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RIEMANNIAN MANIFOLDS REFERRED TO WARPED PRODUCT MODELS

By

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Abstract. We prove a sphere theorem for manifolds referred to spherical warped product models and obtain the optimal result.

1. Introduction

We investigate curvature and topology of Riemannian manifolds referred to warped product models. A *spherical warped product model* $(\tilde{M}, \mathbf{S}^{n-1})$ is by definition a pair $(\tilde{M}, \mathbf{S}^{n-1})$ of compact Riemannian n-manifold $\tilde{M}$ and the standard unit $(n-1)$-sphere $\mathbf{S}^{n-1}$ which is totally geodesically embedded into $\tilde{M}$. Its metric $d\tilde{s}^2$ is expressed in terms of the normal exponential map along $\mathbf{S}^{n-1}$ as:

$$d\tilde{s}^2 = dt^2 + f^2(t)\, ds^2_{\mathbf{S}^{n-1}}(\Theta), \quad (t, \Theta) \in (-\tilde{\ell}_-, \tilde{\ell}_+) \times \mathbf{S}^{n-1}. \tag{1.1}$$

Here $\tilde{\ell}_-, \tilde{\ell}_+ < \infty$ are constants and $t : \tilde{M} \to [-\tilde{\ell}_-, \tilde{\ell}_+]$ is the oriented distance function to $\mathbf{S}^{n-1}$. The function $f : (-\tilde{\ell}_-, \tilde{\ell}_+) \to \boldsymbol{R}$ is positive smooth and called *the warping function of* $\tilde{M}$, and satisfies

$$\lim_{t \downarrow -\tilde{\ell}_-} f(t) = \lim_{t \uparrow \tilde{\ell}_+} f(t) = 0$$

and the Jacobi equation

$$f'' + Kf = 0, \quad f(0) = 1, \quad f'(0) = 0.$$

Here the function $K : [-\tilde{\ell}_-, \tilde{\ell}_+] \to \boldsymbol{R}$ is called the *radial curvature function of* $(\tilde{M}, \mathbf{S}^{n-1})$.

We discuss a pair (M, N) of Riemannian manifolds, where N is totally geodesically embedded into M such that the oriented distance function $\rho_N : M \to \boldsymbol{R}$ is well defined. A unit speed geodesic $\gamma : [0, a] \to M$ is called a

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minimizing geodesic from N if and only if $\gamma'(0)$ is normal to N and $|\rho_N(\gamma(t))| = t$, $t \in [0, a]$. A plane section containing $\gamma'(t)$ is called a *radial plane* and the sectional curvature $K_M(\Pi)$ with respect to a radial plane Π is called the *radial curvature* of (M, N). The class of all the pairs (M, N) of complete n-manifold M and compact $(n-1)$-manifold N totally geodesically embedded into M has been classified into seven types (see Theorem 1.1: [6]), if the radial curvature of (M, N) depends only on the (oriented) distance function to N. A similar result for model surfaces of revolution was obtained in [2] (see Theorem A).

The purpose of the present paper is to prove a topological sphere theorem for manifolds referred to spherical warped product models. Moreover, we prove the optimal topological sphere theorem.

Definition 1.1. $(M, \mathbf{S}^{n-1})$ is by definition referred to a spherical warped product model $(\tilde{M}, \mathbf{S}^{n-1})$ if and only if every radial curvature at every point $p \in M$ is bounded below by $K(\rho_N(p))$.

Recently, Kondo and Ohta have obtained the following optimal result when the reference space $(\tilde{M}, \tilde{o})$ is a von Mangoldt surface of revolution, that is, the radial curvature function $K : [0, \tilde{\ell}) \to \boldsymbol{R}$ of $(\tilde{M}, \tilde{o})$ is monotone and non-increasing.

Theorem A [Kondo-Ohta; [3]]. *Let (M, o) be a compact Riemannian n-manifold whose radial curvature is bounded from below by $K : [0, \tilde{\ell}) \to \boldsymbol{R}$ for $\tilde{\ell} < \infty$, and let $\delta(\tilde{o}^*)$ be the convexity radius at $\tilde{o}^*$. If*

$$\ell := \sup_{x \in M} \rho_o(x) > \tilde{\ell} - \delta(\tilde{o}^*)$$

and if $\tilde{o}^* \in M$ is a critical point for the distance function to o, then (M, o) is homeomorphic to $\mathbf{S}^n$. Here, $\tilde{o}^* \in \tilde{M}$ is the point furthest from $\tilde{o}$, that is, $\tilde{\ell} := d(\tilde{o}, \tilde{o}^*)$.

In a previous paper [4], we have proved the following theorem. We don't require a model surface to be a von Mangoldt surface of revolution.

Let $(\tilde{M}, \tilde{o})$ be a model surface of revolution with $\tilde{\ell} < \infty$ and $f : (0, \tilde{\ell}) \to \boldsymbol{R}^+$ be a warping function of $(\tilde{M}, \tilde{o})$. Let $\delta(\tilde{M})$ be the convexity radius of $\tilde{M}$ and $\eta(\tilde{M}) > 0$ be a constant such that

$$\eta(\tilde{M}) := \inf\left\{\eta > 0;\, f(\eta) = \min_{\delta(\tilde{o}) \le t \le \tilde{\ell} - \delta(\tilde{o}^*)} f(t),\, \eta \in [0, \delta(\tilde{o})] \cup [\tilde{\ell} - \delta(\tilde{o}^*), \tilde{\ell}]\right\}.$$

Let $\varepsilon(\tilde{M})$ is defined by

$$\varepsilon(\tilde{M}) := \min\left\{\frac{1}{3}\delta(\tilde{M}), \eta(\tilde{M})\right\}.$$

With these notations, we state as follows.

Theorem B (Lee; [4]). *Let (M, o) be referred to $(\tilde{M}, \tilde{o})$. Then M is homeomorpic to $\mathbf{S}^n$ if*

$$\operatorname{diam}(M) \le \tilde{\ell}, \quad \delta(o) \ge \delta(\tilde{M}), \quad \ell > \tilde{\ell} - \varepsilon(\tilde{M}).$$

We observe that every compact model surface of revolution $(\tilde{M}, \tilde{o})$ can be thought of as a spherical warped product model. Therefore, we discuss when (M, N) is referred to a spherical warped product model.

Some notations are needed to state our theorem.

Let $(\tilde{M}, \mathbf{S}^{n-1})$ be a spherical warped product model. The oriented distance function $t : \tilde{M} \to [-\tilde{\ell}_-, \tilde{\ell}_+]$ attains its extremal value at the points $\tilde{o}_\pm \in \tilde{M}$ such that $t(\tilde{o}_\pm) = \pm\tilde{\ell}_\pm$ respectively. Let $\tilde{\delta}_\pm > 0$ be the convexity radius at $\tilde{o}_\pm$. We then observe that $f'(-\tilde{\ell}_- + \tilde{\delta}_-) = f'(\tilde{\ell}_+ - \tilde{\delta}_+) = 0$ and $f'(t) \ne 0$ on $t \in [-\tilde{\ell}_-, -\tilde{\ell}_- + \tilde{\delta}_-) \cup (\tilde{\ell}_+ - \tilde{\delta}_+, \tilde{\ell}_+]$. Let $(M, \mathbf{S}^{n-1})$ be referred to $(\tilde{M}, \mathbf{S}^{n-1})$. In [5], we have proved that the oriented distance function $\rho_N : M \to \boldsymbol{R}$ attains its minimum and maximum at a unique point, say, $o_\pm \in M$ such that $\rho_N(o_\pm) = \pm\ell_\pm$. Setting $\tilde{M}_\pm := \{\tilde{p} \in \tilde{M} \mid t(\tilde{p}) \gtrless 0\}$ and $M_\pm := \{p \in M \mid \rho_N(p) \gtrless 0\}$, we compare $M_\pm$ to $\tilde{M}_\pm$ respectively. We define positive constants $\eta_\pm(\tilde{M}_\pm)$ on $\tilde{M}_\pm$ by

$$\eta_+(\tilde{M}_+) := \sup\left\{\eta > 0;\ f(\tilde{\ell}_+ - \lambda) = \min_{0 \le t \le \tilde{\ell}_+ - \lambda} f(t) \text{ for } \forall\lambda \in [0, \eta)\right\},$$

$$\eta_-(\tilde{M}_-) := \sup\left\{\eta > 0;\ f(-\tilde{\ell}_- + \lambda) = \min_{-\tilde{\ell}_- + \lambda \le t \le 0} f(t) \text{ for } \forall\lambda \in [0, \eta)\right\}$$

and further,

$$\varepsilon(\tilde{M}) := \min\{\tilde{\delta}_+, \tilde{\delta}_-, \eta_+(\tilde{M}_+), \eta_-(\tilde{M}_-)\}.$$

With this notation our result is stated as:

Theorem I. *Let $(M, \mathbf{S}^{n-1})$ be referred to $(\tilde{M}, \mathbf{S}^{n-1})$. Then M is homeomorphic to $\mathbf{S}^n$ if*

$$\ell_\pm > \tilde{\ell}_\pm - \varepsilon(\tilde{M}).$$

Remark 1.2. The conditions in Theorem I are optimal in the sense that if one of the two inequality is not satisfied, we then have a counter example, as stated later.

The Bishop-Gromov volume comparison theorem is valid for $(M, \mathbf{S}^{n-1})$ referred to $(\tilde{M}, \mathbf{S}^{n-1})$. From the Berger comparison theorem on the focal point distance to N, we observe

$$\ell_{\pm} \leq \tilde{\ell}_{\pm}$$

and equality holds if and only if $\rho_N^{-1}[0, \tilde{\ell}_+]$ is isometric to $t^{-1}[0, \tilde{\ell}_+]$, etc. The Bishop-Gromov volume comparison theorem in our case is stated as follows.

Proposition 1.3. *Let* $(M, \mathbf{S}^{n-1})$ *be referred to* $(\tilde{M}, \mathbf{S}^{n-1})$. *Then*

$$s \mapsto \frac{\mathrm{vol}\, \rho_N^{-1}[0, s]}{\mathrm{vol}\, t^{-1}[0, s]}, \quad s \geq 0$$

and

$$s \mapsto \frac{\mathrm{vol}\, \rho_N^{-1}[s, 0]}{\mathrm{vol}\, t^{-1}[s, 0]}, \quad s \leq 0$$

is monotone non-increasing on $s \in [0, \tilde{\ell}_+]$ *and monotone non-decreasing on* $s \in [-\tilde{\ell}_-, 0]$.

Making use of the above Proposition we have the following

Corollary to Theorem I. *Let* $(M, \mathbf{S}^{n-1})$ *be referred to* $(\tilde{M}, \mathbf{S}^{n-1})$. *Let* $M_+ := \rho_N^{-1}(0, \ell_+]$ *and* $M_- := \rho_N^{-1}[-\ell_-, 0)$. *Then* M *is homeomorphic to* $\mathbf{S}^n$ *if*

$$\mathrm{vol}(M_+) > \mathrm{vol}(t^{-1}[0, \tilde{\ell}_+ - \varepsilon(\tilde{M})])$$

and

$$\mathrm{vol}(M_-) > \mathrm{vol}(t^{-1}[-\tilde{\ell}_- + \varepsilon(\tilde{M}), 0]).$$

We observe that the two volume conditions are optimal by the same reason as in Theorem I.

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2. Review of the Known Results

Because our models do not have constant curvature, we can not apply the spherical trigonometry. The Clairaut relation gives restrictions to the behavior of geodesics on models and plays an essential role for our study. The following proposition is valid for all the warped product models as stated in the classification [6].

Proposition 2.1 (Mashiko-Shiohama; The Clairaut Relation for warped product models). *Let $(\tilde{M}, N)$ be a warped product model with the metric* (1.1). *Let $\tilde{\gamma} : \boldsymbol{R} \to \tilde{M}$ be a geodesic transversal to a meridian. If we set*

$$\alpha(s) := \angle(\tilde{\gamma}'(s), \nabla t(\tilde{\gamma}(s))) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad s \in \boldsymbol{R}$$

then there exists a constant $\mathscr{C}(\tilde{\gamma})$ depending only on $\tilde{\gamma}$ such that

$$f(t(\tilde{\gamma}(s))) \sin \alpha(s) = \mathscr{C}(\tilde{\gamma}), \quad s \in \boldsymbol{R}. \tag{2.1}$$

We next see that the axiom of plane holds for all the warped product models as stated (see [7]):

Theorem 2.2 (Mashiko-Shiohama; The Axiom of plane for warped product models). *Let $(\tilde{M}, N)$ be a warped product model and $\tilde{\gamma} : [0, a) \to \tilde{M}$ a unit speed geodesic which is transversal to a meridian. Then, $\mathscr{S}(\tilde{\gamma}) \subset \tilde{M}$ is totally geodesic. Moreover, the inner distance of $\mathscr{S}(\tilde{\gamma})$ coincides with that of $\tilde{M}$ if $\tilde{\gamma}$ is minimizing. Here $\mathscr{S}(\tilde{\gamma})$ is the ruled surface consisting of all the meridians passing through points on $\tilde{\gamma}[0, a)$.*

We finally introduce the Toponogov comparison theorem for generalized narrow triangles on $(\tilde{M}, N)$. We assume that (M, N) is referred to $(\tilde{M}, N)$. *A generalized geodesic triangle $\triangle(Nxy) \subset M$* is defined by a triple of minimizing geodesics $\alpha, \beta, \gamma : [0, 1] \to M$ such that

$$\alpha'(0), \beta'(0) \in N^{\perp}, \quad \alpha(1) = \gamma(1) = y, \quad \beta(1) = \gamma(0) = x.$$

Here $x, y \in M \backslash N$ are taken in the same component of $M \backslash N$ and α, β are minimizing geodesics from N. A $\triangle(Nxy)$ is called *a generalized narrow triangle* if and only if $\alpha(t) \in B(\beta(t), \delta(M))$, $t \in [0, 1]$. Here $\delta(M)$ is the convexity radius of M. The following theorem has been established in [6] and valid for pointed manifolds referred to model surfaces of revolution (see [1]).

Theorem 2.3 (Mashiko-Shiohama; Generalized narrow triangle comparison). *Assume that* (M, N) *is referred to* $(\tilde{M}, N)$. *Assume further that a generalized narrow triangle* $\triangle(Nxy) \subset M$ *admits the corresponding generalized narrow triangle* $\triangle(N\tilde{x}\tilde{y}) \subset \tilde{M}$ *such that*

$$d(N, x) = d(N, \tilde{x}), \quad d(N, y) = d(N, \tilde{y}), \quad d(x, y) = d(\tilde{x}, \tilde{y}). \tag{2.2}$$

Then we have

$$\angle Nxy \geq N\tilde{x}\tilde{y}, \quad \angle Nyx \geq N\tilde{y}\tilde{x}. \tag{2.3}$$

Notice that the existence of the corresponding generalized narrow triangle $\triangle(N\tilde{x}\tilde{y}) \subset \tilde{M}$ is ensured by the Berger comparison theorem for focal point distance.

3. Example of Spherical Warped Product Model

Let $a > 0$ be a constant. A point $(x_1, \ldots, x_{n+1}) \in \hat{M} \subset \mathbf{R}^{n+1}$ on a convex C^1-hypersurface in $\mathbf{R}^{n+1}$ is expressed as:

$$\begin{cases} (x_{n+1} + a)^2 + \sum_{i=1}^{n} x_i^2 = 1, & x_{n+1} \leq -a \\ \sum_{i=1}^{n} x_i^2 = 1, & -a \leq x_{n+1} \leq a \\ (x_{n+1} - a)^2 + \sum_{i=1}^{n} x_i^2 = 1, & x_{n+1} \geq a \end{cases}$$

Let $\iota : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be the symmetry with respect to the origin and set $M := \hat{M}/\{\iota, \iota^2 = \mathrm{id.}\}$. We denote by $\pi : \hat{M} \to M$ the covering projection and set $N := \pi(x_{n+1}^{-1}(\{-a\})) \subset M$. Clearly, N is a standard unit $(n-1)$-sphere. For the pair (M, N), we define a warped product model $(\tilde{M}, N)$ as follows. A point $(x_1, x_2, \ldots, x_{n+1}) \in \tilde{M} \subset \mathbf{R}^{n+1}$ is expressed as:

$$\begin{cases} (x_{n+1} + a)^2 + \sum_{i=1}^{n} x_i^2 = 1, & x_{n+1} \leq -a \\ \sum_{i=1}^{n} x_i^2 = 1, & -a \leq x_{n+1} \leq 0 \\ \sum_{i=1}^{n+1} x_i^2 = 1, & x_{n+1} \geq 0 \end{cases}$$

Let $N := x_{n+1}^{-1}(\{-a\}) \subset \tilde{M}$. The radial curvature function $K : [-\pi/2, a + \pi/2] \to \mathbf{R}$ is given by

$$K(t) = \begin{cases} 1, & -\dfrac{\pi}{2} \le t \le 0,\ a \le t \le a + \dfrac{\pi}{2} \\ 0, & 0 < t < a \end{cases}$$

Clearly, the radial curvature of (M, N) is $K|_{[-\pi/2, a]}$. We then observe that

$$\ell_+ = \tilde{\ell}_+ - \varepsilon(\tilde{M}), \quad \ell_- > \tilde{\ell}_- - \varepsilon(\tilde{M}).$$

Here, $\ell_+ = a$, $\ell_- = \pi/2$, $\tilde{\ell}_+ = a + \pi/2$ and $\tilde{\ell}_- = \varepsilon(\tilde{M}) = \pi/2$. Therefore we see that the assumptions in Theorem I are optimal.

From the above example, we also see the volume conditions in Corollary to Theorem I are optimal. To simplify a calculation, we put $n = 2$. Then since $\tilde{\ell}_+ - \varepsilon(\tilde{M}) = (a + \pi/2) - \pi/2 = a$, we get

$$\mathrm{vol}(t^{-1}[0, \tilde{\ell}_+ - \varepsilon(\tilde{M})]) = \int_0^a \int_0^{2\pi} d\theta dt = 2\pi a = \mathrm{vol}(M_+).$$

Also, we have from $-\tilde{\ell}_- + \varepsilon(\tilde{M}) = 0$

$$\mathrm{vol}(t^{-1}[-\tilde{\ell}_- + \varepsilon(\tilde{M}), 0]) = 0 < 2\pi = \int_0^{2\pi} d\theta \int_{-\pi/2}^{0} \cos t\, dt = \mathrm{vol}(M_-).$$

4. Proofs

The crucial point of the proof of our theorem is to verify that if $(M, \mathbf{S}^{n-1})$ is referred to a spherical warped product model $(\tilde{M}, \mathbf{S}^{n-1})$ then M can be decomposed into two disks. This is achieved by showing that the distance function to $N := \mathbf{S}^{n-1}$ has exactly two critical points.

We first apply the Clairaut relation to a broken geodesic on a model surface $\tilde{M}$.

Let $\triangle(Nxy)$ be a generalized geodesic triangle in M. We choose a division $0 = u_0 < u_1 < \cdots < u_k = 1$ of $[0, 1]$ as follows. Let $\gamma : [0, 1] \to M$ be the edge of $\triangle(Nxy)$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $x_i := \gamma(u_i)$, $i = 0, \ldots, k$. Then the sequence $\{\triangle(Nx_{i-1}x_i)\}_{i=1,\ldots,k}$ has the following properties:

(1) $\triangle_i := \triangle(Nx_{i-1}x_i)$ is a narrow triangle for $i = 1, \ldots, k$,

(2) Each $\triangle_i$ admits the corresponding narrow triangle

$$\tilde{\triangle}_i := \triangle(N\tilde{x}_{i-1}\tilde{x}_i).$$

By using Theorem 2.3, we see

$$\angle N x_{i-1} x_i \geq \angle N \tilde{x}_{i-1} \tilde{x}_i, \quad \angle N x_i x_{i-1} \geq \angle N \tilde{x}_i \tilde{x}_{i-1}, \quad i = 1, \dots, k. \tag{4.1}$$

Thus we obtain a broken geodesic with vertices $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k$. We observe from (4.1) that

$$\angle \tilde{x}_{i-1} \tilde{x}_i \tilde{x}_{i+1} \leq \pi, \quad i = 1, \dots, k-1, \tag{4.2}$$

and hence $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_k$ forms a *convex* broken geodesic. Let $\tilde{\gamma} : [0,1] \to \tilde{M}$ be the broken geodesic and set $\tilde{\gamma}_i : [u_{i-1}, u_i] \to \tilde{M}$, $\tilde{\gamma}_i = \tilde{\gamma}|_{[u_{i-1}, u_i]}$, $1 \leq i \leq k$. The Clairaut constant $\mathscr{C}(\tilde{\gamma}_i)$ for $i = 1, \dots, k$ satisfies, setting $(t_i, \theta_i) := \tilde{\gamma}(u_i)$, where $t_i = d(N, \tilde{\gamma}(u_i))$,

$$\begin{aligned} \mathscr{C}(\tilde{\gamma}_i) &= f(t_{i-1}) \sin \angle(\nabla t(\tilde{\gamma}_i(u_{i-1})), \tilde{\gamma}_i'(u_{i-1})) \\ &= f(t_i) \sin \angle(\nabla t(\tilde{\gamma}_i(u_i)), \tilde{\gamma}_i'(u_i)). \end{aligned} \tag{4.3}$$

Hence from (4.2), we get the following

$$\angle(\nabla t(\tilde{\gamma}_{i+1}(u_i)), \tilde{\gamma}_{i+1}'(u_i)) \geq \angle(\nabla t(\tilde{\gamma}_i(u_i)), \tilde{\gamma}_i'(u_i)), \tag{4.4}$$

for $i = 0, \dots, k-1$.

Summing up the above discussion, we have proved the following.

Proposition 4.1. *If a broken geodesic $\tilde{\gamma} : [0,1] \to \tilde{M}$ satisfies* (4.4) *and if $t \circ \tilde{\gamma} : [0,1] \to \boldsymbol{R}$ is monotone, then $\{\mathscr{C}(\tilde{\gamma}_i)\}_{i=1,\dots,k}$ is monotone.*

Remark 4.2. Proposition 4.1 plays an important role for the proof of the non-existence of critical points of distance function to $\mathbf{S}^{n-1}$.

Proposition 4.3. *The distance function ρ_N has the following properties*:

(1) *The oriented distance functions $\pm\rho_N$ are concave on the sets $\rho_N^{-1}[\tilde{\ell}_+ - \varepsilon(\tilde{M}), \ell_+]$ and $\rho_N^{-1}[-\ell_-, -\tilde{\ell}_- + \varepsilon(\tilde{M})]$.*

(2) $\mathrm{Crit}(\rho_N) = \{o_-\} \cup \{o_+\}$, *where $o_\pm \in M$ satisfies*

$$\rho_N^{-1}(\{-\ell_-\}) = \{o_-\} \quad \textit{and} \quad \rho_N^{-1}(\{\ell_+\}) = \{o_+\}.$$

Remark 4.4. The concavity of two functions in (1) is a direct consequence of the second variation formula along every minimizing geodesic from N to a point $\rho_N^{-1}[\tilde{\ell}_+ - \varepsilon(\tilde{M}), \ell_+]$ and $\rho_N^{-1}[-\ell_-, -\tilde{\ell}_- + \varepsilon(\tilde{M})]$ respectively. Therefore we only prove the statement (2) in Proposition 4.3.

Proof of Proposition 4.3-(2). Suppose we have two points $o_+^1, o_+^2 \in M_+$ such that $\rho_N(o_+^1) = \ell_+ = \rho_N(o_+^2)$. Let $\gamma : [0,1] \to M_+$ be a minimizing geodesic with $\gamma(0) = o_+^1$ and $\gamma(1) = o_+^2$. We may assume that $\rho_N \circ \gamma : [0,1] \to \boldsymbol{R}^+$ takes a minimum at an interior point $u_* \in (0,1)$. Choose a finite division, $0 = u_0 < u_1 < \cdots < u_i < \cdots < u_k = 1$, of $[0,1]$ such that $u_* = u_i$ and such that for each $j = 1, \dots, k$ the generalized geodesic triangle $\triangle(N\gamma(u_{j-1})\gamma(u_j))$ is a narrow triangle. The narrow triangle comparison theorem (See Theorem 2.3) implies that

$$\angle\tilde{\gamma}(u_{j-1})\tilde{\gamma}(u_j)\tilde{\gamma}(u_{j+1}) \le \pi \quad \text{for } j = 1, \dots, k.$$

In particular, we have $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i-1}) \le \pi/2$ and $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}) \le \pi/2$ since $\angle N\gamma(u_i)\gamma(u_{i-1}) = \angle N\gamma(u_i)\gamma(u_{i+1}) = \pi/2$. Suppose $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}) < \pi/2$. Then (2.2) implies that

$$\rho_N(\gamma(u_{i+1})) = t(\tilde{\gamma}(u_{i+1})) < t(\tilde{\gamma}(u_i)) = \rho_N(\gamma(u_i)),$$

a contradiction. Therefore, we have $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}) = \pi/2$. Also, we have $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i-1}) = \pi/2$.

Setting $\tilde{\gamma}_j$ the edge of $\triangle(N\tilde{\gamma}(u_{j-1})\tilde{\gamma}(u_j))$ opposite to N, we have convex broken geodesics $\tilde{\gamma}_{i+1} \cup \cdots \cup \tilde{\gamma}_k$ and $\tilde{\gamma}_i \cup \cdots \cup \tilde{\gamma}_1$ with corners $\tilde{\gamma}(u_i), \tilde{\gamma}(u_{i+1}), \dots, \tilde{\gamma}(u_k) = \tilde{\gamma}(1)$ and $\tilde{\gamma}(u_i), \tilde{\gamma}(u_{i-1}), \dots, \tilde{\gamma}(u_0) = \tilde{\gamma}(0)$, respectively. They are transversal to the meridian at every point on it. The transversality follows from the Clairaut relation. Therefore, Proposition 4.1 implies that $\{\mathscr{C}(\tilde{\gamma}_j)\}_{j=i,\dots,1}$ and $\{\mathscr{C}(\tilde{\gamma}_j)\}_{j=i+1,\dots,k}$ are monotone non-decreasing and hence we have $\mathscr{C}(\tilde{\gamma}_{i+1}) \le \cdots \le \mathscr{C}(\tilde{\gamma}_k)$. Since o_+^2 is a critical point of ρ_N, we see that $\angle N\tilde{\gamma}(u_k)\tilde{\gamma}(u_{k-1}) \le \pi/2$. Thus we have, using (2.1)

$$\mathscr{C}(\tilde{\gamma}_k) = f(\rho_N(o_+^2)) \sin \angle N\tilde{\gamma}(u_k)\tilde{\gamma}(u_{k-1}) \le f(\rho_N(o_+^2)).$$

On the other hand, since $\angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}) = \pi/2$, this implies that

$$\mathscr{C}(\tilde{\gamma}_{i+1}) = f(\rho_N(\gamma(u_i))) \sin \angle N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}) = f(\rho_N(\gamma(u_i)))$$

and hence we get

$$f(\rho_N(\gamma(u_i))) \le f(\rho_N(o_+^2)). \tag{4.5}$$

From (1) in Proposition 4.3 we observe that $\rho_N \circ \gamma : [0,1] \to \boldsymbol{R}$ is concave. Therefore (4.5) implies that it is constant. In particular, we have $\angle N\gamma(u_i)\gamma(u_{i+1}) = \pi/2$. However, the corresponding generalized narrow triangle $\triangle(N\tilde{\gamma}(u_i)\tilde{\gamma}(u_{i+1}))$ has its edge angle at $\tilde{\gamma}(u_i)$ greater than $\pi/2$. This contradicts to the generalized narrow triangle comparison theorem. We have proved the uniqueness of the maximal set $\rho_N^{-1}(\{\ell_+\}) =: \{o_+\}$.

The uniqueness of the critical point of ρ_N on M_+ has already been shown in the above argument. In fact, suppose that $q \in M_+ \setminus \{o_+\}$ is a critical point of ρ_N. For a minimizing geodesic $\gamma : [0,1] \to M_+$ with $\gamma(0) = o_+$, $\gamma(1) = q$, we choose minimizing geodesics $\alpha, \beta : [0,1] \to M_+$ from N such that $\alpha(0), \beta(0) \in N$, $\alpha(1) = \gamma(0)$, $\beta(1) = \gamma(1)$ and such that

$$\angle(-\dot{\alpha}(1), \dot{\gamma}(0)) \leq \frac{\pi}{2}, \quad \angle(\dot{\beta}(1), \dot{\gamma}(1)) \leq \frac{\pi}{2}.$$

Choose a finite division $0 = u_0 < u_1 < \cdots < u_k = 1$ and $u_i \in [0,1]$ such that

$$(\rho_N \circ \gamma)(u_i) = \min_{0 \leq u \leq 1} (\rho_N \circ \gamma)(u).$$

We then observe that $(\rho_N \circ \gamma)|_{[0,u_i]}$ is constant. A contradiction is derived by the same reason. Thus we have proved that $\mathrm{Crit}(\rho_N) \cap M_+ = \{o_+\}$. □

REMARK 4.5. We can obtain the same consequence to M_- by using a method similar to that which is used in Proposition 4.3. From these facts, we see that $(M, \mathbf{S}^{n-1})$ is composed with two topological disks.

PROOF OF COROLLARY TO THEOREM I. Since $\mathrm{vol}(M_+) > \mathrm{vol}(t^{-1}[0, \tilde{\ell}_+ - \varepsilon(\tilde{M})])$ $= \mathrm{vol}(M_+) - \mathrm{vol}(t^{-1}[\tilde{\ell}_+ - \varepsilon(\tilde{M}), \tilde{\ell}_+])$, we have $\ell_+ > \tilde{\ell}_+ - \varepsilon(\tilde{M})$. □

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