TSUKUBA J. MATH.
Vol. 31 No. 2 (2007), 233–252

RICCI RECURRENT *CR* SUBMANIFOLDS OF A COMPLEX SPACE FORM

By

Mayuko Kon

Abstract. We show that there is no *CR* submanifold with semi-flat normal connection and with recurrent Ricci tensor in a complex space form of nonzero constant holomorphic sectional curvature, if the dimension of its holomorphic distribution is greater than 2.

1. Introduction

There are many results about real hypersurfaces immersed in a complex space form with additional conditions for the curvature tensor and the Ricci tensor. In [7] Kon proved that there are no Einstein real hypersurfaces of a complex projective space CP^m and determined connected complete pseudo-Einstein real hypersurfaces in CP^m (see also Cecil and Ryan [1]). Moreover, Ki [4] proved the nonexistence of real hypersurfaces with parallel Ricci tensor of a nonflat complex space form.

If the Ricci tensor S of a Riemannian manifold M satisfies the condition $\nabla S = S \otimes \alpha$ for some 1-form α, then M is said to be *Ricci recurrent*. In the theory of Ricci recurrent manifolds, Patterson proved some important formulas in [11] and [12], which are developed by Roter [13] and Olszak [10] and are useful for our theory.

Recently, Hamada [3] showed that there are no real hypersurfaces with recurrent Ricci tensor of CP^m under the condition that the structure vector field ξ of the real hypersurface is a principal curvature vector field. Moreover, Loo [8] proved the theorem above without the assumption that the structure vector field ξ of the real hypersurface is a principal curvature vector field.

A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M sat-

2000 *Mathematics Subject Classification*. Primary 53C40, 53C55 Secondary 53C25.
Key words and phrases. CR submanifold, generic submanifold, complex space form, recurrent Ricci tensor, pseudo-Einstein real hypersurface.
Received February 6, 2006.

isfying the conditions that H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and the complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is anti-invariant, i.e. $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

Any real hypersurface of a Kählerian manifold is a CR submanifold.

The main purpose of the present paper is to prove the following theorem.

Theorem. *Let M be an n-dimensional CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. If* $\dim H_x > 2$, *then M is never Ricci recurrent.*

In section 2, we prepare some definitions and basic formulas for CR submanifolds of a complex space form $M^m(c)$. In section 3, we give an equation about the Ricci tensor of a CR submanifold with semi-flat normal connection of a complex space form. In section 4, we give a useful proof of a proposition about a Ricci recurrent manifold in Olszak [10] for our calculation of a Ricci recurrent CR submanifold with semi-flat normal connection. Combining this with the equation given in section 3, we prove our main theorem. In the last section, we give a characterization of pseudo-Einstein real hypersurfaces of complex space forms using the results of section 3.

2. Preliminaries

Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature $4c$. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by G.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in $M^m(c)$. We denote by g the Riemannian metric induced on M from G, and by p the codimension of M, that is, $p = 2m - n$.

We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M respectively.

Definition. A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is anti-invariant, i.e. $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

If $JT_x(M)^{\perp} \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of $\tilde{M}$. Any real hypersurface of $\tilde{M}$ is obviously a generic submanifold of $\tilde{M}$.

In the following, we put dim $H_x = h$, dim $H_x^{\perp} = q$ and codimension $M = p$. If $q = 0$ (resp. $h = 0$) for any $x \in M$, then the CR submanifold M is a holomorphic submanifold (resp. anti-invariant submanifold or totally real submanifold) of $\tilde{M}$. If $p = q$ for any $x \in M$, then the CR submanifold M is a generic submanifold of $\tilde{M}$ (see [15]).

We denote by $\tilde{\nabla}$ the covariant differentiation in $M^m(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V,$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. We call both A and B the *second fundamental form* of M and are related by $G(B(X, Y), V) = g(A_V X, Y)$. The second fundamental form A and B are symmetric. A_V can be considered as a (n, n)-matrix.

The covariant derivative $(\nabla_X A)_V Y$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of the normal vector* V. If the second fundamental form is parallel in any direction, it is said to be *parallel*. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M.

In the sequel, we assume that M is a CR submanifold of $M^m(c)$. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x + H_x^{\perp}$ at each point x of M, where $H_x^{\perp}$ denotes the orthogonal complement of H_x in $T_x(M)$. Similarly, we see that $T_x(M)^{\perp} = JH_x^{\perp} + N_x$, where N_x is the orthogonal complement of $JH_x^{\perp}$ in $T_x(M)^{\perp}$.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$.

For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV. Then we see that $FP = 0$, $fF = 0$, $tF = 0$ and $Pt = 0$.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$ respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X, Y),$$

$$(\nabla_X F)Y = -B(X, PY) + fB(X, Y),$$

$$(\nabla_X t)V = -PA_V X + A_{fV}X,$$

$$(\nabla_X f)V = -FA_V X - B(X, tV).$$

For any vector fields X and Y in $H_x^{\perp} = tT(M)^{\perp}$ we obtain

$$A_{FX}Y = A_{FY}X.$$

We notice that $P^3 + P = 0$, and hence P defines an f-structure on M (see [14]).

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z = c\{&g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX \\ &- g(PX, Z)PY - 2g(PX, Y)PZ\} + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \end{aligned}$$

for any X, Y and Z tangent to M.

We denote by S the Ricci tensor field of M. Then

$$\begin{aligned} g(SX, Y) = {}&(n-1)cg(X, Y) + 3cg(PX, PY) \\ &+ \sum_a \operatorname{Tr} A_a g(A_a X, Y) - \sum_a g(A_a^2 X, Y), \end{aligned}$$

where A_a is the second fundamental form in the direction of v_a, $\{v_1, \dots, v_p\}$ being an orthonormal frame for $T_x(M)^{\perp}$, and Tr denotes the trace of an operator. From this the scalar curvature r of M is given by

$$r = (n-1)nc + 3(n-p)c + \sum_a (\operatorname{Tr} A_a)^2 - \sum_a \operatorname{Tr} A_a^2,$$

where p is the codimension of M, that is, $p = 2m - n$.

The *equation of Codazzi* of M is given by

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z)$$
$$= c\{g(Y, PZ)g(X, JV) - g(X, PZ)g(Y, JV) - 2g(X, PY)g(Z, JV)\}.$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X, Y]}V.$$

Then we have the *equation of Ricci*

$$G(R^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y)$$
$$= c\{g(Y, JV)g(X, JU) - g(X, JV)g(Y, JU) - 2g(X, PY)g(V, JU)\}.$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be *flat*. We can see that the normal connection of M is flat if and only if there exist locally p mutually orthogonal unit normal vector fields v_a such that each v_a is parallel. If $R^{\perp}(X, Y)V = 2cg(X, PY)fV$, then the normal connection of M is said to be *semi-flat* (see [15]). The justification of this definition, see [15]. We notice that, if M is a generic submanifold of $M^m(c)$, then f vanishes identically, and hence $R^{\perp} = 0$.

A nonzero tensor field K of type (r, s) on M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$. M is said to be *Ricci recurrent* if the Ricci tensor S of M is recurrent, that is, S is nonzero and $(\nabla_X S)Y = \alpha(X)SY$ for any vector fields X and Y.

Any real hypersurface M of $M^m(c)$ ($m \geq 3$, $c \neq 0$) is not Einstein. Therefore, the Ricci tensor S of a real hypersurface M of $M^m(c)$ ($m \geq 3$, $c \neq 0$) is nonzero (see [7], [9]).

3. Ricci Tensor of *CR* Submanifolds

In this section, we give some results about the Ricci tensor of a CR submanifolds of a complex space form $M^m(c)$.

THEOREM 3.1. *Let M be an n-dimensional CR submanifold of a complex space form $M^m(c)$, $c \neq 0$,* dim $H_x > 2$, *with semi-flat normal connection. Suppose that the curvature tensor R and the Ricci tensor S satisfy $g((R(X, Y)S)Z, W) = 0$ for any tangent vectors $X, Y, Z, W \in H_x$. Then we have*

$$g(SX, Y) = \frac{1}{h}\left(r - \sum_{a=1}^{q} g(Stv_a, tv_a)\right)g(X, Y)$$

for any vectors $X, Y \in H_x$, *where* r *denotes the scalar curvature of* M *and* $\{v_1, \dots, v_q\}$ *is an orthonormal basis of* $JH_x^{\perp}$.

Proof. Since $g((R(X,Y)S)Z, W) = 0$ for any tangent vectors $X, Y, Z, W \in H_x$, the first Bianchi identity gives

$$g(R(X,Y)SZ + R(Y,Z)SX + R(Z,X)SY, W) = 0.$$

We take an orthonormal basis $\{e_1, \dots, e_h, tv_1 := e_{h+1}, \dots, tv_q := e_n\}$ of $T_x(M)$, where $\{e_1, \dots, e_h\}$ is an orthonormal basis of H_x and $\{v_1, \dots, v_q\}$ is an orthonormal basis of $JH_x^{\perp}$. Then we have

$$g\left(\sum_{i=1}^{h} R(e_i, Pe_i)SX + \sum_{i=1}^{h} R(Pe_i, X)Se_i + \sum_{i=1}^{h} R(X, e_i)SPe_i, Y\right) = 0.$$

Since $Ptv_a = 0$ for $a = 1, \dots, q$, we have

$$g\left(\sum_{i=1}^{n} R(e_i, Pe_i)SX + \sum_{i=1}^{n} R(Pe_i, X)Se_i + \sum_{i=1}^{n} R(X, e_i)SPe_i, Y\right) = 0.$$

Since we have

$$g\left(\sum_{i=1}^{n} R(Pe_i, X)Se_i, Y\right) = -g\left(\sum_{i=1}^{n} R(e_i, X)SPe_i, Y\right),$$

it follows that

$$\sum_{i=1}^{n} g(R(e_i, Pe_i)SX, Y) = 2\sum_{i=1}^{n} g(R(e_i, X)SPe_i, Y).$$

On the other hand, by the equation of Gauss, we have

$$\sum_i g(R(e_i, Pe_i)SX, Y) = (-2h-4)cg(PSX, Y) + \sum_i g(A_{B(Pe_i, SX)}e_i, Y) - \sum_i g(A_{B(e_i, SX)}Pe_i, Y),$$

$$2\sum_i g(R(e_i, X)SPe_i, Y) = c\Big\{-2g(PSX, Y) + 2g(PSPX, PY) + 4g(PX, PSPY) - 2\sum_i g(SPe_i, Pe_i)g(PX, Y)\Big\} + 2\sum_i g(A_{B(X, SPe_i)}e_i, Y) - 2\sum_i g(A_{B(e_i, SPe_i)}X, Y).$$

Thus we have

$$c\{(-2h-2)g(PSX,Y)-2g(PSPX,PY)-4g(PX,PSPY)\}$$
$$=-2c\sum_i g(SPe_i,Pe_i)g(PX,Y)+2\sum_{i,a} g(A_ae_i,Y)g(A_aX,SPe_i)$$
$$-2\sum_{i,a} g(A_aX,Y)g(A_ae_i,SPe_i)-2\sum_{i,a} g(A_ae_i,Y)g(A_aPe_i,SX).$$

Since the Ricci tensor S of M is given by

$$SX=(n-1)cX-3cP^2X+\sum_a \mathrm{Tr}\,A_a\cdot A_aX-\sum_a A_a^2X,$$

we obtain, for $X,Y\in H_x$,

$$\sum_{i,a} g(A_ae_i,Y)g(A_aX,SPe_i)-\sum_{i,a} g(A_aX,Y)g(A_ae_i,SPe_i)$$
$$-\sum_{i,a} g(A_ae_i,Y)g(A_aPe_i,SX)$$
$$=\sum_{i,a,b} \mathrm{Tr}\,A_b g(A_ae_i,Y)g(A_aX,A_bPe_i)-\sum_{i,a,b} g(A_ae_i,Y)g(A_aX,A_b^2Pe_i)$$
$$-\sum_{i,a,b} \mathrm{Tr}\,A_b g(A_ae_i,Y)g(A_aPe_i,A_bX)+\sum_{i,a,b} g(A_ae_i,Y)g(A_aPe_i,A_b^2X)$$
$$-\sum_{i,a}(n-1)cg(A_aX,Y)g(A_ae_i,Pe_i)+3\sum_{i,a} cg(A_aX,Y)g(A_ae_i,Pe_i)$$
$$-\sum_{i,a,b} \mathrm{Tr}\,A_b g(A_aX,Y)g(A_ae_i,A_bPe_i)+\sum_{i,a,b} g(A_aX,Y)g(A_ae_i,A_b^2Pe_i)$$
$$=-\sum_{a,b} \mathrm{Tr}\,A_b g(A_aY,PA_bA_aX)+\sum_{a,b} g(A_aY,PA_b^2A_aX)$$
$$+\sum_{a,b} \mathrm{Tr}\,A_b g(A_aY,PA_aA_bX)-\sum_{a,b} g(A_aY,PA_aA_b^2X)$$
$$-\sum_{i,a,b} \mathrm{Tr}\,A_b g(A_aX,Y)g(A_ae_i,A_bPe_i)+\sum_{i,a,b} g(A_aX,Y)g(A_ae_i,A_b^2Pe_i).$$

Since the normal connection of M is semi-flat, the equation of Ricci gives

$$A_aA_bX=A_bA_aX$$

for any $X \in H_x$. Therefore, the equation above vanishes identically. From these equations and the assumption $c \neq 0$, we have

$$(h+1)g(PSX, Y) + g(PSPX, PY) + 2g(PX, PSPY) = \sum_i g(SPe_i, Pe_i)g(PX, Y),$$

for any $X, Y \in H_x$. This implies

$$(h-1)g(PSX, Y) + g(SPX, Y) = \sum_i g(SPe_i, Pe_i)g(PX, Y).$$

Since $PX, PY \in H_x$, we also have

$$(h-1)g(PSPX, PY) + g(SP^2X, PY) = \sum_i g(SPe_i, Pe_i)g(PX, Y),$$

and hence

$$(h-1)g(SPX, Y) + g(PSX, Y) = \sum_i g(SPe_i, Pe_i)g(PX, Y).$$

From these equations, we obtain

$$(h-2)g(SPX, PY) = (h-2)g(SX, Y).$$

Since $h > 2$, we have $g(SPX, PY) = g(SX, Y)$. Thus, by the definition of the scalar curvature r of M, we get

$$\begin{aligned} hg(SX, Y) &= \sum_i g(PSe_i, Pe_i)g(X, Y) \\ &= \left(r - \sum_{a=1}^{q} g(Stv_a, tv_a)\right)g(X, Y), \end{aligned}$$

which proves our assertion. □

When M is a generic submanifold, the normal connection of M is flat if M is semi-flat. Let p be the codimension of submanifold M in $M^m(c)$ and $\{v_1, \dots, v_p\}$ be an orthonormal basis of $T_x(M)^{\perp}$. Then we have the following theorem.

Theorem 3.2. *Let M be an n-dimensional generic submanifold of a complex space form $M^m(c)$, $c \neq 0$, $n - p > 2$, with flat normal connection. Suppose that the* *curvature tensor R and the Ricci tensor S satisfy $g((R(X,Y)S)Z,W)=0$ for any tangent vectors $X,Y,Z,W \in H_x$. Then we have*

$$g(SX,Y)=\frac{1}{n-p}\left(r-\sum_{a=1}^{p} g(SJv_a,Jv_a)\right)g(X,Y),$$

for any vectors $X,Y \in H_x$.

Let M be a real $(2m-1)$-dimensional hypersurface immersed in $M^m(c)$. We take the unit normal vector field N of M in $M^m(c)$ and define a tangent vector field ξ by $\xi=-JN$, which is called the structure vector field. As a corollary of Theorem 3.1, we have

Corollary 3.3. *Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the curvature tensor R and the Ricci tensor S of M satisfy $g((R(X,Y)S)Z,W)=0$ for any tangent vectors X, Y, Z and W orthogonal to ξ. Then we have*

$$g(SX,Y)=\frac{1}{2m-2}(r-g(S\xi,\xi))g(X,Y),$$

for any tangent vectors X and Y orthogonal to ξ, where r denotes the scalar curvature of M.

4. Ricci Recurrent *CR* Submanifolds

In this section, we prove our main theorem. First, we give a useful proof of the proposition given by Olszak [10].

Proposition 4.1. *Let M be a Ricci recurrent manifold of dimension n with $\alpha \neq 0$, where α is the recurrent form of the Ricci tensor. Then we have*

$$S^2=\frac{r}{2}S,$$

where r denotes the scalar curvature of M.

Proof. By the definition of the Ricci recurrent manifold, the Ricci tensor S of M satisfies $\nabla S = S \otimes \alpha$. Then we have

$$(\nabla_X\nabla_Y S)Z = (\nabla_X\alpha)(Y)SZ + \alpha(Y)(\nabla_X S)Z + \alpha(\nabla_X Y)SZ$$
$$= (\nabla_X\alpha)(Y)SZ + \alpha(Y)\alpha(X)SZ + \alpha(\nabla_X Y)SZ,$$
$$(\nabla_Y\nabla_X S)Z = (\nabla_Y\alpha)(X)SZ + \alpha(X)\alpha(Y)SZ + \alpha(\nabla_Y X)SZ,$$
$$(\nabla_{[X,Y]}S)Z = \alpha([X,Y])SX.$$

So we obtain

$$(R(X,Y)S)Z = (\nabla_X\alpha)(Y)SZ - (\nabla_Y\alpha)(X)SZ. \tag{4.1}$$

Since S is symmetric and nonzero, we can choose some nonzero function λ and a vector field Z such that $SZ = \lambda Z$. Then

$$(R(X,Y)S)Z = \lambda\{(\nabla_X\alpha)(Y)Z - (\nabla_Y\alpha)(X)Z\}.$$

On the other hand, we have

$$g((R(X,Y)S)Z,Z) = g(R(X,Y)SZ,Z) - g(SR(X,Y)Z,Z)$$
$$= \lambda\{g(R(X,Y)Z,Z) - g(R(X,Y)Z,Z)\}$$
$$= 0.$$

Thus we obtain

$$(\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X) = 0. \tag{4.2}$$

By (4.1) and (4.2), we have $R(X,Y)S = 0$. So we obtain, $R(X,Y)SZ - SR(X,Y)Z = 0$, and hence

$$0 = (\nabla_W R)(X,Y)SZ + R(X,Y)(\nabla_W S)Z - (\nabla_W S)R(X,Y)Z - S(\nabla_W R)(X,Y)Z$$
$$= (\nabla_W R)(X,Y)SZ + \alpha(W)R(X,Y)SZ - \alpha(W)SR(X,Y)Z - S(\nabla_W R)(X,Y)Z$$
$$= (\nabla_W R)(X,Y)SZ - S(\nabla_W R)(X,Y)Z.$$

We take a basis $\{e_1,\dots,e_n\}$ of $T_x(M)$. Generally we have

$$\sum_i g((\nabla_{e_i}R)(e_i,X)Y,Z) = \sum_i g((\nabla_{e_i}R)(Z,Y)X,e_i)$$
$$= -\sum_i g((\nabla_Z R)(Y,e_i)X,e_i) - \sum_i g((\nabla_Y R)(e_i,Z)X,e_i)$$
$$= g((\nabla_Z S)Y,X) - g((\nabla_Y S)Z,X).$$

Using this, we obtain

$$0 = \sum_i \{g((\nabla_{e_i} R)(e_i, Y)SZ, X) - g(S(\nabla_{e_i} R)(e_i, Y)Z, X)\}$$
$$= g((\nabla_X S)SZ, Y) - g((\nabla_{SZ} S)X, Y) - g((\nabla_{SX} S)Z, Y) + g((\nabla_Z S)SX, Y)$$
$$= \alpha(X)g(S^2Z, Y) - \alpha(SX)g(SZ, Y) + \alpha(Z)g(S^2X, Y) - \alpha(SZ)g(SX, Y).$$

On the other hand, we have

$$\alpha(SX) = \sum_i \alpha(e_i)g(Se_i, X) = \sum_i g((\nabla_{e_i} S)e_i, X)$$
$$= \frac{1}{2}Xr = \frac{1}{2}\sum_i Xg(Se_i, e_i) = \frac{1}{2}\sum_i g((\nabla_X S)e_i, e_i)$$
$$= \frac{1}{2}\alpha(X)r,$$

where the third equality is given by the second Bianchi identity. That is, we have the following

$$\alpha(X)\left\{g(S^2Z, Y) - \frac{1}{2}rg(SZ, Y)\right\} + \alpha(Z)\left\{g(S^2X, Y) - \frac{1}{2}rg(SX, Y)\right\} = 0.$$

If $\alpha(X) \neq 0$, setting $X = Z$, we have $S^2 = (r/2)S$. If $\alpha(X) = 0$, taking Z such that $\alpha(Z) \neq 0$, $S^2 = (r/2)S$. Consequently we have $S^2 = (r/2)S$. □

In the proof of Proposition 4.1, we have

Lemma 4.2. *Let M be a Ricci recurrent manifold of dimension n. Then the curvature tensor R and the Ricci tensor S satisfy $R(X, Y)S = 0$ for any vector fields X and Y.*

Lemma 4.2 gives the relation between Ricci recurrent condition and Ricci semi-symmetry.

Remark 4.3. From Lemma 4.2 and a theorem of [5], we see that there are no real hypersurfaces with recurrent Ricci tensor of $M^m(c)$, $m \geq 3$, (Loo [8]).

Theorem 4.4. *Let M be an n-dimensional CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. If $\dim H_x > 2$, then M is not Ricci recurrent.*

Proof. We suppose that M is a Ricci recurrent CR submanifold of $M^m(c)$, $h > 2$, with semi-flat normal connection. Since S is symmetric, by Theorem 3.1, we can choose an orthonormal basis $\{e_1, \dots, e_h, tv_1, \dots, tv_q\}$ of $T_x(M)$ such that the Ricci tensor S is represented by a matrix form

$$S = \begin{pmatrix} a & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & * & \\ 0 & \cdots & a & & & \\ & & & \lambda_1 & \cdots & 0 \\ & * & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & \lambda_p \end{pmatrix},$$

where we have put

$$a = \frac{1}{h}\left(r - \sum_a g(Stv_a, tv_a)\right).$$

By Lemma 3.1, we see that eigenvalues of S are $r/2$ and 0, and that rank $S = 2$. Since $h > 2$, we can assume that S is represented by a matrix form

$$S = \begin{pmatrix} & & & & & & h_{11} & h_{21} \\ & & & & & & \vdots & \vdots \\ & & & 0 & & & h_{1h} & h_{2h} \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & \vdots \\ & & & & & & 0 & 0 \\ h_{11} & \cdots & h_{1h} & 0 & \cdots & 0 & \lambda & 0 \\ h_{21} & \cdots & h_{2h} & 0 & \cdots & 0 & 0 & \mu \end{pmatrix}.$$

Thus we have

$$\operatorname{tr} S^2 = \lambda^2 + \mu^2 + \sum_{i=1}^h h_{1i}^2 + \sum_{i=1}^h h_{2i}^2,$$

$$\operatorname{tr}\left(\frac{r}{2}S\right) = \frac{1}{2}(\lambda + \mu)^2.$$

Since $S^2 = (r/2)S$, we have $\lambda = \mu = r/2$, $h_{1i} = 0$ and $h_{2i} = 0$ for $i = 1, \dots, h$. Thus we see that $\{X \in TM \mid SX = (r/2)X\} \subset H_x^{\perp}$. Then there is a vector field

$v \in JH_x^{\perp}$ such that $Stv = (r/2)tv$. We notice that $Jv = tv \in H_x$ and $fv = 0$. We obtain

$$(\nabla_X S)tv + S\nabla_X tv = \frac{1}{2}(Xr)tv + \frac{r}{2}\nabla_X tv.$$

On the other hand, in the proof of Proposition 3.1, we have $Xr = \alpha(X)r$. Then

$$(\nabla_X S)tv = \alpha(X)Stv = \frac{r}{2}\alpha(X)tv = \frac{1}{2}(Xr)tv.$$

So we obtain

$$S\nabla_X tv = \frac{r}{2}\nabla_X tv.$$

Thus we see that $\nabla_X tv \in H_x^{\perp}$. From the equations $\nabla_X tv - tD_X v = (\nabla_X t)v = -PA_v X + A_{fv} X$ and $fv = 0$, we see that $\nabla_X tv - tD_X v = -PA_v X$. Since the left-hand side is in $H_x^{\perp}$ and the right-hand side is in H_x, we have $\nabla_X tv = tD_X v$. So we obtain

$$\nabla_Y \nabla_X tv = \nabla_Y (tD_X v) = tD_Y D_X v,$$

$$\nabla_X \nabla_Y tv = \nabla_X (tD_Y v) = tD_X D_Y v,$$

$$\nabla_{[X,Y]} tv = tD_{[X,Y]} v.$$

Since the normal connection of M is semi-flat, we have

$$R(X,Y)tv = tR^{\perp}(X,Y)v = 2cg(X,PY)tfv = 0.$$

By the definition of the Ricci tensor S, we see

$$\frac{r}{2} = g(Stv, tv) = \sum_i g(R(e_i, tv)tv, e_i) = 0.$$

So we have $S = 0$. This is a contradiction. □

From Theorem 4.4, we have the following theorem about generic submanifold.

THEOREM 4.5. *Let M be an n-dimensional generic submanifold of a complex space form $M^m(c)$, $c \neq 0$, with flat normal connection. If $n - p > 2$, then M is not Ricci recurrent.*

5. A Characterization of Pseudo-Einstein Real Hypersurfaces

In this section, we give a characterization of pseudo-Einstein real hypersurfaces of a complex space form by using Corollary 3.3.

Let M be a real $(2m-1)$-dimensional hypersurface immersed in a complex space form $M^m(c)$. We take the unit normal vector field N of M in $M^m(c)$. For any vector field X tangent to M, we define P, η and ξ by

$$JX = PX + \eta(X)N, \quad \xi = -JN,$$

where PX is the tangential part of JX, P is a tensor field of type $(1,1)$, η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$P^2X = -X + \eta(X)\xi, \quad P\xi = 0, \quad \eta(PX) = 0$$

for any vector field X tangent to M. Moreover, we have

$$g(PX, Y) + g(X, PY) = 0, \quad \eta(X) = g(X, \xi),$$
$$g(PX, PY) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (P, ξ, η, g) defines an almost contact metric structure on M.

The *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y tangent to M. We call A the *shape operator* (*second fundamental form*) of M.

For the contact metric structure on M we have

$$\nabla_X \xi = PAX, \quad (\nabla_X P)Y = \eta(Y)AX - g(AX, Y)\xi.$$

The *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z = c\{&g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX \\ &- g(PX, Z)PY - 2g(PX, Y)PZ\} + g(AY, Z)AX - g(AX, Z)AY. \end{aligned}$$

By the equation of Gauss, the Ricci tensor S of type $(1,1)$ of M is given by

$$SX = (2n+1)cX - 3c\eta(X)\xi + hAX - A^2X,$$

where h denotes the *mean curvature* of M given by the trace of the shape operator A. Moreover, the scalar curvature r of M is given by

$$r = 4(n^2 - 1)c + h^2 - \mathrm{Tr}\, A^2.$$

If the Ricci tensor S of M is of the form $g(SX,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ for some functions a and b, then M is said to be *pseudo-Einstein*. Then a and b are constant when $m \geq 3$.

Theorem 5.1. *Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$. Then the curvature tensor R and the Ricci tensor S of M satisfy $g((R(X,Y)S)Z,W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ if and only if M is pseudo-Einstein.*

Proof. We suppose that M satisfies $g((R(X,Y)S)Z,W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ. We can choose an orthonormal basis $\{X_1,\dots,X_{2m-2},\xi\}$ of M such that the shape operator A is represented by a matrix form

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_{2m-2} & h_{2m-2} \\ h_1 & \cdots & h_{2m-2} & \alpha \end{pmatrix}.$$

Then, we have

$$\begin{aligned} SX_i &= (2n+1)cX_i - 3c\eta(X_i)\xi + hAX_i - A^2X_i \\ &= ((2n+1)c + h\lambda_i - \lambda_i^2)X_i + h_i(h - \lambda_i - \alpha)\xi - \sum_{k=1}^{2m-2} h_ih_kX_k, \\ S\xi &= (2m+1)c\xi - 3c\eta(\xi)\xi + hA\xi - A^2\xi \\ &= (2m-2)c\xi + h\left(\sum_{k=1}^{2m-2} h_kX_k + \alpha\xi\right) - A\left(\sum_{k=1}^{2m-2} h_kX_k + \alpha\xi\right) \\ &= \sum_{k=1}^{2m-2} h_k(h - \lambda_k - \alpha)X_k + \left((2m-2)c + \alpha h - \sum_{k=1}^{2m-2} h_k^2 - \alpha^2\right)\xi. \end{aligned}$$

By Corollary 3.3, we have

$$g(SX_i, X_j) = -h_ih_j = 0 \quad (i \neq j), \tag{5.1}$$

$$g(SX_i, X_i) = \frac{1}{2n-2}(r - g(S\xi,\xi)) \quad (i = 1,\dots,2m-2). \tag{5.2}$$

Equation (5.1) shows that at most one h_i does not vanish. Thus we can assume that $h_i = 0$ for $i = 2, \dots, 2m-2$. We set $a = g(SX_i, X_i)$. Then we have

$$\begin{aligned} SX_1 &= aX_1 + h_1(h - \lambda_1 - \alpha)\xi, \\ SX_i &= aX_i \quad (i = 2, \dots, 2n-2), \\ S\xi &= h_1(h - \lambda_1 - \alpha)X_1 + ((2m-2)c + \alpha h - h_1^2 - \alpha^2)\xi. \end{aligned} \tag{5.3}$$

Since $g((R(X, Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ, we have

$$g(R(X, Y)SZ - SR(X, Y)Z, W) = 0.$$

By the equation of Gauss, for any $j \geq 2$, we obtain

$$\begin{aligned} 0 &= g(R(X_1, X_j)SX_1, X_j) - g(SR(X_1, X_j)X_1, X_j) \\ &= ag(R(X_1, X_j)X_1, X_j) + h_1(h - \lambda_1 - \alpha)g(R(X_1, X_j)\xi, X_j) - ag(R(X_1, X_j)X_1, X_j) \\ &= h_1(h - \lambda_1 - \alpha)g(R(X_1, X_j)\xi, X_j). \end{aligned}$$

By the equation of Gauss, we have

$$\begin{aligned} g(R(X_1, X_j)\xi, X_j) &= g(AX_j, \xi)g(AX_1, X_j) - g(AX_1, \xi)g(AX_j, X_j) \\ &= -h_1\lambda_j. \end{aligned}$$

Thus we see that $h_1^2\lambda_j(h - \lambda_1 - \alpha) = 0$ for $j \geq 2$. If $h_1(h - \lambda_1 - \alpha) \neq 0$, then we have $\lambda_j = 0$ for $j \geq 2$. Since $h = \mathrm{Tr}\, A$, we have $h = \lambda_1 + \alpha$. This is a contradiction. So we have $h_1(h - \lambda_1 - \alpha) = 0$. By (5.3), we see that M is pseudo-Einstein and that $h_1 = 0$ (see [7]). Thus we see that, if $g((R(X, Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ, then M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, we have $SZ = aZ + b\eta(Z)\xi = aZ$ and $SW = aW$ for any tangent vectors Z and W orthogonal to ξ. Then we have $g((R(X, Y)S)Z, W) = g(R(X, Y)SZ, W) - g(SR(X, Y)Z, W) = 0$. □

We need the following two theorems of pseudo-Einstein real hypersurfaces in a complex projective space CP^m with constant holomorphic sectional curvature 4 (Cecil and Ryan [1], Kon [7]) and a complex hyperbolic space CH^m with constant holomorphic sectional curvature -4 (Montiel [9]).

Theorem A. *Let M be a complete and connected real hypersurface in CP^m, $m \geq 3$, which is pseudo-Einstein. Then M is congruent to one of the following spaces*:

(a) *a geodesic hypersphere,*

(b) *a tube of radius r over a totally geodesic CP^k, $0 < k < m-1$, where $0 < r < \pi/2$ and $\cot^2 r = k/(m-k-1)$,*

(c) *a tube of radius $\pi/4 - \theta$ over a complex quadric Q^{m-1} where $0 < \theta < \pi/4$ and $\cot^2 2r = m-2$.*

Theorem B. *Let M be a complete and connected real hypersurface of CH^m, $m \geq 3$, which is pseudo-Einstein. Then M is congruent to one of the following spaces*:

(a) *a geodesic hypersphere.*

(b) *a tube of radius $r > 0$ over a complex hyperbolic hyperplane CH^{m-1}.*

(c) *a self-tube M_m^*.*

Using Theorem A and Theorem B, Theorem 5.1 implies the following theorems.

Theorem 5.2. *Let M be a complete and connected real hypersurface of CP^m, $m \geq 3$. Suppose that the curvature tensor R and the Ricci tensor S satisfy $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ. Then M is congruent to one of the following spaces*:

(a) *a geodesic hypersphere,*

(b) *a tube of radius θ over a totally geodesic CP^k, $0 < k < m-1$, where $0 < \theta < \pi/2$ and $\cot^2 \theta = k/(m-k-1)$,*

(c) *a tube of radius $\pi/4 - \theta$ over a complex quadric Q^{m-1} where $0 < \theta < \pi/4$ and $\cot^2 2\theta = m-2$.*

Theorem 5.3. *Let M be a complete and connected real hypersurface of CH^m, $m \geq 3$. Suppose that the curvature tensor R and the Ricci tensor S satisfy $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ. Then M is congruent to one of the following spaces*:

(a) *a geodesic hypersphere $M_{0,m-1}^h(\tanh^2 \theta)$ of radius $r > 0$,*

(b) *a tube $M_{m-1,0}^h(\tanh^2 \theta)$ of radius $\theta > 0$ over a complex hyperbolic hyperplane,*

(c) *a self-tube M_m^*.*

As an application of Theorem 5.1, we prove the following theorem (see [5], [6]).

Theorem 5.4. *There are no real hypersurfaces with* $R(X,Y)S = 0$, *semi-symmetric Ricci tensor, of a complex space form* $M^m(c)$, $c \neq 0$, $m \geq 3$.

Proof. We suppose that the Ricci tensor S of the real hypersurface M is semi-symmetric, that is, the curvature tensor and the Ricci tensor satisfy $R(X,Y)S = 0$ for any tangent vector fields X and Y. Then by Theorem 5.1, the real hypersurface M is pseudo-Einstein. Consequently, the Ricci tensor S satisfies $SX_i = aX_i$ for $i = 1, \dots, 2m-2$ and $S\xi = (c(2n-2) + \alpha h - \alpha^2)\xi := b\xi$. Then, for any $i = 1, \dots, 2m-2$, we have

$$\begin{aligned}
0 &= R(\xi, X_i)S\xi - SR(\xi, X_i)\xi \\
&= bR(\xi, X_i)\xi - SR(\xi, X_i)\xi \\
&= b\{-cg(\xi,\xi)X_i - g(A\xi,\xi)AX_i\} - S\{-cg(\xi,\xi)X_i - g(A\xi,\xi)AX_i\} \\
&= -bcX_i - b\alpha\lambda_i X_i + acX_i + a\alpha\lambda_i X_i \\
&= (a-b)(c + \alpha\lambda_i)X_i.
\end{aligned}$$

Since $b \neq a$, we have $\lambda_i = -c/\alpha$, $i = 1, \dots, 2m-2$. We put $\lambda = -c/\alpha$. Suppose that X is a unit vector field orthogonal to ξ. Then we have

$$\begin{aligned}
\nabla_X \nabla_\xi \xi &= \nabla_X PA\xi = 0, \\
\nabla_\xi \nabla_X \xi &= \nabla_\xi PAX = \lambda \nabla_\xi PX \\
&= \lambda(\nabla_\xi P)X + \lambda P \nabla_\xi X \\
&= \lambda(\eta(X)A\xi - g(A\xi, X)\xi) + \lambda P \nabla_\xi X \\
&= \lambda P \nabla_\xi X, \\
\nabla_{[X,\xi]}\xi &= PA[X,\xi] \\
&= PA\nabla_X \xi - PA\nabla_\xi X \\
&= PAPAX - PA\nabla_\xi X \\
&= \lambda^2 P^2 X - PA\nabla_\xi X \\
&= -\lambda^2 X - PA\nabla_\xi X.
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
R(X,\xi)\xi &= \nabla_X\nabla_\xi\xi - \nabla_\xi\nabla_X\xi - \nabla_{[X,\xi]}\xi \\
&= -\lambda P\nabla_\xi X + \lambda^2 X + PA\nabla_\xi X.
\end{aligned}
$$

So we have

$$
\begin{aligned}
g(R(X,\xi)\xi, X) &= -\lambda g(P\nabla_\xi X, X) + \lambda^2 g(X,X) + g(PA\nabla_\xi X, X) \\
&= \lambda g(\nabla_\xi X, PX) + \lambda^2 g(X,X) - \lambda g(\nabla_\xi X, PX) \\
&= \lambda^2 g(X,X) = \lambda^2.
\end{aligned}
$$

By the equation of Gauss, we have $g(R(X,\xi)\xi, X) = c + \alpha\lambda = 0$. These equations imply $\lambda = 0$ and $c = 0$. This is a contradiction. So we have our theorem. □

Remark 5.5. We can see that the totally η-umbilical pseudo-Einstein real hypersurfaces of CP^m and CH^m satisfies $c + \alpha\lambda \neq 0$ by a straightforward computation using principal curvatures of examples (see [6]). Here, we proved Theorem 5.4 by a slight general method.

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Department of Mathematics
Hokkaido University
Kita 10 Nishi 8
Sapporo 060-0810, Japan
E-mail address: mayuko_k13@math.sci.hokudai.ac.jp