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ASYMPTOTIC CONDITIONAL DISTRIBUTIONS RELATED TO ONE-DIMENSIONAL GENERALIZED DIFFUSION PROCESSES

By

Masaru Iizuka, Miyuki Maeno and Matsuyo Tomisaki

Abstract. For a one-dimensional generalized diffusion process $\{X(t) : t \geq 0\}$ on an interval I, we consider an expectation conditional on no hitting the end points of I. If the end points are not accessible, we take two sequences $\{\xi_n\}$ and $\{\eta_n\}$ which converge to the end points as $n \to \infty$, instead of end points. We obtain the asymptotic behavior of this conditional expectation as $t \to \infty$ and $n \to \infty$. As an application of our results, we discuss the asymptotic conditional distribution and related quantities in population genetics.

1 Introduction

Let I be an open interval, $\mathscr{L} = (d/dm)(d/ds)$ be a generalized diffusion operator on I, and $\mathbf{D} = [X(t) : t \geq 0, P_x : x \in I_m]$ be a one-dimensional generalized diffusion process with the generator $\mathscr{L}$, where I_m is the support of the speed measure dm. We set $r_1 = \inf I_m$ and $r_2 = \sup I_m$. In this paper we study the asymptotic behavior of the conditional expectations

$$(1.1) \qquad E_x[f(X(\tau t)) \mid t < \sigma_{r_1} \wedge \sigma_{r_2}], \quad E_x[f(X(\tau t)) \mid t < \sigma_{r_2} < \sigma_{r_1}],$$

as $t \to \infty$ for measurable functions f and $0 < \tau \leq 1$, where σ_a is the first hitting time to a point $a \in I_m$ and $\xi \wedge \eta = \min\{\xi, \eta\}$.

In the case that the scale is natural, $|r_1| + |m(r_1)| < \infty$, $r_2 = \infty$, and the speed measure is regularly varying at infinity, Li *et al.* ([19]) studied that the probability law of $\{v(t)X(\tau t) : 0 < \tau \leq 1\}$ conditioned by $\{\sigma_{r_1} > \tau t\}$ converges as $t \to \infty$ to a conditioned Bessel excursion where $v(t)$ is a suitable function. However it seems to be hard to deduce the asymptotic behavior of (1.1) from the results in [19].

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When **D** is recurrent and $f \in L^1(I,m)$, Minami *et al.* ([22]) showed a global asymptotic estimate of the elementary solution of the generalized diffusion equation

$$\partial u(x,t)/\partial t = \mathscr{L}u(x,t), \quad t > 0,\ x \in I, \tag{1.2}$$

and obtained the asymptotic behavior of $E_x[f(X(t))]$ for large t. In [24] and [30], the problem of asymptotic behavior is studied that corresponds to the case that **D** is transient and $f \in L^1_{loc}(I,m)$. In this paper we follow the same method as in [22], [24], [30] to discuss the asymptotic behavior of (1.1) for f satisfying some conditions.

The asymptotic behavior of (1.1) is also discussed in the theory of population genetics. For a diffusion process $[X(t) : t \geq 0, P_x : 0 \leq x \leq 1]$ with the generator

$$\mathscr{L} = \frac{1}{4N}x(1-x)\frac{d^2}{dx^2},$$

Ewens ([5], [6]) obtained the nontrivial limits

$$\lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_0 \wedge \sigma_1) = \int_E dy, \tag{1.3}$$

$$\lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_1 < \sigma_0) = \int_E 2y\, dy, \tag{1.4}$$

for $x \in (0,1)$ and a Borel set E. These limits are referred to as the asymptotic conditional distributions by Ewens. We should notice that these limit distributions are derived from our results by putting $f(x) = 1_E(x)$ or $f(x) = 1_E(x)P_x(\sigma_1 < \sigma_0)$, where $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ if $x \notin E$ (see Sec. 6 for details). Note that $\lim_{t\to\infty} P_x(X(t) \in E) = 0$ if $E \cap \{0,1\} = \emptyset$. For this diffusion process, we will also see that

$$\lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_0 \wedge \sigma_1) = \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0)$$
$$= 6\int_E y(1-y)\, dy, \quad 0 < \tau < 1,$$

(see Example 6.1). This result is different from those of (1.3) and (1.4). The cases that $\tau = 1$ and $0 < \tau < 1$ are referred to as the regularly divergent limit and the slowly divergent limit, respectively. The asymptotic conditional distributions in the regularly divergent limit ($\tau = 1$) correspond to the behavior when the time from $t = 0$ is large enough to be close to the stationary state if it exists. On the other hand, those in the slowly divergent limit ($0 < \tau < 1$) correspond to the behavior when the time from $t = 0$ is very large but it is not enough to be close to the stationary state.

We will state our main results in Sect. 2. The definition of the elementary solution will be given in Sect. 3. The proofs of the main results are presented in Sect. 4. We will see some examples in Sect. 5. Finally we will consider population genetics models in Sect. 6.

2 Main Results

Let $\overline{\mathbf{R}} = [-\infty, +\infty]$ and m be a nondecreasing right continuous function from $\overline{\mathbf{R}}$ into $\overline{\mathbf{R}}$. We set

$$I = (l_1, l_2), \quad l_1 = \inf\{x \in \overline{\mathbf{R}} : m(x) > -\infty\}, \quad l_2 = \sup\{x \in \overline{\mathbf{R}} : m(x) < \infty\}.$$

Let s be a real valued continuous increasing function on I. We sometimes use the same symbols m and s for the induced measures $dm(x)$ and $ds(x)$, respectively. They are called the speed measure and the scale function, respectively. For a function u on I, we set $u(l_i) = \lim_{x \to l_i, x \in I} u(x)$ if there exists the limit, for $i = 1, 2$. We set

$$\begin{aligned} I_m = \{x : x \in I \ &\text{with } m(x_1) < m(x_2) \text{ for every } x_1 < x < x_2, \\ &\text{or } x = l_i \text{ with } |m(l_i)| + |s(l_i)| < \infty,\ i = 1, 2\}, \\ I_* = (r_1, r_2), \quad &r_1 = \inf I_m, \quad r_2 = \sup I_m. \end{aligned}$$

We assume $I_m \cap I_* \neq \emptyset$ throughout this paper. Let us fix a point $c_o \in I_*$ arbitrarily and set

$$I(x) = \int_{(c_o, x]} ds(y) \int_{(c_o, y]} dm(z), \quad J(x) = \int_{(c_o, x]} dm(y) \int_{(c_o, y]} ds(z), \quad x \in I, \tag{2.1}$$

where the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [7], we call the boundary l_i to be

$$\begin{aligned} &\text{regular} && \text{if } I(l_i) < \infty,\ J(l_i) < \infty, \\ &\text{exit} && \text{if } I(l_i) < \infty,\ J(l_i) = \infty, \\ &\text{entrance} && \text{if } I(l_i) = \infty,\ J(l_i) < \infty, \\ &\text{natural} && \text{if } I(l_i) = \infty,\ J(l_i) = \infty. \end{aligned}$$

Note that $l_i = r_i$ if l_i is not entrance. For $0 < p < \infty$, let $L^p(I, m)$ be the space of all functions f on I satisfying $\int_I |f|^p \, dm < \infty$. Let $D(\mathscr{L})$ be the space of all functions $u \in L^2(I,m)$ which have continuous versions u (we use the same symbol) satisfying the following conditions:

i) There exist two complex constants A, B and a function $h_u \in L^2(I,m)$ such that

$$u(x) = A + Bs(x) + \int_{(c_o,x]} \{s(x) - s(y)\} h_u(y)\, dm(y), \quad x \in I. \tag{2.2}$$

ii) If l_i is regular, then $u(l_i) = 0$ for each $i = 1,2$.

We define the generalized diffusion operator $\mathscr{L}$ from $D(\mathscr{L})$ into $L^2(I,m)$ by $\mathscr{L}u = h_u$. We sometimes use the symbol $\mathscr{L} = (d/dm)(d/ds)$. Due to S. Watanabe's argument, the above setting includes all cases of sticky elastic boundary conditions for regular boundaries as well (see [18], [31]). In the following, for a measurable functions u on I, $u^+(x)$ stands for the right derivative $\lim_{\varepsilon \downarrow 0}\{u(x+\varepsilon) - u(x)\}/\{s(x+\varepsilon) - s(x)\}$, provided it exists. We denote by m_* the restriction of m to I_*. Namely,

$$m_*(x) = \begin{cases} -\infty, & x \le r_1, \\ m(x), & x \in I_*, \\ \infty, & x \ge r_2. \end{cases}$$

The generalized diffusion operator $\mathscr{L}_* = (d/dm_*)(d/ds)$ on I_* is defined in the same way as above. Let Σ_* be the spectrum of $-\mathscr{L}_*$. We put $\lambda_* = \inf \Sigma_*$, which is nonnegative because $-\mathscr{L}_*$ is nonnegative in $L^2(I_*,m)$.

Let $\mathbf{D} = [X(t) : t \ge 0, P_x : x \in I_m]$ be a one-dimensional generalized diffusion process having $\mathscr{L}$ as the generator ([12]). We denote by σ_a the first hitting time to a point $a \in I_m$, that is, $\sigma_a = \inf\{t > 0 : X(t) = a\}$ if $\{t > 0 : X(t) = a\} \ne \emptyset$, $\sigma_a = \infty$ otherwise. It is known that $P_x(\sigma_{r_i} < \infty) > 0$, $x \in I_m \cap I_*$ if and only if $I(r_i) < \infty$ ([12]).

First we consider the asymptotic behavior of (1.1) in the case that l_1 and l_2 are not natural. Let $(Ak)_i$, $k = 1,2,\ldots,5$, $i = 1,2$, be the following conditions.

$(A1)_i$ $\quad |s(r_i)| < \infty$ and $|m(r_i)| < \infty$.

$(A2)_i$ $\quad |s(r_i)| < \infty$, $|m(r_i)| = \infty$, and $\left| \int_{(c_i,r_i)} \{\mu(x)/m(x)\}\, dm(x) \right| < \infty$

for some $c_i \in I_*$, where $\mu(x) = \sup_{x \wedge r_i < y < x \vee r_i} |m(y)\{s(r_i) - s(y)\}|$.

$(A3)_i$ $\quad |s(r_i)| < \infty$, $|m(r_i)| = \infty$, and $\left| \int_{(c_o,r_i)} |s(r_i) - s(x)|^{1/2}\, dm(x) \right| < \infty$.

$(A4)_i$ $\quad l_i$ is entrance and $l_i = r_i$.

$(A5)_i$ $\quad |s(r_i)| = \infty, \quad |m(r_i)| < \infty, \quad$ and $\quad \left| \int_{(c_i, r_i)} \{v(x)/s(x)\}\, ds(x) \right| < \infty$

$$\text{for some } c_i \in I_*, \text{ where } \quad v(x) = \sup_{x \wedge r_i < y < x \vee r_i} |s(y)\{m(r_i) - m(y)\}|.$$

Here $\xi \vee \eta$ denotes $\max\{\xi, \eta\}$. We use the usual conventions $a \vee (-\infty) = a$, $a \wedge \infty = a$. The condition $(A1)_i$ is satisfied if and only if l_i is regular, or entrance with $l_i \neq r_i$. The condition $(A2)_i$ or $(A3)_i$ implies that l_i is exit. The condition $(A5)_i$ implies $(A4)_i$. We denote by $\mathscr{H}$ the set of all measurable functions f on I_* satisfying the following conditions.

$$\left| \int_{(c_o, r_i)} |\{s(r_i) - s(y)\} f(y)|\, dm(y) \right| < \infty \quad \text{if } (A1)_i \text{ or } (A2)_i \text{ is satisfied.} \tag{2.3}$$

$$\left| \int_{(c_o, r_i)} |s(r_i) - s(y)|^{1/2} |f(y)|\, dm(y) \right| < \infty \quad \text{if } (A3)_i \text{ is satisfied.} \tag{2.4}$$

$$\left| \int_{(c_o, r_i)} |s(y) f(y)|\, dm(y) \right| < \infty \quad \text{if } (A4)_i \text{ is satisfied.} \tag{2.5}$$

$$\left| \int_{(c_o, r_i)} |f(y)|\, dm(y) \right| < \infty \quad \text{if } (A5)_i \text{ is satisfied.} \tag{2.6}$$

THEOREM 2.1. *Let $i, j \in \{1, 2\}$ and $i \neq j$. Assume one of $(A1)_i$, $(A2)_i$, $(A3)_i$, and one of $(Ak)_j$, $k = 1, 2, \ldots, 5$. Then there exists a unique function ψ_* on I_* satisfying the following properties.*

(i) *ψ_* is positive and continuous on I_*, and satisfies $\psi_*(c_o) = 1$ and*

$$\psi_*^+(y) - \psi_*^+(x) = -\lambda_* \int_{(x, y]} \psi_*(z)\, dm(z), \quad x, y \in I_*, \tag{2.7}$$

$$\psi_*(r_i) = 0 \quad \textit{if } (A1)_i,\ (A2)_i \textit{ or } (A3)_i \textit{ is satisfied}, \tag{2.8}$$

$$\psi_*^+(l_i) = 0 \quad \textit{if } (A4)_i \textit{ or } (A5)_i \textit{ is satisfied}, \tag{2.9}$$

$$\psi_* \in L^1(I_*, m) \cap L^2(I_*, m), \tag{2.10}$$

$$\psi_* f \in L^1(I_*, m), \quad \psi_*^2 f \in L^1(I_*, m) \quad \textit{for } f \in \mathscr{H}. \tag{2.11}$$

(ii) *It holds that*

$$(2.12)\qquad \lim_{t\to\infty} E_x[f(X(\tau t))\,|\,t<\sigma_{r_1}\wedge\sigma_{r_2}]$$

$$= \begin{cases} \left(\int_{I_*}\psi_*(y)^2\,dm(y)\right)^{-1}\int_{I_*}\psi_*(y)^2 f(y)\,dm(y), & 0<\tau<1, \\ \left(\int_{I_*}\psi_*(y)\,dm(y)\right)^{-1}\int_{I_*}\psi_*(y)f(y)\,dm(y), & \tau=1, \end{cases}$$

for $x\in I_m\cap I_*$ *and* $f\in\mathscr{H}$.

COROLLARY 2.2. *Assume one of* $(Ak)_1$, $k=1,2,\dots,5$, *and one of* $(A1)_2$, $(A2)_2$, $(A3)_2$. *Let* ψ_* *be the function in Theorem* 2.1. *Then it holds that*

$$\lim_{t\to\infty} E_x[f(X(\tau t))\,|\,t<\sigma_{r_2}<\sigma_{r_1}]$$

$$= \begin{cases} \left(\int_{I_*}\psi_*(y)^2\,dm(y)\right)^{-1}\int_{I_*}\psi_*(y)^2 f(y)\,dm(y), & 0<\tau<1, \\ \left(\int_{I_*}\psi_*(y)\{s(y)-s(r_1)\}\,dm(y)\right)^{-1} & \\ \quad\times\int_{I_*}\psi_*(y)\{s(y)-s(r_1)\}f(y)\,dm(y), & |s(r_1)|<\infty,\ \tau=1, \\ \left(\int_{I_*}\psi_*(y)\,dm(y)\right)^{-1}\int_{I_*}\psi_*(y)f(y)\,dm(y), & |s(r_1)|=\infty,\ \tau=1, \end{cases}$$

for $x\in I_m\cap I_*$ *and* $f\in\mathscr{H}$.

REMARK 2.1. If $(A4)_1$ and $(A4)_2$ are satisfied, then $P_x(\sigma_{r_1}\wedge\sigma_{r_2}=\infty)=1$, $x\in I_m$, and it holds that $\lim_{t\to\infty} E_x[f(X(\tau t))]=\{m(l_2)-m(l_1)\}^{-1}\int_I f(y)\,dm(y)$, $x\in I_m$, $0<\tau\le 1$, $f\in L^1(I,m)$ ([22, Corollary 1], see also Theorem 2.8).

Next we consider the case that l_2 is natural. We divide our argument into two cases. The first case is related to periodic generalized diffusion operators. Let $(A6)$ be the following condition.

$(A6)$ $l_1=r_1=0$ or l_1 is entrance with $l_1<r_1=0$. $l_2=r_2=\infty$. $s(0)=m(0)=0$.

There is a positive constant κ such that for every $x,y\in[0,\infty)$,

$$s(x+1)-s(y+1)=\kappa\{s(x)-s(y)\},\quad m(x+1)-m(y+1)=\kappa^{-1}\{m(x)-m(y)\}.$$

Note that $(A6)$ implies that l_2 is natural.

Theorem 2.3. *Assume* ($A6$) *with* $\kappa \neq 1$. *Then there exists a unique function* ψ_* *on* I_* *satisfying the following properties.*

(i) ψ_* *is positive and continuous on* I_*, *and satisfies* (2.7), $\psi_*(0) = 0$, $\psi_*^+(0) = 1$, *and*

$$\sup_{x \in I_*} \kappa^{-x/2}(1+x)^{-1}\psi_*(x) < \infty, \tag{2.13}$$

$$\psi_* \in L^1(I_*, m) \quad \text{if} \quad \kappa > 1. \tag{2.14}$$

(ii) *Let* $0 < \kappa < 1$ *and* f *satisfy*

$$\int_{I_*} \kappa^{x/2}(1+x)|f(x)|\, dm(x) < \infty. \tag{2.15}$$

Then it holds that

$$\begin{aligned} \lim_{t\to\infty} & t^{3/2}e^{\lambda_*\tau t}E_x[f(X(\tau t)) \,|\, t < \sigma_0] \\ &= C_1\tau^{-3/2}\psi_*(x)s(x)^{-1}\int_{I_*}\psi_*(y)F(y;\tau)f(y)\, dm(y), \end{aligned} \tag{2.16}$$

for $x \in I_m \cap I_*$, *where* C_1 *is a positive constant specified by* (4.17), *and* $F(y;\tau) = s(y)$ *if* $0 < \tau < 1$, *and* $F(y;1) = s(l_2)$ $(\in (0,\infty))$.

(iii) *Let* $\kappa > 1$. *If* $0 < \tau < 1$, *then it holds that*

$$\lim_{t\to\infty} t^{3/2}E_x[f(X(\tau t)) \,|\, t < \sigma_0] = C_1\tau^{-3/2}(1-\tau)^{-3/2}\int_{I_*}\psi_*(y)^2 f(y)\, dm(y), \tag{2.17}$$

for $x \in I_m \cap I_*$ *and* f *satisfying*

$$\int_{I_*} \kappa^{x}(1+x)^2|f(x)|\, dm(x) < \infty. \tag{2.18}$$

If $\tau = 1$, *then it holds that*

$$\lim_{t\to\infty} E_x[f(X(t)) \,|\, t < \sigma_0] = \left(\int_{I_*}\psi_*(y)\, dm(y)\right)^{-1}\int_{I_*}\psi_*(y)f(y)\, dm(y), \tag{2.19}$$

for $x \in I_m \cap I_*$ *and* f *satisfying* (2.15).

Remark 2.2. By means of (2.13), $\psi_* f \in L^1(I_*, m)$ for f satisfying (2.15), and $\psi_*^2 f \in L^1(I_*, m)$ for f satisfying (2.18).

For a measurable function f on I, we set

$$\mathrm{supp}[f] = \left\{ x \in I_m : \int_U |f|\, dm > 0 \text{ for any neighborhood } U \text{ of } x \right\}.$$

THEOREM 2.4. *Assume* $(A6)$ *with* $\kappa = 1$. *If* $0 < \tau < 1$, *then it holds that*

$$\lim_{t\to\infty} t^{3/2} E_x[f(X(\tau t)) \mid t < \sigma_0] \tag{2.20}$$

$$= 2^{-1}\pi^{-1/2} m(1)^{1/2} s(1)^{-1/2} \tau^{-3/2}(1-\tau)^{-1/2} \int_{I_*} s(y)^2 f(y)\, dm(y),$$

for $x \in I_m \cap I_*$ *and* $f \in L^1(I_*, m)$ *such that* $\mathrm{supp}[f]$ *is compact in* $[0, \infty)$. *If* $\tau = 1$ *and* f *satisfies* (2.15), *then* $sf \in L^1(I_*, m)$ *and it holds that*

$$\lim_{t\to\infty} tE_x[f(X(t)) \mid t < \sigma_0] = 2^{-1} \int_{I_*} s(y) f(y)\, dm(y), \tag{2.21}$$

for $x \in I_m \cap I_*$.

Next we consider the case that the speed measure is regularly varying near the boundary l_2. Let $0 < \beta < 1$ and L be a positive slowly varying function at infinity, that is, $\lim_{x\to\infty} L(cx)/L(x) = 1$, $c > 0$. Let $(A7)$ and $(A8)$ be the following conditions.

$(A7)$ $l_1 = r_1 = 0$ or l_1 is entrance with $l_1 < r_1 = 0$. $l_2 = r_2 < \infty$. $s(x) = x$, $x \in I_*$. m satisfies $m(0) = 0$ and

$$\lim_{x\to\infty} x^{-1-1/\beta} L(x)^{-1} m(l_2 - 1/x) = 1. \tag{2.22}$$

$(A8)$ $l_1 = r_1 = 0$ or l_1 is entrance with $l_1 < r_1 = 0$. $l_2 = r_2 = \infty$. $s(x) = x$, $x \in I_*$. m satisfies $m(0) = 0$ and

$$\lim_{x\to\infty} x^{1-1/\beta} L(x)^{-1} m(x) = 1. \tag{2.23}$$

Note that l_2 is natural if $(A7)$ or $(A8)$ is satisfied. It is known that, under the assumption (2.22) or (2.23), there is a slowly varying function $\tilde{L}$ satisfying

$$\lim_{x\to\infty} \tilde{L}(x)^{1/\beta} L(x^\beta \tilde{L}(x)) = \lim_{x\to\infty} L(x)^\beta \tilde{L}(x^{1/\beta} L(x)) = 1, \tag{2.24}$$

(cf. [27]). Let denote by C_i, $i = 2, 3$, positive constants given by

$$C_2 = \{\beta(1+\beta)\}^\beta / \Gamma(\beta), \quad C_3 = \{\beta(1-\beta)\}^\beta / \Gamma(\beta),$$

where $\Gamma(z)$ is the gamma function defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$.

THEOREM 2.5. *Let* $x \in I_m \cap I_*$ *and* f *satisfy* $yf(y) \in L^1(I_*, m)$.

(i) *Assume* (*A*7). *Also assume that* $\operatorname{supp}[f]$ *is compact in* $[0, l_2)$. *If* $0 < \tau < 1$, *then it holds that*

$$\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(\tau t)) \mid t < \sigma_0] = C_2\tau^{-1-\beta}l_2^{-2}\int_{I_*} y^2 f(y)\,dm(y). \tag{2.25}$$

If $\tau = 1$, *then it holds that*

$$\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(t)) \mid t < \sigma_0] = C_2 l_2^{-1}\int_{I_*} yf(y)\,dm(y). \tag{2.26}$$

(ii) *Assume* (*A*8). *If* $0 < \tau < 1$ *and* $\operatorname{supp}[f]$ *is compact in* $[0, \infty)$, *then it holds that*

$$\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(\tau t)) \mid t < \sigma_0] = C_3\tau^{-1-\beta}(1-\tau)^{-\beta}\int_{I_*} y^2 f(y)\,dm(y). \tag{2.27}$$

If $\tau = 1$, *then it holds that*

$$\lim_{t\to\infty} tE_x[f(X(t)) \mid t < \sigma_0] = \beta\int_{I_*} yf(y)\,dm(y). \tag{2.28}$$

If $I(l_i) = \infty$ with $l_i = r_i$ for $i = 1$ and 2, then $P_x(\sigma_{r_1} \wedge \sigma_{r_2} = \infty) = 1$, $x \in I_m$. Including this situation, we consider other asymptotic conditional expectations. Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences satisfying

$$\xi_n, \eta_n \in I_m,\ \xi_n < c_o < \eta_n \quad (n \in \mathbf{N}), \quad \xi_n \downarrow r_1,\ \eta_n \uparrow r_2 \quad \text{as } n \to \infty. \tag{2.29}$$

THEOREM 2.6. *There exist subsequences* $\{\xi_n\}$ *and* $\{\eta_n\}$ (*denoted by the same symbols*), *a function* ψ_* *on* I_*, *and three sequences* $\{V_n^{(j)}\}$, $j = 1, 2, 3$, *satisfying the following properties.*

(i) ψ_* *is positive and continuous on* I_*, *and satisfies* (2.7) *and* $\psi_*(c_o) = 1$.

(ii) $\{V_n^{(j)}\}$, $j = 1, 2, 3$, *are sequences of positive numbers and it holds that*

$$\begin{aligned}\lim_{n\to\infty} V_n^{(1)} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] &= \lim_{n\to\infty} V_n^{(1)} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}] \\ &= \int_{I_*} \psi_*(y)^2 f(y)\,dm(y), \quad 0 < \tau < 1,\end{aligned} \tag{2.30}$$

$$\lim_{n\to\infty} V_n^{(2)} \lim_{t\to\infty} E_x[f(X(t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] = \int_{I_*} \psi_*(y)f(y)\,dm(y), \tag{2.31}$$

(2.32) $$\lim_{n\to\infty} V_n^{(3)} \lim_{t\to\infty} E_x[f(X(t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$$
$$= \begin{cases} \int_{I_*} \psi_*(y)\{s(y) - s(r_1)\} f(y)\, dm(y), & |s(r_1)| < \infty, \\ \int_{I_*} \psi_*(y) f(y)\, dm(y), & |s(r_1)| = \infty, \end{cases}$$

for $x \in I_m$ *and* $f \in L^1(I_*, m)$ *with* supp$[f]$ *being compact in* I_*.

REMARK 2.3. If $\ell_i = r_i$ and $|s(l_i)| = \infty$ for $i = 1$ or 2, then $\psi_* = 1$ in Theorem 2.6 (see Proposition 4.3).

We note that Theorem 2.6 is a kind of extension of Theorem 2.1 and Corollary 2.2. We also note that if $(A1)_1$ and $(A1)_2$ are satisfied, then the double limits ($n \to \infty$ and $t \to \infty$) are commutable. More precisely we obtain the following.

COROLLARY 2.7. *Assume* $(A1)_1$ *and* $(A1)_2$. *Then it holds that*

(2.33) $$\lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]$$
$$= \lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$$
$$= \left(\int_{I_*} \psi_*(y)^2\, dm(y)\right)^{-1} \int_{I_*} \psi_*(y)^2 f(y)\, dm(y), \quad 0 < \tau < 1,$$

(2.34) $$\lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]$$
$$= \left(\int_{I_*} \psi_*(y)\, dm(y)\right)^{-1} \int_{I_*} \psi_*(y) f(y)\, dm(y),$$

(2.35) $$\lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$$
$$= \left(\int_{I_*} \psi_*(y)\{s(y) - s(r_1)\}\, dm(y)\right)^{-1}$$
$$\times \int_{I_*} \psi_*(y)\{s(y) - s(r_1)\} f(y)\, dm(y),$$

for any sequences $\{\xi_n\}$ *and* $\{\eta_n\}$ *satisfying* (2.29), f *satisfying* (2.3), *and the function* ψ_* *given in Theorem* 2.1.

In some cases we can obtain asymptotic conditional expectations of $\lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]$ and $\lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$ as $t \to \infty$.

Theorem 2.8. *Assume* $l_i = r_i$, $|s(l_i)| = \infty$ *and* $|m(l_i)| < \infty$ *for* $i = 1, 2$. *Let* $x \in I_m$, $0 < \tau \le 1$ *and* $f \in L^1(I, m)$. *Then it holds that*

$$\lim_{t\to\infty} \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] = \{m(l_2) - m(l_1)\}^{-1} \int_I f(y)\, dm(y). \tag{2.36}$$

Assume that there exists the limit $s_* = \lim_{n\to\infty} |s(\xi_n)|/s(\eta_n) \in (0, \infty]$. *Then it holds that*

$$\lim_{t\to\infty} \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}] = \{m(l_2) - m(l_1)\}^{-1} \int_I f(y)\, dm(y). \tag{2.37}$$

If $s_* \in (0, \infty)$, *then it also holds that*

$$\begin{aligned} &\lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] \\ &= \lim_{n\to\infty} \lim_{t\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}] \\ &= \{m(l_2) - m(l_1)\}^{-1} \int_I f(y)\, dm(y). \end{aligned} \tag{2.38}$$

This theorem implies that the double limits ($n \to \infty$ and $t \to \infty$) are commutable if we take appropriate sequences $\{\xi_n\}$ and $\{\eta_n\}$, when there exists a stationary distribution. However this commutability does not hold in general as we can see in the following.

We consider again the case that the speed measure is regularly varying near the boundary ℓ_2. Let $0 < \beta < 1$ and L be a positive slowly varying function at infinity. Let $(A9)$ and $(A10)$ be the following conditions.

$(A9)$ $\ell_i = r_i$, $|\ell_i| = \infty$ for $i = 1, 2$. $s(x) = x$, $x \in I$, $m(x)$ satisfies (2.23) and there exists the limit

$$\theta = \lim_{t\to\infty} k_2(t)/k_1(t) \in [0, \infty), \tag{2.39}$$

where $k_1(t)$ and $k_2(t)$ are the inverse functions of the mapping $[0, \infty) \ni x \mapsto -xm(-x)$ and $[0, \infty) \ni x \mapsto xm(x)$, respectively.

$(A10)$ $\ell_1 = r_1 > -\infty$ and $\ell_2 = r_2 = \infty$. $s(x) = x$, $x \in I$, and $m(x)$ satisfies (2.23). Further assume that

$$\lim_{x\to\infty} |x^2 m(x)/m(l_1 + 1/x)| = \infty.$$

Since (2.23) is satisfied, there is a slowly varying function $\tilde{L}$ satisfying (2.24).

THEOREM 2.9. *Assume* (*A*9). *Let* $x \in I_m$, $0 < \tau \le 1$ *and* $f \in L^1(I, m)$. *Then it holds that*

$$\lim_{t\to\infty} t^{1-\beta}\tilde{L}(t)^{-1} \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] = C_4 \tau^{-1+\beta} \int_I f(y)\, dm(y), \tag{2.40}$$

where

$$C_4 = \beta\{\beta(1-\beta)\}^{-\beta}\Gamma(1-\beta)^{-1}(1+\theta)^{-1}.$$

If there exists the limit $s_* = \lim_{n\to\infty}|\xi_n|/\eta_n \in (0, \infty]$, *then*

$$\lim_{t\to\infty} t^{1-\beta}\tilde{L}(t)^{-1} \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}] = C_4 \tau^{-1+\beta} \int_I f(y)\, dm(y). \tag{2.41}$$

Further assume that $s_* \in (0, \infty)$. *Then there exist subsequences* $\{\xi_n\}$ *and* $\{\eta_n\}$ (*denoted by the same symbols*) *and positive constants* $v_*^{(j)}$, $j = 1, 2, 3$, *such that the statement* (ii) *of Theorem* 2.6 *holds with* $V_n^{(j)} = v_*^{(j)} m(\eta_n)$, $\psi_* = 1$ *and* $f \in L^1(I, m)$.

REMARK 2.4. Note that $v_*^{(j)}$, $j = 1, 2, 3$, are not necessarily the same (see Sect. 6.3).

THEOREM 2.10. *Assume* (*A*10). *Let* $x \in I_m \cap I$, $0 < \tau \le 1$ *and* $f \in L^1(I, m)$ *with* supp$[f]$ *being compact in* I. *Then it holds that*

$$\begin{aligned} &\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t) \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] \\ &= \lim_{t\to\infty} t^{1+\beta}\tilde{L}(t) \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} < \sigma_{\eta_n}] \\ &= C_3 \tau^{-1-\beta}(x - l_1) \int_I (y - l_1) f(y)\, dm(y). \end{aligned} \tag{2.42}$$

There exist subsequences $\{\xi_n\}$ *and* $\{\eta_n\}$ (*denoted by the same symbols*) *such that the statement* (ii) *of Theorem* 2.6 *holds with a function* $\psi_*(y) = B(y - c_o) + 1$, $y \in I$, *where* B *is a real number satisfying* $0 \le B \le (c_o - l_1)^{-1}$. *If* $\limsup_{n\to\infty}|m(\xi_n)|/\eta_n m(\eta_n) < \infty$, *then* $\{V_n^{(j)}\}$, $j = 1, 2, 3$, *satisfy the following properties.*

(2.43) $$\liminf_{n\to\infty} V_n^{(j)}/m(\eta_n) > 0, \quad j = 1, 2,$$

(2.44) $$\limsup_{n\to\infty} V_n^{(1)}/\eta_n^2 m(\eta_n) < \infty, \quad \limsup_{n\to\infty} V_n^{(2)}/\eta_n m(\eta_n) < \infty,$$

(2.45) $$\liminf_{n\to\infty} V_n^{(3)}/\eta_n m(\eta_n) > 0, \quad \limsup_{n\to\infty} V_n^{(3)}/\eta_n^3 m(\eta_n) < \infty.$$

Remark 2.5. In the same way as in the proof of Theorem 2.8 we can show that

$$\lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}] = E_x[f(X(\tau t))\{X(t) - l_1\}]/E_x[X(t) - l_1],$$

under the assumption of Theorem 2.10. By means of Proposition 3.1 in the next section,

$$0 < E_x[X(t) - l_1] < \infty, \quad x \in I_m \cap I, \ t > 0.$$

But it seems to be difficult to consider the asymptotic behavior of $E_x[X(t) - l_1]$ as $t \to \infty$ since $x - l_1 \notin L^1(I, m)$. So that there are no results on the conditional asymptotic behavior of $\lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$ as $t \to \infty$.

3 Preliminaries

In this section we define the elementary solution $p(t, x, y)$ of the generalized diffusion equation (1.2) following [12], [21] and [33]. Then we study its properties.

Let I, m, s, etc. be those given in the preceding section. Let $\varphi_i(x, \lambda)$, $i = 1, 2$, $\lambda \in \mathbf{C}$, be the solutions of the integral equations

(3.1) $$\varphi_1(x, \lambda) = 1 + \lambda \int_{(c_o, x]} \{s(x) - s(y)\}\varphi_1(y, \lambda)\, dm(y), \quad x \in I,$$

(3.2) $$\varphi_2(x, \lambda) = s(x) - s(c_o) + \lambda \int_{(c_o, x]} \{s(x) - s(y)\}\varphi_2(y, \lambda)\, dm(y), \quad x \in I,$$

where $c_o \in I_*$ is fixed arbitrarily. Then for each $\alpha > 0$, there exist the limits

$$h_1(\alpha) = -\lim_{x \downarrow l_1} \varphi_2(x, \alpha)/\varphi_1(x, \alpha), \quad h_2(\alpha) = \lim_{x \uparrow l_2} \varphi_2(x, \alpha)/\varphi_1(x, \alpha).$$

Define the function $h(\alpha)$ by the equality

$$1/h(\alpha) = 1/h_1(\alpha) + 1/h_2(\alpha).$$

We set

$$h_{11}(\alpha) = h(\alpha), \quad h_{22}(\alpha) = -\{h_1(\alpha) + h_2(\alpha)\}^{-1},$$
$$h_{12}(\alpha) = h_{21}(\alpha) = -h(\alpha)/h_2(\alpha).$$

The functions $h_{ij}(\alpha)$, $i, j = 1, 2$, can be analytically continued to $\mathbf{C}\backslash(-\infty, 0]$. The spectral measures σ_{ij}, $i, j = 1, 2$ are defined by

$$\sigma_{ij}([\lambda_1, \lambda_2]) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im\, h_{ij}(-\lambda + \sqrt{-1}\varepsilon)\, d\lambda,$$

for all continuity points λ_1 and λ_2 of σ_{ij} $(\lambda_1 < \lambda_2)$. The matrix valued measure $[\sigma_{ij}(d\lambda)]_{i,j=1,2}$ is symmetric and nonnegative definite. We define the elementary solution of the generalized diffusion equation (1.2) by

$$p(t, x, y) = \sum_{i,j=1,2} \int_{[0,\infty)} e^{-\lambda t} \varphi_i(x, -\lambda) \varphi_j(y, -\lambda) \sigma_{ij}(d\lambda), \quad t > 0,\ x, y \in I. \tag{3.3}$$

Note that $p(t, x, y) = p(t, y, x)$ and $p(t, x, y)$ is positive and continuous for $t > 0$, $x, y \in I$.

Next we give the definition of the Green function $G(\alpha, x, y)$ of the generalized diffusion equation (1.2). Define the functions $u_i(x, \alpha)$, $i = 1, 2$, $\alpha > 0$, $x \in I$, by

$$u_i(x, \alpha) = \varphi_1(x, \alpha) + (-1)^{i+1} \varphi_2(x, \alpha)/h_i(\alpha). \tag{3.4}$$

It is well known that $u_1(x, \alpha)$ [resp. $u_2(x, \alpha)$] is positive and nondecreasing [resp. nonincreasing] in $x \in I$, and $u_i(x, \alpha)$, $i = 1, 2$, satisfy

$$u_1(l_2, \alpha) < \infty \quad \text{if } I(l_2) < \infty, \tag{3.5}$$

$$u_2(l_1, \alpha) < \infty \quad \text{if } I(l_1) < \infty, \tag{3.6}$$

$$\lim_{x \to l_i, x \in I} u_i(x, \alpha) = 0, \qquad \lim_{x \to l_i, x \in I} u_i^+(x, \alpha) = 0 \quad \text{if } l_i \text{ is natural}, \tag{3.7}$$

$$\left| \int_{(c_o, l_i)} u_i(y, \alpha)\, dm(y) \right| < \infty, \tag{3.8}$$

(see [12], [18]). We define $G(\alpha, x, y)$ by

$$G(\alpha, x, y) = G(\alpha, y, x) = h(\alpha) u_1(x, \alpha) u_2(y, \alpha), \quad \alpha > 0,\ x \le y,\ x, y \in I. \tag{3.9}$$

We note the following facts ([21], [22]).

$$\sup_{t>0} p(t,x,y) < \infty, \quad \text{if } m(x) < m(y),\ \{(l_1,x)\cup(y,l_2)\}\cap I_m \neq \emptyset. \tag{3.10}$$

$$p(t,x,y) = p(t, x\vee r_1, r_1), \quad t>0,\ x\in I,\ y\in(l_1,r_1] \quad \text{if } l_1 < r_1. \tag{3.11}$$

$$p(t,x,y) = p(t, x\wedge r_2, r_2), \quad t>0,\ x\in I,\ y\in[r_2,l_2) \quad \text{if } r_2 < l_2. \tag{3.12}$$

$$p(t,x,y) \le p(t,x,x)^{1/2} p(t,y,y)^{1/2}, \quad t>0,\ x,y\in I. \tag{3.13}$$

$$p(t,x,x) \le t^{-1} G(t^{-1},x,x), \quad t>0,\ x\in I. \tag{3.14}$$

$$G(\alpha,x,x) \le G(\alpha,y,y) + |s(x)-s(y)|, \quad \alpha>0,\ x,y\in I. \tag{3.15}$$

$$\lim_{x\to l_i, x\in I} G(\alpha,x,x) = 0, \quad \alpha>0, \quad \text{if } I(l_i)<\infty. \tag{3.16}$$

$$\lim_{x\to l_i, x\in I} p(t,x,y) = 0, \quad t>0,\ y\in I, \quad \text{if } I(l_i)<\infty. \tag{3.17}$$

Let $\mathbf{D} = [X(t) : t\ge 0, P_x : x\in I_m]$ be a one-dimensional generalized diffusion process with the generator $\mathscr{L}$ as in the preceding section. Then it holds that

$$P_x(X(t)\in E) = \int_E p(t,x,y)\, dm(y), \quad x\in I_m,\ t>0,\ E\in\mathscr{B}(I), \tag{3.18}$$

([12]). We find the following properties from [12], [21] and [22].

$$E_a[e^{-\alpha\sigma_b}] = \frac{u_1(a,\alpha)}{u_1(b,\alpha)}, \quad E_b[e^{-\alpha\sigma_a}] = \frac{u_2(b,\alpha)}{u_2(a,\alpha)}, \quad \alpha>0, \tag{3.19}$$

$$\text{if } a,b\in I_m,\ a<b.$$

$$\begin{aligned} p(t,x,y) &= \int_0^t p(t-u,a,y) P_x(\sigma_a\in du) \\ &= \int_0^t p(t-u,x,a) P_y(\sigma_a\in du), \quad t>0, \end{aligned} \tag{3.20}$$

$$\text{if } x,y,a\in I_m,\ x<a<y.$$

Obviously the expectation $E_x[f(X(t))]$ of $f(X(t))$ with respect to the probability measure P_x is finite for bounded measurable functions f on I. It is easy to show that it is finite for $f\in L^1(I,m)$. Now we observe that $E_x[s(X(t))]$ is finite.

Proposition 3.1. *It holds that*

$$E_x[|s(X(t))|] < \infty, \tag{3.21}$$

for $x\in I_m\cap I_*$ *and* $t>0$.

PROOF. By means of (3.18),

$$E_x[|s(X(t))|] = \int_I p(t,x,y)|s(y)|\,dm(y) \tag{3.22}$$

$$= \int_{(l_1,c_o]} p(t,x,y)|s(y)|\,dm(y) + \int_{(c_o,l_2)} p(t,x,y)|s(y)|\,dm(y).$$

We only show

$$\int_{(c_o,l_2)} p(t,x,y)|s(y)|\,dm(y) < \infty. \tag{3.23}$$

If $s \in L^1((c_o,l_2),m)$ or $|s(l_2)| < \infty$, then (3.23) holds obviously. Assume $s \notin L^1((c_o,l_2),m)$ and $s(l_2) = \infty$, which implies that l_2 is natural. We note that

$$\int_{(c_o,l_2)} |s(y)|u_2(y,\alpha)\,dm(y) < \infty, \quad \alpha > 0. \tag{3.24}$$

Indeed, since by (3.7), $u_2(x,\alpha)$ satisfies

$$-u_2^+(x,\alpha) = \alpha \int_{(x,l_2)} u_2(y,\alpha)\,dm(y),$$

we see by (3.7) again that

$$\infty > u_2(c_o,\alpha) = -\int_{(c_o,l_2)} u_2^+(x,\alpha)\,ds(x)$$

$$= \alpha \int_{(c_o,l_2)} ds(x) \int_{(x,l_2)} u_2(y,\alpha)\,dm(y)$$

$$= \alpha \int_{(c_o,l_2)} \{s(y) - s(c_o)\}u_2(y,\alpha)\,dm(y).$$

Combining this with (3.8) we obtain (3.24). Let $l_1 \le a < b \le l_2$. Then by means of (3.9), (3.13) and (3.14),

$$\sup_{a<y<b} p(t,x,y) \le t^{-1/2} p(t,x,x)^{1/2} \sup_{a<y<b} G(t^{-1},y,y)^{1/2}$$

$$= t^{-1/2} h(t^{-1})^{1/2} p(t,x,x)^{1/2} \sup_{a<y<b} u_1(y,t^{-1})^{1/2} u_2(y,t^{-1})^{1/2}$$

$$\le t^{-1/2} h(t^{-1})^{1/2} p(t,x,x)^{1/2} u_1(b,t^{-1})^{1/2} u_2(a,t^{-1})^{1/2} \equiv F(t,x,a,b).$$

Since l_2 is natural, $(x, l_2) \cap I_m$ is an infinite set. We can take $a_o, b_o \in I_m$ such that $m(x) < m(a_o) < m(b_o)$, so that $M_o \equiv \sup_{u>0} p(u, x, a_o) < \infty$ in view of (3.10). By means of (3.19) and (3.20) we see that for $y \in [b_o, l_2) \cap I_m$,

$$p(t, x, y) \leq M_o P_y(\sigma_{a_o} < t) \leq M_o E_y[e^{1-\sigma_{a_o}/t}] = M_o e u_2(y, t^{-1})/u_2(a_o, t^{-1}).$$

This estimate coupled with (3.24) implies that

$$\begin{aligned}
&\int_{(c_o, l_2)} p(t, x, y)|s(y)|\, dm(y) \\
&\quad = \int_{(c_o, b_o)} p(t, x, y)|s(y)|\, dm(y) + \int_{[b_o, l_2)} p(t, x, y)|s(y)|\, dm(y) \\
&\quad \leq F(t, x, c_o, b_o) \int_{(c_o, b_o)} |s(y)|\, dm(y) \\
&\qquad + M_o e u_2(a_o, t^{-1})^{-1} \int_{[b_o, l_2)} u_2(y, t^{-1})|s(y)|\, dm(y) < \infty.
\end{aligned}$$

This shows (3.23). □

Let $p_*(t, x, y)$ be the elementary solution of the generalized diffusion equation (1.2) with $\mathscr{L}$ and I replaced by $\mathscr{L}_*$ and I_*, respectively. Note that $p_*(t, x, y)$ coincides with $p(t, x, y)$ whenever $l_1 = r_1$ and $l_2 = r_2$. Also note that

$$\begin{aligned}
(3.25) \qquad & P_x(X(t \wedge \sigma_{r_1} \wedge \sigma_{r_2}) \in E) \\
& \quad = \int_E p_*(t, x, y)\, dm(y), \quad x \in I_m \cap I_*,\ t > 0,\ E \in \mathscr{B}(I_*).
\end{aligned}$$

The conditional expectation $E_x[f(X(t)) \,|\, t < \sigma_{r_1} \wedge \sigma_{r_2}]$ is finite for $x \in I_m \cap I_*$, $t > 0$ and f which is bounded on I_* or $f \in L^1(I_*, m)$.

In the rest of this section, we assume that both of l_i, $i = 1, 2$, are not natural. Then the spectrum Σ_* of $-\mathscr{L}_*$ only consists of point spectrum, λ_* $(= \inf \Sigma_*) \in \Sigma_*$, and (3.3) corresponding to $p_*(t, x, y)$ is reduced to

$$(3.26) \qquad p_*(t, x, y) = \sum_{\lambda \in \Sigma_*} e^{-\lambda t} \psi(x, -\lambda) \psi(y, -\lambda), \quad t > 0,\ x, y \in I_*,$$

where the functions $\psi(x, -\lambda)$, $\lambda \in \Sigma_*$, satisfy

$$(3.27) \qquad \psi^+(y, -\lambda) - \psi^+(x, -\lambda) = -\lambda \int_{(x, y]} \psi(z, -\lambda)\, dm(z), \quad x, y \in I_*,$$

$$\psi(r_i, -\lambda) = 0 \quad \text{if } I(l_i) < \infty, \text{ or } l_i \text{ is entrance with } l_i \neq r_i, \tag{3.28}$$

$$\psi^+(l_i, -\lambda) = 0 \quad \text{if } l_i \text{ is entrance with } l_i = r_i, \tag{3.29}$$

$$\int_{I_*} \psi(x, -\lambda)^2 \, dm(x) = 1, \tag{3.30}$$

$$\int_{I_*} \psi(x, -\lambda)\psi(x, -\mu) \, dm(x) = 0 \quad \text{if } \lambda \neq \mu. \tag{3.31}$$

In view of [14, Theorem 1],

$$\psi(x, -\lambda_*) > 0, \quad x \in I_*. \tag{3.32}$$

Proposition 3.2. *Let $i, j \in \{1, 2\}$ and $i \neq j$. Assume one of $(A1)_i$, $(A2)_i$, $(A3)_i$, and one of $(Ak)_j$, $k = 1, 2, \dots, 5$. Then $\psi(\cdot, -\lambda_*)$ belongs to $L^1(I_*, m)$, and $\psi(\cdot, -\lambda_*)f(\cdot)$, $\psi(\cdot, -\lambda_*)^2 f(\cdot)$ also belong to $L^1(I_*, m)$ whenever $f \in \mathscr{H}$. Further there is a nonincreasing function $K(t)$ on $(0, \infty)$ such that*

$$e^{\lambda_* t} p_*(t, x, y) \leq K(t) H(x) H(y), \quad t > 0, \, x, y \in I_*, \tag{3.33}$$

where $H(x) = H_1(x) 1_{(r_1, c_o]}(x) + H_2(x) 1_{(c_o, r_2)}(x)$, and for each $i = 1, 2$,

$$H_i(x) = \begin{cases} |s(x) - s(r_i)|, & \text{if } (A1)_i \text{ or } (A2)_i \text{ is satisfied}, \\ |s(x) - s(r_i)|^{1/2}, & \text{if } (A3)_i \text{ is satisfied}, \\ 1 + |s(x)|^{1/2}, & \text{if } (A4)_i \text{ is satisfied}, \\ 1, & \text{if } (A5)_i \text{ is satisfied}. \end{cases} \tag{3.34}$$

Consequently it holds that

$$\iint_{I_* \times I_*} p_*(u, x, y) p_*(t, y, z) |f(y) g(z)| \, dm(y) dm(z) < \infty, \tag{3.35}$$

for $u, t > 0$, $x \in I_*$ and $f, g \in \mathscr{H}$.

Proof. By means of (3.14), (3.15) and (3.16) corresponding to m_* and I_* in place of m and I, respectively, we see that

$$p_*(t, x, x) \leq t^{-1} |s(x) - s(r_i)|, \quad t > 0, \, x \in I_*, \quad \text{if } I(r_i) < \infty. \tag{3.36}$$

Since $I(r_i) < \infty$ for $i = 1$ or 2, it holds that

$$p_*(t, x, x) \leq t^{-1} \min\{s(x) - s(r_1), s(r_2) - s(x)\}, \quad t > 0, \, x \in I_*. \tag{3.37}$$

We show that

$$\sup_{x \in (r_1, c_o]} p_*(t,x,x)/H_1(x)^2 < \infty, \quad t > 0. \tag{3.38}$$

If $(A3)_1$ is satisfied, then (3.36) implies (3.38). If $(A4)_1$ is satisfied, then $|s(r_1)| = \infty$ and one of $(A1)_2$, $(A2)_2$, $(A3)_2$ is satisfied, and hence $|s(r_2)| < \infty$. Therefore by means of (3.37) we obtain (3.38).

Let us assume $(A5)_1$. Then by using the results in [20, Sect. 4] and [23, (6.4)], we find that $p_*(t,x,x)$ is bounded in $x \in (r_1, c_o]$ for each fixed $t > 0$, which shows (3.38).

If $(A1)_1$ or $(A2)_1$ is satisfied, then in the same way as [20, Sect. 4] we can show that $\sup_{x \in (r_1, c_o]} p_*(t,x,x)\{s(x) - s(r_1)\}^{-2}$ is finite for each $t > 0$. Thus we obtain (3.38).

In the same way as above, we can obtain that

$$\sup_{x \in (c_o, r_2)} p_*(t,x,x)/H_2(x)^2 < \infty, \quad t > 0. \tag{3.39}$$

It follows from (3.38) and (3.39) that

$$\tilde{K}(t) \equiv \sup_{x \in I_*} p_*(t,x,x)/H(x)^2 < \infty, \quad t > 0. \tag{3.40}$$

Since $\psi(x, -\lambda_*)^2 \le e^{\lambda_* t} p_*(t,x,x)$, there is a positive constant C such that

$$0 < \psi(x, -\lambda_*) \le CH(x), \quad x \in I_*. \tag{3.41}$$

Next we show (3.33). We may assume that $\Sigma_* \backslash \{\lambda_*\} \ne \emptyset$. There is a $\delta \in (0,1)$ such that $\lambda_* < \delta\lambda$, $\lambda \in \Sigma_* \backslash \{\lambda_*\}$, from which it holds that $(1-\delta)\lambda < \lambda - \lambda_*$, $\lambda \in \Sigma_* \backslash \{\lambda_*\}$. Then we see that

$$\begin{aligned}
|e^{\lambda_* t} p_*(t,x,y) - \psi(x,-\lambda_*)\psi(y,-\lambda_*)| &= \left| \sum_{\lambda \in \Sigma_* \backslash \{\lambda_*\}} e^{-(\lambda - \lambda_*)t} \psi(x,-\lambda)\psi(y,-\lambda) \right| \\
&\le \sum_{\lambda \in \Sigma_*} e^{-\lambda(1-\delta)t} |\psi(x,-\lambda)|\,|\psi(y,-\lambda)| \\
&\le p_*((1-\delta)t, x, x)^{1/2} p_*((1-\delta)t, y, y)^{1/2}.
\end{aligned}$$

Combining this with (3.40) and (3.41), we find that

$$\begin{aligned}
e^{\lambda_* t} p_*(t,x,y) &\le p_*((1-\delta)t,x,x)^{1/2} p_*((1-\delta)t,y,y)^{1/2} + \psi(x,-\lambda_*)\psi(y,-\lambda_*) \\
&\le \{\tilde{K}((1-\delta)t) + C^2\} H(x) H(y).
\end{aligned}$$

Putting $K(t) = \tilde{K}((1-\delta)t) + C^2$ leads us to (3.33). Since $p_*(t,x,x)$ is non-increasing in t, so is $K(t)$.

It is easy to see that

$$H \in L^1(I_*, m), \quad Hf \in L^1(I_*, m), \quad H^2 f \in L^1(I_*, m), \tag{3.42}$$

for $f \in \mathscr{H}$. Combining (3.41) with (3.42), we obtain that $\psi(\cdot, -\lambda_*)$, $\psi(\cdot, -\lambda_*)f(\cdot)$ and $\psi(\cdot, -\lambda_*)^2 f(\cdot)$ belong to $L^1(I_*, m)$. The result of (3.35) immediately follows from (3.33) and (3.42). $\square$

REMARK 3.1. If one of $(A1)_i$, $(A2)_i$, $(A5)_i$ is satisfied for each $i = 1, 2$, we can show (3.33) with H replaced by $\psi(\cdot, -\lambda_*)$, that is, $\sup_{x,y \in I_*} p_*(t,x,y)/\psi(x,-\lambda_*)\psi(y,-\lambda_*) < \infty$ for each $t > 0$. This implies that $\mathscr{L}$ is intrinsically ultracontractive (cf. [2]).

4 Proof of Theorems

In order to show Theorem 2.1, we prepare the following.

PROPOSITION 4.1. *Under the assumption of Theorem* 2.1, *it holds that*

$$\begin{aligned}\lim_{t\to\infty} & E_x[f(X(\tau t))g(X(t)) \mid t < \sigma_{r_1} \wedge \sigma_{r_2}] \\ &= \left(\int_{I_*} \psi(y, -\lambda_*)\, dm(y)\right)^{-1} \int_{I_*} \psi(y, -\lambda_*)^2 f(y)\, dm(y) \\ &\quad \times \int_{I_*} \psi(z, -\lambda_*) g(z)\, dm(z),\end{aligned} \tag{4.1}$$

for $x \in I_m \cap I_*$, $0 < \tau < 1$ *and* $f, g \in \mathscr{H}$.

PROOF. By virtue of (3.25) and Markov property,

$$\begin{aligned}E_x&[f(X(\tau t))g(X(t)) \mid t < \sigma_{r_1} \wedge \sigma_{r_2}] \\ &= E_x[f(X(\tau t))g(X(t)), t < \sigma_{r_1} \wedge \sigma_{r_2}]/P_x(t < \sigma_{r_1} \wedge \sigma_{r_2}) \\ &= \int_{I_*} p_*(\tau t, x, y) f(y)\, dm(y) \\ &\quad \times \int_{I_*} p_*((1-\tau)t, y, z) g(z)\, dm(z)/P_x(t < \sigma_{r_1} \wedge \sigma_{r_2}),\end{aligned} \tag{4.2}$$

which is finite in view of (3.35). Note that

$$P_x(t < \sigma_{r_1} \wedge \sigma_{r_2}) = \int_{I_*} p_*(t,x,y)\,dm(y). \tag{4.3}$$

It follows from (3.26) that

$$\lim_{t\to\infty} e^{\lambda_* t} p_*(t,x,y) = \psi(x,-\lambda_*)\psi(y,-\lambda_*), \quad x,y \in I_*. \tag{4.4}$$

By means of Proposition 3.2,

$$\begin{aligned} &e^{\lambda_* t} p_*(\tau t,x,y)p_*((1-\tau)t,y,z) \\ &\quad \le K(\tau t_o)K((1-\tau)t_o)H(x)H(y)^2H(z), \quad t \ge t_o,\ x,y \in I_*, \end{aligned}$$

for each $t_o > 0$. Combining this with Lebesgue's dominated convergence theorem, (3.42) and (4.4), we obtain

$$\begin{aligned} &\lim_{t\to\infty} e^{\lambda_* t} \iint_{I_*\times I_*} p_*(\tau t,x,y)p_*((1-\tau)t,y,z)f(y)g(z)\,dm(y)dm(z) \\ &\quad = \psi(x,-\lambda_*) \iint_{I_*\times I_*} \psi(y,-\lambda_*)^2\psi(z,-\lambda_*)f(y)g(z)\,dm(y)dm(z). \end{aligned} \tag{4.5}$$

Since $1 \in \mathscr{H}$, $\int_{I_*} p_*(\tau t,x,y)p_*((1-\tau)t,y,z)\,dm(y) = p_*(t,x,z)$ and $\int_{I_*} \psi(y,-\lambda_*)^2\,dm(y) = 1$, the asymptotic behavior (4.1) is derived from (4.2), (4.3) and (4.5). □

Proof of Theorem 2.1. Put $\psi_*(x) = \psi(c_o,-\lambda_*)^{-1}\psi(x,-\lambda_*)$, $x \in I_*$. It follows from (3.27), (3.28), (3.29), (3.30), (3.32), and Proposition 3.2 that ψ_* satisfies all of the properties in the statement (i). Further by means of Proposition 4.1,

$$\begin{aligned} &\lim_{t\to\infty} E_x[f(X(\tau t))g(X(t)) \mid t < \sigma_{r_1} \wedge \sigma_{r_2}] \\ &\quad = \left(\int_{I_*} \psi_*(y)^2\,dm(y)\right)^{-1} \left(\int_{I_*} \psi_*(y)\,dm(y)\right)^{-1} \\ &\qquad \times \int_{I_*} \psi_*(y)^2 f(y)\,dm(y) \int_{I_*} \psi_*(z)g(z)\,dm(z). \end{aligned} \tag{4.6}$$

We find (2.12) with $0 < \tau < 1$ and $\tau = 1$ by putting $g = 1$ and $f = 1$ in (4.6), respectively. □

Proof of Corollary 2.2. By Markov property, we have

$$E_x[f(X(\tau t))\,|\,t<\sigma_{r_2}<\sigma_{r_1}] = E_x[f(X(\tau t)), t<\sigma_{r_2}<\sigma_{r_1}]/P_x(t<\sigma_{r_2}<\sigma_{r_1})$$
$$= \frac{E_x[f(X(\tau t))\Phi(X(t))\,|\,t<\sigma_{r_1}\wedge\sigma_{r_2}]}{E_x[\Phi(X(t))\,|\,t<\sigma_{r_1}\wedge\sigma_{r_2}]}, \quad 0<\tau\le 1,$$

where

$$(4.7)\quad \Phi(x) = P_x(\sigma_{r_2}<\sigma_{r_1}) = \begin{cases} \{s(x)-s(r_1)\}/\{s(r_2)-s(r_1)\}, & \text{if } |s(r_1)|<\infty, \\ 1, & \text{if } |s(r_1)|=\infty. \end{cases}$$

Since $\Phi \in \mathscr{H}$, we obtain the corollary by means of Proposition 4.1. □

Next we prove Theorems 2.3 and 2.4. In [25] and [29], the asymptotic behavior of elementary solutions of periodic generalized diffusion equations was studied. We list up some results from [25] and [29]. Let $\varphi_i(x,\lambda)$, $i=1,2$, be the solutions of the equations (3.1), (3.2) with $c_o=0$, which exist because of the assumption $(A6)$. It holds that, under the assumption $(A6)$,

$$(4.8)\qquad \lambda_* > 0 \quad \text{if and only if} \quad \kappa \neq 1,$$

$$(4.9)\qquad \varphi_2(x,-\lambda_*) > 0, \quad x\in I_*,$$

$$(4.10)\qquad \sup_{x\in I_*} \kappa^{-x/2}(1+x)^{-1}\varphi_2(x,-\lambda_*) < \infty,$$

$$(4.11)\qquad \int_{I_*} \kappa^{x/2}(1+x)\,dm(x) < \infty \quad \text{if} \quad \kappa>1,$$

$$(4.12)\qquad \varphi_2(\cdot,-\lambda_*) \in L^1(I_*,m) \quad \text{if} \quad \kappa>1,$$

$$(4.13)\qquad sf \in L^1(I_*,m) \quad \text{if} \quad \kappa=1 \text{ and } f \text{ satisfies (2.15)},$$

$$(4.14)\qquad \lim_{t\to\infty} t^{3/2}e^{\lambda_* t}p_*(t,x,y) = C_1\varphi_2(x,-\lambda_*)\varphi_2(y,-\lambda_*), \quad x,y\in I_*,$$

$$(4.15)\qquad \limsup_{t\to\infty} t^{3/2}e^{\lambda_* t}\sup_{x,y\in I_*} \kappa^{-(x+y)/2}(1+x)^{-1}(1+y)^{-1}p_*(t,x,y) < \infty,$$

$$(4.16)\qquad \lim_{t\to\infty}\sup_{x\in I_*} \kappa^{-x/2}(1+x)^{-1} \times \left| t^{3/2}e^{\lambda_* t}\int_{I_*} p_*(t,x,y)f(y)\,dm(y) - C_1\varphi_2(x,-\lambda_*)\int_{I_*}\varphi_2(y,-\lambda_*)f(y)\,dm(y)\right| = 0,$$

for f satisfying (2.15), where C_1 is a positive constant given by

$$C_1 = 2^{-1}\pi^{-1/2}\kappa^{1/4}\{-\Delta'(\lambda_*)\}^{1/2}\varphi_2(1, -\lambda_*)^{-1}, \tag{4.17}$$

$$\Delta(\lambda) = \varphi_1(1, -\lambda) + \kappa\varphi_2^+(1, -\lambda), \quad \Delta'(\lambda_*) = \frac{d}{d\lambda}\Delta(\lambda)\bigg|_{\lambda=\lambda_*} \quad (< 0).$$

Proof of Theorem 2.3. Put $\psi_*(x) = \varphi_2(x, -\lambda_*)$, $x \in I_*$. By means of (3.2), (4.9), (4.10) and (4.12), we see that ψ_* satisfies all of the properties of the statement (i).

Let $0 < \kappa < 1$ and f satisfy (2.15). We note that

$$\lim_{t\to\infty} P_x(t < \sigma_0) = P_x(\sigma_0 = \infty) = s(x)/s(l_2) \in (0, \infty), \quad x \in I_*. \tag{4.18}$$

If $0 < \tau < 1$, then in the same way as in the proof of Proposition 4.1,

$$\begin{aligned} &E_x[f(X(\tau t)) \mid t < \sigma_0] \\ &\quad = \int_{I_*} p_*(\tau t, x, y) f(y) P_y((1-\tau)t < \sigma_0)\, dm(y)/P_x(t < \sigma_0). \end{aligned} \tag{4.19}$$

By using (4.14), (4.15), (4.18), (4.19), and Lebesgue's dominated convergence theorem, we obtain (2.16) with $0 < \tau < 1$. If $\tau = 1$, then

$$E_x[f(X(t)) \mid t < \sigma_0] = \int_{I_*} p_*(t, x, y) f(y)\, dm(y)/P_x(t < \sigma_0). \tag{4.20}$$

Therefore (2.16) with $\tau = 1$ immediately follows from (4.16) and (4.18).

Let $\kappa > 1$. In this case, $f = 1$ satisfies (2.15) by means of (4.11), and hence we obtain (4.16) with $f = 1$. Therefore

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x \in I_m \cap I_*} \kappa^{-x/2}(1 + x)^{-1} \\ &\quad \times \left| t^{3/2} e^{\lambda_* t} P_x(t < \sigma_0) - C_1 \varphi_2(x, -\lambda_*) \int_{I_*} \varphi_2(y, -\lambda_*)\, dm(y) \right| = 0. \end{aligned} \tag{4.21}$$

If $0 < \tau < 1$ and f satisfies (2.18), then (2.17) is obtained by (4.14), (4.15), (4.19), (4.21) and Lebesgue's dominated convergence theorem. If $\tau = 1$ and f satisfies (2.15), then (2.19) follows from (4.16), (4.20) and (4.21). □

Proof of Theorem 2.4. Noting $\kappa = 1$, we find $\lambda_* = 0$, $\varphi_2(x, -\lambda_*) = s(x)$, $C_1 = \{m(1)/4\pi s(1)\}^{1/2}$ ([25]), and

$$\lim_{x\to\infty} x^{-1}s(x) = s(1), \quad \lim_{x\to\infty} x^{-1}m(x) = m(1).$$

Hence by means of [24, Corollary 1],

$$\lim_{t\to\infty} t^{1/2}\int_{I_*} p_*(t,x,y)\,dm(y) = \{m(1)/\pi s(1)\}^{1/2}s(x).$$

Combining this with (4.3), we find that

$$\lim_{t\to\infty} t^{1/2}P_x(t<\sigma_0) = \{m(1)/\pi s(1)\}^{1/2}s(x). \tag{4.22}$$

Since $P_x(t<\sigma_0)$ and $s(x)$ are nondecreasing in $x \in I_m$ and $x \in I_*$, respectively, and $s(x)$ is continuous, the convergence in (4.22) is uniform in $x \in I_m \cap [0,a]$ for each $a \in I_*$. Therefore if $0<\tau<1$, $f \in L^1(I,m)$ and supp$[f]$ is compact in $[0,\infty)$, we obtain (2.20) by means of (4.14), (4.15), (4.19), (4.22), and Lebesgue's dominated convergence theorem. If $\tau = 1$ and f satisfies (2.15), then $sf \in L^1(I_*,m)$ by means of (4.13), and (2.21) follows from (4.16), (4.20) and (4.22). □

We turn to the proof of Theorem 2.5. Under the assumption $(A7)$ or $(A8)$, the asymptotic behavior of the elementary solution $p_*(t,x,y)$ was studied in [30]. We summarize some results which we need below.

PROPOSITION 4.2. *If $(A7)$ is satisfied, then*

$$\lim_{t\to\infty}\sup_{x,y\in(0,a]} |t^{1+\beta}\tilde{L}(t)x^{-1}y^{-1}p_*(t,x,y) - C_2 l_2^{-2}| = 0, \quad a \in I_*. \tag{4.23}$$

If $(A8)$ is satisfied, then

$$\lim_{t\to\infty}\sup_{x,y\in(0,a]} |t^{1+\beta}\tilde{L}(t)x^{-1}y^{-1}p_*(t,x,y) - C_3| = 0, \quad a \in I_*, \tag{4.24}$$

$$\limsup_{t\to\infty} t^{1+\beta}\tilde{L}(t)\sup_{y\in I_*} y^{-1}p_*(t,x,y) < \infty, \tag{4.25}$$

$$\lim_{t\to\infty} t^{\beta}\tilde{L}(t)\int_{I_*} p_*(t,x,y)\,dm(y) = \beta^{-1}C_3 x, \tag{4.26}$$

$$\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)\int_{I_*} p_*(t,x,y)f(y)\,dm(y) = C_3 x\int_{I_*} yf(y)\,dm(y), \tag{4.27}$$

for f satisfying $yf(y) \in L^1(I_*,m)$.

PROOF. In [30], (4.24), (4.25), (4.26) and (4.27) are obtained. We will show (4.23). We note that, under the assumption $(A7)$, there exist the solutions $\varphi_i(x,\lambda)$, $i = 1,2$, of the equations (3.1) and (3.2) with $c_o = 0$. We also note that (3.3) corresponding to $p_*(t,x,y)$ is reduced to

$$(4.28)\quad p_*(t,x,y) = \int_{(0,\infty)} e^{-\lambda t}\varphi_2(x,-\lambda)\varphi_2(y,-\lambda)\sigma_{22}(d\lambda), \quad t>0,\ x,y \in I_*.$$

By virtue of [30, (5.11)],

$$(4.29)\qquad \lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)\int_{(0,\infty)} e^{-\lambda t}\sigma_{22}(d\lambda) = C_2 l_2^{-2}.$$

In view of (4.28) and (4.29), it is enough to show that

$$(4.30)\quad \lim_{t\to\infty}\sup_{x,y\in(0,a]} t^{1+\beta}\tilde{L}(t)\int_{(0,\infty)} e^{-\lambda t}|x^{-1}y^{-1}\varphi_2(x,-\lambda)\varphi_2(y,-\lambda)-1|\sigma_{22}(d\lambda) = 0,$$

for each $a \in I_*$. Due to [13, (2.27)], we have

$$(4.31)\qquad |\varphi_2(x,-\lambda)| \le |x|\exp\{2^{1/2}|\lambda x m(x)|^{1/2}\},$$

$$(4.32)\qquad |\varphi_2(x,\lambda)-x| \le |\lambda x^2 m(x)|\,|\exp\{2^{1/2}|\lambda x m(x)|^{1/2}\},$$

for $x \in I$ and $\lambda \in \mathbf{C}$. Fix a point $a \in I_*$ arbitrarily. By means of (4.31) and (4.32), we have

$$\begin{aligned}
&\sup_{x,y\in(0,a]} |x^{-1}y^{-1}\varphi_2(x,-\lambda)\varphi_2(y,-\lambda)-1| \\
&\quad\le \sup_{x,y\in(0,a]} |x^{-1}\varphi_2(x,-\lambda)-1|\,|y^{-1}\varphi_2(y,-\lambda)| + \sup_{y\in(0,a]} |y^{-1}\varphi_2(y,-\lambda)-1| \\
&\quad\le 2|\lambda| am(a)\exp\{2^{3/2}|\lambda am(a)|^{1/2}\}.
\end{aligned}$$

Fix a positive T arbitrarily and take a positive Λ such that $\Lambda T^2 \ge 32am(a)$. Then it holds that $\sup_{t\ge T,\lambda\ge\Lambda}\exp\{-\lambda t/2 + 2^{3/2}(\lambda am(a))^{1/2}\} \le 1$. Noting $\max_{x\ge 0} xe^{-x} \le e^{-1}$, we see that for every $t \ge T$,

$$\begin{aligned}
&\sup_{x,y\in(0,a]} t^{1+\beta}\tilde{L}(t)\int_{(0,\infty)} e^{-\lambda t}|x^{-1}y^{-1}\varphi_2(x,-\lambda)\varphi_2(y,-\lambda)-1|\sigma_{22}(d\lambda) \\
&\quad\le 2am(a)t^{1+\beta}\tilde{L}(t)\int_{(0,\infty)} \lambda\exp\{-\lambda t + 2^{3/2}(\lambda am(a))^{1/2}\}\sigma_{22}(d\lambda) \\
&\quad\le 2am(a)\{1+\exp\{2^{3/2}(\Lambda am(a))^{1/2}\}\}t^{1+\beta}\tilde{L}(t)\int_{(0,\infty)} \lambda e^{-\lambda t/2}\sigma_{22}(d\lambda) \\
&\quad\le 8e^{-1}am(a)\{1+\exp\{2^{3/2}(\Lambda am(a))^{1/2}\}\}t^{\beta}\tilde{L}(t)\int_{(0,\infty)} e^{-\lambda t/4}\sigma_{22}(d\lambda).
\end{aligned}$$

Combining this with (4.29), we obtain (4.30). □

PROOF OF THEOREM 2.5. (i) Let us assume $(A7)$. Let f satisfy $yf(y) \in L^1(I,m)$ and $\mathrm{supp}[f]$ being compact in $[0,l_2)$. Note that

$$\lim_{t\to\infty} P_x(t<\sigma_0) = P_x(\sigma_0=\infty) = x/l_2. \tag{4.33}$$

If $0<\tau<1$, we notice that (4.19) holds in this case, too. By (4.23), (4.33), and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(\tau t))\,|\,t<\sigma_0] \\
&\quad= \lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)\int_{I_*} p_*(\tau t,x,y)f(y)P_y((1-\tau)t<\sigma_0)\,dm(y)/P_x(t<\sigma_0) \\
&\quad= C_2\tau^{-1-\beta}l_2^{-2}\int_{I_*} y^2 f(y)\,dm(y).
\end{aligned}$$

Let $\tau=1$. By means of (4.20), (4.23), (4.33), and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(t))\,|\,t<\sigma_0] \\
&\quad= \lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)\int_{I_*} p_*(t,x,y)f(y)\,dm(y)/P_x(t<\sigma_0) \\
&\quad= C_2 l_2^{-1}\int_{I_*} yf(y)\,dm(y).
\end{aligned}$$

(ii) Assume $(A8)$ and $yf(y)\in L^1(I_*,m)$. Combining (4.26) with (4.3), we have

$$\lim_{t\to\infty} t^{\beta}\tilde{L}(t)P_x(t<\sigma_0) = \beta^{-1}C_3 x, \tag{4.34}$$

uniformly in x in $[0,a]\cap I_m$ for each $a\in I_*$. If $0<\tau<1$ and $\mathrm{supp}[f]$ is compact in $[0,\infty)$, by using (4.19), (4.24), (4.25), (4.34) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)E_x[f(X(\tau t))\,|\,t<\sigma_0] \\
&\quad= \lim_{t\to\infty} t^{1+\beta}\tilde{L}(t)\int_{I_*} p_*(\tau t,x,y)f(y)P_y((1-\tau)t<\sigma_0)\,dm(y)/P_x(t<\sigma_0) \\
&\quad= C_3\tau^{-1-\beta}(1-\tau)^{-\beta}\int_{I_*} y^2 f(y)\,dm(y).
\end{aligned}$$

If $\tau = 1$, by using (4.20), (4.27), (4.34) and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{t\to\infty} tE_x[f(X(t)) \mid t < \sigma_0]$$
$$= \lim_{t\to\infty} t\int_{I_*} p_*(t,x,y)f(y)\,dm(y)/P_x(t<\sigma_0)$$
$$= \beta\int_{I_*} yf(y)\,dm(y),$$

which completes the proof. □

Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences satisfying (2.29). We set

$$m_n(x) = \begin{cases} -\infty, & x \le \xi_n, \\ m(x), & \xi_n < x < \eta_n, \\ +\infty, & \eta \le x, \end{cases} \quad s_n(x) = 1_{(\xi_n,\eta_n)}(x)s(x), \quad \mathscr{L}_n = \frac{d}{dm_n}\frac{d}{ds_n}.$$

Denote by Σ_n the spectrum of $-\mathscr{L}_n$ and put $\lambda_{*,n} = \inf \Sigma_n$. Since the assumptions $(A1)_i$, $i = 1,2$, are satisfied for m_n, s_n and (ξ_n,η_n), there exists a unique positive continuous function $\psi_{*,n}$ on (ξ_n,η_n) satisfying $\psi_{*,n}(c_o) = 1$ and (2.7), (2.8), (2.10) with ψ_*, λ_* and I_* replaced by $\psi_{*,n}$, $\lambda_{*,n}$ and (ξ_n,η_n), respectively.

Proposition 4.3. *There exist a subsequence* $\{\psi_{*,n}\}$ (*denoted by the same symbol*) *and a positive continuous function* ψ_* *on* I_* *satisfying* (2.7), $\psi_*(c_o) = 1$, *and*

(4.35) $$\lim_{n\to\infty}\sup_{x\in K}|\psi_{*,n}(x) - \psi_*(x)| = 0,$$

for every compact set $K \subset I_*$.

Assume $r_i = l_i$ *and* $|s(l_i)| = \infty$ *for* $i = 1$ *and* 2. *Then* $\psi_*(x) = 1$, $x \in I$. *If* $\{\xi_n\}$ *and* $\{\eta_n\}$ *satisfy*

(4.36) $$0 < \liminf_{n\to\infty}|s(\xi_n)|/s(\eta_n) \le \limsup_{n\to\infty}|s(\xi_n)|/s(\eta_n) < \infty,$$

then

(4.37) $$\sup_{\xi_n \le x \le \eta_n,\, n}\psi_{*,n}(x) < \infty.$$

Assume $\lambda_* = 0$, $r_i = l_i$ *for* $i = 1,2$, $|s(l_1)| < \infty$ *and* $s(l_2) = \infty$. *Then* $\psi_*(x) = B\{s(x) - s(c_o)\} + 1$, $x \in I$, *where* B *is a real number satisfying* $0 \le B \le \{s(c_o) - s(l_1)\}^{-1}$. *For each* $a \in I$, *it holds that*

(4.38) $$\sup_{\xi_n \le x \le a, n} \psi_{*,n}(x) < \infty,$$

(4.39) $$\sup_{a \le x \le \eta_n, n} \psi_{*,n}(x)/s(x) < \infty \quad \textit{if} \ \ s(a) > 0.$$

PROOF. Following the same argument as the proof of [14, Theorem 1], we can show that there exist a subsequence $\{\psi_{*,n}\}$ and a positive continuous function ψ_* satisfying (2.7), $\psi_*(c_o) = 1$ and (4.35).

Assume $r_i = l_i$ and $|s(l_i)| = \infty$ for $i = 1$ and 2. Then $\lambda_* = 0$ by virtue of [18, Theorem 3]. Therefore it follows from (2.7) that $\psi_*(x) = B_1 s(x) + B_2$, $x \in I$, where B_1 and B_2 are constants. Noting that $\psi_*(x) > 0$, $x \in I$ and $\psi_*(c_o) = 1$, we see that $B_1 = 0$ and $B_2 = 1$, that is, $\psi_*(x) = 1$, $x \in I$. Let $\{\xi_n\}$ and $\{\eta_n\}$ satisfy (4.36). We will show (4.37). By virtue of (2.7),

(4.40) $$\psi_{*,n}(x) = 1 + \psi_{*,n}^+(c_o)\{s(x) - s(c_o)\} - \lambda_{*,n} \int_{(c_o, x]} \{s(x) - s(y)\}\psi_{*,n}(y)\, dm(y), \quad x \in (\xi_n, \eta_n),$$

and it holds that

(4.41) $$|\psi_{*,n}^+(c_o)| \le \frac{1 - \psi_{*,n}(\xi_n)}{s(c_o) - s(\xi_n)} \vee \frac{1 - \psi_{*,n}(\eta_n)}{s(\eta_n) - s(c_o)} \le \{s(c_o) - s(\xi_n)\}^{-1} \vee \{s(\eta_n) - s(c_o)\}^{-1}.$$

Combining this with (4.36), we see that

$$M_1 \equiv \sup_{\xi_n \le x \le \eta_n, n} |\psi_{*,n}^+(c_o)\{s(x) - s(c_o)\}| \le \sup_n \{\{s(\eta_n) - s(c_o)\}\{s(c_o) - s(\xi_n)\}^{-1} \vee \{s(c_o) - s(\xi_n)\}\{s(\eta_n) - s(c_o)\}^{-1}\} < \infty.$$

By using (4.40) again, we have

$$\sup_{\xi_n \le x \le \eta_n, n} \psi_{*,n}(x) \le 1 + M_1.$$

Assume $\lambda_* = 0$, $r_i = l_i$ for $i = 1, 2$, $|s(l_1)| < \infty$, and $s(l_2) = \infty$. Noting $\psi_*(l_1) \ge 0$, in the same way as above we find that $\psi_*(x) = B\{s(x) - s(c_o)\} + 1$,

$x \in I$, and $0 \leq B \leq \{s(c_o) - s(l_1)\}^{-1}$. Let $a \in I$. Since $M_2 \equiv \sup_n |\psi^+_{*,n}(c_o)| < \infty$ by means of (4.41), it follows from (4.40) that

$$\sup_{\xi_n \leq x \leq a, n} \psi_{*,n}(x) \leq 1 + M_2 |s(a) - s(c_o)|,$$

$$\sup_{a \leq x \leq \eta_n, n} \psi_{*,n}(x)/s(x) \leq s(a)^{-1} + M_2\{1 + |s(c_o)|/s(a)\} \quad \text{if } s(a) > 0,$$

which shows (4.38) and (4.39). □

Proof of Theorem 2.6. Let $\{\xi_n\}$ and $\{\eta_n\}$ be subsequences corresponding to $\{\psi_{*,n}\}$ in Proposition 4.3. Let $f \in L^1(I_*, m)$ such that supp$[f]$ is compact in I_*. We may assume that supp$[f] \subset (\xi_n, \eta_n)$ for sufficiently large n. Then f satisfies (2.3) with $r_1 = \xi_n$ or $r_2 = \eta_n$. By means of Theorem 2.1 and Corollary 2.2, we see that

$$\lim_{t \to \infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] = \lim_{t \to \infty} E_x[f(X(\tau t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$$

$$= \left(\int_{(\xi_n, \eta_n)} \psi_{*,n}(y)^2 \, dm(y) \right)^{-1} \int_{(\xi_n, \eta_n)} \psi_{*,n}(y)^2 f(y) \, dm(y), \quad 0 < \tau < 1,$$

$$\lim_{t \to \infty} E_x[f(X(t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]$$

$$= \left(\int_{(\xi_n, \eta_n)} \psi_{*,n}(y) \, dm(y) \right)^{-1} \int_{(\xi_n, \eta_n)} \psi_{*,n}(y) f(y) \, dm(y),$$

$$\lim_{t \to \infty} E_x[f(X(t)) \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}]$$

$$= \left(\int_{(\xi_n, \eta_n)} \psi_{*,n}(y) \{s(y) - s(\xi_n)\} \, dm(y) \right)^{-1}$$

$$\times \int_{(\xi_n, \eta_n)} \psi_{*,n}(y) \{s(y) - s(\xi_n)\} f(y) \, dm(y).$$

We put

$$V_n^{(1)} = \int_{(\xi_n, \eta_n)} \psi_{*,n}(y)^2 \, dm(y), \tag{4.42}$$

$$V_n^{(2)} = \int_{(\xi_n, \eta_n)} \psi_{*,n}(y) \, dm(y), \tag{4.43}$$

$$(4.44)\qquad V_n^{(3)} = \begin{cases} \displaystyle\int_{(\xi_n,\eta_n)} \psi_{*,n}(y)\{s(y) - s(\xi_n)\}\, dm(y), & |s(r_1)| < \infty, \\ \displaystyle\int_{(\xi_n,\eta_n)} \psi_{*,n}(y)\{1 - s(y)/s(\xi_n)\}\, dm(y), & |s(r_1)| = \infty. \end{cases}$$

Then we obtain (2.30), (2.31) and (2.32) by virtue of (4.35). □

Proof of Corollary 2.7. We put $\psi_{*,n}(x) = 0$ for $x \in I_* \backslash (\xi_n, \eta_n)$ and $n \in \mathbf{N}$. Then, by means of $(A1)_1$ and $(A1)_2$, we can show that $\psi_{*,n} (n \in \mathbf{N})$ are uniformly bounded and equicontinuous on I_*. Therefore there exist a subsequence $\{\psi_{*,n}\}$ (denoted by the same symbols) and a positive continuous function $\tilde{\psi}_*$ on I_* such that

$$\lim_{n\to\infty} \sup_{x \in I_*} |\psi_{*,n}(x) - \tilde{\psi}_*(x)| = 0.$$

We note that $\tilde{\psi}_*$ satisfies (2.7) with ψ_* replaced by $\tilde{\psi}_*$, $\tilde{\psi}_*(c_o) = 1$ and $\tilde{\psi}_*(r_i) = 0$ $(i = 1, 2)$. Since the function ψ_* in Theorem 2.1 is unique, we see that $\tilde{\psi}_*$ coincides with ψ_*, and

$$(4.45)\qquad \lim_{n\to\infty} \sup_{x \in I_*} |\psi_{*,n}(x) - \psi_*(x)| = 0,$$

for every sequence $\{\psi_{*,n}\}$. It is easy to see that (2.30), (2.31) and (2.32) are valid for every f satisfying (2.3) and $V_n^{(j)}$, $j = 1, 2, 3$, given by (4.42), (4.43), (4.44) with $|s(r_1)| < \infty$. By means of (4.45), $(A1)_1$ and $(A1)_2$, we find that

$$\lim_{n\to\infty} V_n^{(1)} = \int_{I_*} \psi_*(y)^2\, dm(y),$$

$$\lim_{n\to\infty} V_n^{(2)} = \int_{I_*} \psi_*(y)\, dm(y),$$

$$\lim_{n\to\infty} V_n^{(3)} = \int_{I_*} \psi_*(y)\{s(y) - s(r_1)\}\, dm(y).$$

Thus we obtain (2.33), (2.34) and (2.35). □

Proof of Theorem 2.8. Let $0 < \tau \le 1$ and $f \in L^1(I, m)$. It is easy to see that

$$(4.46)\qquad \lim_{n\to\infty} E_x[f(X(\tau t)) \,|\, t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}] = E_x[f(X(\tau t))] = \int_I p(\tau t, x, y) f(y)\, dm(y).$$

This formula together with [22, Corollary 1] implies (2.36).

Assume that there exists the limit $s_* = \lim_{n\to\infty}|s(\xi_n)|/s(\eta_n) \in (0,\infty]$. In the same way as the proof of Corollary 2.2,

$$E_x[f(X(\tau t))\,|\,t < \sigma_{\eta_n} < \sigma_{\xi_n}] = \frac{E_x[f(X(\tau t))\Phi_n(X(t)), t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]}{E_x[\Phi_n(X(t)), t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]}, \tag{4.47}$$

where

$$\Phi_n(x) = P_x(\sigma_{\eta_n} < \sigma_{\xi_n}) = \frac{s(x) - s(\xi_n)}{s(\eta_n) - s(\xi_n)}. \tag{4.48}$$

Since $\lim_{n\to\infty} \Phi_n(x) = s_*(1+s_*)^{-1} \in (0,1]$, we see that

$$\lim_{n\to\infty} E_x[f(X(\tau t))\,|\,t < \sigma_{\eta_n} < \sigma_{\xi_n}] = E_x[f(X(\tau t))], \tag{4.49}$$

from which (2.37) follows.

Next we assume $0 < s_* < \infty$. By means of Proposition 4.3, $\psi_*(x) = 1$, $x \in I$ and it is the unique positive continuous function on I satisfying (2.7) and $\psi_*(c_o) = 1$ because of $\lambda_* = 0$. By (4.37) and the dominated convergence theorem, we see that (2.30), (2.31) and (2.32) are valid for $\psi_* = 1$, any sequences satisfying (2.29) and (4.36), and $f \in L^1(I,m)$. Note that we do not need the condition that supp$[f]$ is compact in I. Noting (4.37), (4.42), (4.43) and (4.44), we find that

$$\lim_{n\to\infty} V_n^{(j)} = m(l_2) - m(l_1), \quad j = 1,2,3.$$

Combining this with Theorem 2.6, we obtain (2.38). □

Proof of Theorem 2.9. Let $0 < \tau \le 1$ and $f \in L^1(I,m)$. Since (4.46) also holds in this case, by means of [16, Theorem 2] and [22, Remark 2] we obtain (2.40).

Assume that there exists the limit $s_* = \lim_{n\to\infty}|\xi_n|/\eta_n \in (0,\infty]$. In the same way as the proof of Theorem 2.8, we can show (2.41). Further assume $s_* \in (0,\infty)$. Then we see that (2.30), (2.31) and (2.32) hold with $\psi_* = 1$, and $f \in L^1(I,m)$. Here we do not need the condition that supp$[f]$ is compact in I. Since (2.23) and (2.39) imply $\lim_{x\to\infty}|m(-x)|/m(x) \in [0,\infty)$, by means of (4.37), (4.42), (4.43) and (4.44) we also see that the sequences $\{V_n^{(j)}/m(\eta_n)\}$, $j = 1,2,3$, are bounded, and hence there are subsequences $\{\xi_n\}$, $\{\eta_n\}$, $\{V_n^{(j)}\}$, $j = 1,2,3$, (denoted by the same symbols) satisfying $v_*^{(j)} \equiv \lim_{n\to\infty} V_n^{(j)}/m(\eta_n) \in [0,\infty)$, $j = 1,2,3$. We will show that $v_*^{(j)} > 0$, $j = 1,2,3$. It follows from (4.40) that

$$\psi_{*,n}(x) \ge (\eta_n - x)/(\eta_n - c_o), \quad c_o \le x \le \eta_n, \tag{4.50}$$

which implies

$$V_n^{(1)} \geq \int_{c_o}^{\eta_n} \psi_{*,n}(y)^2 \, dm(y) \geq \frac{1}{(\eta_n - c_o)^2} \int_{c_o}^{\eta_n/2} (\eta_n - y)^2 \, dm(y)$$

$$\geq \frac{(\eta_n/2)^2}{(\eta_n - c_o)^2} \{m(\eta_n/2) - m(c_o)\}.$$

Combining this with (2.23) we see that $v_*^{(1)} \geq 2^{-(1+1/\beta)}$. In the same way we obtain $v_*^{(j)} > 0, \ j = 2, 3$. □

Proof of Theorem 2.10. Let $0 < \tau \leq 1$ and $f \in L^1(I, m)$ such that $\mathrm{supp}[f]$ is compact in I. Since (4.46) also holds in this case, by means of [30, Theorem 1] we see that

$$\lim_{t\to\infty} t^{1+\beta} \tilde{L}(t) \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]$$

$$= C_3 \tau^{-1-\beta}(x - l_1) \int_I (y - l_1) f(y) \, dm(y).$$

Next we note that

$$E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} < \sigma_{\eta_n}] = \frac{E_x[f(X(\tau t))\Psi_n(X(t)), t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]}{E_x[\Psi_n(X(t)), t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}]},$$

where

$$\Psi_n(x) = P_x(\sigma_{\xi_n} < \sigma_{\eta_n}) = \frac{\eta_n - x}{\eta_n - \xi_n}.$$

Since $\lim_{n\to\infty} \Psi_n(x) = 1$, we see that

$$\lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} < \sigma_{\eta_n}] = E_x[f(X(\tau t))].$$

Therefore we obtain

$$\lim_{t\to\infty} t^{1+\beta} \tilde{L}(t) \lim_{n\to\infty} E_x[f(X(\tau t)) \mid t < \sigma_{\xi_n} < \sigma_{\eta_n}]$$

$$= C_3 \tau^{-1-\beta}(x - l_1) \int_I (y - l_1) f(y) \, dm(y).$$

Note that $\lambda_* = 0$ by virtue of [18, Theorem 3], and hence by means of Proposition 4.3, $\psi_*(x) = B(x - c_o) + 1, \ x \in I$ with $0 \leq B \leq (c_o - l_1)^{-1}$. Thus there exist subsequences $\{\xi_n\}$ and $\{\eta_n\}$ (denoted by the same symbols) such that the statement (ii) of Theorem 2.6 holds with this ψ_*. Assume that $\limsup_{n\to\infty}|m(\xi_n)|/\eta_n m(\eta_n) < \infty$. Fix a point $a \in I$ such that $s(a) > 0$. By using (4.38) and (4.39) we see that

$$\begin{aligned}
&\limsup_{n\to\infty} V_n^{(1)}/\eta_n^2 m(\eta_n)\\
&\quad = \limsup_{n\to\infty} \frac{1}{\eta_n^2 m(\eta_n)} \left\{ \int_{(\xi_n,a]} \psi_{*,n}(y)^2\, dm(y) + \int_{(a,\eta_n)} \psi_{*,n}(y)^2\, dm(y) \right\}\\
&\quad \le \sup_{\xi_n \le y \le a,n} \psi_{*,n}(y)^2 \limsup_{n\to\infty} \{m(a) - m(\xi_n)\}/\eta_n^2 m(\eta_n)\\
&\qquad + \sup_{a \le y \le \eta_n,n} \psi_{*,n}(y)^2/y^2 < \infty.
\end{aligned}$$

In the same way we obtain

$$\limsup_{n\to\infty} V_n^{(2)}/\eta_n m(\eta_n) < \infty, \quad \limsup_{n\to\infty} V_n^{(3)}/\eta_n^3 m(\eta_n) < \infty.$$

Since (4.50) is also valid in this case, (2.43) follows by the same method as in the proof of Theorem 2.9. We also see that

$$\begin{aligned}
V_n^{(3)} &= \int_{(\xi_n,\eta_n)} \psi_{*,n}(y)(y-\xi_n)\, dm(y)\\
&\ge \frac{1}{\eta_n - c_o} \int_{(\eta_n/3,\eta_n/2)} (\eta_n - y)(y-\xi_n)\, dm(y)\\
&\ge \frac{(\eta_n/2)(\eta_n/3-\xi_n)}{\eta_n - c_o} \{m(\eta_n/2) - m(\eta_n/3)\}.
\end{aligned}$$

Combining this with (2.23) we obtain $\liminf_{n\to\infty} V_n^{(3)}/\eta_n m(\eta_n) > 0$. □

5 Examples

We observe two examples in this section.

5.1 Bessel Processes

Let $\mathbf{D} = [X(t) : t \ge 0, P_x : x \in I]$ be a diffusion process whose generator is given by

$$L = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\alpha - 1}{2x}\frac{d}{dx}, \quad x \in I,$$

where $\alpha \in \mathbf{R}$, and $I = (0, 1)$ or $(0, \infty)$. This process is referred to as the α-dimensional Bessel process on I if $\alpha > 0$. In particular, $\mathbf{D}$ is the Brownian motion on I in the case that $\alpha = 1$. We set

$$(5.1)\quad s_o(x) = x^{1-\alpha}, \quad m_o(x) = 2x^{\alpha-1}, \quad s(x) = \int_{c_o}^{x} s_o(y)\,dy, \quad m(x) = \int_{c_o}^{x} m_o(y)\,dy,$$

where c_o is a fixed point in I arbitrarily. We assume that $m(x) = -\infty$ for $x < 0$, and further assume that if $I = (0, 1)$, then $m(x) = \infty$ for $x > 1$. Thus $l_1 = r_1 = 0$ and $l_2 = r_2 = 1$ if $I = (0, 1)$, or $l_2 = r_2 = \infty$ if $I = (0, \infty)$. Then the generator L is reduced to $\mathscr{L} = (d/dm)(d/ds)$ which is a generalized diffusion operator defined in Sect. 2, and s and m above are the scale function and the speed measure, respectively. We note that $(A1)_1$ or $(A2)_1$ or $(A5)_1$ is valid according to $0 < \alpha < 2$ or $\alpha \le 0$ or $\alpha \ge 2$.

(i) Let us consider the case $I = (0, 1)$. In this case $(A1)_2$ holds. Let $J_\nu(x)$ be the Bessel function defined by

$$J_\nu(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\nu+1)}.$$

We denote by $\mathbf{J}(\nu)$ the set of positive zeros of J_ν, that is, $\mathbf{J}(\nu) = \{x > 0 : J_\nu(x) = 0\}$. It is known that $\mathbf{J}(\nu)$ is a countable infinite set, it has no accumulating points in $[0, \infty)$, and

$$J_{\nu-1}(x) J_{\nu+1}(x) \ne 0 \quad \text{for } x \in \mathbf{J}(\nu),$$

(cf. [32, Ch.15]).

If $\alpha < 2$, then

$$p(t, x, y) = \sum_{\kappa \in \mathbf{J}(1-\alpha/2)} |J_{-\alpha/2}(\kappa) J_{2-\alpha/2}(\kappa)|^{-1} e^{-\kappa^2 t/2} (xy)^{1-\alpha/2} J_{1-\alpha/2}(\kappa x) J_{1-\alpha/2}(\kappa y).$$

Suppose that f satisfies $\int_0^{1/2} x|f(x)|\,dx < \infty$ and $\int_{1/2}^{1} (1 - x^{2-\alpha})|f(x)|\,dx < \infty$. By virtue of Theorem 2.1 and Corollary 2.2, it holds that for $x \in I$,

$$\lim_{t\to\infty} E_x[f(X(\tau t)) \,|\, t < \sigma_0 \wedge \sigma_1] = \int_0^1 f^{(1)}(\tau, y) f(y)\,dy,$$

$$\lim_{t\to\infty} E_x[f(X(\tau t)) \,|\, t < \sigma_1 < \sigma_0] = \int_0^1 f^{(2)}(\tau, y) f(y)\,dy,$$

$$(5.2)\qquad f^{(1)}(\tau,y)=f^{(2)}(\tau,y)$$
$$=2|J_{-\alpha/2}(\epsilon_o)J_{2-\alpha/2}(\epsilon_o)|^{-1}yJ_{1-\alpha/2}(\epsilon_o y)^2,\quad \text{if } 0<\tau<1,$$

$$(5.3)\qquad f^{(1)}(1,y)=y^{\alpha/2}J_{1-\alpha/2}(\epsilon_o y)\left(\int_0^1 y^{\alpha/2}J_{1-\alpha/2}(\epsilon_o y)\,dy\right)^{-1},$$

$$(5.4)\qquad f^{(2)}(1,y)=y^{2-\alpha/2}J_{1-\alpha/2}(\epsilon_o y)\left(\int_0^1 y^{2-\alpha/2}J_{1-\alpha/2}(\epsilon_o y)\,dy\right)^{-1},$$

where $\epsilon_o=\min \mathbf{J}(1-\alpha/2)$. In the case that $\alpha=1$, (5.2), (5.3) and (5.4) are reduced to

$$f^{(1)}(\tau,y)=f^{(2)}(\tau,y)=2\sin^2\pi y,\quad \text{if } 0<\tau<1,$$

$$f^{(1)}(1,y)=(\pi/2)\sin\pi y,$$

$$f^{(2)}(1,y)=\pi y\sin\pi y.$$

If $\alpha\geq 2$, then $P_x(X(\tau t)\in E\,|\,t<\sigma_0\wedge\sigma_1)=P_x(X(\tau t)\in E\,|\,t<\sigma_1)=P_x(X(\tau t)\in E\,|\,t<\sigma_1<\sigma_0)$ and

$$p(t,x,y)=\sum_{\kappa\in\mathbf{J}(\alpha/2-1)}|J_{\alpha/2}(\kappa)J_{\alpha/2-2}(\kappa)|^{-1}e^{-\kappa^2t/2}(xy)^{1-\alpha/2}J_{\alpha/2-1}(\kappa x)J_{\alpha/2-1}(\kappa y).$$

By means of Theorem 2.1, we find that

$$\lim_{t\to\infty}E_x[f(X(\tau t))\,|\,t<\sigma_1]=\int_0^1 f^*(\tau,y)f(y)\,dy,$$

$$(5.5)\qquad f^*(\tau,y)=2|J_{\alpha/2}(\delta_o)J_{\alpha/2-2}(\delta_o)|^{-1}yJ_{\alpha/2-1}(\delta_o y)^2,\quad \text{if } 0<\tau<1,$$

$$(5.6)\qquad f^*(1,y)=y^{\alpha/2}J_{\alpha/2-1}(\delta_o y)\left(\int_0^1 y^{\alpha/2}J_{\alpha/2-1}(\delta_o y)\,dy\right)^{-1},$$

where $\delta_o=\min\mathbf{J}(\alpha/2-1)$ and f satisfies $\int_0^{1/2}x^{\alpha-1}|f(x)|\,dx<\infty$ and $\int_{1/2}^1(1-x^{2-\alpha})|f(x)|\,dx<\infty$.

(ii) Let $I=(0,\infty)$. We set $\tilde{m}(x)=m(s^{-1}(x))$, where $s^{-1}(x)$ stands for the inverse function of $s(x)$. Then $\tilde{m}(x)$ is a continuous increasing function on $(s(0),s(\infty))$. Let $\tilde{\mathscr{L}}=(d/d\tilde{m})(d/d\tilde{s})$ with $\tilde{s}(x)=x$, which is a generalized diffusion operator. Let $\tilde{p}(t,x,y)$ be the elementary solution of the generalized diffusion equation (1.2) with $\mathscr{L}$ replaced by $\tilde{\mathscr{L}}$. Then it holds that

$$\tilde{p}(t,s(x),s(y))=p(t,x,y),\quad x,y\in I.$$

Let $0 < \alpha < 2$. Then $-\infty < s(0)$, $s(\infty) = \infty$, and $|m(0)| < \infty$. Further it holds that

$$\lim_{x\to\infty} \tilde{m}(x)/x^{\alpha/(2-\alpha)} = \lim_{x\to\infty} m(x)/s(x)^{\alpha/(2-\alpha)} = 2\alpha^{-1}(2-\alpha)^{\alpha/(2-\alpha)}.$$

This shows that $\tilde{m}$ satisfies (2.23) with $\beta = 1 - \alpha/2 \in (0,1)$. By virtue of Theorem 2.5 (ii), it holds that for $x \in I$,

$$\text{(5.7)} \quad \lim_{t\to\infty} t^{2-\alpha/2} E_x[f(X(\tau t)) \mid t < \sigma_0]$$
$$= 2^{\alpha/2-1}\Gamma(2-\alpha/2)^{-1}\tau^{-2+\alpha/2}(1-\tau)^{-1+\alpha/2}\int_0^\infty y^{3-\alpha}f(y)\,dy, \quad 0 < \tau < 1,$$

$$\text{(5.8)} \quad \lim_{t\to\infty} tE_x[f(X(t)) \mid t < \sigma_0] = \int_0^\infty yf(y)\,dy,$$

where f satisfies $\int_0^\infty x|f(x)|\,dx < \infty$ and in particular for (5.7) it is necessary that $\mathrm{supp}[f]$ is compact in $[0,\infty)$.

Let $\alpha \le 0$. Since $s(\infty) = \infty$, $m(\infty) < \infty$ in this case, we can not apply our theorems to see the asymptotic behavior of (1.1) as $t \to \infty$. Noting (5.1), we deduce from [1] that

$$\text{(5.9)} \qquad p(t,x,y) = (2t)^{-1}(xy)^{1-\alpha/2}e^{-(x^2+y^2)/2t}I_{|1-\alpha/2|}(xy/t),$$

where I_ν is the modified Bessel function defined by

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(x/2)^{2n}}{n!\Gamma(n+\nu+1)}.$$

By using this, we can obtain the asymptotic behavior of (1.1) as $t \to \infty$. Indeed,

$$P_x(t < \sigma_0) = \int_0^\infty p(t,x,y)m_o(y)\,dy$$
$$= 2^{-1+\alpha/2}\Gamma(2-\alpha/2)^{-1}t^{\alpha/2-1}x^{2-\alpha}e^{-x^2/2t}F(1, 2-\alpha/2; x^2/2t),$$

where $F(a,b;z)$ is the hypergeometric function defined by

$$F(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^\infty \frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{z^n}{n!}.$$

Here we used the following relation.

$$\int_0^\infty e^{-a^2x^2} x^{\mu-1} I_\nu(bx)\, dx$$
$$= \frac{\Gamma((\mu+\nu)/2) b^\nu}{2^{\nu+1} a^{\mu+\nu} \Gamma(\nu+1)} F((\mu+\nu)/2, \nu+1; b^2/4a^2), \quad \text{if } \mu+\nu > 0.$$

Therefore we find that

$$\lim_{t\to\infty} \sup_{0<x\le a} |t^{1-\alpha/2} x^{\alpha-2} P_x(t<\sigma_0) - 2^{-1+\alpha/2}\Gamma(2-\alpha/2)^{-1}| = 0, \quad a \in I.$$

If $\mathrm{supp}[f]$ is compact in $[0,\infty)$ and $\int_0^\infty x^{3-\alpha}|f(x)|\,dx < \infty$, then

$$(5.10)\quad \lim_{t\to\infty} t^{2-\alpha/2} E_x[f(X(\tau t)) \mid t<\sigma_0]$$
$$= \lim_{t\to\infty} t^{2-\alpha/2} \int_0^\infty p(\tau t, x, y) P_y((1-\tau)t<\sigma_0) f(y) m_o(y)\, dy / P_x(t<\sigma_0)$$
$$= 2^{-1+\alpha/2}\Gamma(2-\alpha/2)^{-1} \tau^{-2+\alpha/2} (1-\tau)^{\alpha/2-1} \int_0^\infty y^{3-\alpha} f(y)\, dy, \quad 0<\tau<1.$$

If $\int_0^\infty x|f(x)|\,dx < \infty$, then

$$\lim_{t\to\infty} tE_x[f(X(t)) \mid t<\sigma_0]$$
$$= \lim_{t\to\infty} t\int_0^\infty p(t,x,y) f(y) m_o(y)\, dy / P_x(t<\sigma_0) = \int_0^\infty y f(y)\, dy.$$

The above argument is still valid in the case that $0<\alpha<2$, and the asymptotic behavior coincides with (5.7) and (5.8). We note that it is enough for (5.10) that $x^{3-\alpha}f(x)$ is integrable near the end point 0. This fact is also valid for (5.7).

Let $\alpha \ge 2$. In this case we also find that (5.9) holds by virtue of [1], and $P_x(\sigma_0 \wedge \sigma_\infty < \infty) = 0$. Then we derive the following asymptotic behavior.

$$\lim_{t\to\infty} t^{\alpha/2} E_x[f(X(t)) \mid t<\sigma_0\wedge\sigma_\infty] = \lim_{t\to\infty} t^{\alpha/2} E_x[f(X(t))]$$
$$= 2^{1-\alpha/2}\Gamma(\alpha/2)^{-1} \int_0^\infty y^{\alpha-1} f(y)\, dy,$$

for f such that $\int_0^\infty x^{\alpha-1}|f(x)|\,dx < \infty$ and $\mathrm{supp}[f]$ is compact in $[0,\infty)$.

5.2 Birth and Death Processes

For given $0<\kappa,\ \mu<\infty$, we put

$$s(x) = \begin{cases} (\kappa-1)^{-1}(\kappa^x - 1) & \text{if } \kappa \ne 1, \\ x & \text{if } \kappa = 1, \end{cases} \quad x \in [0,\infty),$$

$$m(x) = \begin{cases} (\kappa - 1)^{-1}(\kappa + 1)\mu \sum_{n=0}^{\infty}(1 - \kappa^{-n})1_{[n,n+1)}(x), & \text{if } \kappa \neq 1, \\ 2\mu \sum_{n=0}^{\infty} n1_{[n,n+1)}(x), & \text{if } \kappa = 1, \end{cases} \quad x \in [0, \infty),$$

$$m(x) = -\infty, \quad x \in (-\infty, 0).$$

Then $\mathscr{L} = (d/dm)(d/ds)$ is a generalized diffusion operator and it is a periodic diffusion operator

$$\mathscr{L}f(n) = \{f^+(n) - f^+(n-1)\}/\{m(n) - m(n-1)\},$$

where $f^+(n) = \{f(n+1) - f(n)\}/\{s(n+1) - s(n)\}$. Feller ([8]) pointed out that the generator of a birth and death process can be represented as a difference operator as above. Since $(A6)$ is satisfied, we find that all of the statements of Theorems 2.3 and 2.4 hold. The spectrum of $-\mathscr{L}$ has been studied in [25] and [29], from which

$$\lambda_* = (\kappa + 1)^{-1}(\kappa^{1/2} - 1)^2\mu^{-1}, \quad C_1 = 2^{-1}\pi^{-1/2}\kappa^{1/4}(\kappa + 1)^{1/2}\mu^{1/2},$$

$$\varphi_2(x, -\lambda_*) = s(x), \quad x \in [0, 1],$$

$$\varphi_2(x + n, -\lambda_*) = (\kappa^{-1} + 1)^{-1/2}|\kappa^{1/2} - 1|^{-1}\kappa^{(n-1)/2} \times [\sin n\theta + \{(2\kappa - 2\kappa^{1/2} + 1) \sin n\theta - \kappa^{1/2} \sin(n-1)\theta\}s(x)], \quad x \in [0, 1),\ n \in \mathbf{N}.$$

where $\varphi_2(x, -\lambda)$ is the solution of the equation (3.2) with $c_o = 0$, and θ is the positive number satisfying $\sin\theta = (\kappa^{-1} + 1)^{1/2}|\kappa^{1/2} - 1|$ and $\cos\theta = \kappa^{1/2} + \kappa^{-1/2} - 1$. By means of Theorems 2.3 and 2.4, we obtain the following. Let $k \in \mathbf{N}$.

Assume that $0 < \kappa < 1$ and f satisfies $\sum_{n=1}^{\infty} \kappa^{n/2}(1 + n)|f(n)|\{m(n) - m(n-1)\} < \infty$. If $0 < \tau < 1$, then

$$\lim_{t\to\infty} t^{3/2} \exp\{(\kappa + 1)^{-1}(\kappa^{1/2} - 1)^2\mu^{-1}\tau t\}E_k[f(X(\tau t)) \mid t < \sigma_0]$$
$$= 2^{-1}\pi^{-1/2}\kappa^{1/4}(\kappa + 1)^{1/2}\mu^{1/2} \times \tau^{-3/2}(1 - \kappa^k)^{-1}\varphi_2(k, -\lambda_*) \sum_{n=1}^{\infty}(1 - \kappa^n)\varphi_2(n, -\lambda_*)f(n)\{m(n) - m(n-1)\}.$$

If $\tau = 1$, then

$$\lim_{t\to\infty} t^{3/2} \exp\{(\kappa+1)^{-1}(\kappa^{1/2}-1)^2\mu^{-1}t\}E_k[f(X(t)) \mid t < \sigma_0]$$
$$= 2^{-1}\pi^{-1/2}\kappa^{1/4}(\kappa+1)^{1/2}\mu^{1/2}$$
$$\times (1-\kappa^k)^{-1}\varphi_2(k,-\lambda_*)\sum_{n=1}^{\infty}\varphi_2(n,-\lambda_*)f(n)\{m(n)-m(n-1)\}.$$

Assume $\kappa = 1$. If $0 < \tau < 1$, then

$$\lim_{t\to\infty} t^{3/2}E_k[f(X(\tau t)) \mid t < \sigma_0]$$
$$= (2\pi)^{-1/2}\mu^{1/2}\tau^{-3/2}(1-\tau)^{-1/2}\sum_{n=1}^{\infty}n^2 f(n)\{m(n)-m(n-1)\},$$

for every f such that $\sum_{n=1}^{\infty}|f(n)|\{m(n)-m(n-1)\} < \infty$ and supp$[f]$ is compact in $[0,\infty)$. If $\tau = 1$, then

$$\lim_{t\to\infty} tE_k[f(X(t)) \mid t < \sigma_0] = 2^{-1}\sum_{n=1}^{\infty}nf(n)\{m(n)-m(n-1)\},$$

for every f satisfying $\sum_{n=1}^{\infty}(1+n)|f(n)|\{m(n)-m(n-1)\} < \infty$.

Assume $\kappa > 1$. If $0 < \tau < 1$, then

$$\lim_{t\to\infty} t^{3/2}E_k[f(X(\tau t)) \mid t < \sigma_0]$$
$$= 2^{-1}\pi^{-1/2}\kappa^{1/4}(\kappa+1)^{1/2}\mu^{1/2}$$
$$\times \tau^{-3/2}(1-\tau)^{-3/2}\sum_{n=1}^{\infty}\varphi_2(n,-\lambda_*)^2 f(n)\{m(n)-m(n-1)\},$$

for every f satisfying $\sum_{n=1}^{\infty}\kappa^n(1+n)^2|f(n)|\{m(n)-m(n-1)\} < \infty$. If $\tau = 1$, then

$$\lim_{t\to\infty} E_x[f(X(t)) \mid t < \sigma_0]$$
$$= \left(\sum_{n=1}^{\infty}\varphi_2(n,-\lambda_*)\{m(n)-m(n-1)\}\right)^{-1}\sum_{n=1}^{\infty}\varphi_2(n,-\lambda_*)f(n)\{m(n)-m(n-1)\},$$

for every f satisfying $\sum_{n=1}^{\infty}\kappa^{n/2}(1+n)|f(n)|\{m(n)-m(n-1)\} < \infty$.

6 Application to Population Genetics

In this section, we consider the asymptotic conditional distributions in population genetics since this concept was first introduced in population genetics (see [4]). We consider a locus with two alleles in a randomly mating population of N diploid individuals. We denote by A_1 the wide-type allele and by A_2 the mutant allele. Let $X(n)$ be the relative frequency (gene frequency) of A_1 at the n-th generation in the population $(n = 0, 1, 2, \ldots)$. Mutation, selection and random genetic drift are the factors which change gene frequency $X(n)$. The Wright-Fisher model and the stochastic selection model are the fundamental stochastic models in population genetics. The Wright-Fisher model is a stochastic model due to random genetic drift and this stochastic force has no correlation between distinct generations. On the other hand, in the stochastic selection model stochastic force of selection has autocorrelation from generation to generation in general. These models are described by discrete time stochastic processes because we regard the generation as the time unit. It is difficult, however, to analyze these discrete time models. Then diffusion approximations are employed for the original discrete time models. In other words, we approximate a discrete time stochastic process in population genetics by an appropriate diffusion process by introducing a new time scaling. For approximating methods and applicability of diffusion approximations, see [3], [11] and cited therein. A general stochastic model may be obtained by combining these diffusion models. We will deal with a diffusion process $\mathbf{D} = [X(t) : t \geq 0, P_x : x \in I]$ that is the diffusion model with random genetic drift and stochastic selection, where I is the interval with end points 0 and 1. Further we introduce two deterministic factors of mutation and selection in this diffusion model.

It is known that the generator of the diffusion process $\mathbf{D}$ is given by

$$
\begin{aligned}
L &= \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \\
a(x) &= \frac{1}{2N}x(1-x) + \gamma x^2(1-x)^2, \qquad (6.1)\\
b(x) &= v - (u+v)x + \frac{\gamma}{2}\rho x(1-x)(1-2x) \\
&\quad + \{(S_{11} - 2S_{12} + S_{22})x + S_{12} - S_{22}\}x(1-x)
\end{aligned}
$$

(see [10]). The meaning of each variable and parameter are as follows. The variable x is the gene frequency of A_1 $(0 \leq x \leq 1)$. The parameter N is the population size $(1 \leq N \leq \infty)$. Note that the case that $N = \infty$ corresponds to that without random genetic drift. Three genotypes A_1A_1, A_1A_2 and A_2A_2 have fitnesses $1 + w_n + S_{11}$, $1 + \frac{1}{2}w_n + S_{12}$ and $1 + S_{22}$ in the original discrete time model. Here w_n is the stochastic part of selection parameters at the n-th generation, and S_{11}, S_{12} and S_{22} are the deterministic part of selection parameters $\left(\min\left\{w_n + S_{11}, \frac{1}{2}w_n + S_{12}, S_{22}\right\} \geq -1\right)$. It is assumed that stochastic selection has no dominance. We assume that $\{w_n : 0, \pm 1, \pm 2, \ldots\}$ is a discrete time stationary process with the mean $E[w_n] = 0$. The parameter $\gamma = \sum_{k=-\infty}^{\infty} E[w_o w_k]/4$ is a degree of autocorrelated stochastic selection $(0 \leq \gamma < \infty)$. The parameter ρ denotes the type of stochastic selection $(\rho \geq 1)$. The case that $\rho = 1$ with $N < \infty$ is called the TIM model ([28]) and the case that $\rho > 1$ with $N = \infty$ is called the SAS-CFF model ([9]). The mutation rate per generation from A_1 to A_2 [resp. from A_2 to A_1] is denoted by u [resp. v] $(u, v \geq 0)$.

In this section, we consider the one-dimensional diffusion process $\mathbf{D} = [X(t) : t \geq 0, P_x : x \in (0,1)]$ with the generator defined by (6.1) and $X(0) = x$. Let us fix a point $c_o \in (0,1)$ arbitrarily and set

$$s(x) = \int_{c_o}^{x} s_o(y)\, dy, \quad m(x) = \int_{c_o}^{x} m_o(y)\, dy,$$

$$s_o(x) = C_o^{-1} \exp\left\{-2\int_{c_o}^{x} \frac{b(y)}{a(y)}\, dy\right\}, \tag{6.2}$$

$$m_o(x) = \frac{2}{a(x)} s_o(x)^{-1} = \frac{2C_o}{a(x)} \exp\left\{2\int_{c_o}^{x} \frac{b(y)}{a(y)}\, dy\right\},$$

for $0 < x < 1$, where C_o is a positive constant. We also set $m(x) = -\infty$, $x < 0$ and $m(x) = \infty$, $x > 1$. Then the generator L is reduced to $\mathscr{L} = (d/dm)(d/ds)$ which is a generalized diffusion operator defined in Sect. 2, and s and m given by (6.2) are the scale function and the speed measure, respectively. The densities s_o and m_o can be expressed as follows. If $N < \infty$ and $\gamma = 0$, then the densities are

$$s_o(x) = x^{-4Nv}(1-x)^{-4Nu} \exp\{-4N(S_{12} - S_{22})x - 2N(S_{11} - 2S_{12} + S_{22})x^2\},$$

$$m_o(x) = 4Nx^{-1}(1-x)^{-1}s_o(x)^{-1}.$$

If $N < \infty$ and $\gamma > 0$, then the densities are

$$s_o(x) = x^{-4Nv}(1-x)^{-4Nu}\{(1-2x-\delta)/(1-2x+\delta)\}^{\delta^{-1}\{-2Nu+2Nv-\gamma^{-1}(S_{11}-S_{22})\}}$$
$$\times \{\gamma x(1-x) + 1/2N\}^{2Nu+2Nv-\rho+\gamma^{-1}(S_{11}-2S_{12}+S_{22})},$$

$$m_o(x) = 2x^{-1}(1-x)^{-1}\{\gamma x(1-x) + 1/2N\}^{-1}s_o(x)^{-1},$$

where $\delta = (1 + 2/N\gamma)^{1/2}$. If $N = \infty$ and $\gamma > 0$, then the densities are

$$s_o(x) = x^{-\rho-2\gamma^{-1}(v-u-S_{22}+S_{12})}(1-x)^{-\rho-2\gamma^{-1}(u-v-S_{11}+S_{12})}\exp[2\gamma^{-1}\{u(1-x)^{-1}+vx^{-1}\}],$$

$$m_o(x) = 2\gamma^{-1}x^{-2}(1-x)^{-2}s_o(x)^{-1}.$$

We classify the states of the end points 0 and 1 in Tables 1 and 2.* We will consider the asymptotic conditional distributions and some related asymptotic properties of the diffusion process with L given by (6.1).

Table 1. The state of the end point 0

	$\lvert s(0)\rvert$	$\lvert m(0)\rvert$	state
$N < \infty,\ v = 0$	$< \infty$	$= \infty$	exit
$N < \infty,\ 0 < 4Nv < 1$	$< \infty$	$< \infty$	regular
$N < \infty,\ 4Nv \geq 1$	$= \infty$	$< \infty$	entrance
$N = \infty,\ v > 0$	$= \infty$	$< \infty$	entrance
$N = \infty,\ v = 0,\ u < S_{12} - S_{22} + \gamma(\rho - 1)/2$	$= \infty$	$< \infty$	natural
$N = \infty,\ v = 0,\ u = S_{12} - S_{22} + \gamma(\rho - 1)/2$	$= \infty$	$= \infty$	natural
$N = \infty,\ v = 0,\ u > S_{12} - S_{22} + \gamma(\rho - 1)/2$	$< \infty$	$= \infty$	natural

Table 2. The state of the end point 1

	$s(1)$	$m(1)$	state
$N < \infty,\ u = 0$	$< \infty$	$= \infty$	exit
$N < \infty,\ 0 < 4Nu < 1$	$< \infty$	$< \infty$	regular
$N < \infty,\ 4Nu \geq 1$	$= \infty$	$< \infty$	entrance
$N = \infty,\ u > 0$	$= \infty$	$< \infty$	entrance
$N = \infty,\ u = 0,\ v < S_{12} - S_{11} + \gamma(\rho - 1)/2$	$= \infty$	$< \infty$	natural
$N = \infty,\ u = 0,\ v = S_{12} - S_{11} + \gamma(\rho - 1)/2$	$= \infty$	$= \infty$	natural
$N = \infty,\ u = 0,\ v > S_{12} - S_{11} + \gamma(\rho - 1)/2$	$< \infty$	$= \infty$	natural

*The states of the end points 0 and 1 in general cases are presented in Appendix (Tables 3 and 4). Tables 1 and 2 are special cases of Tables 3 and 4, respectively.

6.1 The Case that $N < \infty$ with $4Nu < 1$ or $4Nv < 1$

By Tables 1 and 2, this is the case that at least one of the boundaries is regular or exit. It is easy to see that $(A1)_1$ [resp. $(A1)_2$] is satisfied if $0 < 4Nv < 1$ [resp. $0 < 4Nu < 1$], $(A3)_1$ [resp. $(A3)_2$] is satisfied if $v = 0$ [resp. $u = 0$], and $(A4)_1$ [resp. $(A4)_2$] is satisfied if $4Nv \geq 1$ [resp. $4Nu \geq 1$]. For $E \in \mathscr{B}((0,1))$, $f(x) = 1_E(x)$ belongs to $\mathscr{H}$. We apply Theorem 2.1 to find the following asymptotic conditional distribution.

$$\lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_0 \wedge \sigma_1)$$
$$= \begin{cases} \left(\int_{I_*} \psi_*(y)^2 \, dm(y)\right)^{-1} \int_E \psi_*(y)^2 \, dm(y), & 0 < \tau < 1, \\ \left(\int_{I_*} \psi_*(y) \, dm(y)\right)^{-1} \int_E \psi_*(y) \, dm(y), & \tau = 1. \end{cases}$$

If $0 \leq 4Nu < 1 \leq 4Nv$, then $s(0) = -\infty$ by Table 1 and hence $P_x(\sigma_1 < \sigma_0) = P_x(\sigma_1 < \infty) = 1$, and

$$P_x(X(\tau t) \in E \mid t < \sigma_0 \wedge \sigma_1) = P_x(X(\tau t) \in E \mid t < \sigma_1)$$
$$= P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0), \quad 0 < \tau \leq 1.$$

If $0 \leq 4Nv < 1 \leq 4Nu$, then $s(1) = \infty$ by Table 2 and hence $P_x(\sigma_0 < \sigma_1) = P_x(\sigma_0 < \infty) = 1$ and $P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0) = 0$.

If $0 \leq 4Nu$, $4Nv < 1$, then $-\infty < s(0) < s(1) < \infty$, and

$$P_x(\sigma_1 < \sigma_0) = \{s(x) - s(0)\}/\{s(1) - s(0)\},$$

which is the probability that A_1 fixes in the population before it disappears from the population. By putting $f(x) = 1_E(x)$ in Corollary 2.2, we obtain the following.

$$P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0)$$
$$= \begin{cases} \left(\int_{I_*} \psi_*(y)^2 \, dm(y)\right)^{-1} \int_E \psi_*(y)^2 \, dm(y), & 0 < \tau < 1, \\ \left(\int_{I_*} \psi_*(y)\{s(y) - s(0)\} \, dm(y)\right)^{-1} \int_E \psi_*(y)\{s(y) - s(0)\} \, dm(y), & \tau = 1. \end{cases}$$

We consider the special case that $N < \infty$ and $\gamma = S_{11} - S_{12} = S_{22} - S_{12} = 0$ (no selection) in the following examples. The generator is given by

$$L = \frac{1}{4N} x(1-x) \frac{d^2}{dx^2} + \{v - (u+v)x\} \frac{d}{dx}.$$

The density of scale function is given by $s_o(x) = x^{-4Nv}(1-x)^{-4Nu}$ and that of speed measure is given by $m_o(x) = 4Nx^{4Nv-1}(1-x)^{4Nu-1}$.

EXAMPLE 6.1. We consider the case that $0 \le 4Nu,\ 4Nv < 1$. It is easy to see that

$$P_x(\sigma_1 < \sigma_0) = \left(\int_0^1 y^{-4Nv}(1-y)^{-4Nu}\,dy\right)^{-1}\int_0^x y^{-4Nv}(1-y)^{-4Nu}\,dy.$$

The probability density function $p(t,x,y)$ has an eigenfunction expansion (see [17]).

$$\begin{aligned} p(t,x,y) = {} & (4N)^{-1}\{\Gamma(2-4Nv)\}^{-2}(xy)^{1-4Nv}\{(1-x)(1-y)\}^{1-4Nu} \\ & \times \sum_{i=1}^{\infty} F(2-4N(u+v)+i, 1-i, 2-4Nv, x) \\ & \times F(2-4N(u+v)+i, 1-i, 2-4Nv, y)\{(i-1)!\Gamma(i+1-4Nu)\}^{-1} \\ & \times \{1-4N(u+v)+2i\}\Gamma(2-4N(u+v)+i)\Gamma(1-4Nv+i) \\ & \times \exp[-(4N)^{-1}i\{i+1-4N(u+v)\}t], \end{aligned}$$

where $F(\alpha,\beta,\gamma,x)$ is the hypergeometric function defined by

$$F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{x^n}{n!}.$$

The asymptotic conditional distributions are as follows.

$$\lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_0 \wedge \sigma_1) = \int_E f^{(1)}(\tau,y)\,dy,$$

$$\lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0) = \int_E f^{(2)}(\tau,y)\,dy,$$

$$\begin{aligned} f^{(1)}(\tau,y) &= f^{(2)}(\tau,y) \\ &= \{3-4N(u+v)\}\Gamma(3-4N(u+v))\{\Gamma(2-4Nv) \\ &\quad \times \Gamma(2-4Nu)\}^{-1}y^{1-4Nv}(1-y)^{1-4Nu}, \quad \text{if } 0<\tau<1, \end{aligned}$$

$$f^{(1)}(1,y) = 1,$$

$$f^{(2)}(1,y) = \left(\int_0^1 z^{-4Nv}(1-z)^{1-4Nu}\,dz\right)^{-1}\int_0^y z^{-4Nv}(1-z)^{-4Nu}\,dz.$$

Note that this case with $u = v = 0$ and $\tau = 1$ coincides with results due to Ewens ([5], [6]). Also note that, in this case with $u = v = 0$ and $0 < \tau < 1$, $f^{(1)}(\tau, y) = f^{(2)}(\tau, y) = 6y(1 - y)$.

Example 6.2. We consider the case that $0 \le 4Nu < 1 \le 4Nv$. The probability density function $p(t, x, y)$ has an eigenfunction expansion (see [17]).

$$
\begin{aligned}
p(t, x, y) &\\
= (4N)^{-1}&\{\Gamma(4Nv)\}^{-2}\{(1-x)(1-y)\}^{1-4Nu} \\
&\times \sum_{i=1}^{\infty} F(4N(v-u)+i, 1-i, 4Nv, x) F(4N(v-u)+i, 1-i, 4Nv, y) \\
&\times \{(i-1)!\Gamma(i+1)\}^{-1}\{4N(v-u)+2i-1\}\Gamma(4N(v-u)+i)\Gamma(4Nv+i-1) \\
&\times \exp[-(4N)^{-1}\{i^2 + (4N(v-u)-1)i + 4Nu(1-4Nv)\}t].
\end{aligned}
$$

The asymptotic conditional distributions are as follows.

$$
\begin{aligned}
\lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_0 \wedge \sigma_1) &= \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_1 < \sigma_0) \\
&= \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_1) = \int_E f^{(1)}(\tau, y)\, dy,
\end{aligned}
$$

$$
\begin{aligned}
f^{(1)}(\tau, y) &= \{4N(v-u)+1\}\Gamma(4N(v-u)+1)\{\Gamma(2-4Nu)\Gamma(4Nv)\}^{-1} \\
&\quad \times y^{4Nv-1}(1-y)^{1-4Nu}, \quad \text{if } 0 < \tau < 1, \\
f^{(1)}(1, y) &= 4Nvy^{4Nv-1}.
\end{aligned}
$$

6.2 The Case that $|s(\ell)| = \infty$, $|m(\ell)| < \infty$, $\ell = 0, 1$

Note that this is the case that there exists the stationary distribution of $X(t)$. By Tables 1 and 2 this is the case that one of the following conditions is valid.

(6.3) $\quad N < \infty, \quad 4Nu \ge 1, \quad 4Nv \ge 1.$

(6.4) $\quad N = \infty, \quad u > 0, \quad v > 0.$

(6.5) $\quad N = \infty, \quad S_{12} - S_{22} + \gamma(\rho - 1)/2 > u > v = 0.$

(6.6) $\quad N = \infty, \quad S_{12} - S_{11} + \gamma(\rho - 1)/2 > v > u = 0.$

(6.7) $\quad N = \infty, \quad S_{12} + \gamma(\rho - 1)/2 > \max\{S_{11}, S_{22}\}, \quad u = v = 0.$

Note that $\gamma \geq 0$ in (6.3), and $\gamma > 0$ in (6.4), (6.5), (6.6) and (6.7). Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences such that

(6.8) $\quad \xi_n < \eta_n \ (n \in \mathbf{N}), \quad \xi_n \downarrow 0 \quad \text{and} \quad \eta_n \uparrow 1 \quad \text{as } n \to \infty,$

(6.9) $\quad \text{there exists the limit } s_* = \lim_{n\to\infty} |s(\xi_n)|/s(\eta_n).$

Putting $\eta_n = s^{-1}(C|s(\xi_n)|)$, $n \in \mathbf{N}$, with some positive number C leads us to (6.9) with $s_* = C^{-1}$, where s^{-1} denotes the inverse function of s. Thus we may assume that $s_* \in (0, \infty)$. In view of Theorem 2.8, we obtain the following asymptotic conditional distributions. Let $0 < \tau \leq 1$, $0 < x < 1$ and $E \in \mathscr{B}((0,1))$. Then

$$\begin{aligned}
&\lim_{t\to\infty} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) \\
&= \lim_{t\to\infty} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}) \\
&= \lim_{n\to\infty} \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) \\
&= \lim_{n\to\infty} \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}) \\
&= \left(\int_0^1 m_o(y)\, dy\right)^{-1} \int_E m_o(y)\, dy.
\end{aligned}$$

Note that the last quantity of the above formulas is the stationary distribution of $X(t)$. Note also that the double limits ($n \to \infty$ and $t \to \infty$) are commutable for these cases.

6.3 The Case that $|s(\ell)| = \infty$, $|m(\ell)| = \infty$, $\ell = 0$ or 1

By Tables 1 and 2 this is the case that $N = \infty$, $\gamma > 0$, and one of the following conditions is satisfied.

(6.10) $\quad S_{12} - S_{11} + \gamma(\rho - 1)/2 = v > u = 0.$

(6.11) $\quad S_{12} - S_{22} + \gamma(\rho - 1)/2 = u > v = 0.$

(6.12) $\quad S_{22} < S_{11} = S_{12} + \gamma(\rho - 1)/2, \quad u = v = 0.$

(6.13) $\quad S_{11} < S_{22} = S_{12} + \gamma(\rho - 1)/2, \quad u = v = 0.$

(6.14) $\quad S_{11} = S_{22} = S_{12} + \gamma(\rho - 1)/2, \quad u = v = 0.$

(6.15) $\quad S_{11} > S_{22} = S_{12} + \gamma(\rho - 1)/2, \quad u = v = 0.$

(6.16) $\quad S_{22} > S_{11} = S_{12} + \gamma(\rho - 1)/2, \quad u = v = 0.$

Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences satisfying (6.8) and (6.9). We may assume that $s_* \in (0,\infty)$ in above cases except (6.15) and (6.16). Also $s_* = \infty$ in case (6.15), and $s_* = 0$ in case (6.16). We set $\tilde{m}(x) = m(s^{-1}(x))$. Then $\tilde{m}(x)$ is a continuous increasing function on $(s(0), s(1))$. Let $\tilde{\mathscr{L}} = (d/d\tilde{m})(d/d\tilde{s})$ with $\tilde{s}(x) = x$, which is a generalized diffusion operator. Let $\tilde{p}(t,x,y)$ be the elementary solution of the generalized diffusion equation (1.2) with $\mathscr{L}$ replaced by $\tilde{\mathscr{L}}$. Then it holds that

$$\tilde{p}(t, s(x), s(y)) = p(t,x,y), \quad x, y \in (0,1).$$

The double limits ($n \to \infty$ and $t \to \infty$) are not commutable in these cases as it is shown in the following.

We now consider the case (6.10) or (6.12). We see that

$$s(0) = -\infty, \quad s(1) = \infty,$$

$$\lim_{x\to-\infty} \tilde{m}(x) > -\infty, \quad \lim_{x\to\infty} x^{-1}\tilde{m}(x) = \lim_{x\uparrow 1} s(x)^{-1}m(x) = 2\gamma^{-1}e^{-4v\gamma^{-1}}.$$

Denoting by $\tilde{k}_1(t)$ and $\tilde{k}_2(t)$ the inverse functions of the mapping $[0,\infty) \ni x \mapsto -x\tilde{m}(-x)$ and $[0,\infty) \ni x \mapsto x\tilde{m}(x)$, respectively, we find that

$$\theta = \lim_{t\to\infty} \tilde{k}_2(t)/\tilde{k}_1(t) = 0. \tag{6.17}$$

Thus it follows from Theorem 2.9 that

$$\begin{aligned} &\lim_{t\to\infty} t^{1/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) \\ &= \lim_{t\to\infty} t^{1/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}) \\ &= \gamma^{1/2}(2\pi\tau)^{-1/2} e^{2v\gamma^{-1}} \int_E m_o(y)\, dy, \end{aligned} \tag{6.18}$$

for $0 < \tau \le 1$, $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ satisfying $\int_E m_o(y)\,dy < \infty$. In the cases (6.11) and (6.13), we similarly obtain (6.18) where we have to replace v by u in the last formula.

Let us consider the case (6.14). We see that $s(0) = -\infty$, $s(1) = \infty$, and

$$\lim_{x\to-\infty} |x^{-1}\tilde{m}(x)| = \lim_{x\downarrow 0} |s(x)^{-1}m(x)| = 2\gamma^{-1},$$

$$\lim_{x\to\infty} x^{-1}\tilde{m}(x) = \lim_{x\uparrow 1} s(x)^{-1}m(x) = 2\gamma^{-1}.$$

Hence (6.17) holds with $\theta = 1$ in place of $\theta = 0$. By using Theorem 2.9 we obtain that

$$\lim_{t\to\infty} t^{1/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$$
$$= \lim_{t\to\infty} t^{1/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n})$$
$$= \gamma^{1/2}(8\pi\tau)^{-1/2} \int_E m_o(y)\, dy,$$

for $0 < \tau \le 1$, $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ satisfying $\int_E m_o(y)\, dy < \infty$.

Let us assume (6.15). Then it holds that $s(0) = -\infty$, $s(1) < \infty$, $\lim_{x\to\infty}|x^{-1}\tilde{m}(-x)| = 2\gamma^{-1}$, $\lim_{x\uparrow 1}\{s(1) - s(x)\}(1-x)^{-2(S_{11}-S_{22})/\gamma} = \gamma/2(S_{11} - S_{22})$, and $\lim_{x\uparrow 1} m(x)(1-x)^{2(S_{11}-S_{22})/\gamma} = 1/(S_{11} - S_{22})$. Therefore

$$\lim_{x\to\infty} |x^2\tilde{m}(-x)t/\tilde{m}(s(1) - 1/x)| = (2/\gamma) \lim_{x\uparrow 1}\{s(1) - s(x)\}^{-3}/m(x) = \infty.$$

By exchanging the role of l_1 and l_2 in Theorem 2.10, we obtain that

$$\lim_{t\to\infty} t^{3/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$$
$$= \lim_{t\to\infty} t^{3/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n})$$
$$= \gamma^{1/2}(8\pi)^{-1/2}\tau^{-3/2}\{s(1) - s(x)\} \int_E \{s(1) - s(y)\} m_o(y)\, dy,$$

for $0 < \tau \le 1$, $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ with $\bar{E} \subset (0,1)$.

Let us assume (6.16). Then it holds that $|s(0)| < \infty$, $s(1) = \infty$, $\lim_{x\to\infty} \tilde{m}(x)/x = 2/\gamma$, $\lim_{x\downarrow 0}\{s(x) - s(0)\}x^{-q} = 1/q$ and $\lim_{x\downarrow 0}|m(x)|x^q = 2/q\gamma$, where $q = 1 - \rho - 2(S_{12} - S_{22})/\gamma > 0$. Therefore

$$\lim_{x\to\infty} |x^2\tilde{m}(x)/\tilde{m}(s(0) + 1/x)| = 2\gamma^{-1} \lim_{x\downarrow 0}\{s(x) - s(0)\}^{-3}/|m(x)| = \infty.$$

By means of Theorem 2.10, we obtain that

$$\lim_{t\to\infty} t^{3/2} \lim_{n\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$$
$$= \gamma^{1/2}(8\pi)^{-1/2}\tau^{-3/2}\{s(x) - s(0)\} \int_E \{s(y) - s(0)\} m_o(y)\, dy,$$

for $0 < \tau \le 1$, $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ with $\bar{E} \subset (0,1)$.

Next, we consider the other order of limits. In cases (6.10) to (6.14), in view of Theorem 2.9, there exist subsequences $\{\xi_n\}$ and $\{\eta_n\}$ (denoted by the same symbols) and positive constants $\mu_*^{(j)}$, $j = 1, 2, 3$, such that

$$\lim_{n\to\infty} V_n \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$$
$$= \lim_{n\to\infty} V_n \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}) = \mu_*^{(1)} \int_E m_o(y)\,dy, \quad 0 < \tau < 1,$$

$$\lim_{n\to\infty} V_n \lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) = \mu_*^{(2)} \int_E m_o(y)\,dy,$$

$$\lim_{n\to\infty} V_n \lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n}) = \mu_*^{(3)} \int_E m_o(y)\,dy,$$

for $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ satisfying $\int_E m_o(y)\,dy < \infty$, where $V_n = \int_{\xi_n}^{\eta_n} m_o(y)\,dy$. We note that $\mu_*^{(j)}$, $j = 1, 2, 3$ are not necessarily the same. For example, let us consider the case that $u = v = S_{11} - S_{12} = S_{22} - S_{12} = \rho - 1 = 0$. Then $s(x)$ and $m(x)$ defined by (6.2) reduce to

$$s(x) = \log\{x/(1-x)\}, \quad m(x) = 2\gamma^{-1} \log\{x/(1-x)\}.$$

Putting $\eta_n = s^{-1}(C|s(\xi_n)|)$, $n \in \mathbf{N}$, implies (6.9) with $s_* = C^{-1}$, where C is a positive number. It is easy to see that $\mu_*^{(1)} = 2$, $\mu_*^{(2)} = 2^{-1}\pi$, $\mu_*^{(3)} = (1+C)^{-1}\pi$.

By applying Theorem 2.6, in cases (6.15) and (6.16), we see that there are subsequences $\{\xi_n\}$, $\{\eta_n\}$ (denoted by the same symbols), and sequences of positive numbers $\{V_n^{(j)}\}$, $j = 1, 2, 3$, and a positive continuous function ψ_* satisfying (2.7) with $\lambda_* = 0$ and $\psi_*(c_o) = 1$ such that

$$\text{(6.19)} \qquad \lim_{n\to\infty} V_n^{(1)} \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$$
$$= \lim_{n\to\infty} V_n^{(1)} \lim_{t\to\infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n})$$
$$= \int_E \psi_*(y)^2 m_o(y)\,dy, \quad 0 < \tau < 1,$$

$$\text{(6.20)} \qquad \lim_{n\to\infty} V_n^{(2)} \lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n}) = \int_E \psi_*(y) m_o(y)\,dy,$$

$$\text{(6.21)} \qquad \lim_{n\to\infty} V_n^{(3)} \lim_{t\to\infty} P_x(X(t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n})$$
$$= \begin{cases} \int_E \psi_*(y)\{s(y) - s(0)\} m_o(y)\,dy, & |s(0)| < \infty, \\ \int_E \psi_*(y) m_o(y)\,dy, & |s(0)| = \infty, \end{cases}$$

for $0 < x < 1$ and $E \in \mathscr{B}((0,1))$ with $\bar{E} \subset (0,1)$.

6.4 Other Cases

Let us consider the case that $N = \infty$, $\gamma > 0$ and one of the following conditions is satisfied

(6.22) $$S_{12} - S_{11} + \gamma(\rho - 1)/2 < v, \quad v > u = 0.$$

(6.23) $$S_{12} - S_{22} + \gamma(\rho - 1)/2 < u, \quad u > v = 0.$$

(6.24) $$S_{12} + \gamma(\rho - 1)/2 < \min\{S_{11}, S_{22}\}, \quad u = v = 0.$$

(6.25) $$S_{11} > S_{12} + \gamma(\rho - 1)/2 > S_{22}, \quad u = v = 0.$$

(6.26) $$S_{11} < S_{12} + \gamma(\rho - 1)/2 < S_{22}, \quad u = v = 0.$$

We should notice that $\lambda_* > 0$ in these cases by means of [18, Theorem 3]. Since the end points are entrance or natural in these cases, $P_x(\sigma_0 \wedge \sigma_1 = \infty) = 1$, $0 < x < 1$. We note that $|s(0)| < \infty$ in cases (6.23), (6.24) and (6.26), $|s(0)| = \infty$ in cases (6.22) and (6.25). By virtue of Theorem 2.6, we see that there are subsequences $\{\xi_n\}$, $\{\eta_n\}$ (denoted by the same symbols), and sequences of positive numbers $\{V_n^{(j)}\}$, $j = 1, 2, 3$, and a positive continuous function ψ_* satisfying (2.7) with $\lambda_* > 0$ and $\psi_*(c_o) = 1$, for which (6.19), (6.20) and (6.21) are satisfied. In these cases we can only show that

$$\lim_{t \to \infty} e^{\lambda_* t} p_*(t, x, y) = 0, \quad x, y \in I.$$

Therefore we can not obtain the asymptotic behavior of $\lim_{n \to \infty} P_x(X(\tau t) \in E \mid t < \sigma_{\xi_n} \wedge \sigma_{\eta_n})$ and $\lim_{n \to \infty} P_x(X(\tau t) \in E \mid t < \sigma_{\eta_n} < \sigma_{\xi_n})$ as $t \to \infty$.

Appendix

Let $\mathbf{D} = [X(t) : t \geq 0, P_x : x \in (0, 1)]$ be the diffusion process with the generator

$$L = \frac{1}{2} a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx},$$

where $a(x), b(x) \in C((0, 1))$ and $a(x)$ is positive on $(0, 1)$. Let us fix a point $c_o \in (0, 1)$ arbitrarily and set

$$s_o(x) = \exp\left\{-2 \int_{c_o}^{x} \frac{b(y)}{a(y)} \, dy\right\}, \quad m_o(x) = \frac{2}{a(x)} \exp\left\{2 \int_{c_o}^{x} \frac{b(y)}{a(y)} \, dy\right\},$$

$$s(x) = \int_{c_o}^{x} s_o(y) \, dy, \quad m(x) = \int_{c_o}^{x} m_o(y) \, dy.$$

We classify the state of the end points 0 and 1 according to the asymptotic behavior of a and b near the end points. We set the following.

Assumption. (i) *For some real numbers p and q there exist the limits*

$$a_0 = \lim_{x \downarrow 0} \frac{a(x)}{x^p} \in (0, \infty), \quad a_1 = \lim_{x \uparrow 1} \frac{a(x)}{(1-x)^q} \in (0, \infty).$$

(ii) *If $b \neq 0$, then for some real numbers μ and ν there exist the limits*

$$b_0 = \lim_{x \downarrow 0} \frac{b(x)}{x^\mu} \in \mathbf{R} \setminus \{0\}, \quad b_1 = \lim_{x \uparrow 1} \frac{b(x)}{(1-x)^\nu} \in \mathbf{R} \setminus \{0\}.$$

(ii-1) *If $\mu - p + 1 = 0$, and one of*

$$p = 2, \quad b_0/a_0 = 1/2, \quad b_0/a_0 = (p-1)/2,$$

is satisfied, then there exists a real number A_0 such that

$$\lim_{\varepsilon \downarrow 0} \sup_{0 < x < \varepsilon} \left| \frac{b(x)}{a(x)} - \frac{b_0}{a_0} \cdot \frac{1}{x} - A_0 \right| = 0.$$

(ii-2) *If $\nu - q + 1 = 0$, and one of*

$$q = 2, \quad b_1/a_1 = -1/2, \quad b_1/a_1 = -(q-1)/2,$$

is satisfied, then there exists a real number A_1 such that

$$\lim_{\varepsilon \downarrow 0} \sup_{1-\varepsilon < x < 1} \left| \frac{b(x)}{a(x)} - \frac{b_1}{a_1} \cdot \frac{1}{1-x} - A_1 \right| = 0.$$

(ii-3) *If $\mu - p + 1 < 0$, then a and b are differentiable near the end point* 0, *and satisfy* $\lim_{x \downarrow 0} (a(x)/b(x))' = 0$.

(ii-4) *If $\nu - q + 1 < 0$, then a and b are differentiable near the end point* 1, *and satisfy* $\lim_{x \uparrow 1} (a(x)/b(x))' = 0$.

Under the Assumption we obtain Tables 3 and 4. It follows from the definition of the classification of boundary that

$$|s(\ell)| < \infty, \ |m(\ell)| < \infty \quad \text{if the end point } \ell \text{ is regular,}$$

$$|s(\ell)| < \infty, \ |m(\ell)| = \infty \quad \text{if the end point } \ell \text{ is exit,}$$

$$|s(\ell)| = \infty,\ |m(\ell)| < \infty \quad \text{if the end point } \ell \text{ is entrace,}$$

$$|s(\ell)| + |m(\ell)| = \infty \quad \text{if the end point } \ell \text{ is natural,}$$

where $\ell = 0$ or 1. Therefore it is quite easy to see the results on the cases ①, ④, ⑧, ⑪, ⑭ on both tables. In order to obtain those on the other cases, we have to observe the convergence or the divergence of $I(\ell)$ and $J(\ell)$ for the functions I and J defined in Sect. 2. After a tedious calculation, we obtain results on all cases in Tables 3 and 4.

Table 3. The state of the end point 0

		$\|s(0)\|$	$\|m(0)\|$	state	
$b = 0$	$p < 1$	$< \infty$	$< \infty$	regular	①
	$1 \le p < 2$	$< \infty$	$= \infty$	exit	②
	$p \ge 2$	$< \infty$	$= \infty$	natural	③
$b \ne 0$	$p < 1$	$< \infty$	$< \infty$	regular	④
	$1 \le p < \mu + 1,\ p < 2$	$< \infty$	$= \infty$	exit	⑤
	$1 \le p < \mu + 1,\ p \ge 2$	$< \infty$	$= \infty$	natural	⑥
	$p = \mu + 1,\ p < 2,\ 2b_0/a_0 \le p - 1$	$< \infty$	$= \infty$	exit	⑦
	$p = \mu + 1,\ p < 2,\ p - 1 < 2b_0/a_0 < 1$	$< \infty$	$< \infty$	regular	⑧
	$p = \mu + 1,\ p < 2,\ 2b_0/a_0 \ge 1$	$= \infty$	$< \infty$	entrance	⑨
	$p = \mu + 1 = 2,\ 2b_0/a_0 < 1$	$< \infty$	$= \infty$	natural	⑩
	$p = \mu + 1 = 2,\ 2b_0/a_0 = 1$	$= \infty$	$= \infty$	natural	⑪
	$p = \mu + 1 = 2,\ 2b_0/a_0 > 1$	$= \infty$	$< \infty$	natural	⑫
	$p = \mu + 1,\ p > 2,\ 2b_0/a_0 < 1$	$< \infty$	$= \infty$	natural	⑬
	$p = \mu + 1,\ p > 2,\ 1 \le 2b_0/a_0 \le p - 1$	$= \infty$	$= \infty$	natural	⑭
	$p = \mu + 1,\ p > 2,\ 2b_0/a_0 > p - 1$	$= \infty$	$< \infty$	natural	⑮
	$p > \mu + 1,\ b_0 < 0,\ \mu < 1$	$< \infty$	$= \infty$	exit	⑯
	$p > \mu + 1,\ b_0 < 0,\ \mu \ge 1$	$< \infty$	$= \infty$	natural	⑰
	$p > \mu + 1,\ b_0 > 0,\ \mu < 1$	$= \infty$	$< \infty$	entrance	⑱
	$p > \mu + 1,\ b_0 > 0,\ \mu \ge 1$	$= \infty$	$< \infty$	natural	⑲

Table 4. The state of the end point 1

		$s(1)$	$m(1)$	state	
$b = 0$	$q < 1$	$< \infty$	$< \infty$	regular	①
	$1 \leq q < 2$	$< \infty$	$= \infty$	exit	②
	$q \geq 2$	$< \infty$	$= \infty$	natural	③
$b \neq 0$	$q < 1$	$< \infty$	$< \infty$	regular	④
	$1 \leq q < v + 1,\ q < 2$	$< \infty$	$= \infty$	exit	⑤
	$1 \leq q < v + 1,\ q \geq 2$	$< \infty$	$= \infty$	natural	⑥
	$q = v + 1,\ q < 2,\ 2b_1/a_1 \leq -1$	$= \infty$	$< \infty$	entrance	⑦
	$q = v + 1,\ q < 2,\ -1 < 2b_1/a_1 < -(q-1)$	$< \infty$	$< \infty$	regular	⑧
	$q = v + 1,\ q < 2,\ 2b_1/a_1 \geq -(q-1)$	$< \infty$	$= \infty$	exit	⑨
	$q = v + 1 = 2,\ 2b_1/a_1 < -1$	$= \infty$	$< \infty$	natural	⑩
	$q = v + 1 = 2,\ 2b_1/a_1 = -1$	$= \infty$	$= \infty$	natural	⑪
	$q = v + 1 = 2,\ 2b_1/a_1 > -1$	$< \infty$	$= \infty$	natural	⑫
	$q = v + 1,\ q > 2,\ 2b_1/a_1 < -(q-1)$	$= \infty$	$< \infty$	natural	⑬
	$q = v + 1,\ q > 2,\ -(q-1) \leq 2b_1/a_1 \leq -1$	$= \infty$	$= \infty$	natural	⑭
	$q = v + 1,\ q > 2,\ 2b_1/a_1 > -1$	$< \infty$	$= \infty$	natural	⑮
	$q > v + 1,\ b_1 > 0,\ v < 1$	$< \infty$	$= \infty$	exit	⑯
	$q > v + 1,\ b_1 > 0,\ v \geq 1$	$< \infty$	$= \infty$	natural	⑰
	$q > v + 1,\ b_1 < 0,\ v < 1$	$= \infty$	$< \infty$	entrance	⑱
	$q > v + 1,\ b_1 < 0,\ v \geq 1$	$= \infty$	$< \infty$	natural	⑲

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Masaru Iizuka
Division of Mathematics
Kyushu Dental College
Kitakyushu 803-8580, Japan

Miyuki Maeno
School of Interdisciplinary Research of Scientific Phenomena and Information
Graduate School of Humanities Sciences
Nara Women's University
Nara 630-8506, Japan

Matsuyo Tomisaki
Department of Mathematics
Faculty of Science
Nara Women's University
Nara 630-8506, Japan