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HYPERSPACES OF FINITE SUBSETS OF NON-SEPARABLE HILBERT SPACES

By

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Abstract. Let $\ell_2(\tau)$ be the Hilbert space with weight τ and ℓ_2^f be the linear span of the canonical orthonormal basis of the separable Hilbert space ℓ_2. In this paper, we prove that if a metric space X is homeomorphic to $\ell_2(\tau)$ or $\ell_2(\tau) \times \ell_2^f$ then the hyperspace $\mathrm{Fin}_H(X)$ of non-empty finite subsets of X with the Hausdorff metric is homeomorphic to $\ell_2(\tau) \times \ell_2^f$.

1. Introduction

Let $\mathrm{Cld}_H(X)$ be the space of all non-empty closed subsets of a metric space $X = (X, d)$ which admits the (infinite-valued) Hausdorff metric $d_H : \mathrm{Cld}_H(X)^2 \to [0, \infty]$ defined as follows:

$$d_H(A, B) = \max\left\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\right\},$$

where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. By $\mathrm{Fin}_H(X)$, we denote the subspace of $\mathrm{Cld}_H(X)$ consisting of all finite subsets of X, where the topology of $\mathrm{Fin}_H(X)$ coincides with the Vietoris topology. For an infinite cardinality τ, let $\ell_2(\tau)$ be the Hilbert space with weight τ, that is,

$$\ell_2(\tau) = \left\{(x_\alpha)_{\alpha \in \tau} \in \mathbf{R}^\tau \,\middle|\, \sum\nolimits_{\alpha \in \tau} x_\alpha^2 < \infty\right\}.$$

Let ℓ_2^f be the linear span of the canonical orthonormal basis of the separable Hilbert space $\ell_2 = \ell_2(\aleph_0)$, that is,

$$\ell_2^f = \{(x_i)_{i \in \mathbf{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbf{N}\}.$$

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In [5], D. Curtis and Nguyen To Nhu proved that $\mathrm{Fin}_H(X)$ is homeomorphic to ($\approx$) the space ℓ_2^f if and only if X is non-degenerate, strongly countable-dimensional, connected, locally path-connected and σ-compact. Recently, the hyperspace $\mathrm{Fin}_{AW}(X)$ with the Attouch-Wets topology and $\mathrm{Fin}_W(X)$ with the Wijsman topology have been studied. In [11], it has been shown that if X is an infinite-dimensional Banach space with weight $w(X) = \tau$ then $\mathrm{Fin}_{AW}(X) \approx \ell_2(\tau) \times \ell_2^f$, and in [6] that if X is an infinite-dimensional separable Banach space then $\mathrm{Fin}_W(X) \approx \ell_2 \times \ell_2^f$.

Let $\mathrm{Comp}_H(X)$ be the subspace of $\mathrm{Cld}_H(X)$ consisting of all compact sets in X. In [4], it is proved that $\mathrm{Comp}_H(\ell_2) \approx \ell_2$.

In this paper, we prove the following:

THEOREM 1.1. *Let τ be an infinite cardinal. If a metric space X is homeomorphic to $\ell_2(\tau)$ or $\ell_2(\tau) \times \ell_2^f$ then* $\mathrm{Fin}_H(X) \approx \ell_2(\tau) \times \ell_2^f$. *Moreover, in case $X \approx \ell_2(\tau)$,* $\mathrm{Comp}_H(X) \approx \ell_2(\tau)$ *and* $\mathrm{Fin}_H(X)$ *is homotopy dense in* $\mathrm{Comp}_H(X)$.

2. The Characterization of $\ell_2(\tau) \times \ell_2^f$

Let $\mathbf{S}_X$ be the unit sphere in a normed linear space $X = (X, \|\cdot\|)$. For each $x \in X$ and $r \in (0, \infty)$, let $\mathrm{B}(x, r) = \{x' \in X \mid \|x - x'\| < r\}$. For a subset $A \subset X$, $\mathrm{cl}\, A$ is the closure of A, $\mathrm{card}\, A$ is the cardinality of A, and $\mathrm{diam}\, A = \sup\{\|a - b\| \mid a, b \in A\}$.

To prove Theorem 1.1, we use the characterization of the space $\ell_2(\tau) \times \ell_2^f$ which is obtained in [10]. Before introducing this characterization, we need several definitions.

A *σ-completely metrizable* space is a metrizable space which is a countable union of completely metrizable closed subsets.

For each open cover $\mathcal{U}$ of Y, two maps $f, g : X \to Y$ are *$\mathcal{U}$-close* (or f is *$\mathcal{U}$-close* to g) if each $\{f(x), g(x)\}$ is contained in some $U \in \mathcal{U}$. When $Y = (Y, d)$ is a metric space, there exists a map $\alpha : Y \to (0, \infty)$ such that each open ball $\mathrm{B}(y, \alpha(y)) = \{z \in Y \mid d(y, z) < \alpha(y)\}$ is contained in some $U \in \mathcal{U}$, whence if g is *α-close* to f, that is, $d(f(x), g(x)) < \alpha(f(x))$ for each $x \in X$, then g is $\mathcal{U}$-close to f.

A closed set $A \subset X$ is called a (*strong*) *Z-set* in X provided, for each open cover $\mathcal{U}$ of X, there is a map $f : X \to X$ such that f is $\mathcal{U}$-close to id_X and $f(X) \cap A = \emptyset$ ($\mathrm{cl}\, f(X) \cap A = \emptyset$). The union of countably many (strong) Z-sets in X is called a (*strong*) *Z_σ-set* in X. When X itself is a (strong) Z_σ-set in X, we call X a (*strong*) *Z_σ-space*. A *Z-embedding* is an embedding whose image is a Z-set.

A space X is said to be *universal for a class* $\mathscr{C}$ (simply, $\mathscr{C}$*-universal*) if every map $f : C \to X$ of $C \in \mathscr{C}$ is approximated by Z-embeddings, that is, for each $C \in \mathscr{C}$, each map $f : C \to X$, and for each open cover $\mathscr{U}$ of X, there is a Z-embedding $g : C \to X$ such that g is $\mathscr{U}$-close to f.

It is said that X is *strongly universal for* $\mathscr{C}$ (simply, *strongly* $\mathscr{C}$*-universal*) when the following condition is satisfied:

($\mathrm{su}_{\mathscr{C}}$) for each $C \in \mathscr{C}$ and each closed set $D \subset C$, if $f : C \to X$ is a map such that $f|D$ is a Z-embedding, then, for each open cover $\mathscr{U}$ of X, there is a Z-embedding $h : C \to X$ such that $h|D = f|D$ and h is $\mathscr{U}$-close to f.

Let $\mathfrak{M}_1(\tau)$ be the class of completely metrizable spaces with weight $\leq \tau$. The next proposition is the characterization of $\ell_2(\tau) \times \ell_2^f$:

Proposition 2.1. *A metrizable space X is homeomorpic to $\ell_2(\tau) \times \ell_2^f$ if and only if X is a strongly $\mathfrak{M}_1(\tau)$-universal AR, which is a σ-completely metrizable strong Z_σ-space of $w(X) = \tau$.*

3. AR-property

The following is due to D. Curtis and Nguyen To Nhu. In fact, it is a combination of Lemmas 3.5, 2.3 and the proof of Theorem 2.4 in [5].

Proposition 3.1. *The hyperspace* $\mathrm{Fin}_H(X)$ *is an ANR* (*an AR*) *if and only if X is locally path-connected* (*and connected*).

Here, we shall prove a result stronger than Proposition 3.1 above. In [8], Michael introduced uniform AR's and uniform ANR's. A *uniform ANR* is a metric space X with the property: for an arbitrary metric space $Z = (Z, d)$ containing X isometrically as a closed subset, there exist a uniform neighborhood U of X in Z (i.e., $U = N(X, \gamma)$ for some $\gamma > 0$) and a retraction $r : U \to X$ which is uniformly continuous at X, that is, for each $\varepsilon > 0$, there is some $\delta > 0$ such that if $x \in X$, $z \in U$ and $d(x, z) < \delta$ then $d(x, r(z)) < \varepsilon$. When $U = Z$ in the above, X is called a *uniform AR*.

Proposition 3.2. *The hyperspace* $\mathrm{Fin}_H(X)$ *is a uniform ANR* (*a uniform AR*) *if and only if X is uniformly locally path-connected* (*and connected*).

Proof. Since $\mathrm{Fin}_H(X)$ is a Lawson semilattice, by Theorem 3.4 in [7], it suffices to show that $\mathrm{Fin}_H(X)$ is uniformly locally path-connected (and connected) if and only if so is X.

To see the "if" part, let $\varepsilon > 0$. Then we have $\delta > 0$ such that each δ-close points $x, y \in X$ can be connected by a path with diam $< \varepsilon/2$. If $A, B \in \mathrm{Fin}_H(X)$ with $d_H(A, B) < \delta$ then for each $a \in A$ there is $b_a \in B$ such that $d(a, b_a) < \delta$, hence we have a path $f_a : [0,1] \to X$ such that $f(0) = a$, $f(1) = b_a \in B$ and diam $f_a([0,1]) < \varepsilon/2$. Now we define $f : [0,1] \to \mathrm{Fin}_H(X)$ as follows:

$$f(t) = B \cup \{f_a(t) \mid a \in A\} \quad \text{for each } t \in [0,1].$$

Since A is finite, it is easy to see that f is continuous. Note that $f(0) = A \cup B$ and $f(1) = B$. Thus, f is a path from $A \cup B$ to B with diam $< \varepsilon/2$. Similarly, we can construct a path f' in $\mathrm{Fin}_H(X)$ from $A \cup B$ to A with diam $< \varepsilon/2$. Therefore, by connecting f and f', we have a path in $\mathrm{Fin}_H(X)$ from A to B with diam $< \varepsilon$.

Next, we show the "only if" part. By the uniform local path-connectedness of $\mathrm{Fin}_H(X)$, for each $\varepsilon > 0$, we have $\delta > 0$ such that each δ-close $A, B \in \mathrm{Fin}_H(X)$ can be connected by an ε-path in $\mathrm{Fin}_H(X)$. Now, let $x, y \in X$ with $d(x, y) < \delta$. Then, there is a path $f : [0,1] \to \mathrm{Fin}_H(X)$ such that $\mathrm{diam}_{d_H} f([0,1]) < \varepsilon/2$, $f(0) = \{x\}$ and $f(1) = \{y\}$. It suffices to show that x and y can be connected by a path in $\bigcup f([0,1]) = \bigcup_{t \in [0,1]} f(t)$ because $\mathrm{diam}_d \bigcup f([0,1]) < \varepsilon$. By Lemma 2.2 in [5], $\bigcup f([0,1])$ is compact and locally connected. Moreover, $\bigcup f([0,1])$ is connected. Otherwise, there would be disjoint open sets U and V in X such that both U and V meet $\bigcup f([0,1])$ and $\bigcup f([0,1]) \subset U \cup V$. Then, $[0,1]$ could be separated into non-empty open sets $U' = \{t \in [0,1] \mid f(t) \subset U\}$ and $V' = \{t \in [0,1] \mid f(t) \cap V \neq \emptyset\}$, which contradicts to the connectedness of $[0,1]$. Thus, $\bigcup f([0,1])$ is a Peano continuum, so $x, y \in \bigcup f([0,1])$ are connected by a path in $\bigcup f([0,1])$.

By replacing ε by ∞, it is shown that X is path-connected if and only if $\mathrm{Fin}_H(X)$ is path-connected. □

For a normed linear space X, $\mathrm{Fin}_H(X)$ is a uniform AR by 3.2. Observe that $\mathrm{Fin}_H(X)$ is dense in $\mathrm{Comp}_H(X)$. Then, by Theorem 2 in [9], we have the following:

COROLLARY 3.3. *For every normed linear space X, $\mathrm{Fin}_H(X)$ and $\mathrm{Comp}_H(X)$ are uniform AR's and $\mathrm{Fin}_H(X)$ is homotopy dense in $\mathrm{Comp}_H(X)$.*

4. Weight of $\mathrm{Fin}_H(X)$

For each $k \in \mathbf{N}$, let $\mathrm{Fin}^k(X) = \{A \in \mathrm{Fin}(X) \mid \mathrm{card}\, A \le k\}$. The following proposition is similarly proved as Proposition 5.1 of [11].

Proposition 4.1. *For every metric space X, $\mathrm{Fin}_H(X)$ has the same weight as X.*

Proof. Let D be a dense set in X with $\operatorname{card} D = w(X)$. Then, $\operatorname{card}\mathrm{Fin}(D) = w(X)$ because

$$\operatorname{card} D \le \operatorname{card}\mathrm{Fin}(D) = \operatorname{card}\bigcup_{k\in\mathbf{N}}\mathrm{Fin}^k(D) \le \aleph_0 \operatorname{card} D = w(X).$$

For each $A \in \mathrm{Fin}_H(X)$ and $\varepsilon > 0$, we have $B \in \mathrm{Fin}(D)$ such that $d_H(A,B) < \varepsilon$. Therefore $\mathrm{Fin}_H(D)$ is dense in $\mathrm{Fin}_H(X)$. Thus $\mathrm{Fin}_H(X)$ has the same weight as X. □

Since $\mathrm{Fin}_H(X)$ is dense in $\mathrm{Comp}_H(X)$, we have the following:

Corollary 4.2. *For every metric space X, $\mathrm{Comp}_H(X)$ has the same weight as X.*

5. σ-complete Metrizablity

In this section, we show that the hyperspace $\mathrm{Fin}_H(X)$ is σ-completely metrizable.

Proposition 5.1. *Let $X = (X,d)$ be a complete metric space. Then the hyperspace $\mathrm{Fin}_H(X)$ is σ-completely metrizable.*

Proof. Note that $\mathrm{Cld}_H(X)$ is complete if X is complete [2, Theorem 3.2.4]. Since $\mathrm{Fin}_H(X) = \bigcup_{k\in\mathbf{N}}\mathrm{Fin}^k(X)$, it is enough to prove that $\mathrm{Fin}^k_H(X)$ is closed in $\mathrm{Cld}_H(X)$. For each $B \in \mathrm{Cld}_H(X)\backslash\mathrm{Fin}^k(X)$, we have $k+1$ many distinct points $b_1,\dots,b_{k+1} \in B$ and $r > 0$ such that $\mathrm{B}(b_i,r)\cap\mathrm{B}(b_j,r) = \emptyset$ if $i \ne j$. If $C \in \mathrm{Cld}_H(X)$ satisfies $d_H(B,C) < r$ then there are $c_1,\dots,c_{k+1} \in C$ such that $d(b_i,c_i) < r$, whence $c_i \ne c_j$ if $i \ne j$. Then $\operatorname{card} C \ge \operatorname{card}\{c_1,\dots,c_{k+1}\} > k$. This implies that $C \in \mathrm{Cld}_H(X)\backslash\mathrm{Fin}^k(X)$, hence the complement of $\mathrm{Fin}^k(X)$ is open in $\mathrm{Cld}_H(X)$. □

For each closed subset Y of a metric space X, $\mathrm{Cld}_H(Y)$ can be regarded as a closed subspace of $\mathrm{Cld}_H(X)$.

Corollary 5.2. *If a metric space X is σ-completely metrizable, then so is $\mathrm{Fin}_H(X)$.*

Proof. We can denote $X = \bigcup_{n\in\mathbf{N}} X_n$, where X_n is a completely metrizable closed subset of X with $X_n \subset X_{n+1}$. By Propositon 5.1, $\mathrm{Fin}_H^k(X_k)$ is a completely metrizable closed subset of $\mathrm{Cld}_H(X_k)$. Since $\mathrm{Cld}_H(X_n)$ is closed in $\mathrm{Cld}_H(X)$, it follows that $\mathrm{Fin}_H(X) = \bigcup_{k\in\mathbf{N}} \mathrm{Fin}_H^k(X_k)$ is σ-completely metrizable. □

The following is well-known. For completeness, we give a proof.

Proposition 5.3. *For every complete metric space $X = (X, d)$, $\mathrm{Comp}_H(X)$ is complete.*

Proof. Since $\mathrm{Cld}_H(X)$ is complete, it suffices to show that $\mathrm{Comp}_H(X)$ is closed in $\mathrm{Cld}_H(X)$. Let $A \in \mathrm{Cld}_H(X)\backslash\mathrm{Comp}_H(X)$. Since A is complete, A is not totally bounded. Then there exist $\varepsilon > 0$ and $a_i \in A$ $(i \in \mathbf{N})$ such that $d(a_i, a_j) > \varepsilon$ if $i \neq j$. If $B \in \mathrm{Cld}_H(X)$ and $d_H(A, B) < \varepsilon/3$ then we have $b_i \in B$ $(i \in \mathbf{N})$ such that $d(b_i, a_i) < \varepsilon/3$ for each $i \in \mathbf{N}$, whence $d(b_i, b_j) > \varepsilon/3$ if $i \neq j$. Thus, B is not totally bounded, hnce B is not compact. Therefore, $\mathrm{Cld}_H(X)\backslash\mathrm{Comp}_H(X)$ is open. □

6. Strong Z_σ-space

Proposition 6.1. *Let X be a normed linear space with* $\dim X \geq 1$. *Then,* $\mathrm{Fin}_H(X)$ *is a strong Z_σ-space.*

Proof. Since $\mathrm{Fin}_H(X) = \bigcup_{k\in\mathbf{N}} \mathrm{Fin}^k(X)$, it is sufficient to prove that each $\mathrm{Fin}^k(X)$ is a strong Z-set in $\mathrm{Fin}_H(X)$. As shown in the proof of Proposition 5.1, $\mathrm{Fin}_H^k(X)$ is a closed subset in $\mathrm{Fin}_H(X)$. Let $\alpha : \mathrm{Fin}_H(X) \to (0,1)$ be any map. Take $v \in \mathbf{S}_X$ and define a map $f : \mathrm{Fin}_H(X) \to \mathrm{Fin}_H(X)$ as follows:

$$f(A) = \left\{ a + \frac{j}{k+1}\alpha(A)v \mid a \in A, j = 0, \dots, k \right\}.$$

Then it is easy to see that $\mathrm{card}\, f(A) \geq k+1$ and f is α-close to id.

We will show that $\mathrm{Fin}^k(X) \cap \mathrm{cl}\, f(\mathrm{Fin}_H(X)) = \varnothing$. Assume the contrary, that is, there is a sequence $A_i \in \mathrm{Fin}_H(X)$ $(i \in \mathbf{N})$ such that the sequence $f(A_i)$ has a limit point $A \in \mathrm{Fin}^k(X)$. If $\liminf \alpha(A_i) = 0$ then by taking a subsequence, we can assume that $\alpha(A_i) \to 0$. Since f is α-close to id, it follows that $d_H(A_i, f(A_i)) \to 0$, which implies that A_i converges to A. But this contradicts the continuity of α and $\alpha(A) > 0$. Therefore, we have $\beta = \liminf \alpha(A_i) > 0$. By taking a subsequence, we can assume that $\alpha(A_i) \to \beta$. For each $i \in \mathbf{N}$, let

$$A_i' = \left\{ a + \frac{j}{k+1}\beta v \mid a \in A_i, j = 0, \dots, k \right\}.$$

Then, $A_i' \to A$ because $d_H(f(A_i), A_i') < |\alpha(A_i) - \beta|$.

Let $\eta = \beta/(k+1) > 0$. Then we have an open neighborhood U_a for each $a \in A$ diam $U_a < \eta$ and $U_a \cap U_{a'} = \emptyset$ if $a \neq a'$. Since $d_H(A_i', A) \to 0$, there is $i \in \mathbf{N}$ such that $A_i' \subset \bigcup_{a \in A} U_a$. Take any $x \in A_i$. Then

$$\left\{ x + \frac{j}{k+1}\beta v \mid j = 0, \dots, k \right\} \subset A_i' \subset \bigcup_{a \in A} U_a.$$

Since card $A \leq k$, there are $a \in A$ and $j \neq j' \leq k$ such that

$$x + \frac{j}{k+1}\beta v, \quad x + \frac{j'}{k+1}\beta v \in U_a.$$

Then, it follows that

$$\eta = \frac{1}{k+1}\beta \leq \left\| \left(x + \frac{j}{k+1}\beta v \right) - \left(x + \frac{j'}{k+1}\beta v \right) \right\| \leq \operatorname{diam} U_a < \eta.$$

But this is a contradiction. □

7. Universality

The following is Proposition 2.4 of [11]:

PROPOSITION 7.1. *An ANR X with weight τ is strongly $\mathfrak{M}_1(\tau)$-universal if every open set in X is $\mathfrak{M}_1(\tau)$-universal.*

The following is well-known (cf. [3, Chapter VI, Theorem 5.1]):

LEMMA 7.2. *The unit sphere $\mathbf{S}_X$ of an infinite-dimensional Banach space X with weight τ is homeomorphic to $X \approx \ell_2(\tau)$.* □

PROPOSITION 7.3. *Let X be an infinite-dimensional Hilbert space with weight τ. Then $\mathrm{Fin}_H(X)$ is strongly $\mathfrak{M}_1(\tau)$-universal.*

PROOF. By Corollary 3.3 and Propositon 7.1, it suffices to show that every open subset $W \subset \mathrm{Fin}_H(X)$ is $\mathfrak{M}_1(\tau)$-universal. Let $Y \in \mathfrak{M}_1(\tau)$, $f : Y \to W$ and $\alpha : W \to (0,1)$ be maps. Our purpose is to construct a Z-embedding $g : Y \to W$ which are α-close to f. Define $\beta : W \to (0,1)$ by

$$\beta(A) = \frac{1}{2} \min\{\alpha(A), d_H(A, \mathrm{Fin}_H(X)\backslash W)\}.$$

Note that if $g : Y \to \mathrm{Fin}_H(X)$ is 2β-close to f then g is α-close to f and $g(Y) \subset W$. Each $v \in \mathbf{S}_X$ has an open neighborhood U in $\mathbf{S}_X$ such that $\langle v_1, v_2 \rangle > 0$ for each $v_1, v_2 \in U$, where $\langle v_1, v_2 \rangle$ is the inner product. Since $\mathbf{S}_X \approx \ell_2(\tau)$ is $\mathfrak{M}_1(\tau)$-universal, we have a closed embedding $h : Y \to \mathbf{S}_X$ such that $\langle h(y), h(y') \rangle > 0$ for each $y, y' \in Y$.

First, we define $p : Y \to \mathrm{Fin}_H(\mathbf{R})$ by

$$p(y) = \{\langle h(y), a \rangle \mid a \in f(y)\} \quad \text{for each } y \in Y.$$

To see the continuity of p, let $\varepsilon > 0$ and $y \in Y$. For each $a \in f(y)$, there is $\delta_a > 0$ such that

$$v \in \mathbf{S}_X, \quad b \in X, \quad \|h(y) - v\|, \quad \|a - b\| < \delta_a \Rightarrow |\langle h(y), a \rangle - \langle v, b \rangle| < \varepsilon.$$

Since $f(y)$ is finite, we have $\delta = \min\{\delta_a \mid a \in f(y)\} > 0$. By the continuity of h and f, we have $\eta > 0$ such that if $y' \in Y$ and $d(y, y') < \eta$ then

$$\|h(y) - h(y')\| < \delta \quad \text{and} \quad d_H(f(y), f(y')) < \delta.$$

The last inequality implies that for each $a \in f(y)$, there is $b_a \in f(y')$ with $\|a - b_a\| < \delta \le \delta_a$, whence

$$d(\langle h(y), a \rangle, p(y')) \le |\langle h(y), a \rangle - \langle h(y'), b_a \rangle| < \varepsilon.$$

Conversely, for each $b \in f(y')$, there is $a_b \in f(y)$ with $\|b - a_b\| < \delta \le \delta_{a_b}$, whence

$$d(\langle h(y'), b \rangle, p(y)) \le |\langle h(y'), b \rangle - \langle h(y), a_b \rangle| < \varepsilon.$$

Therefore, $d(y, y') < \eta$ implies $d_H(p(y), p(y')) < \varepsilon$, so p is continuous.

Next, define $q, r : Y \to \mathrm{Fin}_H(\mathbf{R})$ by

$$q(y) = \{0\} \cup \{s_i - s_{i-1} \mid 2 \le i \le m\},$$

$$r(y) = \{0, \beta(f(y))\} \cup \{x \in q(y) \mid x \le \beta(f(y))\},$$

where $s_1 < \cdots < s_m$ with $p(y) = \{s_i \mid i \le m\}$. For each $y \in Y$, let

$$u(y) = \min\{x > 0 \mid x \in r(y)\}.$$

To see the continuity of q, let $\varepsilon > 0$ and $y \in Y$. Assume that $y' \in Y$ is sufficiently close to y so that $p(y')$ satisfies $d_H(p(y), p(y')) < \eta$, where $\eta = \min\{\varepsilon/2, u(y)/3\} > 0$. Denote $p(y') = \{t_j \mid j \le n\}$, where $t_1 < \cdots < t_n$. Then, for each $i \le m$, we have $j \le n$ such that $|s_i - t_j| < \eta$. Since $p(y') \subset \bigcup_{i \le m} \mathrm{B}(s_i, \eta)$ and η-balls $\mathrm{B}(s_i, \eta)$ are pairwise disjoint, for each $i \le m$, there is $k \le n$ such that

$$k = \max\{j \le n \mid t_j \in \mathrm{B}(s_i, \eta)\} = \min\{j \le n \mid t_j \in \mathrm{B}(s_{i+1}, \eta)\} - 1.$$

Then, it follows that

$$|(s_{i+1} - s_i) - (t_{k+1} - t_k)| \le |s_{i+1} - t_{k+1}| + |t_k - s_i| < 2\eta \le \varepsilon.$$

This means that $d((s_{i+1} - s_i), q(y')) < \varepsilon$. On the other hand, for each $j \le n$, we have $i, i' \le m$ such that $|t_j - s_i|, |t_{j+1} - s_{i'}| < \eta$. Then, it is easy to see that $i \le i' \le i + 1$. If $i' = i$ then

$$|(t_{j+1} - t_j) - 0| = |t_{j+1} - t_j| \le |t_{j+1} - s_{i'}| + |t_j - s_i| < 2\eta \le \varepsilon.$$

If $i' = i + 1$ then

$$|(t_{j+1} - t_j) - (s_{i+1} - s_i)| \le |t_{j+1} - s_{i'}| + |s_i - t_j| < 2\eta \le \varepsilon.$$

These mean that $d((t_{j+1} - t_j), q(y)) < \varepsilon$. Thus, we have $d_H(q(y), q(y')) < \varepsilon$. Consequently, q is continuous.

To see the continuity of r, let $\varepsilon > 0$ and $y \in Y$. By the continuity of q and β, we have $\delta > 0$ such that if $y' \in Y$ and $d(y, y') < \delta$ then

$$|\beta(f(y)) - \beta(f(y'))| < \varepsilon \quad \text{and} \quad d_H(q(y), q(y')) < \varepsilon.$$

For each $a \in q(y)$ with $a < \beta(f(y))$, there is $b_a \in q(y')$ such that $|a - b_a| < \varepsilon$. If $a \le \beta(f(y')) - \varepsilon$ then $b_a \le \beta(f(y'))$, whence $d(a, r(y')) \le |a - b_a| < \varepsilon$. If $\beta(f(y')) - \varepsilon < a$ then $d(a, r(y')) \le |a - \beta(f(y'))| < \varepsilon$ because $a < \beta(f(y)) < \beta(f(y')) + \varepsilon$. On the other hand, for each $b \in q(y')$ with $b < \beta(f(y'))$, there is $a_b \in q(y)$ such that $|b - a_b| < \varepsilon$. If $b \le \beta(f(y)) - \varepsilon$ then $a_b \le \beta(f(y))$, i.e., $a_b \in r(y)$. Hence, $d(b, r(y)) \le |b - a_b| < \varepsilon$. If $\beta(f(y)) - \varepsilon < b$ then $d(b, r(y)) \le |b - \beta(f(y))| < \varepsilon$ because $b < \beta(f(y')) < \beta(f(y)) + \varepsilon$. Therefore, $d(y, y') < \delta$ implies $d_H(r(y), r(y')) < \varepsilon$, hence r is continuous.

Next, we define a map $g : Y \to \mathrm{Fin}_H(X)$ as follows:

$$g(y) = \{a + bh(y) \mid a \in f(y), b \in r(y)\}.$$

Since f and r are continuous, it is easy to see that $g : Y \to \mathrm{Fin}_H(X)$ is continuous. Since $\operatorname{diam} r(y) = \beta(f(y))$, it follows that $d_H(f(y), g(y)) < 2\beta(f(y))$. It should be remarked that

$$(*) \quad \langle h(y), x \rangle - \min p(y) \in \{0\} \cup [u(y), \infty) \quad \text{for each } x \in g(y).$$

Indeed, let $x = a + bh(y) \in g(y)$, where $a \in f(y)$ and $b \in r(y)$. Then,

$$\min p(y) \le \langle h(y), a\rangle \le \langle h(y), a\rangle + b\|h(y)\| = \langle h(y), a + bh(y)\rangle = \langle h(y), x\rangle.$$

Since $b = 0$ or $b \ge u(y) > 0$, we have $(*)$.

To see that g is injective, assume that there are $y \ne y' \in Y$ with $g(y) = g(y')$. Since $h(y) \ne h(y')$, it follows that

$$\begin{aligned}0 < \|h(y) - h(y')\|^2 &= \|h(y)\|^2 + \|h(y')\|^2 - 2\langle h(y), h(y')\rangle \\ &= 2(1 - \langle h(y), h(y')\rangle),\end{aligned}$$

hence $0 < \langle h(y), h(y')\rangle < 1$. Let $a \in f(y)$ with $\langle h(y), a\rangle = \min p(y)$. Note $a \in g(y')$ because $f(y) \subset g(y) = g(y')$. If $a \notin f(y')$ then there are $a' \in f(y') \subset g(y') = g(y)$ and $0 < b \le \beta(f(y'))$ such that $a = a' + bh(y')$, whence

$$\langle h(y), a'\rangle = \langle h(y), (a - bh(y'))\rangle = \langle h(y), a\rangle - b\langle h(y), h(y')\rangle < \min p(y).$$

This contradicts to $(*)$. Therefore, $a \in f(y')$, hence $a + u(y')h(y') \in g(y') = g(y)$. On the other hand, we have no points $c \in g(y)$ with $\min p(y) < \langle h(y), c\rangle < \min p(y) + u(y)$ by $(*)$. Then,

$$\begin{aligned}\min p(y) + u(y) &\le \langle h(y), (a + u(y')h(y'))\rangle \\ &= \langle h(y), a\rangle + u(y')\langle h(y), h(y')\rangle \\ &= \min p(y) + u(y')\langle h(y), h(y')\rangle.\end{aligned}$$

Therefore, $0 < u(y)/u(y') \le \langle h(y), h(y')\rangle < 1$. By replacing y and y' by each others, we get $0 < u(y')/u(y) < 1$ but this is impossible.

To see that g is a closed map, let $A \subset Y$ be a closed set in Y and $y_i \in A$, $i \in \mathbf{N}$, such that $g(y_i)$ converges to $G \in W$. Then $\liminf \beta(f(y_i)) > 0$. Otherwise, by taking a subsequence, we could assume that $\operatorname{diam} r(y_i) = \beta(f(y_i)) \to 0$, hence $d_H(f(y_i), g(y_i)) \to 0$ $(i \to \infty)$. In this case, $f(y_i)$ converges to G, hence $\beta(f(y_i)) \to \beta(G) > 0$, which is a contradiction. Now, for each $i \in \mathbf{N}$, let

$$x_i \in f(y_i) \quad \text{and} \quad x_i' = x_i + \beta(f(y_i))h(y_i) \in g(y_i).$$

Since $x_i \in g(y_i)$ and $g(y_i) \to G$, we have $z_i \in G$ such that $d(x_i, z_i) \to 0$. Since G is finite, by taking a subsequence, it can be assumed that all z_i are the same point $z \in G$, whence $x_i \to z$. By the same way, we can assume that there is $z' \in G$ such that $x_i' \to z'$. Note that $z \ne z'$ because $\liminf \beta(f(y_i)) > 0$. Hence, this implies that $h(y_i)$ converges to $(z' - z)/\|z' - z\| \in \mathbf{S}_X$. Since h is a closed embedding, y_i converges to some $y \in A$, which implies that $G = g(y) \in g(A)$.

To see that $g(Y)$ is a Z-set in W, for each a map $\alpha : W \to (0,1)$, take $y_0 \in Y$ and let

$$\gamma(A) = \frac{1}{2}\min\{\alpha(A), d_H(A, \mathrm{Fin}_H(X)\setminus W), u(y_0)\} > 0.$$

Define maps $p', q', r' : W \to \mathrm{Fin}_H(\mathbf{R})$ and $\varphi : W \to W$ as follows:

$$p'(A) = \{\langle h(y_0), a\rangle \mid a \in A\},$$

$$q'(A) = \{0\} \cup \{s_i - s_{i-1} \mid 2 \le i \le m\},$$

$$r'(A) = \{0, \gamma(A)\} \cup \{x \in q'(A) \mid x \le \gamma(A)\},$$

$$\varphi(A) = \{a + bh(y_0) \mid a \in A, b \in r'(y)\},$$

where $s_1 < \cdots < s_m$ with $p'(A) = \{s_i \mid i \le m\}$. If $g(Y) \cap \varphi(W) \neq \emptyset$ then this intersection is $\{g(y_0)\}$ by the same way as above which shows the injectivity of g. If there is $A \in W$ such that $\varphi(A) = g(y_0)$ then for each $a \in A$ with $\langle h(y_0), a\rangle = \min p(y_0)$, we have $a' = a + \gamma(A)h(y_0) \in \varphi(A)$. But this is impossible because $\gamma(A) < u(y_0)$. □

Remark. In the above proof, when α is extended to a map $\tilde{\alpha} : \tilde{W} \to (0, 1)$ of an open set $\tilde{W}$ in $\mathrm{Comp}_H(X)$ such that $W = \tilde{W} \cap \mathrm{Fin}_H(X)$, it can be seen that $g(Y)$ is closed in $\tilde{W}$ as follows: In this case, β has the natural extension $\tilde{\beta} : \tilde{W} \to (0, 1)$. If $g(y_i)$ converges to $G \in \tilde{W}$, we have $\liminf \beta(f(y_i)) > 0$ by the same arguments. Moreover, even if G is not finite, there is a subsequence of $(z_i)_{i\in\mathbf{N}}$ converging to some $z \in G$ because G is compact. Then, the corresponding subsequence of $(x_i)_{i\in\mathbf{N}}$ converges to z. Thus, we can assume that $x_i \to z$. Similarly, we can assume that $(x_i')_{i\in\mathbf{N}}$ converges to some $z' \in G$. Hence, we have $G \in g(Y)$ by the same way.

Proposition 7.4. *Let X be an infinite-dimensional Hilbert space with weight τ. Then* $\mathrm{Comp}_H(X)$ *is strongly* $\mathfrak{M}_1(\tau)$*-universal.*

Proof. The proof is similar to Propositon 7.3. Let $f : Y \to W$ be a map from $Y \in \mathfrak{M}_1(\tau)$ to an open set $W \subset \mathrm{Comp}_H(X)$. For each open cover $\mathscr{U}$ of W, let $\mathscr{V}$ be an open star-refinement of $\mathscr{U}$. Since $\mathrm{Fin}_H(X)$ is homotopy dense in $\mathrm{Comp}_H(X)$, it easily follows that $W \cap \mathrm{Fin}_H(X)$ is homotopy dense in W. Then, f is $\mathscr{V}$-close to a map $f' : Y \to W \cap \mathrm{Fin}_H(X)$. By Propositon 7.3, f' is $\mathscr{V}$-close to a Z-embedding $g : Y \to W \cap \mathrm{Fin}_H(X)$, where $g(Y)$ is closed in W by the above remark. Then, it follows that $g : Y \to W$ is a Z-embedding which is $\mathscr{U}$-close to f. □

Theorem 7.5. *If a metric space X is homeomorphic to $\ell_2(\tau)$ then*

$$\mathrm{Fin}_H(X) \approx \ell_2(\tau) \times \ell_2^f \quad \textit{and} \quad \mathrm{Comp}_H(X) \approx \ell_2(\tau).$$

Proof. Since the topology of $\mathrm{Comp}_H(X)$ coincides with the Vietoris topology, we have $\mathrm{Fin}_H(X) \approx \mathrm{Fin}_H(\ell_2(\tau))$ and $\mathrm{Comp}_H(X) \approx \mathrm{Comp}_H(\ell_2(\tau))$. It has been proved that $\mathrm{Fin}_H(\ell_2(\tau))$ satisfies the all conditions in Propositon 2.1. Then $\mathrm{Fin}_H(X) \approx \ell_2(\tau) \times \ell_2^f$. On the other hand, $\mathrm{Comp}_H(\ell_2(\tau))$ is a strongly $\mathfrak{M}_1(\tau)$-universal complete metric AR with weight τ. By Toruńczyk's characterization of $\ell_2(\tau)$ [12, Proposition 2.1] (cf. [13]), we have $\mathrm{Comp}_H(X) \approx \ell_2(\tau)$. □

For a dense subspace $Z = \ell_2(\tau) \times \ell_2^f$ of $\ell_2(\tau) \times \ell_2 \approx \ell_2(\tau)$, the unit sphere $\mathbf{S}_Z$ contains a copy $\mathbf{S}_{\ell_2(\tau)} \times \{0\}$ of $\mathbf{S}_{\ell_2(\tau)}$ as closed set. Then there is a closed embedding $h : Y \to \mathbf{S}_Z$ for each $Y \in \mathfrak{M}_1(\tau)$. By the same proof as Propositon 7.3, we can show the $\mathfrak{M}_1(\tau)$-universality of $\mathrm{Fin}_H(Z)$. Consequently, we have the following:

Theorem 7.6. *If a metric space X is homeomorphic to $\ell_2(\tau) \times \ell_2^f$ then* $\mathrm{Fin}_H(X) \approx \ell_2(\tau) \times \ell_2^f$. □

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