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A GENERALIZATION OF THE BOOTHBY-WANG THEOREM*

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Abstract. We consider a Riemannian manifold M with an f-structure. With some additional properties such a manifold is called a $\mathscr{K}$, $\mathscr{C}$ or $\mathscr{S}$-manifold. The considered structures determine a Riemannian foliation, whose leaf closures form a singular Riemannian foliation. We give conditions under which the foliation of the principal stratum is again associated to a structure of the type we consider. The manifold can be partitioned into strata on which the leaf closures are given by toroidal fiber bundles. This theorem is a topological generalization of the classical Boothby-Wang theorem for the contact manifolds.

Introduction

Recent years have seen a renewed interest in contact geometry. The beginnings of the contact geometry can be found in the late fifties. Of particular interest and influence were the contact manifolds defined on $\boldsymbol{S}^1$-bundles and a theorem by M. W. Boothby and H. C. Wang published in 1958, cf. [6] and [4]. These contact manifolds have been generalized in many directions, for a fine review of notions and results see [3, 4]. In this paper we study one of such classes of manifolds, called $\mathscr{K}$-manifolds, cf. [2]. Any $\mathscr{K}$-manifold admits the underlying foliation. We prove that the closures of leaves of this foliation form a new foliation, which outside a closed nowhere dense subset is given by the fibres of a

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toroidal principal fibre bundle over a Kähler manifold. More generally, the manifold itself can be partitioned into submanifolds, which are regularly foliated by the closures of leaves of the underlying foliation, and these regular foliations are given by the fibres of toroidal principal fibre bundles over Kähler manifolds. These theorems can be considered as generalizations of the Boothby-Wang theorem.

All manifolds, maps, distributions considered here are smooth, i.e., of the class C^∞; we denote by $\Gamma(-)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

1 Preliminaries

Let $(M^{2n+s}, \varphi, \xi_i, \eta_j)$, $(i, j = 1, \dots, s)$, be a manifold equipped with an *f-structure φ with a parallelizable kernel*, in short, an *f.pk-manifold*; this means that there exist s global vector fields $\xi_1, \dots, \xi_s \in \Gamma(TM)$ and 1-forms $\eta_1, \dots, \eta_s \in \Gamma(T^*M)$ satisfying the following conditions

$$\varphi(\xi_i) = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^2 = -\mathrm{Id} + \sum_{i=1}^{s} \eta_i \otimes \xi_i, \quad \eta_i(\xi_j) = \delta_{ij} \tag{1.1}$$

for all $i, j = 1, \dots, s$. On such a manifold, there always exists a *compatible* Riemannian metric g, in the sense that for each $X, Y \in \Gamma(TM)$

$$g(X, Y) = g(\varphi(X), \varphi(Y)) + \sum_{i=1}^{s} \eta_i(X)\eta_i(Y). \tag{1.2}$$

We fix such a metric on M, then the obtained structure is called a *metric f.pk-manifold*. Let F_φ be the Sasaki form of φ defined by $F_\varphi(X, Y) := g(X, \varphi Y)$ for $X, Y \in TM$. We denote by $\mathscr{D}$ the bundle $\mathrm{Im}(\varphi)$, which is the orthogonal complement of the bundle $\ker(\varphi) = \mathrm{span}\{\xi_1, \dots, \xi_s\}$. Then the manifold M is equipped with the structure consisting of an f-structure φ, the complemented frame $\xi_1, \dots, \xi_s$, the 1-forms $\eta_1, \dots, \eta_s$, a compatible metric g and the Sasaki 2-from F_φ. We put $\mathscr{X} := (M, g, \varphi, \xi_i, \eta_j)$ for $(i, j = 1, \dots, s)$. There is also a tensor $\mathscr{N}_\varphi$ of type $(2, 1)$ defined in the following way: $\mathscr{N}_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta_i \otimes \xi_i$ where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. We recall some definitions of certain types of metric $f.pk$-manifolds which are studied in the present paper, cf. [2, 7, 11].

Definition 1.1. It is said that

- $\mathcal{X}$ is *normal* if $\mathcal{N}_\varphi = 0$
- $\mathcal{X}$ is a *$\mathcal{C}$-manifold* if $d\eta_i = 0$ for $i = 1, \dots, s$, $dF_\varphi = 0$ and $\mathcal{X}$ is normal
- $\mathcal{X}$ is a *$\mathcal{S}$-manifold* if $d\eta_i = F_\varphi$ for $i = 1, \dots, s$ and $\mathcal{X}$ is normal
- $\mathcal{X}$ is a *$\mathcal{K}$-manifold* if $dF_\varphi = 0$ and $\mathcal{X}$ is normal.

The *f.pk*-manifolds may be seen from a different point of view. Namely as a certain type of almost CR-manifolds. In fact, given an *f.pk*-manifold $\mathcal{X}$, we may define an almost CR-structure by considering $(M, \mathrm{Im}(\varphi), \varphi|_{\mathrm{Im}(\varphi)})$. This structure is usually far from being integrable. However, the conditions on $\mathcal{X}$ for being $\mathcal{C}$, $\mathcal{S}$, $\mathcal{K}$ may be expressed in the language of the CR-geometry. Vice versa, given an almost CR-structure (M, H, J) with a parallelization of the transverse bundle to H is also given, we may obtain an *f.pk*-manifold. However, we shall not use in this paper the language of the CR-geometry.

It is clear that $\{\mathcal{S}\text{-manifolds}\} \subset \{\mathcal{K}\text{-manifolds}\} \supset \{\mathcal{C}\text{-manifolds}\}$. Moreover, if $\mathcal{X}$ is a $\mathcal{K}$-manifold, then for each $i, j \in \{1, \dots, s\}$ the following identities hold:

$$[\xi_i, \xi_j] = 0 \quad \text{and} \quad \mathcal{L}_{\xi_i}\eta_j = 0, \tag{1.3}$$

cf. [16, 2]. The vector fields ξ_i, $i = 1 \cdots s$, define a totally geodesic transversally Kähler foliation on M, cf. [10].

Lemma 1.1. *If $\mathcal{X}$ is a $\mathcal{K}$-manifold then*: 1). $\mathcal{L}_{\xi_i}\varphi = 0$, 2). $\xi_i \lrcorner\, d\eta_j = 0$ *and* 3). $[\xi_i, X] \in \Gamma(\mathcal{D})$ *for any* $i, j = 1, \dots, s$ *and* $X \in \Gamma(\mathcal{D})$.

Proof. Let $i, j = 1, \dots, s$ and $X \in \Gamma(\mathcal{D})$. We have $(\mathcal{L}_{\xi_i}\varphi)(\xi_j) = \mathcal{L}_{\xi_i}\varphi(\xi_j) - \varphi(\mathcal{L}_{\xi_i}\xi_j) = 0$. On the other hand, from the normality of $\mathcal{X}$ we have $0 = \mathcal{N}_\varphi(\xi_i, \varphi(X)) = -\varphi[\xi_i, \varphi^2(X)] - [\xi_i, \varphi(X)] = -(\mathcal{L}_{\xi_i}\varphi)(X)$. Therefore 1). holds. We have $\xi_i \lrcorner\, d\eta_j = \mathcal{L}_{\xi_i}\eta_j - d(\xi_i \lrcorner\, \eta_j) = 0$ which proves 2). Moreover, $\eta_j([\xi_i, X]) = (\mathcal{L}_{\xi_i}X) \lrcorner\, \eta_j = \mathcal{L}_{\xi_i}(X \lrcorner\, \eta_j) - X \lrcorner\, (\mathcal{L}_{\xi_i}\eta_j) = 0$. Hence, $[\xi_i, X]$ annihilates η_j; this implies 3). □

2 Extension of $\mathcal{K}$-manifolds

Throughout all of this section, we suppose that $\mathcal{X} := (M, g, \varphi, \xi_i, \eta_j)$, $(i, j = 1, \dots, s)$, is an *f.pk*-manifold, $\mathcal{D} = \mathrm{Im}(\varphi)$ is the distribution orthogonal to $\mathrm{span}\{\xi_1, \dots, \xi_s\} = \ker(\varphi)$, and F_φ is the associated Sasaki 2-form. Moreover, we suppose that there are given orthonormal sections $\zeta_1, \dots, \zeta_r$ $(r \geq 1)$ of the distribution $\mathcal{D}$ such that the subbundle $\mathrm{span}\{\zeta_1, \dots, \zeta_r\}$ is invariant by φ. We denote by $\mu_1, \dots, \mu_r$ the g-dual 1-forms to the vector fields $\zeta_1, \dots, \zeta_r$. Then we can define a new f-structure

$$\psi := \varphi - \sum_{\alpha=1}^{r} \varphi(\zeta_\alpha) \otimes \mu_\alpha. \tag{2.1}$$

Denote by $\mathscr{D}'$ the distribution orthogonal to $\operatorname{span}\{\xi_1, \dots, \xi_s, \zeta_1, \dots, \zeta_r\} = \ker(\psi)$. It is clear that $\mathscr{D}' = \operatorname{Im}(\psi)$. The following lemma is an easy consequence of the definitions.

Lemma 2.1. *The set* $\mathscr{L}' := (M, g, \psi, \xi_i, \zeta_\alpha, \eta_j, \mu_\beta)$, $(i, j = 1, \dots, s$ *and* $\alpha, \beta = 1, \dots, r)$, *is an f.pk-manifold.* $\square$

For the rest of this section, we assume that for each $i, j \in \{1, \dots, s\}$ and $\alpha, \beta \in \{1, \dots, r\}$

$$[\xi_i, \xi_j] = [\zeta_\alpha, \zeta_\beta] = [\xi_i, \zeta_\alpha] = 0. \tag{2.2}$$

Straightforward but tedious calculations give the following two lemmas.

Lemma 2.2. *Let* $\mathscr{L} := (M, g, \varphi, \xi_i, \eta_j)$, $(i, j = 1, \dots, s)$, *be a* $\mathscr{K}$*-manifold,* ζ_α *be Killing vector fields,* $(\alpha = 1, \dots, r)$, *and* $\mathscr{L}_{\zeta_\alpha}\varphi = 0$ *for all* $\alpha \in \{1, \dots, r\}$, *then for any* $i, j \in \{1, \dots, s\}$, $\alpha, \beta \in \{1, \dots, r\}$, *and any* $X, Y \in \Gamma(\mathscr{D}')$

(i) $\mathscr{L}_{\xi_i}\mu_\alpha = 0$, $\mathscr{L}_{\zeta_i}\eta_i = 0$, $\mathscr{L}_{\zeta_\alpha}\mu_\beta = 0$, $\mathscr{L}_{\xi_i}\eta_j = 0$;

(ii) $\xi_i \lrcorner\, d\mu_\alpha = 0$, $\zeta_\alpha \lrcorner\, d\eta_i = 0$, $\zeta_\alpha \lrcorner\, d\mu_\beta = 0$;

(iii) $[\xi_i, X] \in \Gamma(\mathscr{D}')$, $[\zeta_\alpha, X] \in \Gamma(\mathscr{D}')$;

(iv) $\mathscr{L}_{\xi_i}F_\varphi = 0$, $\mathscr{L}_{\xi_i}F_\psi = 0$, $\mathscr{L}_{\zeta_\alpha}F_\varphi = 0$, $\mathscr{L}_{\zeta_\alpha}F_\psi = 0$;

(v) $\xi_i \lrcorner\, dF_\varphi = 0$, $\xi_i \lrcorner\, dF_\psi = 0$, $\zeta_\alpha \lrcorner\, dF_\psi = 0$;

(vi) F_ψ *is closed*;

(vii) $[X, Y] \in \Gamma(\mathscr{D}' \oplus \ker(\varphi))$, $d\mu_\alpha = 0$;

(viii) *if* $J_{\alpha\beta}$ *are functions on* M *such that* $\varphi(\zeta_\alpha) = \sum_{\beta=1}^{r} J_{\beta\alpha}\zeta_\beta$ *then* $J_{\alpha\beta}$ *are locally constant*;

(ix) $\mathscr{L}_{\varphi(\zeta_\alpha)}\varphi = 0$, $\mathscr{L}_{\varphi(\zeta_\alpha)}F_\varphi = 0$, $\mathscr{L}_{\varphi(\zeta_\alpha)}g = 0$.

Proof. We will prove only the first identities in (i), (ii), (iii), (iv). The other can be proved in the similar way.

For each $i, j \in \{1, \dots, s\}$, $\alpha, \beta \in \{1, \dots, r\}$, and $X \in \Gamma(\mathscr{D}')$, we have: $(\mathscr{L}_{\xi_i}\mu_\alpha)(\xi_j) = \mathscr{L}_{\xi_i}\mu_\alpha(\xi_j) - \mu_\alpha([\xi_i, \xi_j]) = 0$, $(\mathscr{L}_{\xi_i}\mu_\alpha)(\zeta_\beta) = \mathscr{L}_{\xi_i}\mu_\alpha(\zeta_\beta) - \mu_\alpha([\xi_i, \zeta_\beta]) = 0$, and

$$\begin{aligned}(\mathscr{L}_{\xi_i}\mu_\alpha)(X) &= \mathscr{L}_{\xi_i}\mu_\alpha(X) - \mu_\alpha(\mathscr{L}_{\xi_i}X) = -g(\zeta_\alpha, [\xi_i, X]) \\ &= -\xi_i(g(\zeta_\alpha, X)) + (\mathscr{L}_{\xi_i}g)(\zeta_\alpha, X) + g(\mathscr{L}_{\xi_i}\zeta_\alpha, X) = 0,\end{aligned}$$

as ξ_i is Killing and (2.2) holds. Therefore the first equation in (i) follows.

Because of (i) and $\mu_\alpha(\xi_i) = 0$, we have that $\xi_i \lrcorner d\mu_\alpha = \mathscr{L}_{\xi_i}\mu_\alpha - d(\xi_i \lrcorner \mu_\alpha) = 0$. Therefore the first equation in (ii) follows.

We have that

$$\eta_j(\mathscr{L}_{\xi_i}X) = \mathscr{L}_{\xi_i}\eta_j(X) - (\mathscr{L}_{\xi_i}\eta_j)(X) = 0,$$

$$\mu_\alpha(\mathscr{L}_{\xi_i}X) = \mathscr{L}_{\xi_i}\mu_\alpha(X) - (\mathscr{L}_{\xi_i}\mu_\alpha)(X) = 0,$$

because of (ii). It follows that $\mathscr{L}_{\xi_i}X$ annihilates all η_j and μ_α, therefore the first identity in (iii) follows.

Properties (iv) follow immediately from the fact that ξ_i and ζ_α are Killing and $\mathscr{L}_{\xi_i}\varphi = \mathscr{L}_{\zeta_\alpha}\varphi = 0$.

In fact, we have that $\xi_i \lrcorner dF_\varphi = \mathscr{L}_{\xi_i}F_\varphi - d(\xi_i \lrcorner F_\varphi) = 0$ because of (iv). Therefore the first equation in (v) follows.

Since $\xi_i \lrcorner dF_\psi = \zeta_\alpha \lrcorner dF_\psi = 0$, then it is enough to show that for all $X, Y, Z \in \Gamma(\mathscr{D}')$, $(dF_\psi)(X, Y, Z)$ vanishes. It is easy to observe that dF_φ and dF_ψ coincide when restricted to $(\mathscr{D}')^3$, therefore F_ψ is closed. Hence (vi) follows.

Let $X, Y \in \Gamma(\mathscr{D}')$ and $\alpha \in \{1, \dots, r\}$, then (vii) is equivalent to the fact that $g(\zeta_\alpha, [X, Y])$ vanishes. Since $\mathscr{L}_{\zeta_\alpha}\Gamma(\mathscr{D}') \subset \Gamma(\mathscr{D}')$ and the map $\Gamma(\mathscr{D}') \ni (X, Y) \mapsto g(\zeta_\alpha, [X, Y])$ is tensorial, then we may assume that $\mathscr{L}_{\zeta_\alpha}X_p = \mathscr{L}_{\zeta_\alpha}Y_p = 0$ where $p \in M$. Since F_φ is closed and (iv) holds, then

$$\begin{aligned} 0 &= 3(dF_\varphi)(\zeta_\alpha, X, Y)_p \\ &= (\mathscr{L}_{\zeta_\alpha}F_\varphi)(X, Y)_p - F_\varphi(\mathscr{L}_{\zeta_\alpha}X, Y)_p + F_\varphi(X, \mathscr{L}_{\zeta_\alpha}Y)_p - F_\varphi([X, Y], \zeta_\alpha)_p \\ &= g(\zeta_\alpha, \varphi([X, Y]))_p. \end{aligned}$$

It follows that $\varphi([X, Y]) \in \Gamma(\mathscr{D}')$ and then $[X, Y] \in \Gamma(\mathscr{D}' \oplus \ker(\varphi))$. From (ii) it follows that $d\mu_\alpha$ vanishes on each pair of vectors such that one of the vectors belongs to $\ker(\psi)$. On the other hand, if $X, Y \in \Gamma(\mathscr{D}')$, then $2d\mu_\alpha(X, Y) = -\mu([X, Y]) = 0$ because $[X, Y] \in \Gamma(\mathscr{D}' \oplus \ker(\varphi))$. This ends the proof of (vii).

We observe that $J_{\alpha\beta} = g(\zeta_\alpha, \varphi(\zeta_\beta))$ because ζ_α $(\alpha = 1, \dots, r)$ are orthonormal. Then we have

$$\begin{aligned} 2dJ_{\alpha\beta} &= 2d(g(\zeta_\alpha, \varphi(\zeta_\beta))) = 2d(F_\varphi(\zeta_\alpha, \zeta_\beta)) = d(\zeta_\beta \lrcorner (\zeta_\alpha \lrcorner F_\varphi)) \\ &= \mathscr{L}_{\zeta_\beta}(\zeta_\alpha \lrcorner F_\varphi) - \zeta_\beta \lrcorner d(\zeta_\alpha \lrcorner F_\varphi) \\ &= [\zeta_\beta, \zeta_\alpha] \lrcorner F_\varphi + \zeta_\alpha \lrcorner (\mathscr{L}_{\zeta_\beta}F_\varphi) - \zeta_\beta \lrcorner (\mathscr{L}_{\zeta_\alpha}F_\varphi) + \zeta_\beta \lrcorner (\zeta_\alpha \lrcorner dF_\varphi) = 0. \end{aligned}$$

which implies (viii). We observe that from the above equation it follows that for each $\alpha, \beta = 1, \dots, r$ we have

$$d(F_\varphi(\zeta_\alpha, \zeta_\beta)) = 0. \tag{2.3}$$

Since $\mathscr{L}_{\zeta_\alpha}\varphi = 0$, $\mathscr{L}_{\zeta_\alpha}F_\varphi = 0$, $\mathscr{L}_{\zeta_\alpha}g = 0$, and $\varphi(\zeta_\alpha)$ are $C^\infty(M)$-linear combinations of $\zeta_1, \dots, \zeta_r$ with coefficients which are locally constant, then (ix) follows immediately. □

Lemma 2.3. *Let $\mathscr{Z}$ be a $\mathscr{K}$-manifold, ζ_α are Killing, and $\mathscr{L}_{\zeta_\alpha}\varphi = 0$ for all $\alpha \in \{1, \dots, s\}$, then $\mathscr{Z}'$ is normal.*

Proof. Let $i, j \in \{1, \dots, s\}$, $\alpha, \beta \in \{1, \dots, r\}$, and $X, Y \in \Gamma(\mathscr{D}')$. We will proceed by studying the values of $\mathscr{N}_\psi$ on the all possible couples taken from among ξ_j, ξ_j, ζ_α, ζ_β, X, Y.

From the assumption (2.2) and Definition (2.1), it follows that $\mathscr{N}_\psi(\xi_i, \xi_j) = \mathscr{N}_\psi(\xi_i, \zeta_k) = \mathscr{N}_\psi(\zeta_k, \zeta_l) = 0$.

From Definition (2.1) and from (iv) of Lemma 2.2, we have that

$$\begin{aligned}\mathscr{N}_\psi(\xi_i, X) &= \mathscr{N}_\psi(\xi_i, X) - \mathscr{N}_\varphi(\xi_i, X)\\ &= \sum\nolimits_{\alpha=1}^{r} \varphi(\zeta_\alpha)\mu_\alpha([\xi_i, \varphi(X)]) - \sum\nolimits_{\alpha=1}^{r} \varphi(\zeta_\alpha) 2d\mu_\alpha(\xi_i, \varphi(X))\\ &= -\sum\nolimits_{\alpha=1}^{r} (\varphi(\zeta_\alpha)\varphi(X) \lrcorner (\xi_i \lrcorner d\mu_\alpha)) = 0.\end{aligned}$$

Again from Definition (2.1) and from (iv) of Lemma 2.2, we have that

$$\begin{aligned}\mathscr{N}_\psi(\zeta_\alpha, X) &= -\psi[\zeta_\alpha, \varphi(X)] - [\zeta_\alpha, X]\\ &= -\varphi[\zeta_\alpha, \varphi(X)] + \sum\nolimits_{\beta=1}^{r} \varphi(\zeta_\beta)\mu_\beta([\zeta_\alpha, \varphi(X)] - [\zeta_\alpha, X]\\ &= \varphi((\mathscr{L}_{\zeta_\alpha}\varphi)(X)) - \varphi^2[\zeta_\alpha, X] - [\zeta_\alpha, X] + \sum\nolimits_{\beta=1}^{r} \varphi(\zeta_\beta) X \lrcorner (\zeta_\alpha \lrcorner d\mu_\beta)) = 0.\end{aligned}$$

Finally, it is easy to observe that

$$\mathscr{N}_\psi(X, Y) = \mathscr{N}_\varphi(X, Y) + \sum_{\alpha=1}^{r} \varphi(\zeta_\alpha)\mu_\alpha([\varphi(X), Y] + [X, \varphi(Y)]). \tag{2.4}$$

The right hand side in (2.4) vanishes since $\mathscr{Z}$ is normal and $[\varphi(X), Y]$, $[X, \varphi(Y)]$ are sections of $\mathscr{D}' \oplus \ker(\varphi)$, cf. (vii) of Lemma 2.2. □

From Lemmas 2.2 and 2.3 we get the following theorem

Theorem 2.1. *Let* $\mathscr{X}$ *be a* $\mathscr{K}$*-manifold. If*
(i) ζ_α *are Killing and orthonormal,*
(ii) $\mathscr{L}_{\zeta_\alpha}\varphi = 0$ *for all* $\alpha \in \{1, \dots, s\}$, *then* $\mathscr{X}'$ *is also a* $\mathscr{K}$*-manifold.*

Moreover, we have the following theorem.

Theorem 2.2. *Let* $\mathscr{X}$ *is a* $\mathscr{C}$*-manifold. If*
(i) ζ_α *are Killing and orthonormal,*
(ii) $\mathscr{L}_{\zeta_\alpha}\varphi = 0$ *for all* $\alpha \in \{1, \dots, s\}$, *then* $\mathscr{X}'$ *is also a* $\mathscr{C}$*-manifold.*

Proof. Theorem 2.1 implies that $\mathscr{X}'$ is a $\mathscr{K}$-manifold. Let $i \in \{1, \dots, s\}$, $\alpha, \beta \in \{1, \dots, r\}$, and $X, Y \in \Gamma(\mathscr{D}')$. Therefore $d\eta_i = 0$ since $\mathscr{X}$ is a $\mathscr{K}$-manifold. From (v) of Lemma 2.2, we get that $\xi_i \lrcorner d\mu_\alpha = \zeta_\alpha \lrcorner d\mu_\beta = 0$. Finally, from (vii) of Lemma 2.2, we get that $2d\mu_\alpha(X, Y) = -\mu_\alpha([X, Y]) = 0$. It follows that all the 1-forms η_i, μ_α vanish and then our assertion follows. □

Observation 2.1. If $\mathscr{X}$ is an $\mathscr{S}$-manifold, then $\mathscr{X}'$ is never an $\mathscr{S}$-manifold. In fact, we have that $F_\psi \neq F_\varphi = d\eta_1$.

Observation 2.2. The assumption that the vector fields ξ_i and ζ_j are orthonormal is very important. Using the classical formula for the Levi-Civita connection one can show the following property:

Let $\Psi : \boldsymbol{R}^r \times M \to M$ be smooth locally free action on some Riemannian manifold (M, g). The foliation $\mathscr{F}_\Psi$ defined by the action is totally geodesic iff there exits a basis $\{v_i\}$ of $\boldsymbol{R}^r$ such that the corresponding vector fields $X_i = v_i^*$ on M are orthonormal.

In fact, $g(\nabla_{X_i} X_j, X) = \partial_X g(X_i, X_j)$ for any local vector field X commuting with X_i. As the action is isometric and X_i commute, locally, we have such sections which span the tangent bundle to M. Therefore $g(X_i, X_j)$ are locally constant functions and thus constant as our manifold is connected. Using the standard Gramm-Schmidt orthonormalization procedure at one point, we obtain the global result.

The following example illustrates the necessity of the above assumption.

Example 2.1. We consider $\boldsymbol{R}^3$ with its canonical coordinates (x, y, z). Suppose that there are also given the additive Lie groups $\boldsymbol{Z}^3$, $\boldsymbol{R}^2$, $\boldsymbol{R}$ and the actions of these groups on $\boldsymbol{R}^3$ given by $\rho_0 : \boldsymbol{Z}^3 \times \boldsymbol{R}^3 \to \boldsymbol{R}^3$, $\rho_1 : \boldsymbol{Z}^3 \times \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and $\rho_2 : \boldsymbol{Z}^3 \times \boldsymbol{R}^3 \to \boldsymbol{R}^3$ such that

$$\rho_0((k,m,n),(x,y,z)) := (x+k, y+m, z+n)$$
$$\rho_1((a,b),(x,y,z)) := (x, y+a, z+b)$$
$$\rho_2(a,(x,y,z)) := (x, y+a, z+a\alpha)$$

for each $k,m,n \in \boldsymbol{Z}$ and $a,b \in \boldsymbol{R}$; α is a fixed element from $\boldsymbol{R}\backslash\boldsymbol{Q}$. It is easy to observe that these actions commute with each other. We consider vector fields $\xi := \frac{\partial}{\partial y} + \alpha\frac{\partial}{\partial z}$, $\zeta := \alpha\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$ and the 1-forms

$$\eta := \frac{1}{1+\alpha^2}(dy + \alpha\, dz), \quad \mu := \frac{1}{1+\alpha^2}(\alpha\, dy - dz).$$

It follows from the construction that $\eta(\xi) = \mu(\zeta) = 1$ and $\eta(\zeta) = \mu(\xi) = 0$. We observe that the vector fields ξ, ζ and the 1-forms η, μ are invariant with respect to the actions ρ_0, ρ_1 and ρ_2. We define on $\boldsymbol{R}^3$ the following Riemannian metric

$$g := (dx)^2 + (\eta)^2 + f(x)(\mu)^2, \tag{2.5}$$

where $f : \boldsymbol{R} \to (0,+\infty)$ is a smooth function which is periodic with the period equals to one. We assume also that $f(0) = 1$. The map f factorizes to the map $\bar{f} : \boldsymbol{R}/\boldsymbol{Z} \to (0,+\infty)$ which is also smooth. Again the actions ρ_0, ρ_1, and ρ_2 preserve the metric, i.e., the Lie groups act by isometries. Moreover, the vector fields ξ, ζ and the 1-forms η, μ are also preserved.

We consider the manifold $\boldsymbol{S}^1 \times \boldsymbol{T}^2 = (\boldsymbol{S}^1)^3 = \boldsymbol{R}^3/\boldsymbol{Z}^3$. There is the canonical projection of $\pi : \boldsymbol{R}^3 \to \boldsymbol{S}^1 \times \boldsymbol{T}^2$ such that $\pi(t,x,y) = (\bar{t},\bar{x},\bar{y})$. This projection is a local diffeomorphism and gives also local charts on $\boldsymbol{S}^1 \times \boldsymbol{T}^2$. The metric g, the vector fields ξ, ζ, and the 1-forms η and μ project via π, respectively, to $\bar{g}$, $\bar{\xi}$, $\bar{\zeta}$, $\bar{\eta}$ and $\bar{\mu}$. In particular, $\bar{g}$ is a Riemannian metric such that π is a local isometry.

We consider the Lie groups $H := (\boldsymbol{R},+)$ and $G := (\boldsymbol{T}^2,+)$ where the operation on G is just summing on the components of $\boldsymbol{T}^2 = (\boldsymbol{R}/\boldsymbol{Z})^2$. There are the induced actions $\bar{\rho}_1 : G \times (\boldsymbol{S}^1 \times \boldsymbol{T}^2) \to \boldsymbol{S}^1 \times \boldsymbol{T}^2$ such that $\bar{\rho}_1((\bar{a},\bar{b}),(\bar{t},\bar{x},\bar{y})) := (\bar{t},\overline{a+x},\overline{b+y})$; this is just the natural action of G on the second component of $\boldsymbol{S}^1 \times \boldsymbol{T}^2$. There is also given the induced action $\bar{\rho}_2 : H \times (\boldsymbol{S}^1 \times \boldsymbol{T}^2) \to \boldsymbol{S}^1 \times \boldsymbol{T}^2$ such that $\bar{\rho}_2(a,(\bar{t},\bar{x},\bar{y})) := (\bar{t},\overline{x+a},\overline{y+a\alpha})$. We observe that both actions $\bar{\rho}_1$ and $\bar{\rho}_2$ are isometric with respect to the metric $\bar{g}$, i.e., $H, G \subset \mathrm{Isom}(\boldsymbol{S}^1 \times \boldsymbol{T}^2, \bar{g})$. The group H is actually a subgroup of G by considering the monomorphism $u : H \hookrightarrow G$ such that $u(a) := (\bar{a},\overline{a\alpha})$. Moreover, we have that $\rho_1(u(a),(\bar{t},\bar{x},\bar{y})) = \rho_2(a,(\bar{t},\bar{x},\bar{y}))$. Hence $H \subset G \subset \mathrm{Isom}(\boldsymbol{S}^1 \times \boldsymbol{T}^2, \bar{g})$. It is well known that H is dense in G since α is irrational.

The Lie algebra $\mathfrak{g}$ of G is $\boldsymbol{R}^2$ with the vanishing bracket. The Lie algebra of H is just $\boldsymbol{R}(1,\alpha)$. The exponential map $\mathfrak{g} \to G$ is given by $\exp(A,B) = (\bar{0},\bar{A},\bar{B})$. We observe that $(1,\alpha) \in \mathfrak{h}$ determines the infinitesimal automorphism $\bar{\bar{\xi}}$ and $(\alpha,-1)$ determines the infinitesimal automorphism $\bar{\zeta}$ of $\boldsymbol{S}^1 \times \boldsymbol{T}^2$. Then $\bar{g}(\bar{\bar{\xi}},\bar{\bar{\xi}}) = 1$, $\bar{g}(\bar{\bar{\xi}},\bar{\zeta}) = 0$ and $\bar{g}(\bar{\zeta},\bar{\zeta}) = \bar{f}(\bar{x})$. In particular, the vector fields $\bar{\bar{\xi}}$, $\bar{\zeta}$ are orthonormal when restricted to the orbit $\{\bar{0}\} \times \boldsymbol{T}^2$ but they are not orthonormal throughout all of the manifold $\boldsymbol{S}^1 \times \boldsymbol{T}^2$ unless $\bar{f}$ is constant.

3 Structures Determined by the Closure of the Leaves

Throughout all of this section we assume that $\mathscr{X} = (M^{2n+s}, g, \varphi, \xi_i, \eta_j)$, $(i,j = 1,\dots,s)$, is a compact $\mathscr{K}$-manifold.

In [10], it was shown that a $\mathscr{K}$-manifold is a particular Riemannian foliation, a transverse Kähler foliation $\mathscr{F}_{\mathscr{K}}$, cf. [18, 19, 8]. Therefore the closures of the leaves of the foliation $\mathscr{F}_{\mathscr{K}}$ form a new Riemannian foliation $\mathscr{F}_b$ which can be singular, i.e. the leaves can have different dimensions. Using the dimensions of leaves and holonomy of $\mathscr{F}$ we can partition the manifold M into submanifolds Σ_i, cf. [15], on which the foliation $\mathscr{F}_b$ is regular and without holonomy. It implies that, on each Σ_i, $\mathscr{F}_b$ is given by a global submersion $h_i : \Sigma_i \to W_i$ onto some smooth Riemannian manifold W_i. As the fibres are compact, each submersion h_i is a locally trivial fibre bundle. This partition is, in fact, a stratification of the manifold M. The stratum M_0 corresponding to leaves of the greatest dimension and without holonomy is open and dense, the other strata form a closed, nowhere dense subset Σ. Therefore outside a closed subset of measure 0, the foliation $\mathscr{F}_b$ is given by the fibres of a locally trivial fibre bundle.

Let us apply these considerations to a particular class of $\mathscr{K}$-manifolds on compact manifolds, those whose underlying foliation $\mathscr{F}$ has all leaves compact. In this case $\mathscr{F}_b = \mathscr{F}$. In addition to being Riemannian, our foliation is transversely Kähler, therefore the submersion $h : M_0 \to W_0$ induces a Kähler structure on W_0. We call such submersions transverse Kähler. Summing up, we have the following proposition.

Proposition 3.1. *Let $\mathscr{X}$, $(i,j = 1,\dots,s)$ be a $\mathscr{K}$-manifold on a compact manifold M whose underlying foliation $\mathscr{F}$ has all leaves compact. Then there exists a closed nowhere dense saturated subset Σ of M such that the restriction of the $\mathscr{K}$-manifold to $M - \Sigma$ is given by a global locally trivial Riemannian submersion $h : M - \Sigma \to W_0$ onto a Kähler manifold W_0.*

On the other hand, if our manifold is a $\mathscr{C}$-manifold the following is true.

Proposition 3.2. *Let* $\mathscr{X}$, $(i, j = 1, \dots, s)$ *be a* $\mathscr{C}$*-manifold on a compact manifold* M, *whose underlying foliation* $\mathscr{F}$ *has all leaves compact. Then there exists a closed nowhere dense saturated subset* Σ *of* M *such that the open set* $M - \Sigma$, *locally, is the Riemannian product of a leaf of* $\mathscr{F}$ *on* $M - \Sigma$ *and a Kähler manifold, which is a leaf of the transverse foliation restricted to* $M - \Sigma$.

Proof. It is a simple consequence of the considerations preceding Proposition 3.1 and results of [5]. □

The underlying foliation $\mathscr{F}$ of a $\mathscr{K}$-manifold is given by a smooth isometric action of the abelian group $\boldsymbol{R}^s$. Therefore we have a representation of $\boldsymbol{R}^s$ into the group $\mathrm{Isom}(M, g, \varphi)$ of g isometries preserving the tensor φ. Since $\mathrm{Isom}(M, g, \varphi)$ is compact and $\mathrm{Im}(\rho)$ is abelian, then K is an abelian compact subgroup. Let K_0 be the identity component of K. Hence K_0 is a certain torus $\boldsymbol{T}^{s+r}$. The connected components of the orbits of the action of K on M are just orbits of $\boldsymbol{T}^{s+r}$, and these orbits are just the closures of leaves of $\mathscr{F}$. However, this action may not be locally free.

The stratification defined by this action, cf. [14], is the stratification we have introduced earlier. For any point $p \in M$, the orbit $K_0 p$ is diffeomorphic to K_0/H_p, where H_p is the isotropy group of the action at p. As the group K_0 is abelian, $H_p = H_q = H$ for any two points p, q of a given stratum Σ_ρ, and K_0/H is also a Lie group. Therefore the foliation $\mathscr{F}_b$ on Σ_ρ is given by a locally free action of the connected abelian Lie group $K_0/H = \boldsymbol{T}^k$. Moreover, the space of orbits of $\boldsymbol{T}^k$ on Σ_ρ is a smooth manifold and the natural projection $p_0 : \Sigma_\rho \to \Sigma_\rho/\boldsymbol{T}^k$ is a principal $\boldsymbol{T}^k$-bundle. In particular, the foliation $\mathscr{F}_b|\Sigma_\rho$ is defined by a locally free action of $\boldsymbol{R}^k$ on Σ_ρ which extends the original action of $\boldsymbol{R}^s$ on Σ_ρ. Of course $s \leq k$. Therefore it is quite reasonable to ask under which conditions this extended action defines a new $\mathscr{K}$-manifold. Let $p_0 \in M$ be a point belonging to a leaf of maximal dimension. Without loss of generality we can assume that the action of $\boldsymbol{R}^{s+r}$ on the principal stratum is locally free. The Lie algebra $L(K)$ of K is isomorphic to $\boldsymbol{R}^{s+r}$. Then each element v of $L(K) \cong \boldsymbol{R}^{s+r}$ defines a global vector field v^* on M. The vector fields $\xi_1, \dots, \xi_s$ may be recovered in this way. There exists some elements $v_1, \dots, v_r$ in $\boldsymbol{R}^{s+r}$ such that the corresponding vector fields $\zeta_1 := v_1^*, \dots, \zeta_r := v_r^*$ are orthonormal when restricted to $T_{p_0}M$.

For the rest of this section we restrict our attention to the principal stratum which we denote by the same letter M. We denote by $\mathscr{D}'$ the subbundle of TM which is an orthogonal complement of $\mathrm{span}\{\xi_1, \dots, \xi_s, \zeta_1, \dots, \zeta_r\}$.

Lemma 3.1. *The following holds*: (1) *the vector fields* $\xi_1, \dots, \xi_s$, $\zeta_1, \dots, \zeta_r$ *commute with each other*; (2) ζ_α *is Killing and* $\mathcal{L}_{\zeta_\alpha}\varphi = 0$ *for each* $\alpha \in \{1, \dots, r\}$.

Proof. (1) and (2) follow immediately from the fact that $K \subset \mathrm{Isom}(M, g, \varphi)$. □

Then we define $\mu_\alpha(-) := g(\zeta_\alpha, -)$ for $\alpha \in \{1, \dots, r\}$. Moreover, we define an endomorphism $\psi \in \mathrm{End}(TM)$ using formula (2.1) and we put $\mathscr{L}' := (M, g, \psi, \xi_i, \zeta_\alpha, \eta_j, \mu_\beta)$. Moreover, we have the following theorem.

Theorem 3.1. *Let* $\mathscr{L}$ *be a* $\mathscr{K}$*-manifold. If*
(i) $\varphi(\mathscr{D}') \subset \mathscr{D}'$,
(ii) *the vector fields* ζ_j *are orthonormal*,
then $\mathscr{L}'$ *is also a* $\mathscr{K}$*-manifold*; *if in addition*, $\mathscr{L}$ *is a* $\mathscr{C}$*-manifold then* $\mathscr{L}'$ *is so.*

Proof. The condition that $\varphi(\mathscr{D}') \subset \mathscr{D}'$ is equivalent to the one that

$$\varphi(\mathrm{span}\{\zeta_1, \dots, \zeta_s\}) \subset \mathrm{span}\{\zeta_1, \dots, \zeta_s\}. \tag{3.1}$$

Hence from Lemma 2.1, we get that $\mathscr{L}'$ is an *f.pk*-structure. From Lemma 3.1 and Theorem 2.1 it follows that $\mathscr{L}'$ is $\mathscr{K}$-manifold. If in addition $\mathscr{L}$ is a $\mathscr{C}$-manifold, then from Lemma 3.1 and Theorem 2.2 follows that $\mathscr{L}'$ is a $\mathscr{C}$-manifold too. □

The assumption that $\varphi(\mathscr{D}') \subset \mathscr{D}'$, or equivalently condition (3.1), in Theorem 3.1 is essential as shows the following example.

Example 3.1. We consider $M_0 := \boldsymbol{C}^n$ $(n \geq 1)$ and its standard global coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. The manifold carries the standard metric g_{st} and the standard complex structure $J_{\mathrm{st}} \in \mathrm{End}(TM_0)$ such that $J_{\mathrm{st}}\left(\frac{\partial}{\partial x_k}\right) := \frac{\partial}{\partial y_k}$. Suppose that there are given the real numbers $a_1, \dots, a_n$, and $b_1, \dots, b_n$ such that $b_n = 1$. Then we define two vector fields:

$$\xi_1 := \frac{\partial}{\partial x_n}, \quad \xi_2 := \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial y_k}.$$

The vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}, \xi_1, \xi_2$ are linearly independent at each point of M_0. We define a new Riemannian metric g_0 on M_0 by supposing that: $g_0|_{T\boldsymbol{C}^{n-1}} := g_{\mathrm{st}}|_{T\boldsymbol{C}^{n-1}}$, $g_0(\xi_k, \xi_l) := \delta_{kl}$, $(k, l = 1, 2)$, and that $T\boldsymbol{C}^{n-1}$ and

$\mathrm{span}\{\xi_1, \xi_2\}$ are orthogonal. Next, we define an f-structure φ_0 by assuming that $\varphi_0|_{\boldsymbol{C}^{n-1}} := J_{\mathrm{st}}|_{\boldsymbol{C}^{n-1}}$, $\ker(\varphi_0) := \mathrm{span}\{\xi_1, \xi_2\}$, and 1-forms η_1, η_2 as the g_0-duals to ξ_1, ξ_2. It is a standard calculation to verify that $\mathscr{Z}_0 := (M, g_0, \varphi_0, \xi_1, \xi_2, \eta_1, \eta_2)$ is a metric *f.pk*-manifold. Then we have the following explicit formulas

$$g = \sum_{k=1}^{n}(dx_k)^2 + \sum_{k=1}^{n-1}(dy_k)^2 + \|\xi\|_0(dy_n)^2 - \sum_{k=1}^{n} a_k\, dx_k dy_n - \sum_{k=1}^{n-1} b_k\, dy_k dy_n$$

$$\eta_1 = dx_n, \quad \eta_2 = dy_n - \sum_{k=1}^{n} a_k\, dx_k - \sum_{k=1}^{n} b_k\, dy_k$$

$$F_{\varphi_0} = \sum_{k=1}^{n-1} dx_k \wedge dy_k - \sum_{k=1}^{n-1} b_k\, dx_k \wedge dy_n + \sum_{k=1}^{n-1} a_k\, dy_k \wedge dy_n - 2\sum_{k=1}^{n-1} a_k b_k\, dy_n \wedge dy_n,$$

where $\|\xi_k\|_0$ denotes the standard norm in $\boldsymbol{C}^n$ and F_{φ_0} is the Sasaki 2-form. The forms η_1, η_2, F_φ are closed since their coefficients are constants. Moreover, $\mathscr{N}_{\varphi_0}$ vanishes since φ is a linear combination with constant coefficients of the canonical basis of $T^*M_0 \otimes TM_0$. These implies that $\mathscr{Z}_0$ is a $\mathscr{C}$-manifold and in particular a $\mathscr{K}$-manifold. The group $\boldsymbol{Z}^{2n}$ acts properly discontinuously on $\boldsymbol{C}^n$ by translations and all the tensors of $\mathscr{Z}_0$ are $\boldsymbol{Z}^{2n}$-invariant. Hence the $\mathscr{C}$-manifold $\mathscr{Z}_0$ descends to the $\boldsymbol{T}^{2n} = \boldsymbol{C}^n/\boldsymbol{Z}^{2n}$. The underling foliation on $\boldsymbol{C}^n$ defined by the distribution $\mathrm{span}\{\xi_1, \xi_2\}$ consists of parallel 2-dimensional real planes. The induced foliation on $\boldsymbol{T}^{2n}$ is the so-called *linear* 2-*dimensional foliation*. The closures of the leaves have the same dimension. The dimension depends on the dimension of the vector space spanned by $a_1, \dots, a_{n-1}$, $b_1, \dots, b_{n-1}$, 1 over the field of rational numbers $\boldsymbol{Q}$, i.e. if $\dim_{\boldsymbol{Q}}(\mathrm{span}\{a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}, 1\}) = k$ then the dimension of any closure of the leaf of the foliation is equal to $k+1$. If $a_1, \dots, a_{n-1}$ are linearly independent over $\boldsymbol{Q}$ and $b_1, \dots, b_{n-1}$ are rational, then the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are tangent to the closures of the leaves. The images $\varphi(\zeta_\alpha)$ of the added vector fields ζ_α commute with ξ_i and with themselves. However, they can be placed anywhere with respect to the closures of leaves.

The example presented above shows that we cannot expect that the foliation by the closures of leaves of the underlying foliation is also given by a new $\mathscr{K}$-manifold. We have to impose some restrictions.

The suspension construction provides us with a whole family of interesting examples of $\mathscr{C}$-manifolds, cf. [10]. We are going to recall the construction and investigate the properties of the leaf closure foliations of such $\mathscr{C}$-manifolds.

EXAMPLE 3.2. Let k be any positive integer and (W, g_W, J_W) a compact Kähler manifold. The fundamental group of $\boldsymbol{T}^k$ is isomorphic to $\boldsymbol{Z}^k$. Let $\rho_0 : \boldsymbol{Z}^k \to \mathrm{Isom}(W, g_W, J_W)$ be a representation of $\boldsymbol{Z}^k$ into the group of holomorphic isometries of (W, g_W, J_W). On the product $\boldsymbol{R}^k \times W$, we put the Riemannian metric $g_0 + g_W$, (g_0—the Euclidean metric of $\boldsymbol{R}^k$), for which the transformations, $\alpha \in \boldsymbol{Z}^k$, $\tilde{\rho}_0(\alpha)(x, w) := (x + \alpha, \rho_0^{-1}(\alpha)w)$ are isometries. The pull-back by the projection $p : \boldsymbol{R}^k \times W \to W$ of the complex structure J_W defines a complex structure on the bundle p^*TW, which is the normal bundle of the foliation $\ker(dp)$. These objects project onto the compact manifold $M(k, W, \rho_0)$, the quotient of $\boldsymbol{R}^k \times W$ by the isometric action induced by the representation $\tilde{\rho}_0$, and which we have just defined above. They define a $\mathscr{C}$-manifold $\mathscr{X}$ on $M(k, W, \rho_0)$.

Let L be the leaf of the foliation $\mathscr{F}_{\mathscr{X}}$ passing through a point $[x, w] \in M(k, W, h)$. Its trace on the transverse manifold $\{x\} \times W$ is equal to the orbit of $G = \mathrm{Im}(\rho_0)$ at w, i.e. Gw. Therefore the trace of the closure $\overline{L}$ is equal to the orbit Kw, where K is the closure of G in the compact group $\mathrm{Isom}(W, g_W, I_W)$. It is not difficult to see that the foliation by the closures of leaves has the integrable complement iff the action of K on W has the integrable orthogonal complement.

Example 3.2 implies that there is no chance that the *f.pk*-manifold associated with the leaf closure foliation is a $\mathscr{C}$-manifold without any supplementary assumptions.

Example 3.1 has elucidated the complexities of the problems encountered in the study of leaf closures of $\mathscr{K}$-manifolds. To obtain some results, one needs to impose restrictions on the structures under considerations. In view of the example, the only reasonable condition is to assume that the tangent bundle to the closures of leaves is φ-invariant, i.e. the closures of leaves are invariant submanifolds of M, cf. [4, 9]. However, to prove that the leaf closures define a new nice structure we need something more as considerations in Section 3 have shown.

THEOREM 3.2. *Let $\mathscr{X}$ be a $\mathscr{K}$-manifold on a compact manifold M. If the endomorphism φ maps sections of the commuting sheaf into sections of the commuting sheaf, then*

(1) *the closures of leaves of $\mathscr{F}_{\mathscr{X}}$ form a regular foliation;*
(2) *there exists a $\mathscr{K}$-structure on M extending $\mathscr{X}$ whose underlying foliation is $\mathscr{F}_b$;*
(3) *there exists a closed, nowhere dense subset Σ of M such that the leaves of the foliation $\mathscr{F}_b$ on $M - \Sigma$ are the fibres of some $\boldsymbol{T}^k$-principal fibre bundle with $k \geq s$.*

Proof. Let denote by $\mathscr{F}_b$ the foliation by the closures of leaves of $\mathscr{F}_{\mathscr{Y}}$. We know that the foliation $\mathscr{F}_b$ is given by an isometric action of $\boldsymbol{R}^k$ for some $k \geq s$. We have the following orthogonal splitting of TM:

$$TM = T\mathscr{F}_{\mathscr{Y}} \oplus Q_0 \oplus Q_1$$

where $T\mathscr{F}_b = T\mathscr{F}_{\mathscr{Y}} \oplus Q_0$. The splitting together with the assumption that $\varphi(T\mathscr{F}_b) \subset T\mathscr{F}_b$ ensures that $\varphi(Q_1) \subset Q_1$. In fact, the subbundle $T\mathscr{F}_b$ is the foliated orbit space of the commuting sheaf, cf. [15, 19], as the foliation $\mathscr{F}$ is transverse Kähler. Therefore any element X of the commuting sheaf can be understood as an infinitesimal automorphism $\bar{X}$ of a transverse Kähler manifold. The assumption on the commuting sheaf assures that $J(\bar{X})$ is also an infinitesimal automorphism, therefore, according to Lemma on p. 79 of [13], $\bar{X}$ is parallel with respect to the Levi-Civita connection of the Kähler manifold. This fact implies that for any two vector fields $\bar{X}$, $\bar{Y}$ corresponding to sections of X, Y of the commuting sheaf the function $\bar{g}(\bar{X}, \bar{Y})$ is constant. Therefore the function $g(X, Y)$ is constant as well. Thus the vector fields of the commuting sheaf cannot vanish. This fact and the fact that the commuting sheaf is locally constant ensures that the dimension of the foliation $\mathscr{F}_b$ is constant, thus $\mathscr{F}_b$ is a regular foliation with compact leaves. This property permits to choose orthonormal vector fields ζ_j as required by Theorem 3.1. The third part is due to the considerations following Proposition 3.2. □

The structure of the space of leaves of the foliation $\mathscr{F}_b$ is relatively simple. Taking into account the results of the forthcoming paper by the third author, cf. [12], we can formulate the following theorem.

Theorem 3.3. *Let $\mathscr{Y}$ be a $\mathscr{K}$-manifold on a compact manifold M. Then the space $M/\mathscr{F}_b$ of the closures of leaves of the underlying foliation is a stratified Riemannian singular space. If the closures are invariant submanifolds, then $M/\mathscr{F}_b$ is a Kähler singular space.*

By a Kähler singular space we understand a singular stratified space, which is Riemannian singular and symplectic singular at the same time, and on each stratum the induced structures give a Kählerian structure. The geometrical structures on stratified singular spaces can be found in [1] (Riemannian structures on orbit spaces), [17] (symplectic singular spaces). Intuitively speaking the space $M/\mathscr{F}_b$ is a union of Riemannian (symplectic) manifolds with compatible Riemannian (symplectic) structures. Moreover, over each stratum of $M/\mathscr{F}_b$ our foliation $\mathscr{F}_b$ is given by the fibres of a toroidal principal fibre bundle. The total spaces of these fibre bundles are the strata of the stratification of M defined earlier in this section.

The following theorem can be considered a generalization of the Boothby-Wang theorem.

THEOREM 3.4. *Let $\mathscr{X}$ be a $\mathscr{K}$-manifold on a compact manifold M. There exists a stratification $S = \{\Sigma_\alpha\}$ of M such that on each stratum Σ_α the foliation $\mathscr{F}_b$ by the closures of leaves of the underlying foliation $\mathscr{F}$ is given by a Riemannian submersion π_α onto the manifold $\Sigma_\alpha/\mathscr{F}_b = \Sigma_\alpha^b$, which is a principal fibre bundle with a toroidal structure group. The manifolds Σ_α^b form a stratification of the space $M/\mathscr{F}_b$ of leaves of $\mathscr{F}_b$, which is a Riemannian singular space. If the leaves of $\mathscr{F}_b$ are invariant submanifolds, then the space $M/\mathscr{F}_b$ is a singular symplectic space, the strata Σ_α^b carry the induced Kähler structures, and the submersions π_α are transverse Kählerian.*

Taking into account Proposition 3.2, for $\mathscr{C}$-manifolds, we can prove the following:

THEOREM 3.5. *Let $\mathscr{X}$ be a $\mathscr{C}$-manifold on a compact manifold M. There exists a stratification $S = \{\Sigma_\alpha\}$ of M such that on each stratum Σ_α the foliation $\mathscr{F}_b$ by the closures of leaves of the underlying foliation $\mathscr{F}$ is given by a Riemannian submersion π_α onto the manifold $\Sigma_\alpha/\mathscr{F}_b = \Sigma_\alpha^b$, which is a principal fibre bundle with a toroidal structure group. The manifolds Σ_α^b form a stratification of the space $M/\mathscr{F}_b$ of leaves of $\mathscr{F}_b$, which is a Riemannian singular space. If the leaves of $\mathscr{F}_b$ are invariant submanifolds, then the space $M/\mathscr{F}_b$ is a singular symplectic space, the strata Σ_α^b carry the induced Kähler structures, and the submersions π_α are transverse Kählerian. If, additionally, any non-vanishing section of the commuting sheaf is ∇-parallel $(\nabla\zeta = 0)$, then the leaf closure foliation in regular, the extended f.pk-manifold is a $\mathscr{C}$-manifold and the principal stratum, locally, is a Riemannian product flat Riemannian manifold and a Kähler manifold.*

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