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CONTINUITY OF INTERPOLATIONS

By

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Abstract. An interpolation function for a set of finite input-output data is a function which fits the data. Let us say that a topological space X has a continuous interpolation if interpolation functions can be selected continuously, more precisely, if there is a continuous map from a certain subspace of the hyperspace $F(X \times \mathbf{R})$ of finite subsets of $X \times \mathbf{R}$ to the Banach space $C(X)$ of bounded real-valued continuous functions on X. The concept of weakly continuous interpolation is also introduced. The real line has a continuous interpolation. Every metrizable space has a weakly continuous interpolation. On the other hand, ω_1 and $\beta\omega$ do not have weakly continuous interpolations.

1. Introduction

All topological spaces considered here are Tychonoff. Basic terminology is found in [2], [4]. The space of real numbers is denoted by $\mathbf{R}$. Let X be a topological space. The space $C(X)$ is the Banach space of all bounded real-valued continuous functions, with the sup norm: $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. The space $F(X \times \mathbf{R})$ is the hyperspace consisting of all finite subsets of the product space $X \times \mathbf{R}$, with the Vietoris topology [5]. Hence basic neighborhoods of $\{(x_1, r_1), (x_2, r_2), \dots, (x_n, r_n)\} \in F(X \times \mathbf{R})$ are given by:

$$\langle U_1 \times V_1, U_2 \times V_2, \dots, U_n \times V_n \rangle$$

$$= \left\{ D \in F(X \times \mathbf{R}) : D \subset \bigcup_{k=1}^{n} U_k \times V_k, D \cap (U_k \times V_k) \neq \emptyset \ \ (k = 1, 2, \dots, n) \right\},$$

where U_k is a neighborhood of x_k in X and V_k is a neighborhood of r_k in $\mathbf{R}$ for $k = 1, 2, \dots, n$. Let $S(X)$ be the subspace of $F(X \times \mathbf{R})$ defined by

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$$S(X) = \{\{(x_1, r_1), \dots, (x_n, r_n)\} : x_i \neq x_j \text{ for } i \neq j\}.$$

For each $n = 1, 2, \dots,$ define $F_n(X \times \mathbf{R})$ and $S_n(X)$ by:

$$F_n(X \times \mathbf{R}) = \{D \in F(X \times \mathbf{R}) \colon D \text{ has at most } n \text{ points}\},$$

$$S_n(X) = S(X) \cap F_n(X \times \mathbf{R}).$$

Notice that $S_{n-1}(X)$ is closed in $S_n(X)$.

For a point $D = \{(x_1, r_1), (x_2, r_2), \dots, (x_n, r_n)\} \in S(X)$, a function f_D in $C(X)$ is called an interpolation function for D if

$$f_D(x_1) = r_1, f_D(x_2) = r_2, \dots, f_D(x_n) = r_n$$

are satisfied [1]. Suppose that X is the input space and $\mathbf{R}$ is the output space of some system. Then the point D is considered as a set of finite input-output data. The interpolation function f_D is a function which fits the given data. It is obvious that for every $D \in S(X)$ there is an interpolation function f_D for D, since X is Tychonoff. Hence we can consider the map $\Theta : S(X) \to C(X)$ defined by $\Theta(D) = f_D$. Since similar maps under the statistical frameworks are called learning algorithms in learning theory [6], this map Θ might be called an interpolation algorithm in a vague sense. Further we are interested in the case when this interpolation algorithm has some kind of continuity or stability. Let us call the map Θ to be a continuous interpolation of X if Θ is continuous as a map between the topological spaces $S(X)$ and $C(X)$. In case Θ satisfies the weaker condition that the restriction $\Theta|_{S_n(X) - S_{n-1}(X)}$ is continuous for each $n = 1, 2, \dots,$ we call Θ to be a weakly continuous interpolation. That is, the interpolation Θ is weakly continuous if for any $D = \{(x_1, r_1), \dots, (x_n, r_n)\} \in S(X)$ and any $\varepsilon > 0$, there is a neighborhood $W = \langle U_1 \times V_1, \dots, U_n \times V_n \rangle$ of D such that $\|f_{D'} - f_D\|_\infty < \varepsilon$ for any $D' = \{(x'_1, r'_1), \dots, (x'_n, r'_n)\} \in W \cap S_n(X)$. Hence this weak continuity can be called a topological stability of interpolation algorithms like the stabilities of learning algorithms [6]. Our purpose of this paper is to discuss whether a given topological space has a (weakly) continuous interpolation or not. The following are obvious, but fundamental in our argument.

THEOREM 1. *Every discrete space has a* (*weakly*) *continuous interpolation.*

THEOREM 2. *If X has a* (*weakly*) *continuous interpolation, then every subspace of X has a* (*weakly*) *continuous interpolation.*

THEOREM 3. *Let τ_1 and τ_2 be topologies on a set X. If τ_1 is weaker than*

τ_2 and (X, τ_1) has a (weakly) continuous interpolation, then (X, τ_2) has a (weakly) continuous interpolation.

2. Metrizable Spaces and Continuous Interpolations

In our framework, the following simple fact is also fundamental.

THEOREM 4. *The real line $\mathbf{R}$ has a continuous interpolation.*

PROOF. Let

$$D = \{(x_1, r_1), (x_2, r_2), \dots, (x_n, r_n)\}$$

be an arbitrary point in $S(\mathbf{R})$. We can assume that

$$x_1 < x_2 < \cdots < x_n.$$

Let us consider the function $f_D \in C(\mathbf{R})$ defined by

$$f_D(x) = \begin{cases} r_1 & \text{for } x \le x_1 \\ r_{i-1} + (x - x_{i-1}) \frac{r_i - r_{i-1}}{x_i - x_{i-1}} & \text{for } x_{i-1} < x \le x_i,\ i = 2, \dots, n \\ r_n & \text{for } x_n < x. \end{cases}$$

Obviously f_D is an interpolation function for D. It must be checked that the map $\Theta : S(X) \to C(X)$ defined by $\Theta(D) = f_D$ is continuous.

For $D = \{(x_1, r_1), \dots, (x_n, r_n)\} \in S(\mathbf{R})$, let

$$m = \min\{|x_1 - x_2|, \dots, |x_{n-1} - x_n|\}, \quad M = \max\{|r_1|, \dots, |r_n|\}.$$

In case $n = 1$, let m be an arbitrary positive number. For any ε such that $0 < \varepsilon (< 1)$ let $\delta = \frac{1}{2} \min\left\{\frac{m}{3}, \frac{m\varepsilon}{18(M+1)}\right\}$. Now, consider the following neighborhood of D:

$$W = \langle U_\delta(x_1) \times V_{\varepsilon/3}(r_1), \dots, U_\delta(x_n) \times V_{\varepsilon/3}(r_n) \rangle,$$

where $U_\delta(x_i)$ is the δ-neighborhood of x_i and $V_{\varepsilon/3}(r_i)$ is the $\varepsilon/3$-neighborhood of r_i for $i = 1, \dots, n$. We will show that $\|f_D - f_{D'}\|_\infty < \varepsilon$ for any $D' \in W$. Let $D' = \{(x_1', r_1'), \dots, (x_m', r_m')\}$, where $x_1' < \cdots < x_m'$ is satisfied. Then there is the increasing map $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ which satisfies $(x_j', r_j') \in U_\delta(x_{\sigma(j)}) \times V_{\varepsilon/3}(r_{\sigma(j)})$ for any $j = 1, \dots, m$. Since it suffices to show that $|f_D(x) - f_{D'}(x)| < \varepsilon$ for any $x \in \mathbf{R}$, let x be an arbitrary point in $\mathbf{R}$. (1) First, assume that $x \le x_1 - \delta$. Then $f_D(x) = r_1$. Further it must be satisfied that $x < x_1'$, and hence $f_{D'}(x) = r_1'$. Since $|r_1 - r_1'| < \varepsilon/3$, it is obvious that $|f_D(x) - f_{D'}(x)| < \varepsilon/3$. In the case that $x \ge x_n + \delta$, similar argument above implies that $|f_D(x) - f_{D'}(x)| < \varepsilon/3$. (2) Next, we consider the case when there is some i such that $|x - x_i| < \delta$. Notice that for each $k = 2, \dots, n$ the absolute value of the slope $\frac{r_k - r_{k-1}}{x_k - x_{k-1}}$ of the line connecting (x_{k-1}, r_{k-1}) and (x_k, r_k) is less than $\frac{2(M+1)}{m}$. Therefore if $x_{k-1} \le y \le z \le x_k$ and $|y - z| < \delta$ are satisfied, then we obtain that $|f_D(y) - f_D(z)| < \frac{m\varepsilon}{18(M+1)} \frac{2(M+1)}{m} = \varepsilon/9$. Hence in the present case $|f_D(x) - r_i| < \varepsilon/9$ is satisfied. On the other hand, there is some j such that $x'_j \le x \le x'_{j+1}$. If $\sigma(j) = \sigma(j+1) = i$, then $|r'_j - r_i|$, $|r'_{j+1} - r_i| < \varepsilon/3$. Since $r_i - \varepsilon/3 < \min\{r'_j, r'_{j+1}\} \le f_{D'}(x) \le \max\{r'_j, r'_{j+1}\} < r_i + \varepsilon/3$, the inequality $|f_{D'}(x) - r_i| < \varepsilon/3$ is also satisfied. Hence $|f_D(x) - f_{D'}(x)| < 2\varepsilon/3$. If $\sigma(j) = i$ and $\sigma(j+1) = i+1$, then $|x'_j - x'_{j+1}| \ge m - 2\delta \ge 2m/3$. Hence the absolute value of the slope of the line connecting (x'_j, r'_j) and (x'_{j+1}, r'_{j+1}) is less than $\frac{3(M+1)}{m}$. It follows that $|f_{D'}(x) - r'_j| \le \varepsilon/6$. This implies that $|f_D(x) - f_{D'}(x)| \le |f_D(x) - r_i| + |r_i - r'_j| + |r'_j - f_{D'}(x)| < \varepsilon/9 + \varepsilon/3 + \varepsilon/6 < \varepsilon$. Similarly, if $\sigma(j) = i - 1$ and $\sigma(j+1) = i$, it is proved that $|f_D(x) - f_{D'}(x)| < \varepsilon$. (3) Finally, assume that $x_i + \delta \le x \le x_{i+1} - \delta$ for some $i = 1, \dots, n-1$. The number $k = \max \sigma^{-1}(i)$ is settled and it must be satisfied that $\sigma(k+1) = i+1$. Since $x'_k < x_i + \delta$ and $x_{i+1} - \delta < x'_{k+1}$, it is satisfied that $x'_k < x < x'_{k+1}$. Let $x_i^r = \max\{x_i, x'_k\}$, $x_{i+1}^l = \min\{x_{i+1}, x'_{k+1}\}$. Since $|x_i^r - x_i|, |x_{i+1}^l - x_{i+1}| < \delta$, it follows that $|f_D(x_i^r) - f_{D'}(x_i^r)|$, $|f_D(x_{i+1}^l) - f_{D'}(x_{i+1}^l)| < \varepsilon$ by using the result of the case (2). Since f_D, $f_{D'}$ are linear on the interval $x_i^r \le x \le x_{i+1}^l$, it is obvious that $|f_D(x) - f_{D'}(x)| < \varepsilon$ for any x such that $x_i + \delta \le x \le x_{i+1} - \delta$.

Corollary 1. *The Sorgenfrey line and the Michael line have continuous interpolations.*

It seems difficult to extend the result of Theorem 4 to higher dimensional Euclidean spaces $\mathbf{R}^n$. However, we can show that $\mathbf{R}^n$ has a weakly continuous interpolation. More generally the following is obtained.

Theorem 5. *Every metrizable space has a weakly continuous interpolation.*

Proof. Let (X, d) be a metric space. For any $D = \{(x_1, r_1), \dots, (x_n, r_n)\} \in S(X)$, let

$$M = \max\{|r_1|, \dots, |r_n|\}, \quad m = \min\{d(x_i, x_j) : i \ne j\}.$$

Then the function $f_D \in C(X)$ is defined by

$$f_D(x) = \begin{cases} 0 & \text{if } d(x, x_i) \ge m/4 \text{ for each } i = 1, \dots, n \\ r_i - \frac{4r_i}{m} d(x, x_i) & \text{if } d(x, x_i) < m/4 \text{ for some } i = 1, \dots, n. \end{cases}$$

In case $D = \{(x_1, r_1)\} \in S_1(X)$, let $m = \infty$ and hence $f_D(x) = r_1$ for each $x \in X$. It is obvious that f_D is an interpolation function for D. We will show that the map $\Theta : S(X) \to C(X)$ defined by $\Theta(D) = f_D$ is weakly continuous. Since the continuity of $\Theta|_{S_1(X)}$ is obvious, we can assume that $n > 1$. For the above D and an arbitrary $(1 >)\varepsilon > 0$, let $\delta > 0$ be a real number such that

$$\delta < \min\left\{\frac{m}{8}, \frac{m\varepsilon}{32(M+1)}\right\}.$$

Since the absolute value $\left|\frac{4r_i}{m}\right|$ of the coefficient of $d(x, x_i)$ used in the definition of f_D is less than $\frac{4(M+1)}{m}$, the inequality $\frac{4(M+1)}{m}2\delta < \varepsilon/4$ implies the following.

Claim. If $x, y \in X$ satisfy $d(x, y) < 2\delta$, then $|f_D(x) - f_D(y)| < \varepsilon/4$.

It suffices to show that $\|f_{D'} - f_D\|_\infty < \varepsilon$ for $D' = \{(x_1', r_1'), \dots, (x_n', r_n')\} \in S_n(X)$ which satisfies

$$d(x_i', x_i) < \delta, \quad |r_i' - r_i| < \varepsilon/4 \quad \text{for } i = 1, \dots, n.$$

For this D', the numbers $M' = \max\{|r_1'|, \dots, |r_n'|\}$, $m' = \min\{d(x_i', x_j') : i \neq j\}$ are also defined. The inequalities $M' < M + 1$, $m - 2\delta < m' < m + 2\delta$ are obvious. Let x be an arbitrary point in X. Assume that $d(x, x_i) \geq m/4$ for each i, then $f_D(x) = 0$. On the other hand, for this point x it is satisfied that $f_{D'}(x) = 0$ or $0 < |f_{D'}(x)| \leq \left|r_i' - \frac{4r_i'}{m'}d(x, x_i')\right|$ for some i. Even in the latter case, since $\frac{m'}{4} > d(x, x_i') \geq d(x, x_i) - d(x_i', x_i) > \frac{m'}{4} - \frac{3}{2}\delta$ and hence $|f_{D'}(x)| \leq \left|r_i' - \frac{4r_i'}{m'}\left(\frac{m'}{4} - \frac{3}{2}\delta\right)\right| \leq \left|\frac{6r_i'}{m'}\delta\right| < \frac{6(M+1)}{m-2\delta}\delta < \varepsilon/4$, it follows that $|f_D(x) - f_{D'}(x)| < \varepsilon/4$. Next, assume that $d(x, x_i) < m/4$ for some i. If $|r_i| \leq \varepsilon/4$, then $|f_D(x)| \leq \varepsilon/4$. Further the inequlity $|r_i'| \leq \varepsilon/2$ is satisfied. Then $|f_{D'}(x)| \leq \varepsilon/2$, and hence $|f_D(x) - f_{D'}(x)| \leq 3\varepsilon/4$. The remaining is the case $|r_i| > \varepsilon/4$. Let

$$a = r_i - \frac{4r_i}{m}d(x, x_i), \quad b = r_i - \frac{4r_i}{m}d(x, x_i'),$$

$$c = r_i - \frac{4r_i}{m - 2\delta}d(x, x_i'), \quad c' = r_i - \frac{4r_i}{m + 2\delta}d(x, x_i'),$$

$$d_1 = r_i - \varepsilon/4 - \frac{4(r_i + \varepsilon/4)}{m - 2\delta}d(x, x_i'), \quad d_1' = r_i + \varepsilon/4 - \frac{4(r_i - \varepsilon/4)}{m + 2\delta}d(x, x_i'),$$

$$d_2 = r_i - \varepsilon/4 - \frac{4(r_i + \varepsilon/4)}{m + 2\delta}d(x, x_i'), \quad d_2' = r_i + \varepsilon/4 - \frac{4(r_i - \varepsilon/4)}{m - 2\delta}d(x, x_i').$$

Since $f_D(x) = a$ and either $d_1 < f_{D'}(x) < d_1'$ or $d_2 < f_{D'}(x) < d_2'$ are satisfied according to $r_i > \varepsilon/4$ or $r_i < -\varepsilon/4$, if it is proved that $|a - d_1'|, |a - d_1|, |a - d_2|$, $|a - d_2'| < \varepsilon$ then we have $|f_D(x) - f_{D'}(x)| < \varepsilon$.

(1) $|a-b| < \varepsilon/8$: In fact, $|a-b| = \left|\frac{4ri}{m}(d(x,x_i) - d(x,x_i'))\right| \le \frac{4(M+1)}{m}\delta < \varepsilon/8$.

(2) $|b-c| < \varepsilon/8$: This follows from $|b-c| = |r_i| d(x,x_i') \left|\frac{4}{m} - \frac{4}{m-2\delta}\right| = |r_i| d(x,x_i') \left|\frac{8\delta}{m(m-2\delta)}\right| < (M+1)\frac{3m}{8}\frac{8}{m-m/4}\frac{m\varepsilon}{32(M+1)} = \varepsilon/8$.

(3) $|b-c'| < \varepsilon/8$: $|b-c'| = |r_i| d(x,x_i') \left|\frac{4}{m} - \frac{4}{m+2\delta}\right| = |r_i| d(x,x_i') \frac{8\delta}{m(m+2\delta)} < (M+1)\frac{3m}{8}\frac{8}{m+m/4}\frac{m\varepsilon}{32(M+1)} < \varepsilon/8$.

(4) $|c-d_1| < \frac{3}{4}\varepsilon$: $|c-d_1| = \left|\varepsilon/4 + \frac{\varepsilon}{m-2\delta} d(x,x_i')\right| \le \varepsilon/4 + \left|\frac{\varepsilon}{m-2\delta}\right| d(x,x_i') < \varepsilon/4 + \frac{4\varepsilon}{3m}\frac{3m}{8} = \varepsilon/4 + \varepsilon/2 = 3\varepsilon/4$.

(5) $|c'-d_1'| < \frac{3}{4}\varepsilon$: $|c'-d_1'| = \left|-\varepsilon/4 - \frac{\varepsilon}{m+2\delta} d(x,x_i')\right| \le \varepsilon/4 + \frac{\varepsilon}{m+2\delta} d(x,x_i') < \varepsilon/4 + \frac{4\varepsilon}{5m}\frac{3m}{8} < 3\varepsilon/4$.

(6) $|c'-d_2| = |c-d_1| < 3\varepsilon/4$.

(7) $|c-d_2'| = |c'-d_1'| < 3\varepsilon/4$.

Hence $|a-d_1'|, |a-d_1|, |a-d_2|, |a-d_2'| < \varepsilon/8 + \varepsilon/8 + 3\varepsilon/4 = \varepsilon$.

Corollary 2. *Let X be a space whose topology is stronger than a metrizable topology. Then X has a weakly continuous interpolation.*

3. Spaces without Continuous Interpolations

As we see in this section, it is delicate whether a given space has a (weakly) continuous interpolation or not.

Theorem 6. *The ordered space ω_1 of the first uncountable ordinal does not have a weakly continuous interpolation.*

Proof. Assume that there is a weakly continuous interpolation $\Theta : S(\omega_1) \to C(\omega_1)$. Let $\alpha_0 = 0$ and W_0 be the set of all limit ordinals in ω_1. For each $\lambda \in W_0$, let $D_\lambda^0 = \{(\alpha_0, 0), (\lambda, 1)\} \in S_2(\omega_1)$. Then the function $f_\lambda^0 = \Theta(D_\lambda^0)$ is obtained. Since this function is continuous at λ and $f_\lambda^0(\lambda) = 1$, there exists $\mu_\lambda^0 < \lambda$ such that $|f_\lambda^0(x) - 1| < 1/4$ for any x which satisfies $\mu_\lambda^0 < x \le \lambda$. Using the pressing down lemma [4] for the function $\lambda \mapsto \mu_\lambda^0$, there exist an ordinal α_1 and a stationary subset W_1 of W_0 such that $\mu_\lambda^0 = \alpha_1$ for any $\lambda \in W_1$. Repeat the similar procedures. Then we obtain a sequence

$$\alpha_0 < \alpha_1 < \cdots$$

of points in ω_1 and a sequence

$$W_0 \supset W_1 \supset \cdots$$

of stationary sets in ω_1 such that for any $i = 1, 2, \ldots$ and any $\lambda \in W_i$, the function $f_\lambda^{i-1} = \Theta(\{(\alpha_{i-1}, 0), (\lambda, 1)\})$ satisfies

$$|f_\lambda^{i-1}(x) - 1| < 1/4 \quad (*)$$

for any x such that $\alpha_i < x \leq \lambda$. Now, let $\tilde{\alpha} = \lim_{n\to\infty} \alpha_n$. We can take another sequence of ordinals $(\tilde{\alpha} <)\beta_0 < \beta_1 < \cdots$ such that $\beta_i \in W_i$ for each i. Let $\tilde{\beta} = \lim_{n\to\infty} \beta_n$ in ω_1. Then for $\tilde{D} = \{(\tilde{\alpha}, 0), (\tilde{\beta}, 1)\} \in S_2(\omega_1)$ there is the corresponding function $f_{\tilde{D}} = \Theta(\tilde{D})$. Since $\Theta|_{S_2(\omega_1)-S_1(\omega_1)}$ is continuous at $\tilde{D}$, there are neighborhoods $U_{\tilde{\alpha}}$ of $\tilde{\alpha}$ and $V_{\tilde{\beta}}$ of $\tilde{\beta}$ which satisfy the following: For any $\alpha \in U_{\tilde{\alpha}}$ and $\beta \in V_{\tilde{\beta}}$, the function $f_{\alpha\beta} = \Theta(\{(\alpha, 0), (\beta, 1)\})$ satisfies $\|f_{\alpha\beta} - f_{\tilde{D}}\|_\infty < 1/4$ and hence $|f_{\alpha\beta}(\tilde{\alpha})| < 1/4$ and $|f_{\alpha\beta}(\tilde{\beta}) - 1| < 1/4$. Since $\alpha_n \in U_{\tilde{\alpha}}$ and $\beta_n \in V_{\tilde{\beta}}$ for sufficiently large n, it follows that $|f_{\alpha_{n-1}\beta_n}(\tilde{\alpha})| < 1/4$ for sufficiently large n. But this is a contradiction, since the above condition $(*)$ implies that $|f_{\alpha_{n-1}\beta_n}(\tilde{\alpha}) - 1| < 1/4$.

Corollary 3. *Every topological space containing ω_1 does not have a weakly continuous interpolation.*

The space ω_1 is first-countable and countably compact. On the other hand, every countably compact space which has a weakly continuous interpolation must be nearly first-countable in the following sense.

Theorem 7. *Let X be a countably compact space which has a weakly continuous interpolation. Then the tightness $\tau(X)$ of X is countable.*

Proof. Assume that $\tau(X) > \omega$ and that X has a weakly continuous interpolation $\Theta : D \mapsto f_D$. Then there are a subset A of X and a point $p \in cl_X A$ such that $p \notin cl_X B$ for any countable subset B of A. We can assume further that $cl_X B \subset A$ for any countable subset B of A.

Let x_0 be an arbitrary point in A and let $D_0 = \{(x_0, 1), (p, 0)\} \in S_2(X)$. Then there is a point $x_1 \in f_{D_0}^{-1}(0) \cap A$, since X is countably compact and has the property futher assumed above. Next, let $D_1 = \{(x_1, 1), (p, 0)\}$. Then there is a point $x_2 \in f_{D_0}^{-1}(0) \cap f_{D_1}^{-1}(0) \cap A$. Continuing this procedure, we obtain a sequence $\{x_i : i \in \omega\}$ of points in A such that for any $n \in \omega$

$$x_n \notin x_{n+1} \in f_{D_0}^{-1}(0) \cap \cdots \cap f_{D_n}^{-1}(0),$$

where $D_i = \{(x_i, 1), (p, 0)\}$ for each $i \in \omega$. Since X is countably compact, there is an accumuration point x_∞ of $\{x_i : i \in \omega\}$. The procedure of constracting $\{x_i : i \in \omega\}$ implies that $x_\infty \neq p$ and $x_\infty \in \bigcap\{f_{D_i}^{-1}(0) : i \in \omega\}$. Consider the point $D_\infty = \{(x_\infty, 1), (p, 0)\} \in S_2(X)$. Then there exists a neighborhood $W = \langle U_\infty \times V_1, U_p \times V_0 \rangle$ of D_∞ such that $\|f_{D'} - f_{D_\infty}\|_\infty < 1/2$ and hence especially $|f_{D'}(x_\infty) - 1| < 1/2$ for any $D' \in W \cap (S_2(X) - S_1(X))$. But this is a contradiction, since there exists n such that $D_n \in W \cap (S_2(X) - S_1(X))$. In fact, for this D_n it must be satisfied that $f_{D_n}(x_\infty) = 0$.

COROLLARY 4. *The ordered space* $\omega_1 + 1$ *does not have a weakly continuous interpolation.*

For the discrete space $D(\omega_1)$ of cardinality ω_1, let $D(\omega_1) \cup \{\infty_A\}$ be the one-point compactification of $D(\omega_1)$, i.e. the complement of every neighborhood of ∞_A is a finite subset of $D(\omega_1)$. The one-point Lindelöfication $D(\omega_1) \cup \{\infty_L\}$ of $D(\omega_1)$ is the space obtained by adding a point ∞_L with the neighborhood base consisting of co-countable sets.

THEOREM 8. *The one-point Lindelöfication* $D(\omega_1) \cup \{\infty_L\}$ *has a continuous interpolation.*

PROOF. We can assume that the underlying set of $D(\omega_1) \cup \{\infty_L\}$ is $\omega_1 + 1$ as $\infty_L = \omega_1$. For $D = \{(\alpha_1, r_1), \dots, (\alpha_n, r_n)\} \in S(D(\omega_1) \cup \{\infty_L\})$, where $\alpha_1 < \cdots < \alpha_n$, let $f_D \in C(D(\omega_1) \cup \{\infty_L\})$ be the function defined by

$$f_D(\alpha) = \begin{cases} r_1 & \text{for } \alpha \le \alpha_1 \\ r_i & \text{for } \alpha_{i-1} < \alpha \le \alpha_i,\ i = 2, \dots, n-1 \\ r_n & \text{for } \alpha_{n-1} < \alpha. \end{cases}$$

It is easy to see that the map Θ defined by $\Theta(D) = f_D$ is a continuous interpolation of $D(\omega_1) \cup \{\infty_L\}$.

THEOREM 9. *The one-point compactification* $D(\omega_1) \cup \{\infty_A\}$ *does not have a weakly continuous interpolation.*

PROOF. The underlying set of the space $X = D(\omega_1) \cup \{\infty_A\}$ is also the well-ordered set $\omega_1 + 1$ as above. Assume that $D(\omega_1) \cup \{\infty_A\}$ has a weakly continuous interpolation $\Theta : D \mapsto f_D$. Since any real-valued continuous function on $D(\omega_1) \cup \{\infty_A\}$ is constant on a co-countable set and Θ is continuous on $S_2(X) - S_1(X)$, there exists $\gamma_0 < \omega_1$ such that

$$f_{D_{\alpha\beta}}(\infty_A) = 0$$

for any $D_{\alpha\beta} = \{(\alpha, 1), (\beta, 0)\}$ such that $\alpha < \omega$ and $\beta > \gamma_0$.

Let $\beta_0 \in D(\omega_1)$ be a point larger than γ_0. Consider $D_0 = \{(\beta_0, 0), (\infty_A, 1)\}$ in $S_2(X)$. Then $f_{D_0}(\infty_A) = 1$. Since the restriction $\Theta|_{S_2(X) - S_1(X)}$ is continuous, there is a neighborhood W of D_0 in $S_2(X)$ such that

$$\|f_{D'} - f_{D_0}\|_\infty < 1/2$$

and hence $|f_{D'}(\infty_A) - 1| < 1/2$ for any $D' \in W$. Since the complement of any neighborhood of ∞_A in $D(\omega_1) \cup \{\infty_A\}$ is finite, there exists $\alpha_0 < \omega$ such that $D_{\alpha_0\beta_0} = \{(\beta_0, 0), (\alpha_0, 1)\} \in W$. Then $f_{D_{\alpha_0\beta_0}}(\infty_A) > 1/2$. However, since $\alpha_0 < \omega$ and $\gamma_0 < \beta_0$, the above condition of γ_0 implies that $f_{D_{\alpha_0\beta_0}}(\infty_A) = 0$. This is a contradiction.

For a point p in a space X, $\psi(p, X)$ is the pseudo-character of X at p. A similar argument to the proof above show the following.

Theorem 10. *Let X be a space with a point p such that $\psi(p, X) > \omega$. Let $X \vee_{p\omega} (\omega + 1)$ be the quotient space of the topological sum $X \oplus (\omega + 1)$, obtained by the identification of p with ω. Then $X \vee_{p\omega} (\omega + 1)$ does not have a weakly continuous interpolation.*

Proof. In $X \vee_{p\omega} (\omega + 1)$, let p_ω be the point corresponding to the set $\{p, \omega\}$ collapsed. Assume that $X \vee_{p\omega} (\omega + 1)$ has a weakly continuous interpolation $\Theta : D \mapsto f_D$. Since any G_δ-set of X containing p has an infinite number of points, the weak continuity of Θ at $D_{ip_\omega} = \{(i, 1), (p_\omega, 0)\}$ for each $i \in \omega$ implies that there exists an infinite G_δ-set B of X containing p with the following property: If $x \in B - \{p\}$ and $i \in \omega$, then

$$f_{D_{ix}}(p_\omega) = 0$$

where $D_{ix} = \{(i, 1), (x, 0)\}$. Let $q \in B$ be a point which is distinct from p. Consider the point $D_{\omega q} = \{(q, 0), (p_\omega, 1)\}$. Then $f_{D_{\omega q}}(p_\omega) = 1$. On the other hand, any neighborhood W of $D_{\omega q}$ in $S_2(X \vee_{p\omega} (\omega + 1)) - S_1(X \vee_{p\omega} (\omega + 1))$ contains $D_{iq} = \{(i, 1), (q, 0)\}$ for some $i \in \omega$. Since $f_{D_{iq}}(p_\omega) = 0$ for such D_{iq}, this contradicts the weak continuity of Θ.

Corollary 5. *Let X be a space such that $X \times (\omega + 1)$ has a weakly continuous interpolation. Then the pseudo-character $\psi(X)$ is countable.*

Proof. Suppose that $\psi(p, X) > \omega$ for a point p in X. The space $X \vee_{p\omega} (\omega + 1)$ having no weakly continuous interpolation is embedded in $X \times (\omega + 1)$ as $X \times \{\omega\} \cup \{p\} \times (\omega + 1)$.

Let X be the one-point Lindelöfication $D(\omega_1) \cup \{\infty_L\}$ and $Y = \omega + 1$. Then we obtain the following.

THEOREM 11. *There are spaces X, Y having continuous interpolations such that $X \times Y$ does not have a weakly continuous interpolation.*

A subset $\mathscr{F} \subset C(X)$ is called a separating family of X if for any distinct points p, q in X there exists $f \in \mathscr{F}$ such that $f(p) \neq f(q)$.

THEOREM 12. *If an infinite space X has a weakly continuous interpolation, then the density $d(X)$ of X is larger than or equal to the minimum cardinality of separating families of X.*

PROOF. There is a weakly continuous interpolation $\Theta : D \mapsto f_D$ of X. Assume that $|\mathscr{F}| > d(X)$ for every separating family $\mathscr{F}$ of X. Let B be a dense subset of X such that $|B| = d(X)$. Consider the subfamily

$$S'(X) = \{D \in S(X)\colon \text{ if } (x, r) \in D, \text{ then } x \in B, r \in \mathbf{Q}\},$$

where $\mathbf{Q}$ is the set of all rational numbers. Let $\mathscr{F}_B = \{f_D : D \in S'(X)\}$. Since

$$|\mathscr{F}_B| \le |S'(X)| = d(X),$$

$\mathscr{F}_B$ is not a separating family of X. Hence there are distinct points p, q in X such that $f(p) = f(q)$ for any $f \in \mathscr{F}_B$. Take $D_0 = \{(p, 0), (q, 1)\} \in S_2(X)$. From the weak continuity of Θ, it follows that there is a neighborhood W of D_0 in $S_2(X) - S_1(X)$ such that $\Theta(W)$ is included in the 1/2-ball $B_{1/2}(f_{D_0})$ of f_{D_0} in $C(X)$. Since $B \times \mathbf{Q}$ is dense in $X \times \mathbf{R}$, there is $D_1 = \{(p', r), (q', s)\} \in W \cap S'(X)$. For this D_1,

$$\|f_{D_1} - f_{D_0}\|_\infty < 1/2$$

must be satisfied. But this is a contradiction, since

$$f_{D_1}(p) = f_{D_1}(q), \quad f_{D_0}(p) = 0, \quad f_{D_0}(q) = 1.$$

COROLLARY 6. *The uncountable product space $\{0, 1\}^{\omega_1}$ does not have a weakly continuous interpolation. Hence every space containing $\{0, 1\}^{\omega_1}$ does not have a weakly continuous interpolation.*

Since $D(\omega_1) \cup \{\infty_A\}$ can be embedded in $\{0, 1\}^{\omega_1}$, this corollary is considered also as a corollary of Theorem 9.

COROLLARY 7. *The Stone-Čech compactification $\beta\omega$ of the countably infinite discrete space ω does not have a continuous interpolation.*

Since the tightness of $\beta\omega$ is uncountable, this corollary is also a corollary of Theorem 7. There are more examples which show the delicacy of having weakly continuous inerpolations. A family $\mathscr{A}$ of infinite subsets of ω is called an almost disjoint family if the intersection of any two distinct element of $\mathscr{A}$ is finite [3, 4]. A maximal almost disjoint family is an almost disjoint family $\mathscr{A}$ with no almost disjoint family $\mathscr{B}$ properly containing $\mathscr{A}$. For each almost disjoint family $\mathscr{A}$ we can define the topological space $\Psi(\mathscr{A}) = \omega \cup \mathscr{A}$, with the following topology: The points of ω are isolated, while a neighborhood of a point $A \in \mathscr{A}$ is any set containing A and all but a finite number of points of $A (\subset \omega)$ [3].

Theorem 13. (1) *There exists an almost disjoint family* $\mathscr{A}$ *of cardinality* 2^ω *such that* $\Psi(\mathscr{A})$ *has a weakly continuous interpolation.*

(2) *There exists an almost disjoint family* $\mathscr{M}$ *of cardinality* 2^ω *such that* $\Psi(\mathscr{M})$ *does not have a weakly continuous interpolation.*

Proof. (1) Let us consider the following topology τ on the upper half-plane $\mathbf{R} \times [0, \infty)$, which is similar to the Niemytzki tangent disc topology: Neighborhoods of all points (x, y) with $y \neq 0$ are unchanged from those of the Euclidean topology and taking as a base at each point $(r, 0)$ the family $\{\{(r, 0)\} \cup U_n(r) : n = 1, 2, \ldots\}$, where

$$U_n(r) = \{(x, y) \in \mathbf{R} \times (0, \infty) : |x - r| < y < 1/n\}.$$

Since τ is stronger than the Euclidean topology, every subspace of this upper half-plane with the topology τ has a weakly continuous interpolation. Let $\{q_n : n \in \omega\}$ be an enumeration of all rational numbers, and let $\phi : \omega \times \mathbf{Z} \to \mathbf{R} \times (0, \infty)$ be the one-to-one map defined by $\phi(n, m) = (q_n + m/(n+1), 1/(n+1))$, where $\mathbf{Z}$ is the set of all integers. Then the subspace

$$X = \{\phi(n, m) : (n, m) \in \omega \times \mathbf{Z}\} \cup \mathbf{R} \times \{0\}$$

of $(\mathbf{R} \times [0, \infty), \tau)$ has a weakly continuous interpolation. Let $\psi : \omega \to \omega \times \mathbf{Z}$ be a bijection. For each $r \in \mathbf{R}$, let $A_r = \{n \in \omega : \phi \circ \psi(n) \in U_1(r)\}$. Then the family $\mathscr{A} = \{A_r : r \in \mathbf{R}\}$ is an almost disjoint family. It is easy to see that $\Psi(\mathscr{A})$ is homeomorphic to X.

(2) It is well known that there exists a maximal almost disjoint family $\mathscr{M}$ of cardinality 2^ω. Since the density of $\Psi(\mathscr{M})$ is countable, it suffices to show that the cardinality of every separating family of $\Psi(\mathscr{M})$ is greater than ω. Assume that there is a countable separating family $\mathscr{F}$ of $\Psi(\mathscr{M})$. Then the product map $\pi\mathscr{F} : \Psi(\mathscr{M}) \to \mathbf{R}^{\mathscr{F}}$ defined by $\pi\mathscr{F}(x) = (f(x))_{f \in \mathscr{F}}$ is one-to-one and continuous.

Since $\Psi(\mathscr{M})$ is pseudocompact and $\mathbf{R}^{\mathscr{F}}$ is metrizable, the image $\pi\mathscr{F}(\Psi(\mathscr{M}))$ of this continuous map must be compact. For any $x \in \Psi(\mathscr{M})$ and any neighborhood U of x in $\Psi(\mathscr{M})$, there is a real-valued continuous function $f_{x,U} : \Psi(\mathscr{M}) \to [0,1]$ such that $f_{x,U}(x) = 0$, $f_{x,U}|_{\Psi(\mathscr{M})-U} = 1$. Now, consider the family $\mathscr{F}' = \mathscr{F} \cup \{f_{x,U}\}$ obtained by adding one more function $f_{x,U}$ to $\mathscr{F}$. Then there exists also the map $\pi\mathscr{F}' : \Psi(\mathscr{M}) \to \mathbf{R}^{\mathscr{F}'}$ and its compact image $\pi\mathscr{F}'(\Psi(\mathscr{M}))$, in which $\pi\mathscr{F}'(U)$ is a neighborhood of $\pi\mathscr{F}'(x)$. Since the natural projection $P : \mathbf{R}^{\mathscr{F}'} \to \mathbf{R}^{\mathscr{F}}$ is continuous, the restriction $P|_{\pi\mathscr{F}'}(\Psi(\mathscr{M})) : \pi\mathscr{F}'(\Psi(\mathscr{M})) \to \pi\mathscr{F}(\Psi(\mathscr{M}))$ is a one-to-one continuous map between compact spaces and hence a homeomorphism. This means that $\pi\mathscr{F}(U)$ is a neighborhood of $\pi\mathscr{F}(x)$ for any $x \in \Psi(\mathscr{M})$ and any neighborhood U of x. It follows that $\Psi(\mathscr{M})$ is homeomorphic to $\pi\mathscr{F}(\Psi(\mathscr{M}))$, but this is a contradiction since $\Psi(\mathscr{M})$ is neither compact nor metrizable.

The following problems seem to be interesting.

Problem 1. *Does every separable metrizable space have a continuous interpolation*?

This is equivalent to the problem: Does the Hilbert cube I^{ω} or the countable product $\mathbf{R}^{\omega}$ have a continuous interpolation?

Problem 2. *Does every space contain a dense subspace which has a* (*weakly*) *continuous interpolation*?

Addendum. The author was recently pointed out by K. Sakai that the answer of Problem 1 is positive.

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