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A GAP THEOREM FOR COMPLETE FOUR-DIMENSIONAL MANIFOLDS WITH $\delta W^+ = 0$

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Abstract. Let M^4 be a complete noncompact oriented four-dimensional Riemannian manifold satisfying $\delta W^+ = 0$, where W^+ is the self-dual part of the Weyl curvature tensor. Suppose its scalar curvature is nonnegative and Sobolev's inequality holds. We show that if the L^2 norm of W^+ is sufficiently small, then $W^+ \equiv 0$.

1. Introduction

Let M^4 be a complete oriented four-dimensional Riemannian manifold. By the Hodge star operator $*$, the bundle of 2-forms Λ^2 splits into the sum of the bundle of self-dual and anti-self-dual 2-forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. According to this decomposition, Weyl curvature tensor W splits as $W = W^+ + W^-$. W^+ is called the self-dual part of the Weyl curvature tensor. We consider the equation $\delta W^+ = 0$. Here δ is the formal divergence defined as

$$\delta W^+(X_1, X_2, X_3) = -\sum_{i=1}^{4} (\nabla_{e_i} W^+)(e_i, X_1, X_2, X_3),$$

where $\{e_i\}$ is an orthonormal basis of TM with positive orientation. When the Ricci tensor is parallel, $\delta W^+ = 0$ ([2], [3]). Therefore, manifolds satisfying $\delta W^+ = 0$ are natural generalizations of Einstein manifolds or symmetric spaces.

Recently Gursky [4], Itoh-Satoh [5] proved L^2 or pointwise isolation theorem of W^+ for compact oriented four-dimensional Riemannian manifolds with $\delta W^+ = 0$.

In this note we give a gap theorem for noncompact manifolds.

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THEOREM 1.1. *Let M^4 be a complete noncompact oriented four-dimensional Riemannian manifold with $\delta W^+ = 0$. Suppose its scalar curvature is nonnegative and Sobolev's inequality holds on M^4. Then there is a constant $C > 0$ depending only on the Sobolev constant such that if $\int_M |W^+|^2\, dv < C$, then $W^+ \equiv 0$.*

2. Proof

The method of proof is a standard way in proving such an isolation theorem (for example, see [1] for minimal submanifold case, [6] for harmonic map case).

By the Weitzenböck formula, we have (cf. (3.11) and (3.12) in [4])

$$\Delta|W^+|^2 \geq 2|\nabla W^+|^2 + R|W^+|^2 - \sqrt{6}|W^+|^3, \tag{1}$$

where R is the scalar curvature of M. By using the following Kato's inequality ((2.1) in [4])

$$|\nabla|W^+|| \leq \sqrt{\frac{3}{5}}|\nabla W^+|$$

and the assumption $R \geq 0$ to (1), we obtain

$$\frac{2}{3}|\nabla|W^+||^2 \leq |W^+|\Delta|W^+| + \frac{\sqrt{6}}{2}|W^+|^3.$$

For simplicity we set $a = |W^+|$ and rewrite the above inequality as

$$\frac{2}{3}|\nabla a|^2 \leq a\Delta a + \frac{\sqrt{6}}{2}a^3.$$

Let $\lambda \in C_0^1(M)$. Multiplying λ^2 and integrating over M, we get

$$\frac{2}{3}\int \lambda^2|\nabla a|^2 \leq \int \lambda^2 a\Delta a + \frac{\sqrt{6}}{2}\int \lambda^2 a^3. \tag{2}$$

Since

$$\int 2\lambda a\nabla\lambda \cdot \nabla a + \int \lambda^2|\nabla a|^2 + \int \lambda^2 a\Delta a = 0,$$

we get from (2)

$$\frac{5}{3}\int \lambda^2|\nabla a|^2 + 2\int \lambda a\nabla\lambda \cdot \nabla a \leq \frac{\sqrt{6}}{2}\int \lambda^2 a^3. \tag{3}$$

We have

$$2\int |\lambda a \nabla\lambda \cdot \nabla a| \le \varepsilon \int \lambda^2 |\nabla a|^2 + \frac{1}{\varepsilon}\int a^2 |\nabla\lambda|^2$$

for any $\varepsilon > 0$. Plugging this into (3), we get

$$\left(\frac{5}{3} - \varepsilon\right)\int \lambda^2 |\nabla a|^2 \le \frac{\sqrt{6}}{2}\int \lambda^2 a^3 + \frac{1}{\varepsilon}\int a^2 |\nabla\lambda|^2. \tag{4}$$

On the other hand we have Sobolev's inequality

$$\left(\int |f|^{4/3}\right)^{3/4} \le C_S \int |\nabla f| \quad \text{for } \forall f \in C_0^1(M).$$

Substituting f^3 into f, we obtain

$$\left(\int |f|^4\right)^{1/4} \le 3C_S\left(\int |\nabla f|^2\right)^{1/2}.$$

We apply this inequality for λa and get

$$\left(\int \lambda^4 a^4\right)^{1/2} \le 18C_S^2\left\{\int \lambda^2 |\nabla a|^2 + \int a^2 |\nabla\lambda|^2\right\}. \tag{5}$$

From (4) and (5), we get

$$\left(\int \lambda^4 a^4\right)^{1/2} \le \frac{18C_S^2}{\frac{5}{3} - \varepsilon}\left\{\frac{\sqrt{6}}{2}\int \lambda^2 a^3 + \frac{1}{\varepsilon}\int a^2 |\nabla\lambda|^2\right\} + 18C_S^2 \int a^2 |\nabla\lambda|^2. \tag{6}$$

Applying Hölder's inequality

$$\int \lambda^2 a^3 \le \left(\int \lambda^4 a^4\right)^{1/2}\left(\int a^2\right)^{1/2}$$

to (6), we get

$$\left\{1 - \frac{9\sqrt{6}}{\frac{5}{3} - \varepsilon} C_S^2\left(\int a^2\right)^{1/2}\right\}\left(\int \lambda^4 a^4\right)^{1/2} \le \left\{\frac{18C_S^2}{\varepsilon\left(\frac{5}{3} - \varepsilon\right)} + 18C_S^2\right\}\int a^2 |\nabla\lambda|^2.$$

If we have

$$1 - \frac{9\sqrt{6}}{\frac{5}{3} - \varepsilon} C_S^2\left(\int a^2\right)^{1/2} > 0, \tag{7}$$

then by using a standard cut-off function argument, we conclude $a = |W^+| \equiv 0$. Since we can choose ε arbitrarily small, (7) is satisfied if

$$\int_M |W^+|^2 < \frac{25}{4374} \cdot \frac{1}{C_S^4}.$$

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