible projective curve of genus $g \geq 4$ over an algebraically closed field of characteristic 0 and L be a line bundle on X. A g_d^r on X be a linear series of degree d and projective dimension r. If $|L| = g_d^r$, then the *Clifford index* of L is defined by $\mathrm{Cliff}(L) = d - 2r$ and the *Clifford index* of X is defined by $\mathrm{Cliff}(X) = \mathrm{Min}\{\mathrm{Cliff}(L) \mid r \geq 1, g - 1 - d + r \geq 1\}$. We say that a $|L| = g_d^r$ on X *computes* $\mathrm{Cliff}(X)$ if $r \geq 1$, $g - 1 - d + r \geq 1$, and $\mathrm{Cliff}(X) = d - 2r$. The following result about $\mathrm{Cliff}(X)$ is known:

Proposition 1.1 ([4]). *Let a $|L| = g_d^r$ on X compute* $\mathrm{Cliff}(X)$, $r \geq 3$. *Then the g_d^r is simple* (*i.e. birationally very ample*) *unless X is hyperelliptic or bi-elliptic.*

Furthermore, the same type results in the case of $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k = 1, 2)$ are known. Those results are as follows:

Proposition 1.2 ([2], (2.2)). *Let a $|L| = g_d^r$ be a base-point-free linear series on X. Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + 1$, $\mathrm{Cliff}(X) \geq 1$, $r \geq 3$, *and* $g - 1 - d + r \geq 1$. *Then the g_d^r is simple.*

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Proposition 1.3 ([2], (2.3)). *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + 2$, $\mathrm{Cliff}(X) \geq 1$, $r \geq 3$, *and* $g - 1 - d + r \geq 1$. *Then the* g_d^r *is simple unless* X *and the* g_d^r *are the following cases*:

(1) X *is a trigonal curve*, $\mathrm{Cliff}(X) = 1$, *and* $|L| = g_9^3$;
(2) X *is a triple covering of an elliptic curve*, $\mathrm{Cliff}(X) = 4$, *and* $|L| = g_{12}^3$;
(3) X *is a double covering of a curve of genus* $g(Y) = 2, 3, 4, 5, 10$.

Our main purpose in this paper is to generalize the propositions stated above, that is to say, to describe $\mathrm{Cliff}(X)$ and $|L| = g_d^r$ which satisfy $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \geq 3)$. We get the same type consequences for the case of $k = 3, 4$. Our results are the following:

Theorem 1.4. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \geq 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + 3$, $r \geq 3$, $g - 1 - d + r \geq 1$. *Then the* g_d^r *is simple unless* X *and the* g_d^r *are the following cases*:

(1) X *is a trigonal curve*, $\mathrm{Cliff}(X) = 1$, *and* $|L| = g_{12}^4 = 4g_3^1$;
(2) X *is a triple covering of an elliptic curve*, $\mathrm{Cliff}(X) = 4$, *and* $|L| = g_{15}^4$;
(3) X *is a double covering of a curve and* $\mathrm{Cliff}(X)$ *is odd.*

Theorem 1.5. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \geq 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + 4$, $r \geq 3$, $g - 1 - d + r \geq 1$. *Then the* g_d^r *is simple unless* X *and the* g_d^r *are the following cases*:

(1) X *is a* 4*-gonal curve*, $\mathrm{Cliff}(X) = 2$, *and* $|L| = g_{12}^3$;
(2) X *is a* 4*-sheeted covering of an elliptic curve*, $\mathrm{Cliff}(X) = 6$, *and* $|L| = g_{16}^3$;
(3) X *is a trigonal curve*, $\mathrm{Cliff}(X) = 1$, *and* $|L| = g_{15}^5$;
(4) X *is a triple covering of an elliptic curve*, $\mathrm{Cliff}(X) = 4$, *and* $|L| = g_{18}^5$;
(5) X *is a double covering of a curve and* $\mathrm{Cliff}(X)$ *is even.*

In the case of $k \geq 5$, the result is similar to the preceding theorems.

Theorem 1.6. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \geq 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \geq 5)$, $r \geq k - 1$, $g - 1 - d + r \geq 1$. *Then the* g_d^r *is simple unless* X *and the* g_d^r *are the following cases*:

(1) X *is a trigonal curve*, $\mathrm{Cliff}(X) = 1$, *and* $|L| = g_{3(k+1)}^{(k+1)}$;
(2) X *is a triple covering of an elliptic curve*, $\mathrm{Cliff}(X) = 4$, *and* $|L| = g_{3(k+2)}^{(k+1)}$;
(3) X *is a double covering of a curve and* $\mathrm{Cliff}(X) + k$ *is even.*

The organization of our paper is as follows. First we shall prove our theorems in §2. Next we shall derive $d \le 3(\mathrm{Cliff}(X)+k)$ under some conditions from the main theorem in §3.

2. On Linear Series $|L|$ with $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$

We shall study about a linear series g_d^r which satisfies $d - 2r = \mathrm{Cliff}(X) + k$ $(k \ge 0)$.

Proposition 2.1. *Let a* $|L| = g_d^r$ *on* X *satisfy* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \ge 0)$, $r \ge 3$, $g - 1 - d + r \ge 1$. *Assume that the* g_d^r *is base-point-free and defines a morphism* $\pi : X \to \mathbf{P^r}$, *of degree* $m \ge 2$ *onto a curve* Y *in* $\mathbf{P^r}$. *If*

$$\deg \pi = m \ge \begin{cases} k/2 + 2 & \text{for } k = 2l \\ (k-1)/2 + 2 & \text{for } k = 2l + 1, \end{cases}$$

then X, $\mathrm{Cliff}(X)$, *and the* g_d^r *are the following cases*:

(a) *if* k *is even i.e.* $k = 2l$, *then*

(i) X *is a* $(l+2)$*-gonal curve*, $\mathrm{Cliff}(X) = l$, *and*

$$|L| = \begin{cases} g_{2r}^r & \text{for } l = 0 \\ g_{3(l+2)}^3 & \text{for } l > 0; \end{cases}$$

(ii) X *is a* $(l+2)$*-sheeted covering of an elliptic curve*, $\mathrm{Cliff}(X) = k + 2$, *and*

$$|L| = \begin{cases} g_{2(r+1)}^r & \text{for } l = 0 \\ g_{4(l+2)}^3 & \text{for } l > 0; \end{cases}$$

(b) *if* k *is odd i.e.* $k = 2l + 1$, *then*

(i) $\pi : X \to Y \subset \mathbf{P^r}$ *is a* $(l+2)$*-sheeted covering of a rational curve* Y, $\mathrm{Cliff}(X) = l - 1$, *and* $|L| = g_{3(l+2)}^3 = 3g_{(l+2)}^1$;

(ii) $k = 3$, X *is a trigonal curve*, $\mathrm{Cliff}(X) = 1$, *and* $|L| = g_{12}^4 = 4g_3^1$;

(iii) $k = 3$, X *is a triple covering of an elliptic curve*, $\mathrm{Cliff}(X) = 4$, *and* $|L| = g_{15}^4$;

(iv) X *is a* $(l+2)$*-sheeted covering of an elliptic curve*, $\mathrm{Cliff}(X) = k$, *and* $|L| = g_{4(l+2)}^3$.

Proof. Let $c = \mathrm{Cliff}(X)$ and $d' = \deg Y$. Then we have

$$d' = d/m = \{(c+k) + 2r\}/m$$

and consider the induced complete linear series $g^r_{d'}$ on the normalization of Y. If $g^r_{d'}$ is not very ample, then there is a $g^{r-1}_{d'-2}$ ([3], IV. 3.1) on the normalization of Y. Hence X admits a $g^{r-1}_{m(d'-2)}$ and we get

$$m(d'-2) - 2(r-1) = c + k - 2(m-1) \leq \begin{cases} c-2 & \text{for } k = 2l \\ c-1 & \text{for } k = 2l+1. \end{cases}$$

This contradicts with the definition of $\mathrm{Cliff}(X)$. Therefore $g^r_{d'}$ is very ample.

Thus Y is a smooth non-degenerate and linearly normal curve of degree d' in $\mathbf{P^r}$. We assume that $d' \geq r+2$. Since it is well-known that any reduced irreducible and non-degenerate curve of degree $\geq r+2$ $(r \geq 3)$ in $\mathbf{P^r}$ has an r-secant-$(r-2)$-plane, we have a projection of Y onto $\mathbf{P^1}$ with center an r-secant-$(r-2)$-plane, and we obtain a $g^1_{d'-r}$ on Y. Hence there is a $g^1_{m(d'-r)}$ on X and we get

$$m(d'-r) - 2 = c + k - 2 - (m-2)r < \begin{cases} c-2 & \text{for } k = 2l \\ c-1 & \text{for } k = 2l+1. \end{cases}$$

This is a contradiction.

Therefore Y in $\mathbf{P^r}$ is the following 3 cases:

(1) Y is a rational normal curve of degree r;

(2) Y is a rational curve of degree $r+1$;

(3) Y is an elliptic curve of degree $r+1$.

Case (1): In this case, X has a g^1_m and $d' = r$. Hence we have

$$m - 2 = \{(c+k) + 2r\}/r - 2 = (c+k)/r \geq c. \tag{I}$$

By deriving from (I), we get $k \geq c(r-1)$. Since $r \geq 3$,

$$c \leq \begin{cases} k/2 = l & \text{for } k = 2l \\ (k-1)/2 = l & \text{for } k = 2l+1. \end{cases}$$

First we shall consider the case of $k = 2l$. Let $c = k/2 = l$. We have $m - 2 = 3l/r \geq l$ by (I). If $l = 0$, then $m = 2$ for any r. If $l > 0$, then $3 \geq r$. Since $r \geq 3$, we get $r = 3$ and $m = l+2$. This case is (i) of (a). Let $c \leq l-1$. In this case, since $c + k \leq l - 1 + 2l = 3l - 1 < 3l$, we get $(c+k)/r < l$, whence $m < l+2$ by (I). This contradicts the assumption on m.

Next we shall consider the case of $k = 2l+1$. Let $c = (k-1)/2 = l$. We have $m - 2 = (3l+1)/r \geq l$ by (I). Hence we get $l = 1$, $r = 4$, and $m = 3$. This case is (ii) of (b). Let $c \leq l-1$, whence we obtain $(c+k)/r \leq 3l/r \leq l$. On the other hand, we have $l \leq (c+k)/r$ by combining the assumption about m with (I). Hence we get $(c+k)/r = l$, $m = l+2$, $r = 3$, and $c = l-1$. This case is (i) of (b).

Case (2): In this case, X has a g^1_m and $d' = r+1$, whence we have

$$m - 2 = \{(c+k) + 2r\}/(r+1) - 2 = (c+k-2)/(r+1) \geq c. \qquad \text{(II)}$$

By deriving from (II), we obtain $k - 2 \geq cr \geq 0$. Since $r \geq 3$, we get $c \leq (k-2)/3 \leq (k-2)/2$. By using (II) again, we have $m - 2 \leq \{(k-2)/2 + k - 2\}/(r+1) \leq k/2 - 1$. This contradicts the assumption about m.

Case (3): In this case, X has a g^1_{2m} and $d' = r+1$. Therefore we have

$$2m - 2 = 2(c + k + 2r)/(r+1) - 2 = 2(c + k - 1 + r)/(r+1) \geq c. \qquad \text{(III)}$$

By deriving from (III), we get $(c-2)(r-1) \leq 2k$. Since $r \geq 3$, we have $c \leq k+2$.

First we shall consider the case of $k = 2l$. Let $c = k+2$. We have $m = 2k/(r+1) + 2$. If $l = 0$, then $m = 2$ for any r. If $l > 0$, then $m \leq 2k/(3+1) + 2 = k/2 + 2$. By virtue of the assumption, we get $m = k/2 + 2$ and $r = 3$. This case is (ii) of (a). Let $c \leq k+1$. We have $m < k/2 + 2$. This contradicts the assumption.

Next we shall consider the case of $k = 2l+1$. By the same way stated above, we have $m \leq k/2 + 2$. Combining this with the assumption, we get $m = (k-1)/2 + 2$ and $c \leq k+1$. If $c = k+1$, then $m = (k-1)/2 + 2 = (2k-1)/(r+1) + 2$, whence we obtain $k = 3$, $r = 4$, and $m = 3$. This case is (iii) of (b). Let $c = k$. We have $m \leq (k-1)/2 + 2$. By virtue of the assumption, we get $m = (k-1)/2 + 2$ and $r = 3$. This case is (iv) of (b). Let $c \leq k-1$. We have $m < (k-1)/2 + 2$. This contradicts the assumption. □

Needless to say, the case of $k = 0$ in (2.1) coincides with (1.1). Relating to this case, we shall prove the following lemma needed later.

Lemma 2.2. *Let a* $|L| = g^r_d$ *on* X *compute* Cliff(X), $r \geq 2$. *Assume that* g^r_d *is not simple and defines a morphism* $\pi : X \to \mathbf{P}^\mathbf{r}$, *of degree* $m \geq 2$ *onto a curve* Y *in* $\mathbf{P}^\mathbf{r}$. *Then* $\deg \pi = m = 2$.

Proof. Let $\deg Y = d'$ and $|L_0| = g^r_{d'}$ be the induced complete linear series on the normalization of Y. If $m \geq 3$, then we have $h^0(L(-\pi^* y)) = h^0(\pi^*(L_0(-y))) \geq h^0(L_0(-y)) = r \geq 2$, $h^1(L(-\pi^* y)) \geq 2$, and $\mathrm{Cliff}(L(-\pi^* y)) < \mathrm{Cliff}(X)$ for any $y \in Y$. This is a contradiction. Therefore $m = 2$. □

Here we shall provide the proof for the case of $k \geq 3$ by using (2.1).

Proof of Theorem 1.4. Thanks to (2.1), we have only to consider the four cases. If X is both a hyperelliptic curve and a trigonal curve, then we have $g \le 2$ by Castelnuovo's lemma ([1], p. 366, C-1). It is a contradiction. If X is a trigonal curve, then we are in case (1). If X is a triple covering of an elliptic curve and $\mathrm{Cliff}(X) = 4$, then we are in case (2). Let X be a triple covering of an elliptic curve and $\mathrm{Cliff}(X) = 3$. Since we assume that $g \ge 16$, X carries g_5^1 by virtue of [5]. Applying Castelnuovo's lemma ([1], p. 366, C-1), we get $g \le 11$. It is a contradiction. □

Proof of Theorem 1.5. Assume that the g_d^r is not simple and defines a morphism $\pi : X \to \mathbf{P}^r$, of degree $m \ge 2$ onto a curve Y in $\mathbf{P}^r$. Let $c = \mathrm{Cliff}(X)$ and $d' = \deg Y$. Then we have

$$d' = d/m = \{(c+4) + 2r\}/m$$

and consider the induced complete linear series $g_{d'}^r$ on the normalization of Y. If $m \ge 4$, we get the case (1) and (2) by means of (2.1). Let $m = 3$. If $g_{d'}^r$ is not very ample, X has a $g_{3(d'-2)}^{r-1}$ and we get $3(d'-2) - 2(r-1) = c$. By virtue of (2.2), X must be a double covering of a curve. This contradicts $m = 3$. Hence $g_{d'}^r$ is very ample and we can get $d' \le r+1$ by repeating the way used in (2.1).

Here we may use the same method stated in (2.1). If Y is a rational normal curve of degree r, then

$$3 - 2 = (c + 4 + 2r)/r - 2 = (c+4)/r \ge c.$$

Since $4 \ge c(r-1)$ and $r \ge 3$, we get $c \le 2$. Let $c = 0$. Since X is both a hyperelliptic and a trigonal curve, we have $g \le 2$ by ([1], p. 366, C-1). It is a contradiction. Let $c = 1$. We get $r = 5$ and we are in case (3). Let $c = 2$. We have $1 = 5/r \ge 2$ and it is a contradiction. If Y is a rational curve of degree $r+1$, then

$$3 - 2 = (c + 4 + 2r)/(r+1) - 2 = (c+2)/(r+1) \ge c.$$

Since $2 \ge rc$ and $r \ge 3$, we obtain $c = 0$. Let $c = 0$. We have $1 = 2/(r+1) \ge 0$, whence $r = 1$. This contradicts the assumption. If Y is an elliptic curve of degree $r+1$, then

$$2 \cdot 3 - 2 = 2(c + 4 + 2r)/(r+1) - 2 = 2(c + 3 + r)/(r+1) \ge c.$$

Since $(c-2)(r-1) \le 8$ and $r \ge 3$, we get $c \le 6$. Let $c = 0$ (resp. $c = 1$). We get $r = 1$ (resp. $r = 2$). It is a contradiction. Let $c = 2$. Since we assume that $g \ge 16 > 10$, X carries g_4^1 by virtue of [5]. Applying ([1], p. 366, C-1), we get $g \le 9$. It is a contradiction. Let $c = 3$. By the same way, we obtain $g \le 11$ and this is a contradiction. Let $c = 4$. We obtain $r = 5$ and we are in case (4). Let $c = 5$ (resp. $c = 6$). We have $4 \geq c = 5$ (resp. $4 \geq c = 6$). It is a contradiction. If $m = 2$, then we are in (5). □

Proof of Theorem 1.6. Assume that the g_d^r is not simple and defines a morphism $\pi : X \to \mathbf{P}^r$, of degree $m \geq 2$ onto a curve Y in $\mathbf{P}^r$. Let $c = \mathrm{Cliff}(X)$ and $d' = \deg Y$. Then

$$d' = d/m = \{(c + k) + 2r\}/m$$

and consider the induced complete linear series $g_{d'}^r$ on the normalization of Y. We shall prove this theorem by induction on $k(\geq 5)$.

First let $k = 5$. If $m = 2$, then we are in case (3). If $m \geq 4$, thanks to (2.1) there doesn't exist the g_d^r which satisfies our condition. Let $m = 3$. If $g_{d'}^r$ is not very ample, X has a $g_{3(d'-2)}^{r-1}$ and $3(d' - 2) - 2(r - 1) = c + 1$. By applying the results of the case of $k \leq 1$, X has no complete linear series $|M|$ which satisfies $\mathrm{Cliff}(M) \leq c + 1$. It is a contradiction. Therefore the $g_{d'}^r$ is very ample and we may apply the same method used in (2.1). If $\deg Y \geq r + 2$, then we have a projection of Y onto $\mathbf{P}^1$ with center an r-secant-$(r - 2)$-plane and obtain a $g_{d'-r}^1$ on Y. Therefore there is a $g_{3(d'-r)}^1$. Since $r \geq 4$, we get

$$3(d' - r) - 2 = c + 3 - r \leq c - 1 < c.$$

This is a contradiction, whence $\deg Y \leq r + 1$.

If Y is a rational normal curve of degree r, then

$$3 - 2 = (c + 5 + 2r)/r - 2 = (c + 5)/r \geq c.$$

Since $5 \geq c(r - 1)$ and $r \geq 4$, we get $c \leq 1$. Let $c = 0$. By using ([1], p. 366, C-1), we get $g \leq 2$. It is a contradiction. Let $c = 1$. We get $r = 6$ and we are in case (1).

If Y is a rational curve of degree $r + 1$, then

$$3 - 2 = (c + 5 + 2r)/(r + 1) - 2 = (c + 3)/(r + 1) \geq c.$$

Since $3 \geq rc$ and $r \geq 4$, we obtain $c = 0$. Let $c = 0$. We have $1 = 3/(r + 1) \geq 0$, whence $r = 2$. This contradicts the assumption.

If Y is an elliptic curve of degree $r + 1$, then

$$2 \cdot 3 - 2 = 2(c + 5 + 2r)/(r + 1) - 2 = 2(c + 4 + r)/(r + 1) \geq c.$$

Since $(c - 2)(r - 1) \leq 10$ and $r \geq 4$, we get $c \leq 5$. Let $c = 0$ (resp. $c = 1$). We get $r = 2$ (resp. $r = 3$). It is a contradiction. Let $c = 2$ (resp. $c = 3$). By the same way we stated in the proof of (1.5), we get $g \leq 9$ (resp. $g \leq 11$). It is a contradiction.

Let $c = 4$. We get $r = 6$ and we are in case (2). Let $c = 5$. We have $4 \geq c = 5$. It is a contradiction.

Next let $k \geq 6$. If $m = 2$, then we are in case (3). If

$$m \geq \begin{cases} k/2 + 2 = l + 2, & \text{for } k = 2l \\ (k-1)/2 + 2 = l + 2, & \text{for } k = 2l + 1, \end{cases}$$

by virtue of (2.1) there doesn't exist the g_d^r which satisfies our condition. Let $m = 3$. If $g_{d'}^r$ is not very ample, X has $g_{3(d'-2)}^{r-1}$ and $3(d'-2) - 2(r-1) = c + k - 4$. Here we may apply the results of the case of $k \leq 5$ and the induction hypothesis to the complete linear series $|N|$ which satisfies $\mathrm{Cliff}(N) \leq c + k - 4$. Since we assumed the X is a triple covering of a curve, we obtain $r - 1 \leq (k-4) + 1 = k - 3$. This contradicts $r - 1 \geq k - 2$. Let $4 \leq m \leq l + 1$. If $g_{d'}^r$ is not very ample, we get a contradiction by the same way. Therefore the $g_{d'}^r$ is very ample and we can repeat the way stated in the the case of $k = 5$. If $\deg Y \geq r + 2$, by virtue of a projection of Y onto $\mathbf{P}^1$ with center an r-secant-$(r-2)$-plane we obtain a $g_{d'-r}^1$ on Y and a $g_{m(d'-r)}^1$ on X. Since $m \geq 3$ and $r \geq k - 1$, we get

$$m(d'-r) - 2 = c + k - 2 - (m-2)r \leq c + k - 2 - (3-2) \cdot (k-1) = c - 1 < c.$$

Since this contradicts the definition of $\mathrm{Cliff}(X)$, we have $\deg Y \leq r + 1$.

If Y is a rational normal curve of degree r, then

$$m - 2 = (c + k + 2r)/r - 2 = (c + k)/r \geq c.$$

Since $k \geq c(r-1)$ and $r \geq k - 1$, we get $c \leq 1$. Let $c = 0$. We get $r = k$, $m = 3$. Applying ([1], p. 366, C-1), we get $g \leq 2$. It is a contradiction. Let $c = 1$. We get $r = k + 1$, $m = 3$, and we are in case (1).

If Y is a rational curve of degree $r + 1$, then

$$m - 2 = (c + k + 2r)/(r+1) - 2 = (c + k - 2)/(r+1) \geq c.$$

Since $k - 2 \geq cr$ and $r \geq k - 1$, we obtain $c = 0$. Therefore we have $1 \leq m - 2 = (k-2)/(r+1)$, whence $r \leq k - 3$. This contradicts $r \geq k - 1$.

If Y is an elliptic curve of degree $r + 1$, then

$$2m - 2 = 2(c + k + 2r)/(r+1) - 2 = 2(c + k - 1 + r)/(r+1) \geq c.$$

Since $(c-2)(r-1) \leq 2k$ and $r \geq k - 1$, we get $c \leq 5$. Let $c = 0$ (resp. $c = 1$). We have $2 \leq (k - 1 + r)/(r+1)$ (resp. $2 \leq (k + r)/(r+1)$), whence $r \leq k - 3$ (resp. $r \leq k - 2$). These contradicts the assumption. Let $c = 2$ (resp. $c = 3$). We get $(r, m) = (k-1, 3)$ (resp. $(r, m) = (k, 3)$). But the same reason we stated above in the case of $k = 5$, we can omit these cases. Let $c = 4$. We obtain $2 \le (k + r + 3)/(r + 1)$, whence $r = k + 1$, $m = 3$, and we are in case (2). Let $c = 5$. We get $2 \le (k + r + 4)/(r + 1)$, whence $r = k + 2$ and $m = 3$. But this contradicts $2(m - 1) \ge c$. □

Furthermore, if d is odd and $r \ge k + 2$, then we can provide a sufficient condition for g_d^r being simple by using (1.2), (1.3), (1.4), (1.5), and (1.6).

Corollary 2.3. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \ge 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \ge 1)$, $g - 1 - d + r \ge 1$. *If* d *is odd and* $r \ge k + 2$, *then* g_d^r *is simple.*

3. Some Corollaries

In this section, we shall derive some corollaries from (2.3).

Corollary 3.1. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \ge 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \ge 1)$ *and* $g - 1 - d + r \ge 1$. *If* d *is odd*, $\mathrm{Cliff}(X) \ge 2k + 1$, *and*

$$3(\mathrm{Cliff}(X) + 2)/2 + 2k < d < 2(\mathrm{Cliff}(X) + 2),$$

then we have $g \le 2(\mathrm{Cliff}(X) + 2 + k)$.

Proof. Let $c = \mathrm{Cliff}(X)$. If $d \ge g$, then this is trivial. Therefore we may assume $d \le g - 1$. Since $d > 3(c + 2)/2 + 2k$ and $c \ge 2k + 1$, we get $2r = d - c - k > 2(k + 1)$, whence $r \ge k + 2$. By using (2.3), we have g_d^r is simple and X may be regarded as a curve of degree d in $\mathbf{P^r}$. Assume that X lies on a quadric $\mathbf{Q}$ in $\mathbf{P^r}$ of rank $s \le 4$. Then the two pencils of $(r - 2)$-planes on $\mathbf{Q}$ sweep out pencil g_a^1 and g_b^1 on X such that $a + b \le d$, $a - 2 \ge c$ and $b - 2 \ge c$. This contradicts $d < 2(c + 2)$. Therefore the space of quadrics containing X does not meet the closed subvariety of quadrics of rank $s \le 4$ in $\mathbf{P^r}$ which has codimension $(r - 3)(r - 2)/2$ in the projective space of all quadrics in $\mathbf{P^r}$. Hence we have $h^0(\mathbf{P}^r, I_X(2)) \le (r - 3)(r - 2)/2$ and

$$\begin{aligned} h^0(X, \mathcal{O}_X(2)) &\ge h^0(\mathbf{P}^r, \mathcal{O}(2)) - h^0(\mathbf{P}^r, I_X(2)) \\ &\ge (r + 1)(r + 2)/2 - (r - 3)(r - 2)/2 = 4r - 2. \end{aligned}$$

Since this means $h^0(2L) \ge 4r - 2$, we get

$$g = 2d + 1 - h^0(2L) + h^1(2L) \le 2d + 1 - (4r - 2) + h^1(2L).$$

If $h^1(2L) \le 1$, then we have $g \le 2(c+2+k)$. If $h^1(2L) \ge 2$, then we have

$$c \le 2d - 2(h^0(2L) - 1) \le 2d - 2(4r - 2) + 2 = 4c + 4k + 6 - 2d.$$

This contradicts our hypothesis on d. □

By applying the famous formula of Castelnuovo ([1], p. 116) to the result stated above, we get the following corollary.

Corollary 3.2. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \ge 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \ge 1)$ *and* $g - 1 - d + r \ge 1$. *If* d *is odd,* $d \le g - 1$, $\mathrm{Cliff}(X) \ge 2k + 1$, *and* $g \ge 3(\mathrm{Cliff}(X) + k + 1)$, *then*

$$d \le 3(\mathrm{Cliff}(X) + 2)/2 + 2k, \quad 2(\mathrm{Cliff}(X) + 2) \le d < 2(\mathrm{Cliff}(X) + 2 + k).$$

Proof. Let $c = \mathrm{Cliff}(X)$. By virtue of (3.1), we may assume that $d \ge 2(c+2+k)$. Since $c \ge 2k+1$, we have $2r = d - c - k \ge c + k + 4 \ge 2(k+1) + k + 3 > 2(k+1)$ *i.e.* $r \ge k+2$, whence g_d^r is simple by (2.3). We remark that $2(r-1) > c + k + 1$. Hence we have $d - 1 = 2(r-1) + c + k + 1 < 4(r-1)$. Therefore we see that $2 < (d-1)/(r-1) < 4$. Let $m = [(d-1)/(r-1)]$. In our case we have m is 2 or 3. Now we use Castelnuovo's bound ([1], p. 116).

If $m = 2$, then we get $g \le 2d - 3r + 1$. Since $[(d-1)/(r-1)] = [\{c+k+1+2(r-1)\}/(r-1)] = 2$, we have $c+k+1 < r-1$, whence $d = 2r + c + k < 2r + (r-2) = 3r - 2 < 3r - 1 \le 2d - g$. Therefore $g < d$. It is a contradiction.

If $m = 3$, then we get $g \le 3(d - 2r + 1) = 3(c+k+1)$. Since $g \ge 3(c+k+1)$, we get $g = 3(c+k+1)$, *i.e.* X has the maximum possible genus. It is known that these curves of maximal genus have a g_4^1 ([1], III, (2.6)). Therefore $\mathrm{Cliff}(X) \le 2$. This contradicts $\mathrm{Cliff}(X) \ge 2k+1$. □

In [4] we find the result that $d \le 3\mathrm{Cliff}(X)$ for $\mathrm{Cliff}(X) \ge 3$. We shall present a similar type result in next corollary.

Corollary 3.3. *Let a* $|L| = g_d^r$ *be a base-point-free linear series on* X *of genus* $g \ge 16$. *Assume that* $\mathrm{Cliff}(L) = \mathrm{Cliff}(X) + k$ $(k \ge 1)$ *and* $g - 1 - d + r \ge 1$. *If* d *is odd,* $d \le g - 1$, *and* $\mathrm{Cliff}(X) \ge 2k + 1$, *then we have* $d \le 3(\mathrm{Cliff}(X) + k)$.

Proof. Let $c = \mathrm{Cliff}(X)$. If $g \ge 3(c+k+1)$, then thanks to (3.2) we obtain $d \le 2(c+k) + 3 \le 3(c+k)$. If $g \le 3(c+k) + 2$, then we have $d \le g - 1 \le$

$3(c+k)+1$. But $d = 3(c+k)+1$ means $d \not\equiv c+k \pmod 2$. Since $d-2r = c+k$, this is a contradiction. □

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