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LEVI-PARALLEL HYPERSURFACES IN A COMPLEX SPACE FORM

By

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Abstract. In this paper, we classify a Hopf hypersurface in a non-flat complex space form whose Levi-form is parallel with respect to the generalized Tanaka-Webster connection.

1. Introduction

Let $\tilde{M} = (\tilde{M}^n, J, \tilde{g})$ be a complex n-dimensional Kählerian manifold with complex structure J and Kählerian metric $\tilde{g}$. Let M be an oriented real hypersurface in $\tilde{M}$, g be the induced metric and η be the 1-form defined by $\eta(X) = g(X, \xi)$ where $\xi = -JN$ and N is a unit normal vector field on M. Then M has an (integrable) CR-structure associated with the complex structure of the ambient space. Let TM be the tangent bundle of M and D be the subbundle of TM (or the $(2n-2)$-dimensional distribution) which is defined by $\eta = 0$. We denote by $\mathbf{C}D = D \otimes \mathbf{C}$ its complexification. Then we see that D is *holomorphic* (or maximally invariant by J) and

$$\mathscr{H} = \{X - iJX : X \in D\}$$

defines an *CR-structure* on M. That is, $\mathscr{H}$ satisfies the following properties:

(i) each fiber $\mathscr{H}_x$ $(x \in M)$ is of complex dimension $n-1$,

(ii) $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$,

(iii) $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$ (integrability).

Furthermore, we have $\mathbf{C}D = \mathscr{H} \oplus \overline{\mathscr{H}}$. We call $\{D, J\}$ the real representation of $\mathscr{H}$. Then for $\{D, J\}$ we define the Levi form by

$$L : D \times D \to \mathscr{F}(M), \quad L(X, Y) = d\eta(X, JY)$$

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where $\mathscr{F}(M)$ denotes the algebra of differentiable functions on M. If the Levi form is hermitian, then the CR-structure is called *pseudo-hermitian*, in addition, in the case that the Levi form is non-degenerate (positive or negative definite, resp.), then the CR structure is called a *non-degenerate* (*strongly pseudo-convex*, resp.) *pseudo-hermitian CR structure*. Recently, Y. T. Siu [14] proved the nonexistence of compact smooth Levi-flat hypersurfaces in complex projective spaces of dimension ≥ 3. Here, it is remarkable that the assumption of compactness in Siu's theorem has a crucial role. Actually, there are non-complete examples which are Levi-flat in a complex projective space (see section 2). Anyway, the examples of Levi-flat hypersurfaces which are known so far are not Hopf. In this situation, we prove that there does not exist a Levi-flat Hopf hypersurface (Theorem 3).

On the other hand, the Tanaka-Webster connection ([19], [20]) is defined as a canonical affine connection on a pseudo-hermitian, non-degenerate, integrable CR manifold. For contact metric manifolds, their associated almost CR structures are pseudo-hermitian and strongly pseudo-convex, but they are not in general integrable. For a non-zero real number k, the author [7] defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}$ for real hypersurfaces in Kählerian manifolds. The g.-Tanaka-Webster connection $\hat{\nabla}$ coincides with the Tanaka-Webster connection if real hypersurfaces satisfy $\phi A + A\phi = 2k\phi$ (Proposition 2). The covariant differentiation of the Levi form L with respect to the g.-Tanaka-Webster connection $\hat{\nabla}$ is well-defined:

$$(\hat{\nabla}_X L)(Y, Z) = XL(Y, Z) - L(\hat{\nabla}_X Y, Z) - L(Y, \hat{\nabla}_X Z)$$

for any $X, Y, Z \in D$. Then we say that M is *Levi-parallel with respect to the g.-Tanaka-Webster connection* or shortly *Levi-parallel* if M satisfies

$$g((\hat{\nabla}_X L)(Y, Z)) = 0$$

for any vector fields $X, Y, Z \in D$. We note that a Levi-flat hypersurface is Levi-parallel (see (2) in Remark 1).

A complex n-dimensional complete and simply connected Kählerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $\tilde{M}_n(c)$. A complex space form consists of a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $E_n\mathbf{C}$ or a complex hyperbolic space $H_n\mathbf{C}$, according as $c > 0$, $c = 0$ or $c < 0$. R. Takagi [16, 17] classified the homogeneous real hypersurfaces of $P_n\mathbf{C}$ into six types. T. E. Cecil and P. J. Ryan [6] extensively studied a real hypersurface whose structure vector ξ is a principal curvature vector, which is realized as tubes over certain submanifolds in $P_n\mathbf{C}$, by using its focal map. A real hypersurface of a complex space form is said to be a *Hopf hypersurface* if its structure vector is a principal curvature vector. By making use of those results and the mentioned work of R. Takagi, M. Kimura [9] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbf{C}$ whose all principal curvatures are constant. For the case $H_n\mathbf{C}$, J. Berndt [3] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

The main purpose of the present paper is to classify real hypersurfaces of $\tilde{M}_n(c)$, $c \neq 0$ whose Levi form is parallel with respect to the generalized Tanaka-Webster connection. More specifically, in section 4, we prove

Main Theorem. *Let M be a Hopf hypersurface of a complex space form $\tilde{M}_n(c)$, $c \neq 0$. Suppose that M is Levi-parallel with respect to the g.-Tanaka-Webster connection. Then we have the following.*

(I) *If $\tilde{M}_n(c) = P_n\mathbf{C}$, then M is locally congruent to one of:*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

(II) *If $\tilde{M}_n(c) = H_n\mathbf{C}$, then M is locally congruent to one of:*

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbf{R}$.*

2. The Generalized Tanaka-Webster Connection for Real Hypersurfaces

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented. First, we give a brief review of several fundamental concepts and formulas on almost contact structure. An odd-dimensional smooth manifold M^{2n+1} has an *almost contact structure* if it admits a vector ξ, a 1-form η and a (1,1)-tensor field φ satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 X = -X + \eta(X)\xi.$$

Then there exists a compatible Riemannian metric g:

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y on M. We call (η, ξ, φ, g) an almost contact metric structure of M and $M = (M; \eta, \xi, \varphi, g)$ an almost contact metric manifold. For an almost contact metric manifold M we define its fundamental 2-form Φ by $\Phi(X, Y) = g(\varphi X, Y)$. If

$$\Phi = d\eta, \tag{1.1}$$

M is called a contact metric manifold. We refer to [4] on contact metric geometry for more detail.

For an almost contact metric manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The restriction $\bar{\varphi} = \varphi|D$ of φ to D defines an almost complex structure to D. If the associated Levi form L, defined by

$$L(X, Y) = d\eta(X, \bar{\varphi}Y),$$

$X, Y \in D$, is hermitian, then $(\eta, \bar{\varphi})$ is called a pseudo-hermitian CR structure and in addition, if its Levi form is non-degenerate (positive or negative definite, resp.), then $(\eta, \bar{\varphi})$ is called a non-degenerate (strongly pseudo-convex, resp.) pseudo-hermitian CR structure. Moreover, if the following conditions are satisfied:

$$[\bar{\varphi}X, \bar{\varphi}Y] - [X, Y] \in D \tag{1.2}$$

and

$$[\bar{\varphi}, \bar{\varphi}](X, Y) = 0 \tag{1.3}$$

for all $X, Y \in D$, where $[\bar{\varphi}, \bar{\varphi}]$ is the Nijenhuis torsion of $\bar{\varphi}$, then the pair $(\eta, \bar{\varphi})$ is called a pseudo-hermitian, non-degenerate, (strongly pseudo-convex, resp.) integrable CR structure associated with the almost contact metric structure (η, ξ, φ, g). In particular, for a contact metric manifold its associated CR structure is pseudo-hermitian, strongly pseudo-convex but is not in general integrable. For further details about CR structures, we refer for example to [2], [5], [18].

Let M be a real hypersurface of a Kählerian manifold $\tilde{M} = (\tilde{M}; J, \tilde{g})$ and N a global unit normal vector on M. By $\tilde{\nabla}$, A we denote the Levi-Civita connection in $\tilde{M}$ and the shape operator with respect to N, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

(2.1) $$JX = \varphi X + \eta(X)N, \quad JN = -\xi.$$

We easily see that the structure (η, ξ, φ, g) is an almost contact metric structure on M. From the condition $\tilde{\nabla} J = 0$, the relations (2.1) and by making use of the Gauss and Weingarten formulas, we have

(2.2) $$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

(2.3) $$\nabla_X \xi = \varphi AX.$$

By using (2.2) and (2.3), we see that a real hypersurface in a Kählerian manifold always satisfies (1.2) and (1.3), the integrability condition of the associated almost CR structure. From (1.1) and (2.3) we have

Proposition 1. *Let $M = (M; \eta, \xi, \varphi, g)$ be a real hypersurface of a Kählerian manifold. The almost contact metric structure of M is contact metric if and only if $\varphi A + A\varphi = \pm 2\varphi$, where $\pm$ is determined by the orientation.*

The Tanaka-Webster connection ([19], [20]) is the canonical affine connection defined in a non-degenerate integrable CR manifold. Tanno ([18]) defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated almost CR structure is integrable. We define the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) for real hypersurfaces of Kählerian manifolds by the naturally extended one of Tanno's generalized Tanaka-Webster connection for contact metric manifolds.

We recall Tanno's generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y$$

for all vector fields X and Y.

Taking account of (2.3), the g.-Tanaka-Webster connection for real hypersurfaces of Kählerian manifolds, which is denoted by the same symbol $\hat{\nabla}$ as the one for contact metric manifolds, is naturally defined by (cf. [7])

(2.4) $$\hat{\nabla}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y,$$

where k is a non-zero real number. We put $F_X Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$. Then the torsion tensor $\hat{T}$ is given by $\hat{T}(X, Y) = F_X Y - F_Y X$. Also, by using (1.2), (1.3), (2.2), (2.3) and (2.4) we can see that

(2.5) $$\hat{\nabla}\eta = 0, \quad \hat{\nabla}\xi = 0, \quad \hat{\nabla}g = 0, \quad \hat{\nabla}\varphi = 0.$$

and

$$\hat{T}(X, Y) = 2\, d\eta(X, Y)\xi, \quad X, Y \in D.$$

We note that the associated Levi form is

$$L(X, Y) = \frac{1}{2} g((\bar{\varphi}\bar{A} + \bar{A}\bar{\varphi})X, \bar{\varphi}Y),$$

where we denote by $\bar{A}$ the restriction A to D. If M satisfies $\varphi A + A\varphi = 2k\varphi$, then we see that the associated CR structure is pseudo-hermitian, strongly pseudo-convex and further satisfies $\hat{T}(\xi, \varphi Y) = -\varphi \hat{T}(\xi, Y)$, and hence the generalized Tanaka-Webster connection $\hat{\nabla}$ coincides with the Tanaka-Webster connection. Namely, we have (cf. [7])

Proposition 2. *Let $M = (M; \eta, \xi, \varphi, g)$ be a real hypersurface of a Kählerian manifold. If M satisfies $\varphi A + A\varphi = 2k\varphi$, then the associated CR-structure is pseudo-hermitian, strongly pseudo-convex, integrable, and further the generalized Tanaka-Webster connection $\hat{\nabla}$ coincides with the Tanaka-Webster connection.*

Since the structure vector field ξ is $\hat{\nabla}$-parallel, we see that $\hat{\nabla}_X Y$ for $X, Y \in D$ still belongs to D. We define the covariant differentiation of the Levi form L with respect to the g.-Tanaka-Webster connection $\hat{\nabla}$ as follows:

$$(\hat{\nabla}_X L)(Y, Z) = XL(Y, Z) - L(\hat{\nabla}_X Y, Z) - L(Y, \hat{\nabla}_X Z) \tag{2.6}$$

for any $X, Y, Z \in D$.

3. Real Hypersurfaces of a Complex Space Form

Let $\tilde{M} = \tilde{M}_n(c)$ be a non-flat complex space form of constant holomorphic sectional curvature $c(\neq 0)$ and let M a real hypersurface of $\tilde{M}$. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y \\ &+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{3.1}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\} \tag{3.2}$$

for any tangent vector fields X, Y, Z on M. We now suppose that M is a Hopf hypersurface, that is, $A\xi = \alpha\xi$. Then we already know that α is constant (see [8]). Differentiating this covariantly along M, and then by using (2.3) we have

$$(\nabla_X A)\xi = \alpha\varphi AX - A\varphi AX,$$

and further by using (3.2) we obtain

$$(\nabla_\xi A)X = \frac{c}{4}\varphi X + \alpha\varphi AX - A\varphi AX$$

for any vector field X on M. From this, we have

$$2A\varphi AX - \frac{c}{2}\varphi X = \alpha(\varphi A + A\varphi)X.$$

Here, we assume that $AX = \lambda X$ for a unit vector field X orthogonal to ξ, then

$$(2\lambda - \alpha)A\varphi X = \left(\alpha\lambda + \frac{c}{2}\right)\varphi X. \tag{3.3}$$

Now, we prove

THEOREM 3. *There does not exist a Levi-flat Hopf hypersurface in a non-flat complex space form.*

PROOF. Suppose that M is Hopf and Levi-flat. Then $A\xi = \alpha\xi$ and we get

$$\varphi AX + A\varphi X = 0$$

for any $X \in D$. We assume $AX = \lambda X$. Since ξ is a principal curvature vector by using (3.3) we have $2\lambda^2 + \frac{c}{2} = 0$, which shows $c < 0$. Then we see that M has at most three constant principal curvatures λ, μ and α, and further we see that $\mu = -\lambda$. But, Corollary 1 in [3] states that $\lambda\mu + c/4 = 0$. Thus, we have a contradiction. □

We remark here that there are examples of Levi-flat hypersurfaces which are not Hopf. We say that M is a *ruled real hypersurface* of $\tilde{M}_n(c)$, $c \neq 0$ if there is a foliation of M by complex hyperplanes $\tilde{M}_{n-1}(c)$. In other words, M is ruled if and only if D is integrable and its integral manifold is a totally geodesic submanifold $\tilde{M}_{n-1}(c)$. Then we easily see that a ruled real hypersurface is Levi-flat. In fact, its shape operator may be written down as following:

$$A\xi = \alpha\xi + \mu U \quad (\mu \neq 0),$$

$$AU = \mu\xi,$$

$$AZ = 0$$

for any $Z \in D, \perp U$, where U is unit vector orthogonal to ξ, α and μ are functions on M. M. Kimura [10] constructed ruled real hypersurfaces in complex projective space. Let $\overline{M}$ be a hypersurface in S^{2n+1} defined by

$$\left\{ (re^{it}\cos\theta, re^{it}\sin\theta, \sqrt{1-r^2}z_2, \ldots, \sqrt{1-r^2}z_n) \in \mathbf{C}^{n+1} \,\middle|\, \sum_{j=2}^{n} |z_j|^2 = 1, 0 < r < 1, 0 \leq t, \theta < 2\pi, \right\}.$$

Then the Hopf image M of $\overline{M}$ is a minimal ruled hypersurface in $\mathbf{C}P^n$. We note that the above example of a ruled real hypersurface is not complete. In a similar way, in [1] the authors gave a minimal ruled real hypersurfaces in complex hyperbolic space. For more details about ruled real hypersurfaces we may refer to [13].

From Proposition 2, together with the results in [12] (in case of $P_n\mathbf{C}$) and [15] (in case of $H_n\mathbf{C}$) we get easily

Theorem 4. *Let M be a real hypersurface of $\tilde{M}_n(c)$, $c \neq 0$. Suppose that M satisfies $\varphi A + A\varphi = 2k\varphi$ for some non-zero constant k. Then the CR-structure is pseudo-hermitian and strongly pseudo-convex. Furthermore we have the following*:

(I) *in the case $\tilde{M}_n(c) = P_n\mathbf{C}$ with the Fubini-Study metric of $c = 4$, then M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$*,

(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

(II) *in the case $\tilde{M}_n(c) = H_n\mathbf{C}$ with the Bergman metric of $c = -4$, then M is locally congruent to one of the following*:

(A_0) *a horosphere*,

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$*,

(B) *a tube over a totally real hyperbolic space $H_n\mathbf{R}$.*

Remark 1. (1) Together with Proposition 1, we see that the almost contact metric structure on M which appears in the above theorem is a contact metric structure only for the very special case determined by $k = \pm 1$, where $\pm$ depends on the orientation. More precisely, with the help of the tables in [3] and [16], we see that the almost contact metric structures are contact metric only for a geodesic hypersphere of radius $\frac{\pi}{4}$ in $P_n\mathbf{C}$, for a horosphere in $H_n\mathbf{C}$. Hence for real hypersurfaces appearing in Theorem 4, except those just mentioned, they do not admit contact structure but their associated CR structures are pseudo-hermitian, strongly pseudo-convex and further the g.-Tanaka-Webster connection $\hat{\nabla}$ defined on them coincides with the Tanaka-Webster connection.

(2) From (2.6), it follows that Levi-flat hypersurface is Levi-parallel. Leaving the Levi-flat case aside, we find that real hypersurfaces stated in Theorem 4 are also Levi-parallel.

We prepare some more results which are needed to prove our Main Theorem.

Theorem 5 ([9]). *Let M be a Hopf hypersurface of $P_n\mathbf{C}$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*

(C) *a tube of radius r over $P_1\mathbf{C} \times P_{(n-1)/2}\mathbf{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\ge 5)$ is odd,*

(D) *a tube of radius r over a complex Grassmann $G_{2,5}\mathbf{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Theorem 6 ([3]). *Let M be a Hopf hypersurface of $H_n\mathbf{C}$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \le k \le n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbf{R}$.*

Theorem 7 ([11], [15]). *Let M be a Hopf hypersurface of a non-flat complex space form $\tilde{M}_n(c)$, $c \neq 0$. Suppose that the shape operator A is η-parallel (i.e., $g((\nabla_X A)Y, Z) = 0$) for any tangent vectors X, Y and Z which are orthogonal to ξ). Then we have the following.*

(I) *In case that $\tilde{M}_n(c) = P_n\mathbf{C}$, then M is locally congruent to one of real hypersurfaces of type (A_1), (A_2) and (B);*

(II) *In case that $\tilde{M}_n(c) = H_n\mathbf{C}$, then M is locally congruent to one of real hypersurfaces of type (A_0), (A_1), (A_2) and (B).*

4. Levi-parallel Hopf Hypersurfaces in a Complex Space Form

In this section we shall prove our Main Theorem. Suppose that M is a Levi-parallel Hopf hypersurface of a complex space form $\tilde{M}_n(c)$ with respect to g.-Tanaka-Webster connection. Then by using (2.5) and (2.6) we have

$$g((\varphi(\hat{\nabla}_Z A) + (\hat{\nabla}_Z A)\varphi)X, \varphi Y) = 0$$

for any vector fields X, Y, Z orthogonal to ξ on M. It follows easily that

$$g((\hat{\nabla}_Z A)X, Y) - \eta((\hat{\nabla}_Z A)X)\eta(Y) + g((\hat{\nabla}_Z A)\varphi X, \varphi Y) = 0$$

for any $X, Y, Z \in D$.

Together with (2.4), we have

$$\begin{aligned} g((\nabla_Z A)X, Y) &- \eta(AX)g(\varphi AZ, Y) - g(\varphi AZ, X)\eta(AY) \\ &+ g((\nabla_Z A)\varphi X, \varphi Y) - \eta(A\varphi X)g(\varphi AZ, \varphi Y) - g(\varphi AZ, \varphi X)\eta(A\varphi Y) = 0 \end{aligned} \tag{4.1}$$

for any $X, Y, Z \in D$. We now suppose that $A\xi = \alpha\xi$. Then (4.1) reduces to

$$g((\nabla_Z A)X, Y) - g(\varphi(\nabla_Z A)\varphi X, Y) = 0 \tag{4.2}$$

where $X, Y, Z \in D$. Assume $X \in V_\lambda$, that is, $AX = \lambda X$, where we denote by V_λ the eigenspace of A associated with a principal curvature λ. Taking account of (3.3), we divide our arguments into two cases: (i) $2\lambda \neq \alpha$ and $2\lambda = \alpha$. First, we consider the case (i). Then for any $Z \in D$, we get

$$\begin{aligned} (\nabla_Z A)X &= \nabla_Z(AX) - A(\nabla_Z X) \\ &= (Z\lambda)X + (\lambda I - A)(\nabla_Z X). \end{aligned}$$

So we have

$$\begin{aligned} g((\nabla_Z A)X, X) &= Z\lambda + g((\lambda I - A)\nabla_Z X, X) \\ &= Z\lambda + g(\nabla_Z X, (\lambda I - A)X) = Z\lambda. \end{aligned} \tag{4.3}$$

Similarly, by using (3.3), we have

$$g((\nabla_Z A)\varphi X, \varphi X) = -(Z\lambda)\frac{\alpha^2 + c}{(2\lambda - \alpha)^2}. \tag{4.4}$$

From (4.2), (4.3) and (4.4) we obtain

$$(Z\lambda)\left(\lambda^2 - \alpha\lambda - \frac{c}{4}\right) = 0.$$

Since α is constant, this shows that

$$Z\lambda = 0 \quad \text{for any } Z \in D. \tag{4.5}$$

Also, it follows from the equation of Codazzi (3.2) that

$$(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\frac{c}{4}\varphi Z \quad \text{for any } Z \in D.$$

On the other hand, from (2.3) and (3.3) we find

$$\begin{aligned}(\nabla_Z A)\xi - (\nabla_\xi A)Z &= \nabla_Z(A\xi) - A\nabla_Z\xi - \nabla_\xi(AZ) + A(\nabla_\xi Z)\\ &= (\alpha I - A)\varphi AZ - (\xi\lambda)Z - (\lambda I - A)\nabla_\xi Z\\ &= \lambda\left(\alpha - \frac{\alpha\lambda + \frac{c}{2}}{2\lambda - \alpha}\right)\varphi Z - (\xi\lambda)Z - (\lambda I - A)\nabla_\xi Z\end{aligned}$$

for any unit vector $Z \in V_\lambda$. From the above two equations, we obtain

$$\xi\lambda = 0 \tag{4.6}$$

where we have used $g(\varphi Z, Z) = 0$ and $g((\lambda I - A)\nabla_\xi Z, Z) = 0$. Hence from (4.5) and (4.6) we see that λ is constant. Next, in the case (ii) $2\lambda = \alpha$, since α_1 is constant, λ must be constant.

Thus, by virtue of Theorems 5 and 6 we can see that M is locally congruent to one of six types (A_1), (A_2), (B), (C), (D) and (E) in $P_n\mathbf{C}$ or (A_0), (A_1), (A_2) and (B) in $H_n\mathbf{C}$. Conversely, by using Theorem 7, we check that real hypersurfaces of types (A_1), (A_2), (B) in $P_n\mathbf{C}$ or (A_0), (A_1), (A_2) and (B) in $H_n\mathbf{C}$ are Levi-parallel (with respect to the g.-Tanaka-Webster connection).

Now, we shall prove M of types (C), (D) and (E) in $P_n\mathbf{C}$ is not Levi parallel. For M of type (C), (D) or (E) in $P_n\mathbf{C}$, M has five distinct constant principal curvatures, say λ_1, λ_2, λ_3, λ_4 and α so that $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \{\xi\}_{\mathbf{R}}$. We put $x = \cot\left(\theta - \frac{\pi}{4}\right)$ $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$. Then we may express (cf. [11])

$$\lambda_1 = x, \quad \lambda_2 = -\frac{1}{x}, \quad \lambda_3 = \frac{x+1}{1-x}, \quad \lambda_4 = \frac{x-1}{x+1}, \quad \alpha = \frac{-4x}{x^2 - 1}. \tag{4.7}$$

We note that

(4.8) $\quad 0 < x < 1 \quad$ and $\quad \varphi V_{\lambda_1} = V_{\lambda_2}, \quad \varphi V_{\lambda_2} = V_{-\lambda_1}, \quad \varphi V_{\lambda_a} = V_{\lambda_a}, \quad a = 3, 4.$

We first prove the following

Lemma 1. *Let M be a real hypersurface M of types (C), (D) and (E) in $P_n\mathbf{C}$. If M is Levi-parallel, then*

(4.9) $\quad$ (1) *for* $X \in V_{\lambda_i}$ $(i = 1, 2)$; $\nabla_Z X = (\nabla_Z X)_{\lambda_i} - g(X, \varphi AZ)\xi,$

$\qquad$ (2) *for* $X \in V_{\lambda_a}$ $(a = 3, 4)$; $\nabla_Z X = (\nabla_Z X)_{\lambda_a} - g(X, \varphi AZ)\xi.$

for any $Z \in D$, where X_λ denotes the V_λ-component of the vector X.

Proof. For $X \in V_\lambda$ and $Y \in V_\mu$, we get

$$g((\nabla_Z A)X, Y) = (\lambda - \mu)g(\nabla_Z X, Y).$$

If we put $\bar{\lambda} = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$, then $\varphi X \in V_{\bar{\lambda}}$ and $\varphi Y \in V_{\bar{\mu}}$. Together with (2.2) we get

$$\begin{aligned} g((\nabla_Z A)\varphi X, \varphi Y) &= (\bar{\lambda} - \bar{\mu})g(\nabla_Z(\varphi X), \varphi Y) \\ &= (\bar{\lambda} - \bar{\mu})g(\varphi(\nabla_Z X), \varphi Y) \\ &= (\bar{\lambda} - \bar{\mu})g(\nabla_Z X, Y) \end{aligned}$$

Suppose that M is Levi-parallel. Then from (4.2) we obtain

(4.10) $$[(\lambda - \mu) + (\bar{\lambda} - \bar{\mu})]g(\nabla_Z X, Y) = 0.$$

From (4.7) and (4.10) we calculate the following:

(4.11) $\quad$ for $X \in V_{\lambda_i}$ $(i = 1, 2)$, $Y \in V_{\lambda_3}$; $\dfrac{(x+1)(x^2+1)}{x(x-1)} g(\nabla_Z X, Y) = 0,$

$\qquad$ for $X \in V_{\lambda_i}$, $Y \in V_{\lambda_4}$; $\dfrac{(x-1)(x^2+1)}{x(x+1)} g(\nabla_Z X, Y) = 0,$

$\qquad$ for $X \in V_{\lambda_1}$, $Y \in V_{\lambda_2}$; $2xg(\nabla_Z X, Y) = 0,$

$\qquad$ for $X \in V_{\lambda_2}$, $Y \in V_{\lambda_1}$; $-2xg(\nabla_Z X, Y) = 0,$

$\qquad$ for $X \in V_{\lambda_3}$, $Y \in V_{\lambda_4}$; $\dfrac{(x+1)(x^2+1)}{x(1-x)} g(\nabla_Z X, Y) = 0,$

$\qquad$ for $X \in V_{\lambda_4}$, $Y \in V_{\lambda_3}$; $\dfrac{(1-x)(x^2+1)}{x(x+1)} g(\nabla_Z X, Y) = 0.$

Since $g(\nabla_Z X, \xi) = -g(X, \varphi AZ)$, from (4.8) and (4.11), we may express $\nabla_Z X$ as (4.9). □

Secondly, we also prove

Lemma 2. *Let M be a real hypersurface M of type (C), (D) and (E) in $P_n\mathbf{C}$. Then we have*

$$\nabla_\xi Z \in V_{\lambda_i} \oplus \{\varphi Z\}_{\mathbf{R}} \quad \text{for } Z \in V_{\lambda_i} \ (i = 1, 2). \tag{4.12}$$

Proof. For any unit vector $Z \in V_\lambda$, from (2.3) and Proposition 3 it follows that

$$\begin{aligned}(\nabla_Z A)\xi - (\nabla_\xi A)Z &= \nabla_Z(A\xi) - A\nabla_Z\xi - \nabla_\xi(AZ) + A(\nabla_\xi Z)\\ &= (\alpha I - A)\varphi AZ - (\xi\lambda)Z - (\lambda I - A)\nabla_\xi Z\\ &= \lambda\left(\alpha - \frac{\alpha\lambda + 2}{2\lambda - \alpha}\right)\varphi Z - (\lambda I - A)\nabla_\xi Z.\end{aligned}$$

On the other hand, from (3.2) we get

$$(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\varphi Z.$$

Hence we obtain

$$(\lambda I - A)\nabla_\xi Z = \left[\lambda\left(\alpha - \frac{\alpha\lambda + 2}{2\lambda - \alpha}\right)\right]\varphi Z \quad \text{for } Z \in V_\lambda. \tag{4.13}$$

Since $\varphi V_{\lambda_1} = V_{\lambda_2}$, from (4.13) we can find (4.12). □

Thus, it follows from Proposition 3 and (4.13) that for $i = 1, 2$,

$$\left\{\lambda_i - \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\right\} g(\nabla_\xi Z, \varphi Z) = \left[\lambda_i\left(\alpha - \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\right)\right] g(\varphi Z, \varphi Z)$$

or

$$2(\lambda_i^2 - \alpha\lambda_i - 1) g(\nabla_\xi Z, \varphi Z) = \alpha(\lambda_i^2 - \alpha\lambda_i - 1) g(\varphi Z, \varphi Z). \tag{4.14}$$

But, for a real hypersurface M which is locally congruent to one of types (C), (D) and (E) we know that $\lambda^2 - \alpha\lambda - 1 \neq 0$. (We note that the equation $\lambda^2 - \alpha\lambda - 1 = 0$ holds if and only if M is locally congruent to a real hypersurface of type (A_1) or (A_2).) Therefore from (4.14) we get

$$g(\nabla_\xi Z, \varphi Z) = \frac{\alpha}{2} g(\varphi Z, \varphi Z) \quad \text{for } Z \in V_{\lambda_i},\ i = 1, 2. \tag{4.15}$$

For $X \in V_{\lambda_1}$ and $Z \in V_{\lambda_3}$, by using (1) and (2) in (4.9), we have

$$\begin{aligned} R(Z, \varphi Z)X &= \nabla_Z(\nabla_{\varphi Z} X) - \nabla_{\varphi Z}(\nabla_Z X) - \nabla_{[Z, \varphi Z]} X \\ &= \nabla_Z\{(\nabla_{\varphi Z} X)_{\lambda_1} - \lambda_3 g(X, \varphi^2 Z)\xi\} \\ &\quad - \nabla_{\varphi Z}\{(\nabla_Z X)_{\lambda_1} - \lambda_3 g(X, \varphi Z)\xi\} \\ &\quad - \nabla_{\{(\nabla_Z \varphi Z)_{\lambda_3} - \lambda_3 \xi\}} X + \nabla_{\{(\nabla_{\varphi Z} Z)_{\lambda_3} + \lambda_3 \xi\}} X \\ &= (\nabla_Z(\nabla_{\varphi Z} X)_{\lambda_1})_{\lambda_1} - \lambda_3 g((\nabla_{\varphi Z} X)_{\lambda_1}, \varphi Z)\xi \\ &\quad - (\nabla_{\varphi Z}(\nabla_Z X)_{\lambda_1})_{\lambda_1} + \lambda_3 g((\nabla_Z X)_{\lambda_1}, \varphi^2 Z)\xi \\ &\quad - (\nabla_{(\nabla_Z \varphi Z)_{\lambda_3}} X)_{\lambda_1} + \lambda_3 g(X, \varphi(\nabla_Z \varphi Z)_{\lambda_3})\xi + \lambda_3 \nabla_\xi X \\ &\quad + (\nabla_{(\nabla_{\varphi Z} Z)_{\lambda_3}} X)_{\lambda_1} - \lambda_3 g(X, \varphi(\nabla_{\varphi Z} Z)_{\lambda_3})\xi + \lambda_3 \nabla_\xi X. \end{aligned} \tag{4.16}$$

The equations (4.15) and (4.16) show that

$$g(R(Z, \varphi Z)X, \varphi X) = 2\lambda_3 g(\nabla_\xi X, \varphi X) = \alpha\lambda_3 g(\varphi X, \varphi X).$$

On the other hand, since $\varphi X \in V_{\lambda_2}$ and $\varphi Z \in V_{\lambda_3}$, the equation of Gauss (3.1) gives

$$g(R(Z, \varphi Z)X, \varphi X) = -2g(\varphi Z, \varphi Z)g(\varphi X, \varphi X). \tag{4.17}$$

From this, together with (4.7), we have $\frac{-4x}{x^2-1} \cdot \frac{1+x}{1-x} = -2$, that is, $x^2 + 1 = 0$. This is a contradiction.

Thus, we have our Main Theorem. □

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