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ON HOCHSCHILD COHOMOLOGY RING OF THE INTEGRAL GROUP RING OF THE QUATERNION GROUP†

By

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Abstract. We will determine the ring structure of the Hochschild cohomology $HH^*(\mathbf{Z}Q_2)$ for the quaternion group Q_2 of order 8 using the ring isomorphism determined by Siegel and Witherspoon.

Introduction

Let RG be a group ring for a finite group G over a commutative ring R. If G is an abelian group, Holm [7] and Cibils and Solotar [3] prove that the Hochschild cohomology ring $HH^*(RG)$ is isomorphic to the tensor product of RG and the ordinary cohomology ring:

$$HH^*(RG) \simeq RG \otimes_R H^*(G, R).$$

If G is a non-abelian group, it seems more difficult to investigate the ring structure of $HH^*(RG)$.

On the other hand, it is well known that the Hochschild cohomology ring $HH^*(RG)$ is isomorphic to the ordinary cohomology ring $H^*(G, {}_{\psi}RG)$. In the above, ${}_{\psi}RG$ is regarded as a left RG-module by conjugation. In fact, there are periodic resolutions of period 2, 4 of cyclic groups, generalized quaternion groups, respectively. So it is theoretically possible to calculate the products of the cohomology using the resolution. Thus we have determined the ring structure of $HH^*(\mathbf{Z}Q_t)$ for arbitrary generalized quaternion group Q_t by calculating the ordinary cup product in $H^*(Q_t, {}_{\psi}\mathbf{Z}Q_t)$ (see [4]).

The Hochschild cohomology $HH^n(RG)$ is isomorphic to the direct sum of

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the ordinary group cohomology of the centralizers of representatives of the conjugacy classes of G (see [1, Theorem 2.11.2], [11, Section 4]):

$$HH^*(RG) \simeq \bigoplus_j H^*(G_j, R).$$

Siegel and Witherspoon [11] define a new product on $\bigoplus_j H^*(G_j, R)$ so that the above additive isomorphism is multiplicative. Besides, they calculate the Hochschild cohomology rings of $\mathbf{F}_3S_3$, $\mathbf{F}_2A_4$, $\mathbf{F}_2D_{2^n}$ using this new product. The author considers that it is interesting to investigate the ring structure of the Hochschild cohomology of various group algebras using this new product. As an example of it, we will consider the ring structure of $HH^*(\boldsymbol{Z}Q_2)$ for the quaternion group Q_2 of order 8.

Our aim in this paper is to determine the ring structure of $HH^*(\boldsymbol{Z}Q_2)$ using the new product determined by Siegel and Witherspoon. This method is different from [4].

In Section 1, as preliminaries, we describe some definitions and properties about the Hochschild cohomology, group cohomology, and the new product determined by Siegel and Witherspoon.

In Section 2, we calculate the cup products on $HH^*(\boldsymbol{Z}Q_2)$. In Section 2.1, we describe the presentation of the integral cohomology rings of the quaternion group and the cyclic group. In Section 2.2, as preliminaries of calculating the cup products, we calculate conjugation, restriction and corestriction between the integral cohomology rings of the subgroups of Q_2. In Section 2.3, by calculating the cup products using the new product determined by Siegel and Witherspoon, we determine the ring structure of $HH^*(\boldsymbol{Z}Q_2)$ (Theorem).

1 Preliminaries

Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. Suppose that M is a $\Lambda^e(= \Lambda \otimes_R \Lambda^{op})$-module. Then the n-th Hochschild cohomology of Λ with coefficients in M is defined by

$$H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda, M).$$

Suppose N is another left Λ^e-module. Then for every pair of integers $p, q \geq 0$ there is a (Hochschild) cup product

$$H^p(\Lambda, M) \otimes_R H^q(\Lambda, N) \xrightarrow{\smile} H^{p+q}(\Lambda, M \otimes_\Lambda N).$$

If we put $M = N = \Lambda$, then the cup product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ the structure of a graded ring with identity $1 \in Z(\Lambda) \simeq HH^0(\Lambda)$, where $HH^n(\Lambda)$

denotes $H^n(\Lambda, \Lambda)$ and $Z(\Lambda)$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. Note that the Hochschild cohomology ring $HH^*(\Lambda)$ is anti-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$ (see [9, Proposition 1.2] for example).

Suppose that G is a finite group and that A is a G-module. Then we have the definition of the n-th cohomology group of G with coefficients in A:

$$H^n(G, A) := \mathrm{Ext}^n_{RG}(R, A).$$

Suppose B is another G-module. Then for every pair of integers $p, q \geq 0$ there exists a homomorphism called (ordinary) cup product

$$H^p(G, A) \otimes H^q(G, B) \xrightarrow{\smile} H^{p+q}(G, A \otimes B).$$

In the following, we state the definition of conjugation, restriction and corestriction. Suppose H is a subgroup of G, and A is a G-module. Let ${}^gH$ denote the conjugacy subgroup gHg^{-1} of H for $g \in G$. If $g \in G$, the maps $\phi : {}^gH \to H$; $h' \mapsto g^{-1}h'g$ and $f : A \to A$; $a \mapsto ga$ induce the homomorphism

$$g^* : H^n(H, A) \to H^n({}^gH, A)$$

and call it conjugation by g. Note that g^* is the identity for $g \in H$. Let $\iota : H \hookrightarrow G$ denote the inclusion map, and let $\mathrm{Id} : A \to A$. Then the induced map

$$\mathrm{res}^G_H : H^n(G, A) \to H^n(H, A)$$

is called restriction. Let $G = \bigcup_{i=1}^m \sigma_i H$ be a left coset decomposition. If (Z, d) is an RG-projective resolution, then it is also an RH-projective resolution. Define

$$S_{H\to G} : \mathrm{Hom}_{RH}(Z_n, A) \to \mathrm{Hom}_{RG}(Z_n, A)$$

$$S_{H\to G}(f)(x) = \sum_{i=1}^m \sigma_i f(\sigma_i^{-1}x) \quad (x \in Z_n),$$

where $f \in \mathrm{Hom}_{RH}(Z_n, A)$. This homomorphism does not depend on the choice of coset representatives and this induces

$$\mathrm{cor}^G_H : H^n(H, A) \to H^n(G, A)$$

which is called corestriction. Note that

$$\mathrm{cor}^G_H \cdot \mathrm{res}^G_H \alpha = |G : H|\alpha = m\alpha \quad \text{for } \alpha \in H^n(G, A). \tag{1.1}$$

These mappings of the cohomology groups are independent of the choice of resolutions.

About the group ring RG, there are close relations between the Hochschild cohomology and the group cohomology. The following isomorphism is well known:

$$H^n(RG, M) \xrightarrow{\sim} H^n(G, {}_{\psi}M).$$

In the above, ${}_{\psi}M$ denotes M regarded as a G-module using a ring homomorphism $\psi : RG \to RG^e$; $x \mapsto x \otimes (x^{-1})^{\circ}$ for $x \in G$ and $H^n(G, {}_{\psi}M)$ denotes the ordinary n-th group cohomology. Note that the above isomorphism preserves cup products, that is, the following diagram is commutative:

$$\begin{array}{ccc} H^p(RG, M) \otimes H^q(RG, N) & \xrightarrow{\smile} & H^{p+q}(RG, M \otimes_{RG} N) \\ \wr\downarrow & & \downarrow\wr \\ H^p(G, {}_{\psi}M) \otimes H^q(G, {}_{\psi}N) & \xrightarrow[\smile_{\mu}]{} & H^{p+q}(G, {}_{\psi}(M \otimes_{RG} N)). \end{array}$$

In the above diagram, $\smile_{\mu}$ denotes the map induced by the (ordinary) cup product and a left RG-homomorphism $\mu : {}_{\psi}M \otimes {}_{\psi}N \to {}_{\psi}(M \otimes_{RG} N)$; $a \otimes b \mapsto a \otimes_{RG} b$. If we put $M = N = RG$ and identify RG with $RG \otimes_{RG} RG$ as an RG^e-module, then we have a ring isomorphism

$$HH^*(RG) \xrightarrow{\sim} H^*(G, {}_{\psi}RG)$$

(*cf.* [11, Proposition 3.2], [10, Section 1] or [8]).

Let $g_1 = 1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of G, and let G_i be the centralizer of g_i. Fix g_i and consider the following two RG_i-homomorphisms

$$\theta_{g_i} : R \to RG; \quad \lambda \mapsto \lambda g_i,$$

$$\pi_{g_i} : RG \to R; \quad \sum_{a \in G} \lambda_a a \mapsto \lambda_{g_i}.$$

These maps induce

$$\theta_{g_i}^* : H^n(G_i, R) \to H^n(G_i, {}_{\psi}RG),$$

$$\pi_{g_i}^* : H^n(G_i, {}_{\psi}RG) \to H^n(G_i, R).$$

We define $\gamma_i : H^n(G_i, R) \to H^n(G, {}_{\psi}RG)$ by

$$\gamma_i(\alpha) = \mathrm{cor}_{G_i}^{G}\, \theta_{g_i}^*(\alpha), \quad \text{for } \alpha \in H^n(G_i, R).$$

Then we have the following isomorphism of graded R-modules

$$\Phi : H^n(G, {}_\psi RG) \xrightarrow{\sim} \bigoplus_i H^n(G_i, R); \quad \zeta \mapsto (\pi_{g_i}^* \operatorname{res}_{G_i}^G(\zeta))_i, \tag{1.2}$$

and its inverse is given by $\Phi^{-1}(\alpha) = \gamma_i(\alpha)$ for $\alpha \in H^n(G_i, R)$ (see [11, Section 4]).

Next, let D be a set of double coset representatives for $G_i \backslash G / G_j$. For each $a \in D$, there is a unique $k = k(a)$ such that

$$g_k = {}^b g_i {}^{ba} g_j \tag{1.3}$$

for some $b \in G$. In the above, ${}^x g$ denotes xgx^{-1} for $x, g \in G$. Siegel and Witherspoon [11] define the following new product on $\bigoplus_j H^*(G_j, R)$ so that the above additive isomorphism is multiplicative:

Lemma 1.1 (Product Formula). *Let $\alpha \in H^*(G_i, R)$, $\beta \in H^*(G_j, R)$. Then the following equation holds in $H^*(G, {}_\psi RG)$:*

$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{a \in D} \gamma_k(\operatorname{cor}_W^{G_k}(\operatorname{res}_W^{{}^b G_i} b^* \alpha \smile \operatorname{res}_W^{{}^{ba} G_j} (ba)^* \beta)), \tag{1.4}$$

where D is a set of double coset representatives for $G_i \backslash G / G_j$, $k = k(a)$ and $b = b(a)$ are chosen to satisfy (1.3), *and $W = {}^{ba} G_j \cap {}^b G_i$.*

Note that the sum in (1.4) is independent of the choices for a and b.

2 Calculations

Let Q_2 denote the quaternion group of order 8:

$$Q_2 = \langle x, y \mid x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle.$$

In this section, we calculate the ring structure of $HH^*(ZQ_2)$ using the Product Formula.

We take representatives of the conjugacy classes of Q_2 as follows:

$$g_1 = 1, \quad g_2 = x^2, \quad g_3 = x, \quad g_4 = y, \quad g_5 = xy.$$

Then the centralizers of them are

$$G_1 = Q_2, \quad G_2 = Q_2, \quad G_3 = \langle x \rangle, \quad G_4 = \langle y \rangle, \quad G_5 = \langle xy \rangle.$$

Note that G_3, G_4 and G_5 are cyclic subgroups of order 4.

2.1 Integral Cohomology Rings of the Quaternion Group and the Cyclic Group

In this subsection, we describe the presentation of the integral cohomology rings of the quaternion group and the cyclic group. First, we consider the quaternion case. In the following, we set $\Lambda = \boldsymbol{Z}Q_2$. Then the following periodic Λ-free resolution of $\boldsymbol{Z}$ of period 4 is well known (see [2, Chapter XII, Section 7], [12, Chapter 3, Periodicity]):

$$(Y,\delta):\quad \cdots \to \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\delta_4} \Lambda \xrightarrow{\delta_3} \Lambda^2 \xrightarrow{\delta_2} \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\varepsilon} \boldsymbol{Z} \to 0,$$

$$\delta_1(c_1,c_2) = c_1(x-1) + c_2(y-1),$$

$$\delta_2(c_1,c_2) = (c_1(x+1) + c_2(xy+1), -c_1(y+1) + c_2(x-1)),$$

$$\delta_3(c) = (c(x-1), -c(xy-1)),$$

$$\delta_4(c) = cN,$$

where Λ^2 denotes the direct sum $\Lambda \oplus \Lambda$ and N denotes $\sum_{\tau \in Q_2} \tau$ $(\in \Lambda)$. Applying the functor $\mathrm{Hom}_\Lambda(-,\boldsymbol{Z})$ to the above periodic resolution (Y,δ), we have the following complex:

$$(\mathrm{Hom}_\Lambda(Y,\boldsymbol{Z}),\delta^{\#}):\quad 0 \to \boldsymbol{Z} \xrightarrow{\delta_1^{\#}} \boldsymbol{Z}^2 \xrightarrow{\delta_2^{\#}} \boldsymbol{Z}^2 \xrightarrow{\delta_3^{\#}} \boldsymbol{Z} \xrightarrow{\delta_4^{\#}} \boldsymbol{Z} \to \cdots,$$

$$\delta_1^{\#}(z) = ((x-1)z, (y-1)z) = (0,0),$$

$$\begin{aligned}\delta_2^{\#}(z_1,z_2) &= ((x+1)z_1 - (y+1)z_2, (xy+1)z_1 + (x-1)z_2)\\ &= 2(z_1-z_2)(1,0) + 2z_1(0,1),\end{aligned}$$

$$\delta_3^{\#}(z_1,z_2) = (x-1)z_1 - (xy-1)z_2 = 0,$$

$$\delta_4^{\#}(z) = Nz = 8z.$$

Clearly the module structure of $H^n(Q_2,\boldsymbol{Z})$ is presented by the form of a subquotient of the complex $\mathrm{Hom}_\Lambda(Y,\boldsymbol{Z})$ as follows:

$$H^n(Q_2,\boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & \text{for } n = 0,\\ \boldsymbol{Z}/8 & \text{for } n \equiv 0 \mod 4,\ n \neq 0,\\ 0 & \text{for } n \equiv 1 \mod 4,\\ \boldsymbol{Z}(1,0)/2 \oplus \boldsymbol{Z}(0,1)/2 & \text{for } n \equiv 2 \mod 4,\\ 0 & \text{for } n \equiv 3 \mod 4.\end{cases}$$

By calculating the products of generators $A := (1,0)$, $B := (0,1) \in H^2(Q_2,\boldsymbol{Z})$ and $C := 1 \in H^4(Q_2,\boldsymbol{Z})$ using a diagonal approximation on (Y,δ), we have

$$H^*(Q_2, \boldsymbol{Z}) = \boldsymbol{Z}[A, B, C]/(2A, 2B, 8C, A^2, B^2, AB - 4C),$$

where $\deg A = \deg B = 2$ and $\deg C = 4$ (see [5, Section 4]).

Next, we describe the integral cohomology ring of the cyclic group. Let $H = \langle a \rangle$ denote the cyclic group of order m for any positive integer $m \geq 2$. We set $\Gamma = \boldsymbol{Z}H$. Then the following periodic Γ-free resolution for $\boldsymbol{Z}$ of period 2 is well known (see [2, Chapter XII, Section 7] for example):

$$(Z_H, \partial_H): \quad \cdots \longrightarrow \Gamma \xrightarrow{(\partial_H)_1} \Gamma \xrightarrow{(\partial_H)_2} \Gamma \xrightarrow{(\partial_H)_1} \Gamma \xrightarrow{(\partial_H)_2} \Gamma \xrightarrow{(\partial_H)_1} \Gamma \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow 0,$$

$$(\partial_H)_1(c) = c(a-1),$$

$$(\partial_H)_2(c) = c\sum_{i=0}^{m-1} a^i.$$

Applying the functor $\mathrm{Hom}_\Gamma(-, \boldsymbol{Z})$ to the above periodic resolution, we have the following complex:

$$(\mathrm{Hom}_\Gamma(Z_H, \boldsymbol{Z}), (\partial_H)^{\#}): \quad 0 \longrightarrow \boldsymbol{Z} \xrightarrow{(\partial_H)_1^{\#}} \boldsymbol{Z} \xrightarrow{(\partial_H)_2^{\#}} \boldsymbol{Z} \xrightarrow{(\partial_H)_1^{\#}} \boldsymbol{Z} \longrightarrow \cdots,$$

$$(\partial_H)_1^{\#}(c) = (a-1)c = 0,$$

$$(\partial_H)_2^{\#}(c) = \sum_{i=0}^{m-1} a^i c = mc.$$

Hence we have

$$H^n(H, \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & \text{for } n = 0, \\ \boldsymbol{Z}/m & \text{for } n \equiv 0 \mod 2,\ n \neq 0, \\ 0 & \text{for } n \equiv 1 \mod 2. \end{cases}$$

By calculating the products of a generator $D := 1 \in H^2(H, \boldsymbol{Z})$, we have

$$H^*(H, \boldsymbol{Z}) = \boldsymbol{Z}[D]/(mD),$$

where $\deg D = 2$ (see [2, Chapter XII, Section 7]).

2.2 Conjugation, Restriction and Corestriction

In the following, we set

$$H^*(G_r, \boldsymbol{Z}) = \boldsymbol{Z}[A, B, C]/(2A, 2B, 8C, A^2, B^2, AB - 4C)$$
$$(\deg A = \deg B = 2, \deg C = 4),$$

$$H^*(G_3, \boldsymbol{Z}) = \boldsymbol{Z}[\lambda]/(4\lambda) \quad (\deg \lambda = 2),$$

$$H^*(G_4, \boldsymbol{Z}) = \boldsymbol{Z}[\mu]/(4\mu) \quad (\deg \mu = 2),$$

$$H^*(G_5, \boldsymbol{Z}) = \boldsymbol{Z}[\nu]/(4\nu) \quad (\deg \nu = 2),$$

where $r = 1, 2$. By (1.2), we have

$$H^n(Q_2, {}_{\psi}\boldsymbol{Z}Q_2) = \begin{cases} \boldsymbol{Z}^5 & \text{for } n = 0, \\ (\boldsymbol{Z}/8)^2 \oplus (\boldsymbol{Z}/4)^3 & \text{for } n \equiv 0 \mod 4,\ n \neq 0, \\ 0 & \text{for } n \equiv 1 \mod 4, \\ (\boldsymbol{Z}/2)^4 \oplus (\boldsymbol{Z}/4)^3 & \text{for } n \equiv 2 \mod 4, \\ 0 & \text{for } n \equiv 3 \mod 4. \end{cases}$$

Furthermore we set

$$H^*(\langle x^2 \rangle, \boldsymbol{Z}) = \boldsymbol{Z}[\sigma]/(2\sigma) \quad (\deg \sigma = 2).$$

In order to calculate the cup products using the Product Formula, we need the calculation of conjugation, restriction and corestriction.

First, we calculate conjugation maps. This is given by calculating the images of the generators of the cohomologies on the cochain level. For the calculation, we need the following two lemmas.

LEMMA 2.1. *Let $H = \langle a \rangle$ denote the cyclic group of order m for any positive integer $m \geq 2$. (Z_H, ∂_H) denotes the periodic resolution of H and (X_H, d_H) denotes the standard resolution of H.*

(i) *An initial part of chain transformation $(v_H)_n : (Z_H)_n \to (X_H)_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$(v_H)_0(1) = [\cdot];$$

$$(v_H)_1(1) = [a];$$

$$(v_H)_2(1) = \sum_{i=0}^{m-1} [a^i | a].$$

(ii) *An initial part of chain transformation $(u_H)_n : (X_H)_n \to (Z_H)_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$(u_H)_0([\cdot]) = 1;$$

$$(u_H)_1([a^i]) = \begin{cases} a^{i-1} + a^{i-2} + \cdots + 1 & (i \geq 1) \\ 0 & (i = 0); \end{cases}$$

$$(u_H)_2([a^i|a^j]) = \begin{cases} 1 & (i+j \geq m) \\ 0 & (i+j < m), \end{cases}$$

for $0 \leq i, j < m$.

PROOF. See [6, Proposition 1] for (i). Note that the notation in this proposition is independent of the notation in [6, Proposition 1].

As for (ii), it suffices to show that $(\partial_H)_n \cdot (u_H)_n = (u_H)_{n-1} \cdot (d_H)_n$ holds for $n = 1, 2$. In fact, for any integer $n \geq 3$, we can define $(u_H)_n$ inductively. For any integer $i \geq 0$, we set

$$M_i = \begin{cases} a^{i-1} + a^{i-2} + \cdots + 1 & (i \geq 1) \\ 0 & (i = 0). \end{cases}$$

Note that the equations $M_i + a^i M_j = M_{i+j}$ and $M_i(a-1) = a^i - 1$ hold for $i, j \geq 0$. In the case $n = 1$ we have the following:

$$(\partial_H)_1 \cdot (u_H)_1([a^i]) = M_i(a-1) = a^i - 1 = (u_H)_0(a^i[\cdot] - [\cdot]) = (u_H)_0 \cdot (d_H)_1([a^i]).$$

In the case $n = 2$, the proof is divided into two cases.

Case $0 \leq i + j < m$: The left hand side is as follows:

$$(\partial_H)_2 \cdot (u_H)_2([a^i|a^j]) = 0.$$

The right hand side is as follows:

$$\begin{aligned} (u_H)_1 \cdot (d_H)_2([a^i|a^j]) &= (u_H)_1(a^i[a^j] - [a^{i+j}] + [a^i]) \\ &= a^i M_j - M_{i+j} + M_i \\ &= 0. \end{aligned}$$

Case $i + j \geq m$: The left hand side is as follows:

$$(\partial_H)_2 \cdot (u_H)_2([a^i|a^j]) = (\partial_H)_2(1) = \sum_{k=0}^{m-1} a^k = M_m.$$

The right hand side is as follows:

$$\begin{aligned} (u_H)_1 \cdot (d_H)_2([a^i|a^j]) &= (u_H)_1(a^i[a^j] - [a^{i+j-m}] + [a^i]) \\ &= a^i M_j - M_{i+j-m} + M_i \\ &= M_{i+j} - M_{i+j-m} \\ &= M_{i+j} - (M_{i+j} - a^{i+j-m} M_m) \\ &= M_m. \end{aligned}$$

This completes the proof. □

LEMMA 2.2. *Suppose H is a subgroup of a finite group G and A is a G-module. (X_H, d_H) and $(X_{({}^gH)}, d_{({}^gH)})$ denote the standard resolutions of H and ${}^gH = gHg^{-1}$ for $g \in G$, respectively. Then the conjugation map $g^* : H^n(H, A) \to H^n({}^gH, A)$ is given by the following on the cochain level*:

$$\tilde{g} : \mathrm{Hom}_{ZH}((X_H)_n, A) \to \mathrm{Hom}_{Z({}^gH)}((X_{({}^gH)})_n, A)$$

$$(\tilde{g}(f))([\cdot]) = gf([\cdot]) \qquad (n = 0),$$

$$(\tilde{g}(f))([\rho_1|\rho_2|\cdots|\rho_n]) = gf([g^{-1}\rho_1 g|g^{-1}\rho_2 g|\cdots|g^{-1}\rho_n g]) \quad (n \geq 1),$$

where $\rho_1, \rho_2, \ldots, \rho_n \in {}^gH$.

PROOF. See [13, Proposition 2-5-1]. □

PROPOSITION 2.3. *The following hold*:

$$y^*(\lambda) = -\lambda, \quad x^*(\mu) = -\mu, \quad x^*(\nu) = -\nu.$$

Moreover, $\tau^*(1) = 1$ holds for $1 \in H^0(G_r, \boldsymbol{Z})$ $(1 \leq r \leq 5)$ and $\tau \in Q_2$.

PROOF. Note that $\langle x\rangle$, $\langle y\rangle$ and $\langle xy\rangle$ are normal subgroups of Q_2. First, we calculate the image of λ by $y^* : H^2(\langle x\rangle, \boldsymbol{Z}) \to H^2(\langle x\rangle, \boldsymbol{Z})$. This is given by the composition of the following maps:

$$\begin{aligned}\boldsymbol{Z} &\xrightarrow{\beta_2^{-1}} \mathrm{Hom}_{\boldsymbol{Z}\langle x\rangle}((Z_{\langle x\rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{(u_{\langle x\rangle})_2^{\#}} \mathrm{Hom}_{\boldsymbol{Z}\langle x\rangle}((X_{\langle x\rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{\tilde{y}} \mathrm{Hom}_{\boldsymbol{Z}\langle x\rangle}((X_{\langle x\rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{(v_{\langle x\rangle})_2^{\#}} \mathrm{Hom}_{\boldsymbol{Z}\langle x\rangle}((Z_{\langle x\rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{\beta_2} \boldsymbol{Z},\end{aligned}$$

where β_2 denotes the isomorphism $\mathrm{Hom}_{\boldsymbol{Z}\langle x\rangle}((Z_{\langle x\rangle})_2, \boldsymbol{Z}) \xrightarrow{\sim} \boldsymbol{Z}$. So we have

$$\begin{aligned}y^*(\lambda) &= \beta_2 \cdot (v_{\langle x\rangle})_2^{\#} \cdot \tilde{y} \cdot (u_{\langle x\rangle})_2^{\#} \cdot \beta_2^{-1}(\lambda) \\ &= \beta_2^{-1}(\lambda) \cdot (u_{\langle x\rangle})_2 \left(\sum_{i=0}^{3} [x^i|x^{-1}]\right) \\ &= -\lambda.\end{aligned}$$

Similarly we calculate the image of μ by $x^* : H^2(\langle y \rangle, \boldsymbol{Z}) \to H^2(\langle y \rangle, \boldsymbol{Z})$ and the image of ν by $x^* : H^2(\langle xy \rangle, \boldsymbol{Z}) \to H^2(\langle xy \rangle, \boldsymbol{Z})$:

$$\begin{aligned} x^*(\mu) &= \beta_2' \cdot (v_{\langle y \rangle})_2^\# \cdot \tilde{x} \cdot (u_{\langle y \rangle})_2^\# \cdot (\beta_2')^{-1}(\mu) \\ &= (\beta_2')^{-1}(\mu) \cdot (u_{\langle y \rangle})_2 \left(\sum_{i=0}^{3} [y^i | x^2 y] \right) \\ &= -\mu, \\ x^*(\nu) &= \beta_2'' \cdot (v_{\langle xy \rangle})_2^\# \cdot \tilde{x} \cdot (u_{\langle xy \rangle})_2^\# \cdot (\beta_2'')^{-1}(\nu) \\ &= (\beta_2'')^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2 \left(\sum_{i=0}^{3} [(xy)^i | x^{-1} y] \right) \\ &= -\nu, \end{aligned}$$

where β_2' and β_2'' are the isomorphisms $\mathrm{Hom}_{\boldsymbol{Z}\langle y \rangle}((Z_{\langle y \rangle})_2, \boldsymbol{Z}) \xrightarrow{\sim} \boldsymbol{Z}$ and $\mathrm{Hom}_{\boldsymbol{Z}\langle xy \rangle}((Z_{\langle xy \rangle})_2, \boldsymbol{Z}) \xrightarrow{\sim} \boldsymbol{Z}$, respectively. The other equation is easily obtained. This completes the proof. □

Next, we calculate restriction maps.

LEMMA 2.4. *Let (Y, δ) be the periodic resolution of Q_2 and (Z_H, ∂_H) the periodic resolution of a cyclic subgroup H of Q_2.*

(i) *A chain transformation $(w_{\langle x \rangle})_n : (Z_{\langle x \rangle})_n \to Y_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$\begin{aligned} (w_{\langle x \rangle})_{4k}(1) &= 1; \\ (w_{\langle x \rangle})_{4k+1}(1) &= (1, 0); \\ (w_{\langle x \rangle})_{4k+2}(1) &= (1 - x^2 y, (x+1)xy); \\ (w_{\langle x \rangle})_{4k+3}(1) &= yx + 1 \quad \text{for } k \geq 0. \end{aligned}$$

(ii) *A chain transformation $(w_{\langle y \rangle})_n : (Z_{\langle y \rangle})_n \to Y_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$\begin{aligned} (w_{\langle y \rangle})_{4k}(1) &= 1; \\ (w_{\langle y \rangle})_{4k+1}(1) &= (0, 1); \end{aligned}$$

$$(w_{\langle y\rangle})_{4k+2}(1) = (-x^2(y+1), (x+1)xy);$$

$$(w_{\langle y\rangle})_{4k+3}(1) = x+1 \quad \textit{for } k \geq 0.$$

(iii) *A chain transformation* $(w_{\langle xy\rangle})_n : (Z_{\langle xy\rangle})_n \to Y_n$ *lifting the identity map on* $\boldsymbol{Z}$ *is given as follows*:

$$(w_{\langle xy\rangle})_{4k}(1) = 1;$$

$$(w_{\langle xy\rangle})_{4k+1}(1) = (1, x);$$

$$(w_{\langle xy\rangle})_{4k+2}(1) = (-x^2(y+1), (x+1)xy + x^2 + 1);$$

$$(w_{\langle xy\rangle})_{4k+3}(1) = x + yx \quad \textit{for } k \geq 0.$$

PROOF. We prove (i) only. The proof of (ii) and (iii) are done similarly. In this proof, we set $(\partial_{\langle x\rangle})_{2k+i} = (\partial_{\langle x\rangle})_i$ for any integer $k \geq 0$ and $i = 1, 2$, since $(Z_{\langle x\rangle}, \partial_{\langle x\rangle})$ is periodic of period 2. It suffices to show that the equation $\delta_n \cdot (w_{\langle x\rangle})_n = (w_{\langle x\rangle})_{n-1} \cdot (\partial_{\langle x\rangle})_n$ holds for $n = 1, 2, 3, 4$. In the case $n = 1$, we have

$$\delta_1 \cdot (w_{\langle x\rangle})_1(1) = \delta_1(1, 0) = x - 1 = (w_{\langle x\rangle})_0(x-1) = (w_{\langle x\rangle})_0 \cdot (\partial_{\langle x\rangle})_1(1).$$

In the case $n = 2$, we have

$$\begin{aligned}
&\delta_2 \cdot (w_{\langle x\rangle})_2(1) \\
&\quad = ((1 - x^2y)(x+1) + (x^2y + xy)(xy+1), \\
&\qquad (x^2y - 1)(y+1) + (x^2y + xy)(x-1)) \\
&\quad = (N_{\langle x\rangle}, 0) \\
&\quad = (w_{\langle x\rangle})_1 \cdot (\partial_{\langle x\rangle})_2(1),
\end{aligned}$$

where $N_{\langle x\rangle} = \sum_{i=0}^{3} x^i$. In the case $n = 3$, we have

$$\begin{aligned}
\delta_3 \cdot (w_{\langle x\rangle})_3(1) &= ((yx+1)(x-1), (yx+1)(1-xy)) \\
&= ((x-1)(1-x^2y), (x-1)(x^2+x)y) \\
&= (w_{\langle x\rangle})_2 \cdot (\partial_{\langle x\rangle})_3(1).
\end{aligned}$$

In the case $n = 4$, we have

$$\delta_4 \cdot (w_{\langle x\rangle})_4(1) = N = N_{\langle x\rangle} \cdot (yx+1) = (w_{\langle x\rangle})_3 \cdot (\partial_{\langle x\rangle})_4(1),$$

where $N = \sum_{\tau \in Q_2} \tau$. This completes the proof. □

LEMMA 2.5. *Let $H = \langle a \rangle$ be the cyclic group of order $2m$ for any positive integer $m \geq 2$. Let (Z_H, ∂_H) denote the periodic resolution of H and let $(Z_{\langle a^2 \rangle}, \partial_{\langle a^2 \rangle})$ denote the periodic resolution for the subgroup $\langle a^2 \rangle$ of order m. Then a chain transformation $s_n : (Z_{\langle a^2 \rangle})_n \to (Z_H)_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows*:

$$s_{2k}(1) = 1;$$

$$s_{2k+1}(1) = a + 1 \quad \text{for } k \geq 0.$$

PROOF. It suffices to show that the equation $(\partial_H)_n \cdot s_n = s_{n-1} \cdot (\partial_{\langle a^2 \rangle})_n$ holds for $n = 1, 2$. In the case $n = 1$, we have

$$(\partial_H)_1 \cdot s_1(1) = (\partial_H)_1(a+1) = a^2 - 1 = s_0(a^2 - 1) = s_0 \cdot (\partial_{\langle a^2 \rangle})_1(1).$$

In the case $n = 2$, we have

$$(\partial_H)_2 \cdot s_2(1) = (\partial_H)_2(1) = N_H = s_1(N_{\langle a^2 \rangle}) = s_1 \cdot (\partial_{\langle a^2 \rangle})_2(1),$$

where $N_H = \sum_{i=0}^{2m-1} a^i$ and $N_{\langle a^2 \rangle} = \sum_{i=0}^{m-1} a^{2i}$. □

PROPOSITION 2.6. *The following hold*:

(i) $\mathrm{res}^{Q_2}_{\langle x \rangle} A = 0$, $\mathrm{res}^{Q_2}_{\langle x \rangle} B = 2\lambda$, $\mathrm{res}^{Q_2}_{\langle x \rangle} C^k = \lambda^{2k}$ *for* $k \geq 0$.
(ii) $\mathrm{res}^{Q_2}_{\langle y \rangle} A = \mathrm{res}^{Q_2}_{\langle y \rangle} B = 2\mu$, $\mathrm{res}^{Q_2}_{\langle y \rangle} C^k = \mu^{2k}$ *for* $k \geq 0$.
(iii) $\mathrm{res}^{Q_2}_{\langle xy \rangle} A = 2\nu$, $\mathrm{res}^{Q_2}_{\langle xy \rangle} B = 0$, $\mathrm{res}^{Q_2}_{\langle xy \rangle} C^k = \nu^{2k}$ *for* $k \geq 0$.
(iv) $\mathrm{res}^{\langle x \rangle}_{\langle x^2 \rangle} \lambda^k = \mathrm{res}^{\langle y \rangle}_{\langle x^2 \rangle} \mu^k = \mathrm{res}^{\langle xy \rangle}_{\langle x^2 \rangle} \nu^k = \sigma^k$ *for* $k \geq 0$.

PROOF. We prove (i) only. First we calculate $\mathrm{res}^{Q_2}_{\langle x \rangle} A$ and $\mathrm{res}^{Q_2}_{\langle x \rangle} B$. Using Lemma 2.4 (i), these are given by the composition of the following maps:

$$\begin{aligned} \boldsymbol{Z} \oplus \boldsymbol{Z} &\xrightarrow{\alpha_2^{-1}} \mathrm{Hom}_\Lambda(Y_2, \boldsymbol{Z}) \\ &\xrightarrow{(w_{\langle x \rangle})_2^\#} \mathrm{Hom}_{\boldsymbol{Z}\langle x \rangle}((Z_{\langle x \rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{\beta_2} \boldsymbol{Z}, \end{aligned}$$

where α_2 denotes the isomorphism $\mathrm{Hom}_\Lambda(Y_2, \boldsymbol{Z}) \xrightarrow{\sim} \boldsymbol{Z} \oplus \boldsymbol{Z}$ and β_2 denotes the isomorphism $\mathrm{Hom}_{\boldsymbol{Z}\langle x \rangle}((Z_{\langle x \rangle})_2, \boldsymbol{Z}) \xrightarrow{\sim} \boldsymbol{Z}$ stated in the proof of Proposition 2.3. Thus we obtain

$$\begin{aligned}\operatorname{res}^{Q_2}_{\langle x\rangle} A &= \beta_2 \cdot (w_{\langle x\rangle})_2^{\#} \cdot \alpha_2^{-1}(A)\\ &= (\alpha_2^{-1}(A)\cdot (w_{\langle x\rangle})_2)(1)\\ &= (\alpha_2^{-1}(A))(1 - x^2 y, (x^2+x)y)\\ &= 0,\\ \operatorname{res}^{Q_2}_{\langle x\rangle} B &= (\alpha_2^{-1}(B))(1 - x^2 y, (x^2+x)y)\\ &= 2\lambda.\end{aligned}$$

Next we calculate $\operatorname{res}^{Q_2}_{\langle x\rangle} C^k$ for $k \geq 0$. This is given by the composition of the following maps:

$$\begin{aligned} \boldsymbol{Z} &\xrightarrow{\alpha_{4k}^{-1}} \operatorname{Hom}_\Lambda(Y_{4k}, \boldsymbol{Z})\\ &\xrightarrow{(w_{\langle x\rangle})_{4k}^{\#}} \operatorname{Hom}_{\boldsymbol{Z}\langle x\rangle}((Z_{\langle x\rangle})_{4k}, \boldsymbol{Z})\\ &\xrightarrow{\beta_{4k}} \boldsymbol{Z},\end{aligned}$$

where α_{4k} denotes the isomorphism $\operatorname{Hom}_\Lambda(Y_{4k}, \boldsymbol{Z}) \tilde{\to} \boldsymbol{Z}$ and β_{4k} denotes the isomorphism $\operatorname{Hom}_{\boldsymbol{Z}\langle x\rangle}((Z_{\langle x\rangle})_{4k}, \boldsymbol{Z}) \tilde{\to} \boldsymbol{Z}$. Then we have $\operatorname{res}^{Q_2}_{\langle x\rangle} C^k = (\alpha_{4k}^{-1}(C^k))(1) = \lambda^{2k}$.

The other computations are similar. These are given by using Lemma 2.4 (ii), (iii) and Lemma 2.5. □

Finally, we calculate corestriction maps.

LEMMA 2.7. *Let (X, d) be the standard resolution of Q_2. An initial part of chain transformation $v_n : Y_n \to X_n$ lifting the identity map on $\boldsymbol{Z}$ is given as follows*:

$$\begin{aligned}&v_0(1) = [\cdot];\\ &v_1(1,0) = [x], \quad v_1(0,1) = [y];\\ &v_2(1,0) = [x|x] - [y|y], \quad v_2(0,1) = [x|y] + [xy|x].\end{aligned}$$

PROOF. See [5, Proposition 1]. □

LEMMA 2.8. *Suppose H is a subgroup of index m of a finite group G and A is a G-module. Fix a set of right coset representatives $S = \{\tau_1(=1), \tau_2, \dots, \tau_m\}$ of H in G, and let $c(\sigma)$ $(\in S)$ denote the representative of the right coset containing $\sigma \in G$. (X_G, d_G) and (X_H, d_H) denote the standard resolutions of G and H, re-*

spectively. Then the corestriction map $\mathrm{cor}_H^G : H^n(H, A) \to H^n(G, A)$ *is given by the following on the cochain level*:

$$T_H^G : \mathrm{Hom}_{ZH}((X_H)_n, A) \to \mathrm{Hom}_{ZG}((X_G)_n, A)$$

$$(T_H^G(u))([\cdot]) = \sum_{\tau \in S} \tau^{-1} u([\cdot]) \quad (n = 0),$$

$$\begin{aligned}(T_H^G(u))([\sigma_1|\sigma_2|\dots|\sigma_n]) \\ = \sum_{\tau \in S} \tau^{-1} u([c(\tau)\sigma_1 c(\tau\sigma_1)^{-1} | c(\tau\sigma_1)\sigma_2 c(\tau\sigma_1\sigma_2)^{-1} | \dots \\ | c(\tau\sigma_1 \cdots \sigma_{n-1})\sigma_n c(\tau\sigma_1 \cdots \sigma_n)^{-1}]) \quad (n \geq 1),\end{aligned}$$

where $u \in \mathrm{Hom}_{ZH}((X_H)_n, A)$ *and* $\sigma_1, \sigma_2, \dots, \sigma_n \in G$.

PROOF. See [13, Proposition 2-5-2]. □

PROPOSITION 2.9. *The following hold*:

(i) $\mathrm{cor}_{\langle x \rangle}^{Q_2} \lambda = A$, $\mathrm{cor}_{\langle x \rangle}^{Q_2} \lambda^{2k} = 2C^k$ *for* $k \geq 0$.
(ii) $\mathrm{cor}_{\langle y \rangle}^{Q_2} \mu = A + B$, $\mathrm{cor}_{\langle y \rangle}^{Q_2} \mu^{2k} = 2C^k$ *for* $k \geq 0$.
(iii) $\mathrm{cor}_{\langle xy \rangle}^{Q_2} \nu = B$, $\mathrm{cor}_{\langle xy \rangle}^{Q_2} \nu^{2k} = 2C^k$ *for* $k \geq 0$.
(iv) $\mathrm{cor}_{\langle x^2 \rangle}^{\langle x \rangle} \sigma^k = 2\lambda^k$, $\mathrm{cor}_{\langle x^2 \rangle}^{\langle y \rangle} \sigma^k = 2\mu^k$, $\mathrm{cor}_{\langle x^2 \rangle}^{\langle xy \rangle} \sigma^k = 2\nu^k$ *for* $k \geq 0$.

PROOF. In this proof, we calculate $\mathrm{cor}_{\langle x \rangle}^{Q_2} \lambda$, $\mathrm{cor}_{\langle y \rangle}^{Q_2} \mu$ and $\mathrm{cor}_{\langle xy \rangle}^{Q_2} \nu$ only. The other equations are easily obtained by (1.1) and Proposition 2.6. For example,

$$\mathrm{cor}_{\langle x \rangle}^{Q_2} \lambda^{2k} = \mathrm{cor}_{\langle x \rangle}^{Q_2} \cdot \mathrm{res}_{\langle x \rangle}^{Q_2} C^k = |Q_2 : \langle x \rangle| C^k = 2C^k, \quad \text{for } k \geq 0.$$

First we calculate $\mathrm{cor}_{\langle x \rangle}^{Q_2} \lambda$. Using Lemmas 2.1, 2.7 and 2.8, this is given by the composition of the following maps:

$$\begin{aligned} \boldsymbol{Z} &\xrightarrow{\beta_2^{-1}} \mathrm{Hom}_{\boldsymbol{Z}\langle x \rangle}((\boldsymbol{Z}_{\langle x \rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{(u_{\langle x \rangle})_2^{\#}} \mathrm{Hom}_{\boldsymbol{Z}\langle x \rangle}((X_{\langle x \rangle})_2, \boldsymbol{Z}) \\ &\xrightarrow{T_{\langle x \rangle}^{Q_2}} \mathrm{Hom}_{\Lambda}(X_2, \boldsymbol{Z}) \\ &\xrightarrow{v_2^{\#}} \mathrm{Hom}_{\Lambda}(Y_2, \boldsymbol{Z}) \\ &\xrightarrow{\alpha_2} \boldsymbol{Z} \oplus \boldsymbol{Z}. \end{aligned}$$

Let $\{1, y\}$ be a set of right coset representatives of $\langle x\rangle$ in Q_2. Then $c(x^i) = 1$ and $c(x^i y) = y$ hold for $0 \le i \le 3$. Since

$$\begin{aligned}
&(T_{\langle x\rangle}^{Q_2}(\beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2))(v_2(1,0)) \\
&\quad = (T_{\langle x\rangle}^{Q_2}(\beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2))([x|x]-[y|y]) \\
&\quad = \beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2([c(1)xc(x)^{-1}|c(x)xc(x^2)^{-1}]+[c(y)xc(yx)^{-1}|c(yx)xc(yx^2)^{-1}] \\
&\qquad\qquad - [c(1)yc(y)^{-1}|c(y)yc(x^2)^{-1}] - [c(y)yc(x^2)^{-1}|c(x^2)yc(x^2y)^{-1}]) \\
&\quad = \beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2([x|x]+[x^3|x^3]-[1|x^2]-[x^2|1]) \\
&\quad = 1, \\
&(T_{\langle x\rangle}^{Q_2}(\beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2))(v_2(0,1)) \\
&\quad = (T_{\langle x\rangle}^{Q_2}(\beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2))([x|y]+[xy|x]) \\
&\quad = \beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2([c(1)xc(x)^{-1}|c(x)yc(xy)^{-1}]+[c(y)xc(yx)^{-1}|c(yx)yc(x)^{-1}] \\
&\qquad\qquad + [c(1)xyc(xy)^{-1}|c(xy)xc(y)^{-1}] + [c(y)xyc(x)^{-1}|c(x)xc(x^2)^{-1}]) \\
&\quad = \beta_2^{-1}(\lambda)\cdot(u_{\langle x\rangle})_2([x|1]+[x^3|x^2]+[x|x^3]+[x|x]) \\
&\quad = 2,
\end{aligned}$$

we have $\mathrm{cor}_{\langle x\rangle}^{Q_2}\lambda = A + 2B = A$.

Next let $\{1, x\}$ be a set of right coset representatives of $\langle y\rangle$ in Q_2. Then $c(x^i y^j) = 1$ $(i = 0, 2)$ and $c(x^i y^j) = x$ $(i = 1, 3)$ hold for $j = 0, 1$. Since

$$\begin{aligned}
&(T_{\langle y\rangle}^{Q_2}(\beta_2'^{-1}(\mu)\cdot(u_{\langle y\rangle})_2))(v_2(1,0)) \\
&\quad = \beta_2'^{-1}(\mu)\cdot(u_{\langle y\rangle})_2([c(1)xc(x)^{-1}|c(x)xc(x^2)^{-1}]+[c(x)xc(x^2)^{-1}|c(x^2)xc(x^3)^{-1}] \\
&\qquad\qquad - [c(1)yc(y)^{-1}|c(y)yc(x^2)^{-1}] - [c(x)yc(xy)^{-1}|c(xy)yc(x^3)^{-1}]) \\
&\quad = \beta_2'^{-1}(\mu)\cdot(u_{\langle y\rangle})_2([1|x^2]+[x^2|1]-[y|y]-[x^2y|x^2y]) \\
&\quad = -1, \\
&(T_{\langle y\rangle}^{Q_2}(\beta_2'^{-1}(\mu)\cdot(u_{\langle y\rangle})_2))(v_2(0,1)) \\
&\quad = \beta_2'^{-1}(\mu)\cdot(u_{\langle y\rangle})_2([c(1)xc(x)^{-1}|c(x)yc(xy)^{-1}]+[c(x)xc(x^2)^{-1}|c(x^2)yc(x^2y)^{-1}] \\
&\qquad\qquad + [c(1)xyc(xy)^{-1}|c(xy)xc(y)^{-1}] + [c(x)xyc(x^2y)^{-1}|c(x^2y)xc(xy)^{-1}])
\end{aligned}$$

$$= \beta_2'^{-1}(\mu) \cdot (u_{\langle y \rangle})_2([1|x^2y] + [x^2|y] + [x^2y|x^2] + [x^2y|1])$$
$$= 1,$$

we have $\mathrm{cor}_{\langle y \rangle}^{Q_2} \mu = A + B$.

Finally let $\{1, x\}$ be a set of right coset representatives of $\langle xy \rangle$ in Q_2. Then $c(x^{i-1}) = c(x^i y) = 1$ $(i = 1, 3)$ and $c(x^{i+1}) = c(x^i y) = x$ $(i = 0, 2)$ hold. Since

$$(T_{\langle xy \rangle}^{Q_2}(\beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2))(v_2(1,0))$$
$$= \beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2([c(1)xc(x)^{-1}|c(x)xc(x^2)^{-1}] + [c(x)xc(x^2)^{-1}|c(x^2)xc(x^3)^{-1}]$$
$$- [c(1)yc(y)^{-1}|c(y)yc(x^2)^{-1}] - [c(x)yc(xy)^{-1}|c(xy)yc(x^3)^{-1}])$$
$$= \beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2([1|x^2] + [x^2|1] - [xy|xy] - [xy|xy])$$
$$= 0,$$

$$(T_{\langle xy \rangle}^{Q_2}(\beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2))(v_2(0,1))$$
$$= \beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2([c(1)xc(x)^{-1}|c(x)yc(xy)^{-1}] + [c(x)xc(x^2)^{-1}|c(x^2)yc(x^2y)^{-1}]$$
$$+ [c(1)xyc(xy)^{-1}|c(xy)xc(y)^{-1}] + [c(x)xyc(x^2y)^{-1}|c(x^2y)xc(xy)^{-1}])$$
$$= \beta_2''^{-1}(\nu) \cdot (u_{\langle xy \rangle})_2([1|xy] + [x^2|xy] + [xy|1] + [x^3y|x^2])$$
$$= 1,$$

we have $\mathrm{cor}_{\langle xy \rangle}^{Q_2} \nu = B$. This completes the proof. □

2.3 Products on $H^*(Q_2, {}_\psi \boldsymbol{Z}Q_2)(\simeq HH^*(\boldsymbol{Z}Q_2))$

In this subsection, we calculate the products on the Hochschild cohomology $H^*(Q_2, {}_\psi \boldsymbol{Z}Q_2)(\simeq HH^*(\boldsymbol{Z}Q_2))$ using the Product Formula (see Lemma 1.1). In the following, we write XY in place of $X \smile Y$ for brevity.

Proposition 2.10. *The following equations hold in $H^0(Q_2, {}_\psi \boldsymbol{Z}Q_2)$ for the generators of $H^0(Q_2, {}_\psi \boldsymbol{Z}Q_2)$:*

$$\gamma_2(1)^2 = 1, \quad \gamma_2(1)\gamma_r(1) = \gamma_r(1), \quad \gamma_r(1)^2 = 2(1 + \gamma_2(1)) \quad (\text{for } r = 3, 4, 5),$$
$$\gamma_3(1)\gamma_4(1) = 2\gamma_5(1), \quad \gamma_3(1)\gamma_5(1) = 2\gamma_4(1), \quad \gamma_4(1)\gamma_5(1) = 2\gamma_3(1).$$

Proof. Note that the relations of degree 0 correspond to the multiplication in the center of $\boldsymbol{Z}Q_2$. So we obtain, for example,

$$\gamma_2(1)\gamma_3(1) = x^2(x + x^{-1}) = x + x^{-1} = \gamma_3(1).$$ □

Next, we compute the cup product of the generators of $H^0(Q_2, {}_\psi ZQ_2)$ and the generators of $H^2(Q_2, {}_\psi ZQ_2)$. The following Table 1 is useful for Propositions 2.11 and 2.12.

Table 1. Data for the Product Formula

i	j	a	$g_i{}^a g_j$	b	k	${}^bG_i$	${}^{ba}G_j$	$W = {}^bG_i \cap {}^{ba}G_j$
1	r $(1 \le r \le 5)$	1	g_r	1	r	Q_2	G_r	G_r
2	2	1	1	1	1	Q_2	Q_2	Q_2
2	3	1	x^{-1}	y	3	Q_2	$\langle x\rangle$	$\langle x\rangle$
2	4	1	x^2y	x	4	Q_2	$\langle y\rangle$	$\langle y\rangle$
2	5	1	$x^{-1}y$	x	5	Q_2	$\langle xy\rangle$	$\langle xy\rangle$
3	3	1 y	x^2 1	1 1	2 1	$\langle x\rangle$ $\langle x\rangle$	$\langle x\rangle$ $\langle x\rangle$	$\langle x\rangle$ $\langle x\rangle$
3	4	1	xy	1	5	$\langle x\rangle$	$\langle y\rangle$	$\langle x^2\rangle$
3	5	1	x^2y	x	4	$\langle x\rangle$	$\langle xy\rangle$	$\langle x^2\rangle$
4	4	1 x	x^2 1	1 1	2 1	$\langle y\rangle$ $\langle y\rangle$	$\langle y\rangle$ $\langle y\rangle$	$\langle y\rangle$ $\langle y\rangle$
4	5	1	x	1	3	$\langle y\rangle$	$\langle xy\rangle$	$\langle x^2\rangle$
5	5	1 x	x^2 1	1 1	2 1	$\langle xy\rangle$ $\langle xy\rangle$	$\langle xy\rangle$ $\langle xy\rangle$	$\langle xy\rangle$ $\langle xy\rangle$

PROPOSITION 2.11. *The following equations hold in* $H^2(Q_2, {}_\psi ZQ_2)$ *for the generators of* $H^0(Q_2, {}_\psi ZQ_2)$ *and the generators of* $H^2(Q_2, {}_\psi ZQ_2)$:

(i) $\gamma_2(1)\gamma_1(A) = \gamma_2(A)$, $\gamma_2(1)\gamma_1(B) = \gamma_2(B)$, $\gamma_2(1)\gamma_3(\lambda) = -\gamma_3(\lambda)$, $\gamma_2(1)\gamma_4(\mu) = -\gamma_4(\mu)$, $\gamma_2(1)\gamma_5(\nu) = -\gamma_5(\nu)$.

(ii) $\gamma_3(1)\gamma_1(A) = 0$, $\gamma_3(1)\gamma_1(B) = 2\gamma_3(\lambda)$, $\gamma_3(1)\gamma_3(\lambda) = \gamma_1(A)(1 + \gamma_2(1))$, $\gamma_3(1)\gamma_4(\mu) = 2\gamma_5(\nu)$, $\gamma_3(1)\gamma_5(\nu) = 2\gamma_4(\mu)$.

(iii) $\gamma_4(1)\gamma_1(A) = \gamma_4(1)\gamma_1(B) = 2\gamma_4(\mu)$, $\gamma_4(1)\gamma_3(\lambda) = 2\gamma_5(\nu)$, $\gamma_4(1)\gamma_4(\mu) = \gamma_1(A + B)(1 + \gamma_2(1))$, $\gamma_4(1)\gamma_5(\nu) = 2\gamma_3(\lambda)$.

(iv) $\gamma_5(1)\gamma_1(A) = 2\gamma_5(\nu)$, $\gamma_5(1)\gamma_1(B) = 0$, $\gamma_5(1)\gamma_3(\lambda) = 2\gamma_4(\mu)$, $\gamma_5(1)\gamma_4(\mu) = 2\gamma_3(\lambda)$, $\gamma_5(1)\gamma_5(\nu) = \gamma_1(B)(1 + \gamma_2(1))$.

In particular, $H^2(Q_2, {}_\psi ZQ_2)$ *is generated by the products of* $\gamma_2(1)$, $\gamma_1(A)$, $\gamma_1(B)$, $\gamma_3(\lambda)$, $\gamma_4(\mu)$ *and* $\gamma_5(\nu)$.

PROOF. These are obtained by using Lemma 1.1, Propositions 2.3, 2.6 and 2.9 and Table 1 as follows:

(i): The calculation of the products of $\gamma_2(1)$ and the generators of $H^2(Q_2, {}_{\psi}\boldsymbol{Z}Q_2)$:

$$\gamma_1(A)\gamma_2(1) = \gamma_2(\mathrm{cor}_{Q_2}^{Q_2}(\mathrm{res}_{Q_2}^{Q_2} A \cdot \mathrm{res}_{Q_2}^{Q_2} 1)) = \gamma_2(A),$$

$$\gamma_1(B)\gamma_2(1) = \gamma_2(\mathrm{cor}_{Q_2}^{Q_2}(\mathrm{res}_{Q_2}^{Q_2} B \cdot \mathrm{res}_{Q_2}^{Q_2} 1)) = \gamma_2(B),$$

$$\gamma_2(1)\gamma_3(\lambda) = \gamma_3(\mathrm{cor}_{\langle x\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x\rangle}^{Q_2} y^*(1) \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} y^*(\lambda))) = -\gamma_3(\lambda),$$

$$\gamma_2(1)\gamma_4(\mu) = \gamma_4(\mathrm{cor}_{\langle y\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle y\rangle}^{Q_2} x^*(1) \cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} x^*(\mu))) = -\gamma_4(\mu),$$

$$\gamma_2(1)\gamma_5(\nu) = \gamma_5(\mathrm{cor}_{\langle xy\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle xy\rangle}^{Q_2} x^*(1) \cdot \mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle} x^*(\nu))) = -\gamma_5(\nu).$$

(ii): The calculation of the products of $\gamma_3(1)$ and the generators of $H^2(Q_2, {}_{\psi}\boldsymbol{Z}Q_2)$:

$$\begin{aligned}
\gamma_1(A)\gamma_3(1) &= \gamma_3(\mathrm{cor}_{\langle x\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x\rangle}^{Q_2} A \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} 1)) \\
&= 0, \\
\gamma_1(B)\gamma_3(1) &= \gamma_3(\mathrm{cor}_{\langle x\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x\rangle}^{Q_2} B \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} 1)) \\
&= 2\gamma_3(\lambda), \\
\gamma_3(1)\gamma_3(\lambda) &= \gamma_2(\mathrm{cor}_{\langle x\rangle}^{Q_2}(\mathrm{res}_{\langle x\rangle}^{\langle x\rangle} 1 \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} \lambda)) + \gamma_1(\mathrm{cor}_{\langle x\rangle}^{Q_2}(\mathrm{res}_{\langle x\rangle}^{\langle x\rangle} 1 \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} y^*(\lambda))) \\
&= \gamma_2(A) - \gamma_1(A), \\
\gamma_3(1)\gamma_4(\mu) &= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} 1 \cdot \mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle} \mu)) \\
&= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle} \sigma) \\
&= 2\gamma_5(\nu), \\
\gamma_3(1)\gamma_5(\nu) &= \gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} x^*(1) \cdot \mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle} x^*(\nu))) \\
&= -\gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle} \sigma) \\
&= 2\gamma_4(\mu).
\end{aligned}$$

(iii): The calculation of the products of $\gamma_4(1)$ and the generators of $H^2(Q_2, {}_{\psi}\boldsymbol{Z}Q_2)$:

$$\begin{aligned}
\gamma_1(A)\gamma_4(1) &= \gamma_4(\mathrm{cor}_{\langle y\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle y\rangle}^{Q_2} A\cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} 1))\\
&= 2\gamma_4(\mu),\\
\gamma_1(B)\gamma_4(1) &= \gamma_4(\mathrm{cor}_{\langle y\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle y\rangle}^{Q_2} B\cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} 1))\\
&= 2\gamma_4(\mu),\\
\gamma_3(\lambda)\gamma_4(1) &= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} \lambda\cdot \mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle} 1))\\
&= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle} \sigma)\\
&= 2\gamma_5(\nu),\\
\gamma_4(1)\gamma_4(\mu) &= \gamma_2(\mathrm{cor}_{\langle y\rangle}^{Q_2}(\mathrm{res}_{\langle y\rangle}^{\langle y\rangle} 1\cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} \mu))\\
&\quad + \gamma_1(\mathrm{cor}_{\langle y\rangle}^{Q_2}(\mathrm{res}_{\langle y\rangle}^{\langle y\rangle} 1\cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} x^*(\mu)))\\
&= \gamma_2(\mathrm{cor}_{\langle y\rangle}^{Q_2} \mu) - \gamma_1(\mathrm{cor}_{\langle y\rangle}^{Q_2} \mu)\\
&= \gamma_2(A+B) - \gamma_1(A+B),\\
\gamma_4(1)\gamma_5(\nu) &= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle} 1\cdot \mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle} \nu))\\
&= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle} \sigma)\\
&= 2\gamma_3(\lambda).
\end{aligned}$$

(iv): The calculation of the products of $\gamma_5(1)$ and the generators of $H^2(Q_2, {}_{\psi}\boldsymbol{Z}Q_2)$:

$$\begin{aligned}
\gamma_1(A)\gamma_5(1) &= \gamma_5(\mathrm{cor}_{\langle xy\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle xy\rangle}^{Q_2} A\cdot \mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle} 1))\\
&= 2\gamma_5(\nu),\\
\gamma_1(B)\gamma_5(1) &= \gamma_5(\mathrm{cor}_{\langle xy\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle xy\rangle}^{Q_2} B\cdot \mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle} 1))\\
&= 0,\\
\gamma_3(\lambda)\gamma_5(1) &= \gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} x^*(\lambda)\cdot \mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle} x^*(1)))\\
&= \gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle} \sigma)\\
&= 2\gamma_4(\mu),
\end{aligned}$$

$$\begin{aligned}\gamma_4(\mu)\gamma_5(1) &= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle}\,\mu\cdot\mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle}\,1))\\ &= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle}\,\sigma)\\ &= 2\gamma_3(\lambda),\\ \gamma_5(1)\gamma_5(\nu) &= \gamma_2(\mathrm{cor}_{\langle xy\rangle}^{Q_2}(\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\,1\cdot\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\,\nu))+\gamma_1(\mathrm{cor}_{\langle xy\rangle}^{Q_2}(\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\,1\cdot\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\,x^*(\nu)))\\ &= \gamma_2(\mathrm{cor}_{\langle xy\rangle}^{Q_2}\,\nu)-\gamma_1(\mathrm{cor}_{\langle xy\rangle}^{Q_2}\,\nu)\\ &= \gamma_2(B)-\gamma_1(B).\end{aligned}$$

This completes the proof. □

REMARK 1. By the Product Formula, it is easy to see that the following equation holds:

$$\gamma_1(C^k)\gamma_r(\beta)=\gamma_r(\mathrm{cor}_{G_r}^{G_r}(\mathrm{res}_{G_r}^{Q_2}\,C^k\cdot\mathrm{res}_{G_r}^{G_r}\,\beta))=\gamma_r(\mathrm{res}_{G_r}^{Q_2}\,C^k\cdot\beta)$$

where $\beta\in H^*(G_r,\boldsymbol{Z})$, $1\le r\le 5$ and $k\ge 0$. Since the equations

$$\mathrm{res}_{\langle x\rangle}^{Q_2}\,C^k=\lambda^{2k},\quad \mathrm{res}_{\langle y\rangle}^{Q_2}\,C^k=\mu^{2k},\quad \mathrm{res}_{\langle xy\rangle}^{Q_2}\,C^k=\nu^{2k}\quad (k\ge 0)$$

hold by Proposition 2.6, the cup product with $\gamma_1(C)$ gives a periodicity isomorphism

$$\gamma_1(C)\smile - : H^n(Q_2,{}_{\psi}\boldsymbol{Z}Q_2)\xrightarrow{\sim} H^{n+4}(Q_2,{}_{\psi}\boldsymbol{Z}Q_2)$$

for all $n\ge 1$. In particular, we have

$$\gamma_2(C)=\gamma_1(C)\gamma_2(1),\quad \gamma_3(\lambda^2)=\gamma_1(C)\gamma_3(1),$$
$$\gamma_4(\mu^2)=\gamma_1(C)\gamma_4(1),\quad \gamma_5(\nu^2)=\gamma_1(C)\gamma_5(1).$$

Finally, we compute the relations in degree 4. These are obtained by a method similar to the above proposition.

PROPOSITION 2.12. *The following equations hold in $H^4(Q_2,{}_{\psi}\boldsymbol{Z}Q_2)$ for the generators of $H^2(Q_2,{}_{\psi}\boldsymbol{Z}Q_2)$ and the generators of $H^4(Q_2,{}_{\psi}\boldsymbol{Z}Q_2)$:*

(i) $\gamma_1(A)^2=\gamma_1(A)\gamma_3(\lambda)=0$, $\gamma_1(A)\gamma_1(B)=4\gamma_1(C)$, $\gamma_1(A)\gamma_4(\mu)=2\gamma_1(C)\gamma_4(1)$, $\gamma_1(A)\gamma_5(\nu)=2\gamma_1(C)\gamma_5(1)$.

(ii) $\gamma_1(B)^2=\gamma_1(B)\gamma_5(\nu)=0$, $\gamma_1(B)\gamma_3(\lambda)=2\gamma_1(C)\gamma_3(1)$, $\gamma_1(B)\gamma_4(\mu)=2\gamma_1(C)\gamma_4(1)$.

(iii) $\gamma_3(\lambda)^2=2\gamma_1(C)(\gamma_2(1)-1)$, $\gamma_3(\lambda)\gamma_4(\mu)=2\gamma_1(C)\gamma_5(1)$, $\gamma_3(\lambda)\gamma_5(\nu)=2\gamma_1(C)\gamma_4(1)$.

(iv) $\gamma_4(\mu)^2 = 2\gamma_1(C)(\gamma_2(1) - 1)$, $\gamma_4(\mu)\gamma_5(\nu) = 2\gamma_1(C)\gamma_3(1)$.
(v) $\gamma_5(\nu)^2 = 2\gamma_1(C)(\gamma_2(1) - 1)$.

PROOF. These are obtained by using Lemma 1.1, Propositions 2.3, 2.6 and 2.9 and Table 1 as follows:

(i) and (ii): First, note that γ_1 is a monomorphism between the cohomology rings (see [11, Section 5]). Then by Section 2.1 the equations $\gamma_1(A)^2 = \gamma_1(B)^2 = 0$ and $\gamma_1(A)\gamma_1(B) = 4\gamma_1(C)$ hold. The other computations are as follows:

$$\gamma_1(A)\gamma_3(\lambda) = \gamma_3(\mathrm{cor}_{\langle x\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x\rangle}^{Q_2} A \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} \lambda)) = 0,$$

$$\gamma_1(A)\gamma_4(\mu) = \gamma_4(\mathrm{cor}_{\langle y\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle y\rangle}^{Q_2} A \cdot \mathrm{res}_{\langle y\rangle}^{\langle y\rangle} \mu)) = \gamma_4(2\mu^2) = 2\gamma_1(C)\gamma_4(1),$$

$$\gamma_1(A)\gamma_5(\nu) = \gamma_5(\mathrm{cor}_{\langle xy\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle xy\rangle}^{Q_2} A \cdot \mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle} \nu)) = \gamma_5(2\nu^2) = 2\gamma_1(C)\gamma_5(1),$$

$$\gamma_1(B)\gamma_3(\lambda) = \gamma_3(\mathrm{res}_{\langle x\rangle}^{Q_2} B \cdot \lambda) = \gamma_3(2\lambda^2) = 2\gamma_1(C)\gamma_3(1),$$

$$\gamma_1(B)\gamma_4(\mu) = \gamma_4(\mathrm{res}_{\langle y\rangle}^{Q_2} B \cdot \mu) = \gamma_4(2\mu^2) = 2\gamma_1(C)\gamma_4(1),$$

$$\gamma_1(B)\gamma_5(\nu) = \gamma_5(\mathrm{res}_{\langle xy\rangle}^{Q_2} B \cdot \nu) = 0.$$

(iii): The calculation of the products of $\gamma_3(\lambda)$ and the generators of $H^2(Q_2, {}_{\psi}\mathbf{Z}Q_2)$:

$$\begin{aligned}
\gamma_3(\lambda)^2 &= \gamma_2(\mathrm{cor}_{\langle x\rangle}^{Q_2}(\mathrm{res}_{\langle x\rangle}^{\langle x\rangle} \lambda \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} \lambda)) + \gamma_1(\mathrm{cor}_{\langle x\rangle}^{Q_2}(\mathrm{res}_{\langle x\rangle}^{\langle x\rangle} \lambda \cdot \mathrm{res}_{\langle x\rangle}^{\langle x\rangle} y^*(\lambda)))\\
&= \gamma_2(\mathrm{cor}_{\langle x\rangle}^{Q_2} \lambda^2) - \gamma_1(\mathrm{cor}_{\langle x\rangle}^{Q_2} \lambda^2)\\
&= 2\gamma_1(C)(\gamma_2(1) - 1),\\
\gamma_3(\lambda)\gamma_4(\mu) &= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} \lambda \cdot \mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle} \mu))\\
&= \gamma_5(\mathrm{cor}_{\langle x^2\rangle}^{\langle xy\rangle} \sigma^2)\\
&= 2\gamma_1(C)\gamma_5(1),\\
\gamma_3(\lambda)\gamma_5(\nu) &= \gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle x\rangle} x^*(\lambda) \cdot \mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle} x^*(\nu)))\\
&= -\gamma_4(\mathrm{cor}_{\langle x^2\rangle}^{\langle y\rangle} \sigma^2)\\
&= 2\gamma_1(C)\gamma_4(1).
\end{aligned}$$

(iv): The calculation of $\gamma_4(\mu)^2$ and $\gamma_4(\mu)\gamma_5(\nu)$:

$$\begin{aligned}\gamma_4(\mu)^2 &= \gamma_2(\mathrm{cor}_{\langle y\rangle}^{Q_2}(\mathrm{res}_{\langle y\rangle}^{\langle y\rangle}\mu\cdot\mathrm{res}_{\langle y\rangle}^{\langle y\rangle}\mu)) + \gamma_1(\mathrm{cor}_{\langle y\rangle}^{Q_2}(\mathrm{res}_{\langle y\rangle}^{\langle y\rangle}\mu\cdot\mathrm{res}_{\langle y\rangle}^{\langle y\rangle}x^*(\mu)))\\ &= \gamma_2(\mathrm{cor}_{\langle y\rangle}^{Q_2}\mu^2) - \gamma_1(\mathrm{cor}_{\langle y\rangle}^{Q_2}\mu^2)\\ &= 2\gamma_1(C)(\gamma_2(1)-1),\\ \gamma_4(\mu)\gamma_5(\nu) &= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x^2\rangle}^{\langle y\rangle}\mu\cdot\mathrm{res}_{\langle x^2\rangle}^{\langle xy\rangle}\nu))\\ &= \gamma_3(\mathrm{cor}_{\langle x^2\rangle}^{\langle x\rangle}\sigma^2)\\ &= 2\gamma_1(C)\gamma_3(1).\end{aligned}$$

(v): Finally, we calculate $\gamma_5(\nu)^2$:

$$\begin{aligned}\gamma_5(\nu)^2 &= \gamma_2(\mathrm{cor}_{\langle xy\rangle}^{Q_2}(\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\nu\cdot\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\nu)) + \gamma_1(\mathrm{cor}_{\langle xy\rangle}^{Q_2}(\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}\nu\cdot\mathrm{res}_{\langle xy\rangle}^{\langle xy\rangle}x^*(\nu)))\\ &= \gamma_2(\mathrm{cor}_{\langle xy\rangle}^{Q_2}\nu^2) - \gamma_1(\mathrm{cor}_{\langle xy\rangle}^{Q_2}\nu^2)\\ &= 2\gamma_1(C)(\gamma_2(1)-1).\end{aligned}$$

This completes the proof. □

We will state the ring structure of $H^*(Q_2, {}_\psi\boldsymbol{Z}Q_2)$ by summarizing Propositions 2.10 through 2.12 and Remark 1.

THEOREM. *The Hochschild cohomology ring* $H^*(Q_2, {}_\psi\boldsymbol{Z}Q_2)(\simeq HH^*(\boldsymbol{Z}Q_2))$ *is commutative, generated by the elements*

$$\gamma_1(1) = 1, \quad \gamma_2(1), \gamma_3(1), \gamma_4(1), \gamma_5(1) \in H^0(Q_2, {}_\psi\boldsymbol{Z}Q_2),$$
$$\gamma_1(A), \gamma_1(B), \gamma_3(\lambda), \gamma_4(\mu), \gamma_5(\nu) \in H^2(Q_2, {}_\psi\boldsymbol{Z}Q_2),$$
$$\gamma_1(C) \in H^4(Q_2, {}_\psi\boldsymbol{Z}Q_2).$$

The relations are given by Table 2.

REMARK 2. If we put

$$A_0 = 1, \quad B_0 = \gamma_2(1), \quad (C_1)_0 = \gamma_3(1), \quad D_0 = \gamma_4(1), \quad E_0 = \gamma_5(1);$$
$$(A_\alpha)_2 = \gamma_1(A), \quad (A_\beta)_2 = \gamma_1(B), \quad (B_\alpha)_2 = \gamma_1(A)\gamma_2(1),$$
$$(B_\beta)_2 = \gamma_1(B)\gamma_2(1), \quad (C_1)_2 = \gamma_3(\lambda), \quad D_2 = \gamma_4(\mu), \quad E_2 = \gamma_5(\nu);$$
$$A_4 = \gamma_1(C),$$

Table 2. Cohomology ring $H^*(Q_{2,\psi}\boldsymbol{Z}Q_2)(\simeq HH^*(\boldsymbol{Z}Q_2))$

	$\gamma_2(1)$	$\gamma_3(1)$	$\gamma_4(1)$	$\gamma_5(1)$	$\gamma_1(A)$	$\gamma_1(B)$	$\gamma_3(\lambda)$	$\gamma_4(\mu)$	$\gamma_5(\nu)$
$\gamma_2(1)$	1	$\gamma_3(1)$	$\gamma_4(1)$	$\gamma_5(1)$	$\gamma_1(A)\gamma_2(1)$	$\gamma_1(B)\gamma_2(1)$	$-\gamma_3(\lambda)$	$-\gamma_4(\mu)$	$-\gamma_5(\nu)$
$\gamma_3(1)$		$2(1+\gamma_2(1))$	$2\gamma_5(1)$	$2\gamma_4(1)$	0	$2\gamma_3(\lambda)$	$\gamma_1(A)+\gamma_2(1)\gamma_1(A)$	$2\gamma_5(\nu)$	$2\gamma_4(\mu)$
$\gamma_4(1)$			$2(1+\gamma_2(1))$	$2\gamma_3(1)$	$2\gamma_4(\mu)$	$2\gamma_4(\mu)$	$2\gamma_5(\nu)$	$\gamma_1(A)+\gamma_2(1)\gamma_1(A)+$ $\gamma_1(B)+\gamma_2(1)\gamma_1(B)$	$2\gamma_3(\lambda)$
$\gamma_5(1)$				$2(1+\gamma_2(1))$	$2\gamma_5(\nu)$	0	$2\gamma_4(\mu)$	$2\gamma_3(\lambda)$	$\gamma_1(B)+\gamma_2(1)\gamma_1(B)$
${}_2\gamma_1(A)$					0	$4\gamma_1(C)$	0	$2\gamma_1(C)\gamma_4(1)$	$2\gamma_1(C)\gamma_5(1)$
${}_2\gamma_1(B)$						0	$2\gamma_1(C)\gamma_3(1)$	$2\gamma_1(C)\gamma_4(1)$	0
${}_4\gamma_3(\lambda)$							$2\gamma_1(C)(\gamma_2(1)-1)$	$2\gamma_1(C)\gamma_5(1)$	$2\gamma_1(C)\gamma_4(1)$
${}_4\gamma_4(\mu)$								$2\gamma_1(C)(\gamma_2(1)-1)$	$2\gamma_1(C)\gamma_3(1)$
${}_4\gamma_5(\nu)$									$2\gamma_1(C)(\gamma_2(1)-1)$
${}_8\gamma_1(C)$	${}_8(\gamma_1(C)\gamma_2(1))$	${}_4(\gamma_1(C)\gamma_3(1))$	${}_4(\gamma_1(C)\gamma_4(1))$	${}_4(\gamma_1(C)\gamma_5(1))$					

${}_lW_m$ means that l is the order of $W_m \in H^m(Q_{2,\psi}\boldsymbol{Z}Q_2)$ as a $\boldsymbol{Z}$-module.

then the relations of the generators of the cohomology $H^*(Q_2, {}_{\psi}ZQ_2)\cdot$ $(\simeq HH^*(ZQ_2))$ stated in Table 2 correspond to the result given by calculating the product using a diagonal approximation on the periodic resolution of Q_2 (see [4, Table 3]).

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