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# CR EINSTEIN-WEYL STRUCTURES

By

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**Abstract.** An Einstein-Weyl structure is a natural generalization of an Einstein structure within the framework of conformal geometry. We are interested in considering an Einstein-Weyl structure on a CR manifold. A CR manifold has a conformal structure only on its hyperdistribution. In this paper, on a CR manifold we naturally define an Einstein-Weyl structure closely related to the conformal structure on the hyperdistribution.

## 0. Introduction

A conformal structure on a differentiable manifold is a conformal equivalence class of Riemannian metrics (or pseudo-Riemannian metrics) on the manifold. On a conformal manifold, the objects which are invariant for every Riemannian metric included in the conformal class are important, or more strictly, the object except for them does not have significance. Weyl conformal curvature tensor is representative one of them. It is interesting to consider whether the results obtained in conformal geometry also hold in CR geometry. In this paper, we study an analogy of Weyl structure in CR geometry. A CR structure on an odd dimensional manifold is a pair $(\mathscr{D}, J)$ of a 1-codimensional subbundle $\mathscr{D}$ of the tangent bundle and a complex structure $J$ on $\mathscr{D}$ with a certain integrability condition. Assuming the nondegenerate property for $\mathscr{D}$, we have a conformal class of fiber metrics on $\mathscr{D}$. It is well-known that Bochner curvature tensor is one of the objects which are invariant for this conformal class on CR manifolds.

In this paper, we discuss a structure analogous to Einstein-Weyl structure on a conformal manifold and especially consider whether we can comfortably define this structure for a conformal class on $\mathscr{D}$. An Einstein-Weyl structure is a natural generalization of an Einstein structure within the framework of conformal ge-

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ometry. Strictly speaking, Einstein-Weyl structure is a pair of $([g], D)$ of a Riemannian metric class $[g]$ and a linear connection $D$, preserving $[g]$, whose Ricci tensor satisfies an equation that the symmetric part is proportional to $g$ pointwise. On a CR manifold there are naturally almost contact structures $(\phi, \xi, \theta)$ which determine a conformal class on $\mathscr{D}$. Therefore almost contact structures $(\phi, \xi, \theta)$ associated with $(\mathscr{D}, J)$ correspond to Riemannian structures in conformal geometry. Furthermore, a connection corresponding to Levi-Civita connection is defined by Tanaka [11], which is called Tanaka connection. We need to define a connection which preserves the conformal class on $\mathscr{D}$. Such connection corresponds to the Weyl connection $D$.

In Section 1, we recall the definition of Einstein-Weyl structure and relation between a Weyl connection $D$ and Levi-Civita connection $\nabla$ of a Riemannian metirc included in a given conformal structure (cf. [7], [8]). This section will be useful to understand the analogy mentioned above. In Section 2, we recall the definition of CR structure, results obtained in [9] and certain cochain complex $\{C^{p,q}(M), d''\}$ defined by Tanaka [11]. In Section 3, we define CR Weyl connection and study the relation between CR Weyl connection $D$ and Tanaka connection $\nabla$, where Tanaka connection $\nabla$ is a unique linear connection associated with almost contact structure $(\phi, \xi, \theta)$ introduced in Section 2. In Section 4, we see a CR Weyl connection from the standpoint of the frame bundle. Section 5 is devoted to the study of curvature tensor of a CR Weyl connection. In Section 6 we study the relation between the curvature tensor of a CR Weyl connection and that of a Tanaka connection. In fact, we obtain an equation including these two tensors, which is similar to the equation appearing in [2]. Using this equation, we define a CR Einstein-Weyl structure in a natural fashion. In the last section, we introduce an example of a CR Einstein-Weyl manifold. In fact, we see that $SO(3)$-bundle over a quaternion Kähler manifold admits a CR Einstein-Weyl structure.

## 1. Einstein-Weyl Structures

Let $M$ be an $n$-dimensional manifold with a conformal class $[g]$. A *Weyl connection* on $M$ is a torsion-free linear connection which satisfies the following condition:

$$Dg = -2p \otimes g \tag{1.1}$$

for some 1-form $p$. If we choose $g' = e^{2\mu}g$ for a smooth function $\mu$ in the conformal class $[g]$, we have a 1-form $p' = p - d\mu$ instead of $p$ for the equation

(1.1). From this we can say that a Weyl connection $D$ *preserves* the conformal class $[g]$. Let $([g], D)$ be a pair of a conformal class $[g]$ and a Weyl connection preserving it. A pair $([g], D)$ is called a *Weyl structure* on $M$ and if $M$ admits a Weyl structure, then $(M, [g], D)$ is called a *Weyl manifold*. We can also say that a Weyl connection is a torsion-free linear connection which is reducible to a connection in $CO(M)$ corresponding to the conformal class $[g]$, where $CO(M)$ is a subbundle in the frame bundle $F(M)$ with a structure group $CO(n)$.

Now let $\nabla$ be the Levi-Civita connection of $g$ on a Weyl manifold $M$. We can write $D = \nabla + H$ where $H$ is a tensor field of type $(1,2)$. Then we have from (1.1)

$$H(X, Y) = p(X)Y + p(Y)X - g(X, Y)P \tag{1.2}$$

for $X, Y \in \mathfrak{X}(M)$, where $P$ is the dual vector field of $p$ with respect to $g$. Conversely if we define $D$ with (1.2) for an arbitrary pair $(p, g)$, $D$ satisfies the equation (1.1). Therefore we see that an arbitrary pair $(p, g)$ determines a Weyl structure on $M$.

Now let $r^D$ be the Ricci tensor of a Weyl connection $D$. Note that as $D$ is not a metric connection, $r^D$ is not necessarily symmetric. A Weyl structure $([g], D)$ is called an *Einstein-Weyl structure* if the symmetric part of $r^D$ is proportional to $g$ pointwise. Note that the proportional factor may be non-constant. If $M$ admits an Einstein-Weyl structure $([g], D)$, then $M$ is called an *Einstein-Weyl manifold*.

Now if we let $r^\nabla$ be the Ricci tensor of the connection $\nabla$, then $r^D$ and $r^\nabla$ are related by the following equation (cf. [7], [8]):

$$\begin{aligned} r^D(X, Y) = {} & (1-n)(\nabla_X p)(Y) + (\nabla_Y p)(X) + (n-2)p(X)p(Y) \\ & + g(X, Y)(\delta p + (n-2)g(P, P)) + r^\nabla(X, Y) \end{aligned} \tag{1.3}$$

for $X, Y \in \mathfrak{X}(M)$, where $\delta p$ denotes the codifferential with respect to $g$.

We have the following local characterization of Einstein-Weyl structures (cf. [7], [8]):

PROPOSITION 1.1. *Let $(p, g)$ be a Weyl structure on $M$. Then $(p, g)$ is an Einstein-Weyl structure if and only if there exists a smooth function $\Lambda$ on $M$ satisfying the equation*

$$\frac{2-n}{2}((\nabla_X p)(Y) + (\nabla_Y p)(X) - 2p(X)p(Y)) + r^\nabla(X, Y) = \Lambda g(X, Y)$$

*for every $X, Y \in \mathfrak{X}(M)$.*

## 2. CR Structure and Tanaka Connection

Let $M$ be a connected differentiable manifold of dimension $2n+1$ $(n \geq 1)$. An *almost contact structure* on $M$ is a triplet of a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and a 1-form $\theta$ satisfying

$$\theta(\xi) = 1, \quad \phi^2 = -I + \theta \otimes \xi \tag{2.1}$$

which imply

$$\phi\xi = 0, \quad \theta \circ \phi = 0 \quad \text{and} \quad \operatorname{rank} \phi = 2n, \tag{2.2}$$

where $I$ denotes the identity transformation. An almost contact structure $(\phi, \xi, \theta)$ naturally corresponds to a reduced bundle in the frame bundle $F(M)$ with structure group

$$\left\{ \begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{0} & C \end{pmatrix} \middle| C \in GL(n; \mathbf{C}) \right\}.$$

Now let $\mathscr{D}$ denote a 1-codimensional subbundle of the tangent bundle $TM$, which is called a hyperdistribution. A cross section $J$ of the bundle $\mathscr{D} \otimes \mathscr{D}^*$ satisfying $J^2 = -I$ is called a complex structure on $\mathscr{D}$, where $\mathscr{D}^*$ is the dual bundle of $\mathscr{D}$.

If $M$ admits a pair $(\mathscr{D}, J)$, there is always a locally defined almost contact structure $(\phi, \xi, \theta)$ satisfying that the 1-form $\theta$ annihilates $\mathscr{D}$ and the restriction of $\phi$ to $\mathscr{D}$ coincides with $J$. In fact, since there always exists a 1-form $\theta$ annihilating $\mathscr{D}$ in each coordinate neighborhood $U$ of $M$, we have a vector field $\xi$ on $U$ in such a way that $\theta(\xi) = 1$. Then we can define, on $U$, a $(1,1)$ tensor field $\phi$ by

$$\phi(V) = J(V - \theta(V)\xi)$$

for $V \in \mathfrak{X}(U)$ because $V - \theta(V)\xi$ belongs to $\mathscr{D}$. We shall denote $V - \theta(V)\xi$ by $V_{\mathscr{D}_\xi}$ and call $\mathscr{D}$-component of $V$ with respect to $\xi$. Then a straightforward calculation shows that $(\phi, \xi, \theta)$ is an almost contact structure on $U$. An almost contact structre $(\phi, \xi, \theta)$ such that the 1-form $\theta$ annihilates $\mathscr{D}$ and the restriction of $\phi$ to $\mathscr{D}$ coincides with $J$ is said that the almost contact structure $(\phi, \xi, \theta)$ *belongs* to the pair $(\mathscr{D}, J)$. In addition, if $M$ is orientable, there are globally defined almost contact structures $(\phi, \xi, \theta)$ belonging to $(\mathscr{D}, J)$. A 1-form $\theta$ annihilating $\mathscr{D}$ is determined up to a non-vanishing smooth function. Moreover we have

$$d(f\theta)(X, Y) = f\, d\theta(X, Y)$$

for every $X, Y \in \Gamma(\mathscr{D})$ and smooth function $f$, where $\Gamma(\mathscr{D})$ denotes the set of cross sections of the vector bundle $\mathscr{D}$ on $M$. Therefore, in virtue of this fact, the following definition is well-defined. If $d\theta$ is nondegenerate on $\mathscr{D}$, then $(\mathscr{D}, J)$ is said to be *nondegenerate*.

A pair $(\mathscr{D}, J)$ is called a *CR structure* if the following two conditions hold:

$$\text{(C.1)} \qquad [JX, JY] - [X, Y] \in \Gamma(\mathscr{D})$$

$$\text{(C.2)} \qquad [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0$$

for every $X, Y \in \Gamma(\mathscr{D})$. If $M$ admits a CR structure $(\mathscr{D}, J)$, then $M$ is called a *CR manifold*. In the sequel, $(\mathscr{D}, J)$ will be a nondegenerate CR structure.

Now let $M$ be a connected orientable manifold furnished with a CR structure $(\mathscr{D}, J)$ and $(\phi, \xi, \theta)$ an almost contact structure belonging to $(\mathscr{D}, J)$. Define $\omega$ by

$$\text{(2.3)} \qquad \omega = -2\, d\theta.$$

Then $\omega$ satisfies

$$\text{(2.4)} \qquad \omega(JX, JY) = \omega(X, Y)$$

for every $X, Y \in \Gamma(\mathscr{D})$ because of the condition (C.1). Moreover define $g : \mathscr{D} \times \mathscr{D} \to \mathbf{R}$ by

$$\text{(2.5)} \qquad g(X, Y) = \omega(JX, Y),$$

which satisfies the equations

$$\text{(2.6)} \qquad g(X, Y) = g(Y, X), \quad g(JX, JY) = g(X, Y)$$

for every $X, Y \in \Gamma(\mathscr{D})$. Therefore $g$ is symmetric, Hermitian and nondegenerate, which is called *Levi metric*.

From a given almost contact structure belonging to $(\mathscr{D}, J)$ we can always make an almost contact structure which belongs to the same $(\mathscr{D}, J)$ and satisfies the following condition

$$(*) \qquad [\xi, \Gamma(\mathscr{D})] \subset \Gamma(\mathscr{D})$$

(cf. [9]). This condition $(*)$ is equivalent to

$$\text{(2.7)} \qquad \mathscr{L}_\xi \theta = 0 \quad \text{or} \quad \omega(\xi, X) = 0$$

for $X \in \mathscr{D}$, where $\mathscr{L}_\xi$ denotes the Lie differentiation with respect to $\xi$. Such an almost contact structure is denoted by $(\phi, \xi, \theta)^*$ and we call it a *$\mathscr{D}$-preserving almost contact structure*. We shall restrict our attention to the family of $\mathscr{D}$-

preserving almost contact structures which belong to CR structure $(\mathscr{D}, J)$. The following result is proved in [9]:

LEMMA 2.1. *If* $(\phi, \xi, \theta)^*$ *and* $(\phi', \xi', \theta')^*$ *belong to* $(\mathscr{D}, J)$, *then they are related by*

$$(2.8) \qquad \theta' = \varepsilon e^{2\mu}\theta, \quad \xi' = \varepsilon e^{-2\mu}(\xi - 2Q^*), \quad \phi' = \phi - 2\theta \otimes P^*$$

*where* $\varepsilon = \pm 1$, $\mu$ *is a smooth function,* $P^* \in \Gamma(\mathscr{D})$ *is defined by* $g(P^*, X) = d\mu(X)$ *for* $X \in \Gamma(\mathscr{D})$ *and* $Q^* = JP^*$.

Next we shall explain Tanaka connection associated with $(\phi, \xi, \theta)^*$ and how the connection changes under (2.8). We don't have to assume the condition (C.2) so far, but we need to assume the condition (C.2) for the next lemma (cf. [9], [12]).

LEMMA 2.2. *Let* $(\phi, \xi, \theta)^*$ *be a* $\mathscr{D}$*-preserving almost contact structure. Then there exists uniquely a linear connection* $\nabla$ *such that* $\nabla\phi = 0$, $\nabla\xi = 0$, $\nabla\theta = 0$, $\nabla^\circ g = 0$, $T_{\mathscr{D}_\xi} = 0$ *and* $T(\xi, X) = -1/2\phi(\mathscr{L}_\xi\phi)X$, *where* $\nabla^\circ$ *denotes the induced connection on the hyperdistribution* $\mathscr{D}$ *and* $T_{\mathscr{D}_\xi}(X, Y)$ *the* $\mathscr{D}$*-component of the torsion tensor* $T(X, Y)$ *of* $\nabla$ *with respect to* $\xi$ *for* $X, Y \in \Gamma(\mathscr{D})$.

REMARK. We put $FV = T(\xi, V)$ for $V \in TM$. Note that $F$ is symmetric with respect to $g$ and anticommutes with $J$ (cf. [9]).

The linear connection stated in the above lemma is called *Tanaka connection* associated with $(\phi, \xi, \theta)^*$. We give the following (cf. [9]).

LEMMA 2.3. *Let* $(\phi, \xi, \theta)^*$ *and* $(\phi', \xi', \theta')^*$ *be two* $\mathscr{D}$*-preserving almost contact structures which belong to the CR structure* $(\mathscr{D}, J)$. *Let* $\nabla$ *and* $\nabla'$ *be Tanaka connections associated with* $(\phi, \xi, \theta)^*$ *and* $(\phi', \xi', \theta')^*$ *respectively. Define the difference* $H$ *between* $\nabla$ *and* $\nabla'$ *by*

$$H(V, W) = \nabla'_V W - \nabla_V W, \quad V, W \in \mathfrak{X}(M).$$

*Then we have*

$$(2.9) \qquad \begin{aligned} H(X, Y) = {} & p^*(X)Y + p^*(Y)X - g(X, Y)P^* \\ & + q^*(X)JY + q^*(Y)JX - g(JX, Y)Q^*, \end{aligned}$$

(2.10) $H(\xi, X) = \nabla_{JX}P^* + \nabla_X Q^* - 2q^*(X)P^* + 2p^*(X)Q^* + 2g(P^*, P^*)JX,$

*for every* $X, Y \in \Gamma(\mathscr{D})$, *where* $p^* = d\mu$ *and* $q^* = -p^* \circ \phi$.

REMARK. We have $g(P^*, X) = p^*(X)$ and $g(Q^*, X) = q^*(X)$ for every $X \in \Gamma(\mathscr{D})$.

Next we shall introduce a cochain complex $\{C^{p,q}, d''\}$ of a CR manifold $M$ with complex coefficients, which corresponds to that in the case of a complex manifold (cf. [11]). We shall use the following fact in Section 6.

Let $(\mathscr{D}, J)$ be a nondegenerate CR structure of a $(2n+1)$-dimensional orientable manifold $M$. Then the complexification $\mathbf{C}TM$ of the tangent bundle $TM$ is decomposed as $\mathbf{C}TM = \mathbf{C}\mathscr{D} \oplus \mathscr{L}$ where $\mathbf{C}\mathscr{D}$ is the complexification of $\mathscr{D}$ and $\mathscr{L}$ is a trivial line bundle isomorphic with $\mathbf{C}TM/\mathbf{C}\mathscr{D}$. The complex structure $J$ on $\mathscr{D}$ can be uniquely extended to a complex linear endomorphism of $\mathbf{C}\mathscr{D}$ and the extended endomorphism will be also denoted by $J$. Let $\mathscr{D}^{1,0}$ (resp. $\mathscr{D}^{0,1}$) be a subbundle of $\mathbf{C}\mathscr{D}$ composed of the eigenvectors corresponding to $i$ (resp. $-i$) of the endomorphism $J$. Note that $\mathscr{D}^{0,1} = \overline{\mathscr{D}}^{1,0}$, where the notation "bar" denotes the conjugate operator. It is clear that conditions (C.1) and (C.2) are equivalent to

(2.11) $$[\Gamma(\mathscr{D}^{1,0}), \Gamma(\mathscr{D}^{1,0})] \subset \Gamma(\mathscr{D}^{1,0}).$$

Now we put $A^k(M) = \Gamma(\Lambda^k(\mathbf{C}TM))$ and denote by $F^p(\Lambda^k(\mathbf{C}TM))$ the subbundle of $\Lambda^k(\mathbf{C}TM)$ consisting of all $\psi \in \Lambda^k(\mathbf{C}TM)$ which satisfy the equality:

(2.12) $$\psi(X_1, \dots, X_{p-1}, \overline{Y}_1, \dots, \overline{Y}_{k-p+1}) = 0$$

for all $X_1, \dots, X_{p-1} \in \mathbf{C}TM$ and $Y_1, \dots, Y_{k-p+1} \in \mathscr{D}^{1,0}$. Note that we define $F^0(\Lambda^k(\mathbf{C}TM)) = \Lambda^k(\mathbf{C}TM)$. Then we have

(2.13) $$F^{p+1}(\Lambda^k(\mathbf{C}TM)) \subset F^p(\Lambda^k(\mathbf{C}TM)), \quad F^{p+1}(\Lambda^p(\mathbf{C}TM)) = 0.$$

Furthermore putting $A^{p,q}(M) = \Gamma(F^p(\Lambda^{p+q}(\mathbf{C}TM)))$, we easily find that

(2.14) $$dA^{p,q}(M) \subset A^{p,q+1}(M),$$

because of (2.11). Moreover putting $C^{p,q}(M) = A^{p,q}(M)/A^{p+1,q-1}(M)$, then we have the well-defined operator $d'' : C^{p,q}(M) \to C^{p,q+1}(M)$ which is naturally induced from the operator $d$ satisfying (2.14). And we obtain the cochain complex

(2.15) $$0 \to \Omega^p \to C^{p,0}(M) \to C^{p,1}(M) \to C^{p,2}(M) \to \cdots,$$

where $\Omega^p$ denotes the kernel of $C^{p,0}(M) \to C^{p,1}(M)$, whose element is called a *holomorphic p-form* in the mean of CR geometry. Since $A^{p,q}(M) = A^{p+1,q-1}(M) \oplus C^{p,q}(M)$, we have the decompositon:

$$A^{p,q}(M) = \bigoplus_{i=0}^{q} C^{p+q-i,i}(M).$$

Now for $\psi \in C^{p,q}(M)$ we have $d\psi \in A^{p,q+1}(M)$ or more precisely the following fact is well-known (cf. [11]):

$$d\psi \in C^{p+2,q-1}(M) \oplus C^{p+1,q}(M) \oplus C^{p,q+1}(M). \tag{2.16}$$

Consequently $d\psi$ can be written uniquely in the form:

$$d\psi = A\psi + d'\psi + d''\psi,$$

where $A\psi \in C^{p+2,q-1}(M)$ and $d'\psi \in C^{p+1,q}(M)$. For any $\psi \in C^{p,q}(M)$, $A\psi$, $d'\psi$ and $d''\psi$ are described as follows:

$$\begin{aligned}(A\psi)&(X_1,\dots,X_{p+2},\bar{Y}_1,\dots,\bar{Y}_{q-1})\\ &= \frac{1}{p+q+1}\sum_{\lambda<\mu}(-1)^{\lambda+\mu+1}\psi(T(X_\lambda,X_\mu),\\ &\qquad X_1,\dots,\hat{X}_\lambda,\dots,\hat{X}_\mu,\dots,X_{p+1},\bar{Y}_1,\dots,\bar{Y}_{q-1})\end{aligned} \tag{2.17}$$

$$\begin{aligned}(d'\psi)&(X_1,\dots,X_{p+1},\bar{Y}_1,\dots,\bar{Y}_q)\\ &= \frac{1}{p+q+1}\sum_{\lambda}(-1)^{\lambda+1}(\nabla_{X_\lambda}\psi)(X_1,\dots,\hat{X}_\lambda,\dots,X_{p+1},\bar{Y}_1,\dots,\bar{Y}_q),\end{aligned} \tag{2.18}$$

$$\begin{aligned}(d''\psi)&(X_1,\dots,X_p,\bar{Y}_1,\dots,\bar{Y}_{q+1})\\ &= \frac{(-1)^p}{p+q+1}\Bigg\{\sum_{\lambda}(-1)^{\lambda+1}(\nabla_{\bar{Y}_\lambda}\psi)(X_1,\dots,X_p,\bar{Y}_1,\dots,\hat{\bar{Y}}_\lambda,\dots,\bar{Y}_{q+1}),\\ &\qquad + \sum_{\lambda,\mu}(-1)^{\lambda+\mu+1}\psi(T(X_\lambda,\bar{Y}_\mu),X_1,\dots,\hat{X}_\lambda,\dots,X_p,\\ &\qquad\qquad \bar{Y}_1,\dots,\hat{\bar{Y}}_\mu,\dots,\bar{Y}_{q+1})\Bigg\}\end{aligned} \tag{2.19}$$

for $Y_\lambda \in \mathscr{D}^{1,0}$ and $X_\lambda \in \mathscr{D}^{1,0} \oplus \mathscr{L}$, where $\nabla$ is a Tanaka connection associated with some $\mathscr{D}$-preserving almost contact structure $(\phi,\xi,\theta)^*$ and $T$ is the torsion tensor of $\nabla$. Note that $\mathscr{L} = \mathbf{C} \otimes \mathrm{span}\{\xi\}$.

## 3. CR Weyl Structures

Let $(M, \mathscr{D}, J)$ be a connected orientable $(2n+1)$-dimensional manifold furnished with a nondegenerate CR structure $(\mathscr{D}, J)$. Under the notation of lemma 2.1, if $g$ and $g'$ are the Levi-metrics made from $\theta$ and $\theta'$ respectively, we have

$$g' = \varepsilon e^{2\mu} g \tag{3.1}$$

Therefore the family of $\mathscr{D}$-preserving almost contact structures which belong to the CR structure $(\mathscr{D}, J)$ induces pseudo conformal geometry only on the hyperdistribution $\mathscr{D}$. We shall naturally define a certain Weyl structure with respect to this pseudo conformal geometry. The word "naturally" of the above sentence means that the relation between a CR Weyl connection of the CR structure $(\mathscr{D}, J)$ and a Tanaka connection of a $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$ is analogous to that between a Weyl connection of a conformal class and Levi-Civita connection of a Riemannian metric in the conformal class.

Definition. Let $(\phi, \xi, \theta)^*$ be an arbitrary $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$. A linear connection $D$ on $M$ is a *CR Weyl connection* if, for every $V \in \mathfrak{X}(M)$, $X, Y \in \Gamma(\mathscr{D})$ and for some 1-form $p$ on $M$, the following conditions are satisfied:

$$D_V\theta = -2p(V)\theta \tag{a}$$

$$D_V\xi_p = 2p(V)\xi_p \tag{b}$$

$$D^\circ_V J = 0 \tag{c}$$

$$D^\circ_V g = -2p(V)g \tag{d}$$

$$T(X, Y) = -\omega(X, Y)\xi_p \tag{e}$$

$$T(\xi_p, X) = -\frac{1}{2}\phi_p(\mathscr{L}_{\xi_p}\phi_p)X, \tag{f}$$

where $D^\circ$ denotes the induced connection on the hyperdistribution $\mathscr{D}$, $T$ the torsion tensor of $D$, $\xi_p = \xi - 2Q$, $\phi_p = \phi - 2\theta \otimes P$, $P$ the cross-section of $\mathscr{D}$ such that $g(P, X) = p(X)$ for every $X \in \Gamma(\mathscr{D})$ and $Q = JP$.

Remark. If $D$ is a CR Weyl connection, we can show that

$$D_V\phi_p = 0, \quad (D_VT)(X, Y) = 0 \tag{3.2}$$

for every $V \in \mathfrak{X}(M)$ and $X, Y \in \Gamma(\mathscr{D})$ by direct calculation. In addition, we note that $(\phi_p, \xi_p, \theta)$ is also an almost contact structure belonging to $(\mathscr{D}, J)$ which may not satisfy condition $(*)$.

The family of almost contact structures belonging to $(\mathscr{D}, J)$ and satisfying $(*)$ is smaller than that of all almost contact structures belonging to $(\mathscr{D}, J)$. However, we can always obtain an almost contact structure satisfying $(*)$ from almost contact structure belonging to the same $(\mathscr{D}, J)$ if it is nondegenerate (cf. [9]). Therefore we may deal with only $\mathscr{D}$-preserving almost contact structures. The following propositoin allows us to call $D$ a CR Weyl connection. By direct computation, we obtain

PROPOSITION 3.1. *The CR Weyl connection $D$ is well defined: the equations from* (a) *to* (f) *in above definition are invariant for the change* (2.8).

REMARK. If we replace $(\phi, \xi, \theta)^*$ by $(\phi', \xi', \theta')^*$, then the 1-form $p$ in the above definition changes to $p' = p - d\mu$.

From this, we can say that a CR Weyl connection $D$ *preserves* the CR structure $(\mathscr{D}, J)$. Let $((\mathscr{D}, J), D)$ be a pair of a CR structure $(\mathscr{D}, J)$ and a CR Weyl connection preserving it. The pair $((\mathscr{D}, J), D)$ is called a *CR Weyl structure* on $M$.

Next we closely observe the conditions of a CR Weyl connection. In fact, we don't have to assume the condition (f) if we add a certain condition to the torsion tensor of a linear connection satisfying from (a) to (e) for a 1-form $p$. To see this, we need the following:

LEMMA 3.2. *Let $D$ be a linear connection satisfying from* (a) *to* (e) *for a 1-form $p$ and $T$ the torsion tensor of $D$. Then $T$ satisfies*

$$\theta(T(\xi_p, V)) = 0, \tag{3.3}$$

$$\phi_p(T(\xi_p, \phi_p V)) + T(\xi_p, V) = -\phi_p(\mathscr{L}_{\xi_p}\phi_p)(V) \tag{3.4}$$

*for every $V \in \mathfrak{X}(M)$.*

PROOF. It is sufficient to show that $T(\xi_p, V)$ belongs to $\Gamma(\mathscr{D})$ for $V = \xi_p$ and $V = X \in \Gamma(\mathscr{D})$. When $V = \xi_p$, $T(\xi_p, \xi_p) = 0$. When $V = X$, we have

$$T(\xi_p, X) = D_{\xi_p}X - D_X\xi_p - [\xi_p, X] = D_{\xi_p}X - 2p(X)\xi_p - [\xi_p, X]$$

because of (b). The condition (a) implies that $D_V\Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D})$. The $\mathscr{D}$-component $[\xi_p, X]_{\mathscr{D}_{\xi_p}}$ with respect to $\xi_p$ is given by

$$\begin{aligned}[\xi_p, X]_{\mathscr{D}_{\xi_p}} &= [\xi_p, X] - \theta([\xi_p, X])\xi_p \\ &= [\xi_p, X] - \theta([\xi - 2Q, X])\xi_p \\ &= [\xi_p, X] + 2\theta([Q, X])\xi_p \\ &= [\xi_p, X] + 2\omega(Q, X)\xi_p \\ &= [\xi_p, X] + 2g(P, X)\xi_p = [\xi_p, X] + 2p(X)\xi_p.\end{aligned}$$

Therefore we have

$$T(\xi_p, X) = D_{\xi_p}X - 2p(X)\xi_p - ([\xi_p, X]_{\mathscr{D}_{\xi_p}} - 2p(X)\xi_p)$$

which proves (3.3). Since

$$\begin{aligned}0 &= (D_{\xi_p}\phi_p)V = D_{\xi_p}(\phi_p V) - \phi_p(D_{\xi_p}V) \\ &= D_{\phi_p V}\xi_p + [\xi_p, \phi_p V] + T(\xi_p, \phi_p V) - \phi_p(D_V\xi_p + [\xi_p, V] + T(\xi_p, V)),\end{aligned}$$

we have

$$T(\xi_p, \phi_p V) - \phi_p(T(\xi_p, V)) = -2p(\phi_p V)\xi_p - (\mathscr{L}_{\xi_p}\phi_p)V$$

because of (b) and the equation $\phi_p\xi_p = 0$. Thus if we apply $\phi_p$ to the both hand sides of the above equation, we obtain (3.4). □

Now put $F_pV = T(\xi_p, V)$ for $V \in \mathfrak{X}(M)$. Then we have

$$\theta \circ F_p = 0, \tag{3.5}$$

$$\phi_p \circ F_p \circ \phi_p + F_p = -\phi_p(\mathscr{L}_{\xi_p}\phi_p). \tag{3.6}$$

We demand for $F_p$ the condition that $F_p$ anticommutes with $\phi_p$. Then $F_p$ must be $-1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$. Conversely we see that $F_p$ anticommutes with $\phi_p$ if $F_p = -1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$. Therefore if we add the condition that $F_p$ anticommutes with $\phi_p$ to the conditions from (a) to (e) for a 1-form $p$, $D$ becomes a CR Weyl connection. For $F_p$, we also have

LEMMA 3.3. *Let $D$ be a connection satisfying from* (a) *to* (e) *for a* 1*-form $p$ and $T$ the torsion tensor of $D$. Then $F_p$ satisfies*

$$g(F_pY, Z) + g(Y, F_pZ) = -g(\phi_p(\mathscr{L}_{\xi_p}\phi_p)Y, Z) - 4\,dp(JY, Z) \tag{3.7}$$

*for every $Y, Z \in \Gamma(\mathscr{D})$.*

PROOF. Since $F_p Y = T(\xi_p, Y)$, we have, from (b),

$$D_{\xi_p} Y = 2p(Y)\xi_p + [\xi_p, Y] + F_p Y.$$

We substitute this equation into the right hand side of $(D^{\circ}_{\xi_p} g)(Y,Z) = \xi_p \cdot \omega(\phi_p Y, Z) - \omega(\phi_p D_{\xi_p} Y, Z) - \omega(\phi_p Y, D_{\xi_p} Z)$. Since $\phi|_{\mathscr{D}} = \phi_p|_{\mathscr{D}} = J$ on $\mathscr{D}$, we consequently obtain

$$\begin{aligned} (3.8) \qquad & g(F_p Y, Z) + g(Y, F_p Z) \\ &= \xi_p \cdot \omega(\phi_p Y, Z) - \omega(\phi_p[\xi_p, Y], Z) \\ &\quad - \omega(\phi_p Y, 2p(Z)\xi_p + [\xi_p, Z]) - (D^{\circ}_{\xi_p} g)(Y, Z). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (3.9) \qquad & -2(d\mathscr{L}_{\xi_p}\theta)(\phi_p Y, Z) \\ &= (\phi_p Y) \cdot \theta([\xi_p, Z]) - Z \cdot \theta([\xi_p, \phi_p Y]) + \xi_p \cdot \omega(\phi_p Y, Z) \\ &\quad - \theta([\phi_p Y, \mathscr{L}_{\xi_p} Z]) - \theta([\mathscr{L}_{\xi_p}(\phi_p Y), Z]) \end{aligned}$$

by using Jacobi identity. Combining (3.9) with (3.8), we obtain

$$\begin{aligned} & g(F_p Y, Z) + g(Y, F_p Z) \\ &= -2(d\mathscr{L}_{\xi_p}\theta)(\phi_p Y, Z) - (\phi_p Y) \cdot \theta([\xi_p, Z]) + Z \cdot \theta([\xi_p, \phi_p Y]) \\ &\quad - \theta([\phi_p Y, 2p(Z)\xi_p]) + \theta([(\mathscr{L}_{\xi_p}\phi_p) Y, Z]) - (D^{\circ}_{\xi_p} g)(Y, Z). \end{aligned}$$

Furthermore by (2.7) and (d), the above equation becomes

$$\begin{aligned} (3.10) \qquad & g(F_p Y, Z) + g(Y, F_p Z) \\ &= 4(d\mathscr{L}_Q\theta)(\phi Y, Z) + 2(\phi Y) \cdot \omega(Q, Z) - 2Z \cdot \omega(Q, \phi Y) \\ &\quad - \theta([\phi Y, 2p(Z)\xi_p]) + \theta([(\mathscr{L}_{\xi_p}\phi_p)(Y), Z]) + 2p(\xi_p)\omega(\phi Y, Z). \end{aligned}$$

Next we shall calculate $4(d\mathscr{L}_Q\theta)(\phi Y, Z)$. If we use (c), we have

$$(3.11) \qquad 2(D^{\circ}_Q g)(Y, Z) = 2Q \cdot \omega(\phi Y, Z) - 2\omega(D_Q(\phi Y), Z) - 2\omega(\phi Y, D_Q Z).$$

We obtain $p(Q) = 0$ since $g$ is Hermitian, so that the left hand side of (3.11) vanishes by (d). Applying this fact and (e) to (3.11), we have

$$\begin{aligned} (3.12) \qquad 0 &= 2Q \cdot \omega(\phi Y, Z) - 2\omega(D_{\phi Y} Q, Z) - 2\theta([\mathscr{L}_Q(\phi Y) - \omega(Q, \phi Y)\xi_p, Z]) \\ &\quad - 2\omega(\phi Y, D_Z Q) - 2\theta([\phi Y, \mathscr{L}_Q Z - \omega(Q, Z)\xi_p]). \end{aligned}$$

On the other hand, a straightforward computation shows

$$4(d\mathscr{L}_Q\theta)(\phi Y, Z) = -2(\phi Y)\cdot\omega(Q, Z) + 2Z\cdot\omega(Q, \phi Y) - 2Q\cdot\omega(\phi Y, Z) + 2\theta([\mathscr{L}_Q(\phi Y), Z]) + 2\theta([\phi Y, \mathscr{L}_Q Z]).$$

Combining this equation with (3.12), we obtain

$$\begin{aligned}(3.13)\qquad & 4(d\mathscr{L}_Q\theta)(\phi Y, Z) \\ &= -2(\phi Y)\cdot\omega(Q, Z) + 2Z\cdot\omega(Q, \phi Y) \\ &\quad - 2\omega(D_{\phi Y}Q, Z) - 2\omega(\phi Y, D_Z Q) \\ &\quad + 2\theta([\omega(Q, \phi Y)\xi_p, Z]) + 2\theta([\phi Y, \omega(Q, Z)\xi_p]).\end{aligned}$$

Moreover, we directly calculate $4\,dp(\phi Y, Z)$. Then we obtain

$$\begin{aligned}(3.14)\qquad 4\,dp(\phi Y, Z) &= 2(\phi Y)\cdot p(Z) - 2Z\cdot p(\phi Y) - 2p([\phi Y, Z]) \\ &= 2g(D_{\phi Y}Q, \phi Z) + 2g(Q, D_{\phi Y}(\phi Z)) - 4p(\phi Y)g(Q, \phi Z) \\ &\quad - 2g(D_Z Q, \phi^2 Y) - 2g(Q, D_Z(\phi^2 Y)) + 4p(Z)g(Q, \phi^2 Y) \\ &\quad - 2p(D_{\phi Y}Z - D_Z(\phi Y) + \omega(\phi Y, Z)\xi_p) \\ &= 2\omega(D_{\phi Y}Q, Z) + 2\omega(\phi Y, D_Z Q) - 2p(\xi_p)\omega(\phi Y, Z).\end{aligned}$$

Substitute (3.13) into (3.10) and use (3.14). Then we have

$$\begin{aligned}(3.15)\qquad & g(F_p Y, Z) + g(Y, F_p Z) \\ &= -4\,dp(\phi Y, Z) + 2\theta([\omega(Q, \phi Y)\xi_p, Z]) + \theta([(\mathscr{L}_{\xi_p}\phi_p)Y, Z]).\end{aligned}$$

Finally since the $\mathscr{D}$-component of $(\mathscr{L}_{\xi_p}\phi_p)Y$ with respect to $\xi_p$ is given by

$$((\mathscr{L}_{\xi_p}\phi_p)Y)_{\mathscr{D}_{\xi_p}} = (\mathscr{L}_{\xi_p}\phi_p)Y + 2\omega(Q, \phi Y)\xi_p,$$

substituting which into (3.15), we obtain (3.7). □

By Lemma 3.3, we have

**Lemma 3.4.** *Let $D$ be a CR Weyl connection and $p$ the corresponding 1-form. Then $p$ satisfies*

$$(3.16)\qquad dp(JX, JY) + dp(X, Y) = 0$$

*for every $X, Y \in \Gamma(\mathscr{D})$.*

PROOF. Applying the assumption (f) or the condition that $F_p$ anticommutes with $J$ to the equation (3.7), we have

$$g(F_pX, Y) - g(X, F_pY) = 4\,dp(JX, Y). \tag{3.17}$$

Thus by anticommutativity of $F_p$ with $J$, we obtain (3.16). □

Now as we deal with $\mathscr{D}$-preserving almost contact structures $(\phi, \xi, \theta)^*$ belonging to a CR structure $(\mathscr{D}, J)$, we have a unique linear connection called Tanaka connection associated with $(\phi, \xi, \theta)^*$. Therefore we have to compute the difference between a CR Weyl connection $D$ and Tanaka connection with respect to a fixed $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$.

PROPOSITION 3.5. *Let $(\phi, \xi, \theta)^*$ be a $\mathscr{D}$-preserving almost contact structure, $D$ a CR Weyl connection and $\nabla$ Tanaka connection associated with $(\phi, \xi, \theta)^*$. Define the difference $H$ between $D$ and $\nabla$ by*

$$H(V, W) = D_V W - \nabla_V W, \quad V, W \in \mathfrak{X}(M)$$

*Then we have*

$$H(X, Y) = p(X)Y + p(Y)X - g(X, Y)P + q(X)JY + q(Y)JX - g(JX, Y)Q, \tag{3.18}$$

$$H(\xi, X) = \nabla_{JX}P + \nabla_X Q - 2q(X)P + 2p(X)Q + 2g(P, P)JX, \tag{3.19}$$

*for every $X, Y \in \Gamma(\mathscr{D})$, where $p$ is the 1-form of $D$ corresponding to $(\phi, \xi, \theta)^*$, $P$ $(\in \Gamma(\mathscr{D}))$ defined by $g(P, X) = p(X)$ for $X \in \Gamma(\mathscr{D})$, $Q = JP$ and $q$ a 1-form defined by $q = -p \circ \phi$.*

PROOF. First we denote the torsion tensor of Tanaka connection by $T^\nabla$ and note that

$$T^\nabla(Y, Z) = -\omega(Y, Z)\xi \tag{3.20}$$

for $Y, Z \in \Gamma(\mathscr{D})$ since $T^\nabla_{\mathscr{D}_\xi} = 0$ and $\theta(T^\nabla(Y, Z))\xi = -\omega(Y, Z)\xi$ by Lemma 2.2.

Computing $H(Y, Z) - H(Z, Y)$ directly, we have

$$\begin{aligned} H(Y, Z) - H(Z, Y) &= D_Y Z - \nabla_Y Z - D_Z Y + \nabla_Z Y \\ &= T(Y, Z) + [Y, Z] - (T^\nabla(Y, Z) + [Y, Z]) \\ &= T(Y, Z) - T^\nabla(Y, Z) \end{aligned}$$

for $Y, Z \in \Gamma(\mathscr{D})$. Using (e) and (3.20), we obtain

$$H(Y,Z) - H(Z,Y) = 2\omega(Y,Z)Q. \tag{3.21}$$

On the other hand, since $(D^\circ_X g)(Y,Z) = -2p(X)g(Y,Z)$ and $(\nabla^\circ_X g)(Y,Z) = 0$ for $X, Y, Z \in \Gamma(\mathscr{D})$, we have

$$g(H(X,Y),Z) + g(Y,H(X,Z)) = 2p(X)g(Y,Z). \tag{3.22}$$

In the equation (3.22) we permute $X$, $Y$ and $Z$ cyclically and subtract one from the sum of the other two. Applying (3.21) to the resulting equation, we have the equation (3.18). Next we compute $H(\xi, X)$ for $X \in \Gamma(\mathscr{D})$. Since $F_p X = D_{\xi_p} X - 2p(X)\xi_p - [\xi_p, X]$ and $FX = \nabla_\xi X - [\xi, X]$, we have

$$\begin{aligned} H(\xi_p, X) &= D_{\xi_p} X - \nabla_{\xi_p} Y = F_p X + [\xi_p, X] + 2p(X)\xi_p - \nabla_\xi X + 2\nabla_Q X \\ &= F_p X + [\xi - 2Q, X] + 2p(X)\xi_p - (FX + [\xi, X]) + 2\nabla_Q X \\ &= F_p X - FX - 2[Q, X] + 2\nabla_Q X + 2p(X)\xi_p. \end{aligned}$$

Furthermore, applying (3.20) to this equation and noting that $\omega(Q,X) = p(X)$, we have

$$H(\xi_p, X) = F_p X - FX + 2\nabla_X Q - 4p(X)Q. \tag{3.23}$$

Now computing $F_p X - FX$ directly by the equation $F_p = -1/2\phi_p(\mathscr{L}_{\xi_p}\phi_p)$ and $F = -1/2\phi(\mathscr{L}_\xi\phi)$, we have

$$\begin{aligned} F_p X - FX &= -\frac{1}{2}\{\phi_p([\xi_p, JX] - \phi_p[\xi_p, X]) - J([\xi, JX] - J[\xi, X])\} \\ &= -\frac{1}{2}\{\phi_p([\xi, JX] - 2[Q, JX] - J[\xi, X] + 2\phi_p[Q, X]) \\ &\quad - J([\xi, JX] - J[\xi, X])\} \\ &= (\phi - 2\theta \otimes P)[Q, JX] - (\phi - 2\theta \otimes P)^2[Q, X] \\ &= \phi(\nabla_Q JX - \nabla_{JX} Q + \omega(Q, JX)\xi) \\ &\quad - 2\theta(\nabla_Q JX - \nabla_{JX} Q + \omega(Q, JX)\xi)P \\ &\quad - (\phi - 2\theta \otimes P)\{\phi(\nabla_Q X - \nabla_X Q + \omega(Q, X)\xi) \\ &\quad - 2\theta(\nabla_Q X - \nabla_X Q + \omega(Q, X)\xi)P\} \\ &= \nabla_{JX} P - \nabla_X Q + 2q(X)P + 2p(X)Q. \end{aligned} \tag{3.24}$$

Therefore we have

$$H(\xi_p, X) = \nabla_{JX} P + \nabla_X Q + 2q(X)P - 2p(X)Q. \tag{3.25}$$

In the equation $H(\xi_p, X) = H(\xi, X) - 2H(Q, X)$, we use (3.18) for $H(Q, X)$ and (3.25) for $H(\xi_p, X)$. Then we obtain the equation (3.19). □

REMARK. We can compute $H(X, \xi)$ and $H(\xi, \xi)$ by the same way as the equation (3.19). They are given by

$$H(X, \xi) = 2\nabla_X Q - 4p(X)Q - 4q(X)P + 2g(P, P)JX + 2p(X)\xi, \tag{3.26}$$

$$H(\xi, \xi) = 2(\nabla_\xi Q - \nabla_P P + \nabla_Q Q - 4g(P, P)P - 2p(\xi)Q) + 2p(\xi)\xi. \tag{3.27}$$

Conversely, one may ask whether given Tanaka connection $\nabla$ and $p$ define a CR Weyl connection. We have the following answer to this question.

PROPOSITION 3.6. *Let $(\phi, \xi, \theta)^*$ be a $\mathscr{D}$-preserving almost contact structure belonging to CR structure $(\mathscr{D}, J)$ and $\nabla$ Tanaka connection associated with $(\phi, \xi, \theta)^*$. If $D$ is defined by $D_V W = \nabla_V W + H(V, W)$ for a given $p$ satisfying* (3.16), *where $H$ is defined by* (3.18), (3.19), (3.26) *and* (3.27), *then it becomes a CR Weyl connection.*

By Propositon 3.6 we see that an arbitrary pair $(p, (\phi, \xi, \theta)^*)$ of a 1-form $p$ satisfying (3.16) and a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ determines a CR Weyl structure.

## 4. The View from G-Structure

Let $M$ be an oriented $(2n+1)$-dimensional manifold and $\pi : F^+(M) \to M$ the principal bundle of positively oriented frames over $M$. Assume that a pair $(\mathscr{D}, J)$ of a hyperdistribution $\mathscr{D}$ and a complex structure $J$ on $\mathscr{D}$ is given on $M$. In addition, we assume that $(\mathscr{D}, J)$ is a nondegenerate CR structure. Now we define the subspace $\mathscr{D}_0$ in $\mathbf{R}^{2n+1}$, the matrix $\tilde{J}_0 \in GL(2n+1; \mathbf{R})$ and the matrix $J_0 \in GL(2n; \mathbf{R})$ by

$$\mathscr{D}_0 = \left\{ \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} \in \mathbf{R}^{2n+1} \,\middle|\, x^0 = 0 \right\}, \quad \tilde{J}_0 = \begin{pmatrix} 0 & {}^t\mathbf{0} \\ \mathbf{0} & J_0 \end{pmatrix} \quad \text{and} \quad J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{4.1}$$

respectively, where $I_n$ is $n \times n$ unit matrix and the boldface denotes a column vector of degree $2n$. We have a principal subbundle $\mathfrak{P}$ of $F^+(M)$:

$$\bar{\mathfrak{P}} = \{u \in F(M) \mid u\mathscr{D}_0 \subset \mathscr{D}, Ju|_{\mathscr{D}_0} = u\tilde{J}_0|_{\mathscr{D}_0}\}$$

whose structure group is

$$\bar{G} = \left\{ \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix} \middle| a > 0, \mathbf{b} \in \mathbf{R}^{2n}, CJ_0 = J_0C \right\},$$

where the linear frame $u$ is considered as a linear map from $\mathbf{R}^{2n+1}$ to $T_{\pi(u)}M$ (cf. [4]). Futhermore we define $\theta_0 \in (\mathbf{R}^{2n+1})^*$ and $\xi_0 \in \mathbf{R}^{2n+1}$ by

$$\theta_0 = (1 \quad {}^t\mathbf{0}) \quad \text{and} \quad \xi_0 = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \tag{4.2}$$

respectively. By using a local cross section $\bar{\sigma}$ of $\bar{\mathfrak{P}}$, we define a 1-form $\theta^{\bar{\sigma}}$ and vector field $\xi^{\bar{\sigma}}$ on an open set $U^{\bar{\sigma}}$ by

$$\theta^{\bar{\sigma}} = \theta_0\bar{\sigma}^{-1} \quad \text{and} \quad \xi^{\bar{\sigma}} = \bar{\sigma}\xi_0 \tag{4.3}$$

respectively. Then we obtain

$$\theta^{\bar{\sigma}}|_{\mathscr{D}} = 0, \quad \theta^{\bar{\sigma}}(\xi^{\bar{\sigma}}) = 1 \tag{4.4}$$

because of their definitions. Note that the definitions of $\theta^{\bar{\sigma}}$ and $\xi^{\bar{\sigma}}$ are dependent of the local section $\bar{\sigma}$. Later on, we shall study between $\theta^{\bar{\sigma}}$ (resp. $\xi^{\bar{\sigma}}$) and $\theta^{\bar{\tau}}$ (resp. $\xi^{\bar{\tau}}$) defined by another local section $\bar{\tau}$ whose domain has non empty intersection with $U^{\bar{\sigma}}$.

Next, we define a 2-form $\omega^{\bar{\sigma}}$ by

$$\omega^{\bar{\sigma}} = -2\, d\theta^{\bar{\sigma}}. \tag{4.5}$$

Then since we assume that $(\mathscr{D}, J)$ is a nondegenerate CR structure, we see that $\omega^{\bar{\sigma}}$ is a nondegenerate and Hermitian 2-form when it is restricted to $\mathscr{D}$:

$$\omega^{\bar{\sigma}}(JX, JY) = \omega^{\bar{\sigma}}(X, Y) \tag{4.6}$$

for $X, Y \in \Gamma(U^{\bar{\sigma}}, \mathscr{D})$, where $\Gamma(U^{\bar{\sigma}}, \mathscr{D})$ denotes the set of cross sections on $U^{\bar{\sigma}}$ of the vector bundle $\mathscr{D}$. By using $\omega^{\bar{\sigma}}$, we define $B^{\bar{\sigma}} \in \Gamma(U^{\bar{\sigma}}, \mathscr{D})$ by

$$\omega^{\bar{\sigma}}(B^{\bar{\sigma}}, X) = -\omega^{\bar{\sigma}}(\xi^{\bar{\sigma}}, X) \tag{4.7}$$

for every $X \in \Gamma(U^{\bar{\sigma}}, \mathscr{D})$. This is uniquely defined since $\omega^{\bar{\sigma}}$ is nondegenerate. Moreover, define a local bilinear form $g^{\bar{\sigma}}$ on $\mathscr{D}$ by

$$g^{\bar{\sigma}}(X, Y) = \omega^{\bar{\sigma}}(JX, Y), \tag{4.8}$$

which satisfies the equations

$$(4.9)\qquad g^{\bar{\sigma}}(X,Y)=g^{\bar{\sigma}}(Y,X),\quad g^{\bar{\sigma}}(JX,JY)=g^{\bar{\sigma}}(X,Y)$$

for $X, Y \in \Gamma(U^{\bar{\sigma}}, \mathscr{D})$. Thus it becomes a fiber pseudo-metric of $\mathscr{D}$ defined on $U^{\bar{\sigma}}$. When we take two local cross sections $\bar{\sigma}$ and $\bar{\tau}$ of $\overline{\mathfrak{P}}$ defined on $U^{\bar{\sigma}}$ and $U^{\bar{\tau}}$ respectively, we suppose that they are related by $\bar{\tau}=\bar{\sigma}\bar{h}$ on $U^{\bar{\sigma}\bar{\tau}}$, where $U^{\bar{\sigma}\bar{\tau}}$ denotes the intersection of $U^{\bar{\sigma}}$ and $U^{\bar{\tau}}$, $\bar{h}$ is a $\bar{G}$-valued function of the form

$$(4.10)\qquad \bar{h}=\begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C\end{pmatrix}=\begin{pmatrix} e^{-2\mu} & {}^t\mathbf{0} \\ \mathbf{b} & C\end{pmatrix},$$

and $\mu$ a function on $U^{\bar{\sigma}\bar{\tau}}$. Then we obtain

$$(4.11)\qquad \theta^{\bar{\tau}}=e^{2\mu}\theta^{\bar{\sigma}}$$

on $U^{\bar{\sigma}\bar{\tau}}$. Thus we have

$$(4.12)\qquad \omega^{\bar{\tau}}=e^{2\mu}\omega^{\bar{\sigma}}-4\,d\mu\wedge\theta^{\bar{\tau}}$$

because of (4.5). Furthermore we have

$$(4.13)\qquad \omega^{\bar{\tau}}|_{\mathscr{D}}=e^{2\mu}\omega^{\bar{\sigma}}|_{\mathscr{D}},\quad g^{\bar{\tau}}|_{\mathscr{D}}=e^{2\mu}g^{\bar{\sigma}}|_{\mathscr{D}}.$$

Therefore we have the conformal structure $[g^{\bar{\sigma}}]$ over $\mathscr{D}$. Let $a(p,\bar{\sigma})$ be the dimension of the maximal subspace in $\mathscr{D}_p$ where $g^{\bar{\sigma}}_p$ is negative definite for each point $p\in M$ and local section $\bar{\sigma}$ defined on a neighborhood of $p$. These numbers are necessary even and we see from the equation (4.13) that $a(p,\bar{\sigma})$ depends only on $p$. So we put $\gamma(p)=a(p,\bar{\sigma})$. Since $\gamma$ is a lower semicontinuous function on $M$ and $M$ is connected, we easily see that it is constant.

Now we define a subbundle $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ by

$$\mathfrak{P}=\left\{u\in\overline{\mathfrak{P}}\;\middle|\; g^{\bar{\sigma}}_{\pi(u)}\left(u\begin{pmatrix}0\\ \mathbf{x}\end{pmatrix},u\begin{pmatrix}0\\ \mathbf{y}\end{pmatrix}\right)={}^t\mathbf{x}\widetilde{E_\gamma}\mathbf{y},\bar{\sigma}(\pi(u))=u\right\}$$

whose structure group is

$$(4.14)\qquad G=\left\{\begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C\end{pmatrix}: a>0, \mathbf{b}\in\mathbf{R}^{2n}, CJ_0=J_0C, {}^tC\widetilde{E_\gamma}C=a\widetilde{E_\gamma}\right\},$$

where

$$\widetilde{E_\gamma}=\begin{pmatrix} E_\gamma & 0 \\ 0 & E_\gamma\end{pmatrix},\quad E_\gamma=\begin{pmatrix} -I_\gamma & 0 \\ 0 & I_{n-\gamma}\end{pmatrix}.$$

We remark that $C\in CU_\gamma=GL(n,\mathbf{C})\cap CO(2\gamma,2n-2\gamma)$. A local cross section $\sigma$ of $\mathfrak{P}$ is witten as

(4.15) $$\sigma = \langle \xi^\sigma, X_1, \dots, X_n, JX_1, \dots, JX_n \rangle,$$

where $\{X_1, \dots, X_n, JX_1, \dots, JX_n\}$ is a local orthonormal frame field of $\mathscr{D}$ with respect to $g^\sigma$. And we can also express (4.15) as follows:

(4.16) $$X_i = \sigma e_i, \quad JX_i = \sigma \tilde{J}_0 e_i \quad (i = 1, \dots, n)$$

where $e_i = {}^t(0|0 \ \cdots \ 1 \ \cdots \ 0|0 \ \cdots \ 0)$.

Let $\mathfrak{g}$ and $\mathfrak{cu}_\gamma$ denote the Lie algebra of $G$ and $CU_\gamma$ respectively. Let $\vartheta = \begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix}$ be a connection form of a linear connection $D$ reducible to $\mathfrak{P}$, where $\phi$ is $\mathbf{R}$-valued 1-form, $\eta$ $\mathbf{R}^{2n}$-valued 1-form and $\alpha$ $\mathfrak{cu}_\gamma$-valued 1-form on $\mathfrak{P}$. The connection form $\vartheta$ satisfies

(4.17) $$\vartheta(A^*) = A, \quad R_h^*\vartheta = Ad(h^{-1})\vartheta,$$

where $A^*$ denotes the fundamental vector field corresponding to $A \in \mathfrak{g}$ and $h$ an element of $G$. Since

$$R_h^*\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix} = Ad(h^{-1})\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix} = \begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix}^{-1}\begin{pmatrix} \phi & {}^t\mathbf{0} \\ \eta & \alpha \end{pmatrix}\begin{pmatrix} a & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix}$$
$$= \begin{pmatrix} 1/a & {}^t\mathbf{0} \\ -(1/a)C^{-1}\mathbf{b} & C^{-1} \end{pmatrix}\begin{pmatrix} \phi a & {}^t\mathbf{0} \\ \eta a + \alpha\mathbf{b} & \alpha C \end{pmatrix} = \begin{pmatrix} \phi & {}^t\mathbf{0} \\ * & C^{-1}\alpha C \end{pmatrix}$$

where $* = C^{-1}(-\phi\mathbf{b} + a\eta + \alpha\mathbf{b})$, we have

(4.18) $$R_h^*\phi = \phi, \quad R_h^*\eta = C^{-1}(-\phi\mathbf{b} + a\eta + \alpha\mathbf{b}), \quad R_h^*\alpha = Ad(C^{-1})\alpha.$$

Now let $\sigma$ and $\tau$ be local cross sections of $\mathfrak{P}$ defined on $U^\sigma$ and $U^\tau$ respectively. Suppose that they are related by $\tau = \sigma h$ on $U^{\sigma\tau}$, where $h$ is a $G$-valued function of the form as (4.10) with $C \in CU_\gamma$. Then, for the differential maps of $\sigma$ and $\tau$, we have

(4.19) $$d\tau(V) = dR_h(d\sigma(V)) + (h^{-1}(dh)(V))^*$$

for $V \in \mathfrak{X}(U^{\sigma\tau})$ (cf. [4]). Applying the connection form $\vartheta$ to (4.19), we obtain, from (4.17),

(4.20) $$\tau^*\vartheta = Ad(h^{-1})\sigma^*\vartheta + h^{-1}\,dh.$$

On the other hand, we have

(4.21) $$h^{-1}\,dh = \begin{pmatrix} 1/a & 0 \\ -(1/a)C^{-1}\mathbf{b} & C^{-1} \end{pmatrix}\begin{pmatrix} da & {}^t\mathbf{0} \\ d\mathbf{b} & dC \end{pmatrix} = \begin{pmatrix} a^{-1}\,da & {}^t\mathbf{0} \\ ** & C^{-1}\,dC \end{pmatrix}$$

where $** = C^{-1}(-a^{-1}\mathbf{b}\, da + d\mathbf{b})$. In particular, we have

$$a^{-1}\, da = -2\, d\mu \tag{4.22}$$

by (4.10), and hence from (4.20) we obtain

$$\tau^*\phi = \sigma^*\phi - 2\, d\mu. \tag{4.23}$$

We put $2p^\sigma = \sigma^*\phi$ and $2p^\tau = \tau^*\phi$ for local cross section $\sigma$ and $\tau$ respectively. Thus we obtain

$$p^\tau = p^\sigma - d\mu. \tag{4.24}$$

We regard local cross sections of $\mathfrak{P}$ as those of $\bar{\mathfrak{P}}$. Then we also have $\theta^\sigma$, $\xi^\sigma$, $\omega^\sigma$ and $g^\sigma$ on $U^\sigma$. We define $P^\sigma, Q^\sigma \in \Gamma(U^\sigma, \mathscr{D})$ by

$$g^\sigma(P^\sigma, X) = p^\sigma(X), \quad Q^\sigma = JP^\sigma \tag{4.25}$$

for every $X \in \Gamma(U^\sigma, \mathscr{D})$. Then we have

LEMMA 4.1. *Let $\sigma$ and $\tau$ be two local cross sections of $\mathfrak{P}$ such that $\tau = \sigma h$ on $U^{\sigma\tau}$. We put $\xi_{p^\sigma} = \xi^\sigma + B^\sigma - 2Q^\sigma$, where $B^\sigma$ is defined by* (4.7). *Then $\xi_{p^\sigma}$ and $\xi_{p^\tau}$ are related by*

$$\xi_{p^\tau} = e^{-2\mu}\xi_{p^\sigma}. \tag{4.26}$$

*It follows that we have a transversal line bundle $\mathscr{L} = \mathrm{span}\{\xi_{p^\sigma}\}$ associated with the connection $D$.*

PROOF. Since $0 = \omega^\tau(\xi^\tau + B^\tau, X) = \omega^\sigma(\xi^\sigma + B^\sigma, X)$ because of (4.7), by using (4.12) we have

$$\begin{aligned} 0 = \omega^\tau(\xi^\tau + B^\tau, X) &= e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) - 4(d\mu \wedge \theta^\tau)(\xi^\tau + B^\tau, X) \\ &= e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) - \omega^\sigma(\xi^\sigma + B^\sigma, X) + 2\, d\mu(X). \end{aligned}$$

Define $(d\mu^\#)^\sigma \in \Gamma(U^{\sigma\tau}, \mathscr{D})$ by

$$g^\sigma((d\mu^\#)^\sigma, X) = d\mu(X) \tag{4.27}$$

for every $X \in \Gamma(U^{\sigma\tau}, \mathscr{D})$. We have

$$\begin{aligned} e^{2\mu}\omega^\sigma(\xi^\tau + B^\tau, X) &= \omega^\sigma(\xi^\sigma + B^\sigma, X) - 2g^\sigma((d\mu^\#)^\sigma, X) \\ &= \omega^\sigma(\xi^\sigma + B^\sigma, X) - 2\omega^\sigma(J(d\mu^\#)^\sigma, X). \end{aligned}$$

Therefore, since $\omega^\sigma$ is nondegenerate, we obtain

$$\xi^\tau + B^\tau = e^{-2\mu}(\xi^\sigma + B^\sigma - 2J(d\mu^\#)^\sigma). \tag{4.28}$$

We have, from (4.24),

$$\begin{aligned} g^\tau(Q^\tau, X) = g^\tau(JP^\tau, X) &= -g^\tau(P^\tau, JX) \\ &= -p^\tau(JX) \\ &= -(p^\sigma - d\mu)(JX) \\ &= -p^\sigma(JX) + d\mu(JX) \\ &= g^\sigma(Q^\sigma - J(d\mu^\#)^\sigma, X). \end{aligned}$$

It follows that

$$Q^\tau = e^{-2\mu}(Q^\sigma - J(d\mu^\#)^\sigma). \tag{4.29}$$

Combining (4.28) with (4.29), we obtain (4.26). □

Next we investigate the covariant derivative $D$ of $TM$ determined by $\vartheta$. We take a fixed local frame field (4.15) of $TM$. Note that, for a fixed $W \in \mathfrak{X}(U^\sigma)$, the local frame field $\sigma$ induces a map $\sigma^{-1}W : x \in U^\sigma \mapsto \sigma(x)^{-1}W(x) \in \mathbf{R}^{2n+1}$. The covariant derivative of $W \in \mathfrak{X}(U^\sigma)$ in the direction $V \in TU^\sigma$ is given by

$$D_V W = \sigma(d(\sigma^{-1}W)(V) + (\sigma^*\vartheta(V))(\sigma^{-1}W)). \tag{4.30}$$

Futhermore, since $\sigma^{-1}X \in \mathbf{R}^{2n}$ for $X \in \Gamma(U^\sigma, \mathscr{D})$, we obtain

$$D_V X = \sigma(d(\sigma^{-1}X)(V) + (\sigma^*\alpha(V))(\sigma^{-1}X)). \tag{4.31}$$

Note that the product of the second term of the right hand side in the equations above is the matrix multiplication. From (4.31), it is clear that

$$D_V\Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D}). \tag{4.32}$$

It follows that $D$ induces the covariant differentiation of the vector bundle $\mathscr{D}$, which is denoted by $D^\circ$.

LEMMA 4.2. *Let $D$ be the covariant derivative of $TM$ determined by $\vartheta$ and $D^\circ$ the covariant derivative on $\mathscr{D}$ determined by $\alpha$. Then $D$ and $D^\circ$ satisfy*

$$D_V\theta^\sigma = -2p^\sigma(V)\theta^\sigma, \quad D^\circ_V J = 0 \quad \textit{and} \quad D^\circ_V g^\sigma = -2p^\sigma(V)g^\sigma \tag{4.33}$$

*for $V \in TU^\sigma$.*

PROOF. Since

$$(D_V\theta^\sigma)(X) = V\cdot\theta^\sigma(X) - \theta^\sigma(D_VX) = -\theta^\sigma(D_V^\circ X) = 0$$

for every $X \in \Gamma(U^\sigma, \mathscr{D})$, we have $(D_V\theta^\sigma)(X) = -2p^\sigma(V)\theta^\sigma(X)$. Futhermore, we have

$$\begin{aligned}(D_V\theta^\sigma)(\xi^\sigma) &= V\cdot(\theta^\sigma\xi^\sigma) - \theta^\sigma(D_V\xi_{p^\sigma}) \\ &= -\theta^\sigma(\sigma((\sigma^*\vartheta V)(\sigma^{-1}\xi^\sigma))) = -\sigma^*\phi(V)\theta^\sigma(\xi^\sigma).\end{aligned}$$

Thus we obtain $(D_V\theta^\sigma)(W) = -2p^\sigma(V)\theta^\sigma(W)$ for every $W \in \mathfrak{X}(U^\sigma)$. Next we have

$$\begin{aligned}J(D_VY_\lambda) &= J\{\sigma((\sigma^*\alpha(V))\sigma^{-1}Y_\lambda)\} \\ &= \sigma\{\widetilde{J}_0(\sigma^*\alpha(V))\sigma^{-1}Y_\lambda\} \\ &= \sigma\{(\sigma^*\alpha(V))\sigma^{-1}(\sigma\widetilde{J}_0\sigma^{-1}Y_\lambda)\} \\ &= \sigma\{(\sigma^*\alpha(V))(\sigma^{-1}JY_\lambda)\} \\ &= D_V(JY_\lambda)\end{aligned}$$

for $\lambda = 1,\dots,2n$, where we have put $Y_i = X_i$, $Y_{n+i} = JX_i$ $(i = 1,\dots,n)$. Therefore, since $(D_V^\circ J)(Y_\lambda) = D_V(JY_\lambda) - J(D_VY_\lambda)$, we obtain $D_V^\circ J = 0$. At last we show that $D_V^\circ g^\sigma = -2p^\sigma(V)g^{\bar{\sigma}}$. Since

$${}^t\alpha\widetilde{E_\gamma} + \widetilde{E_\gamma}\alpha = \phi\widetilde{E_\gamma},$$

we have

$$\begin{aligned}g^\sigma(D_V^\circ Y_\lambda, Y_\mu) &= {}^t\{(\sigma^*\alpha(V))\sigma^{-1}Y_\lambda\}\widetilde{E_\gamma}(\sigma^{-1}Y_\mu) \\ &= {}^t(\sigma^{-1}Y_\lambda){}^t(\sigma^*\alpha(V))\widetilde{E_\gamma}(\sigma^{-1}Y_\mu) \\ &= {}^t(\sigma^{-1}Y_\lambda)\{-\widetilde{E_\gamma}(\sigma^*\alpha(V)) + \sigma^*\phi(V)\widetilde{E_\gamma}\}(\sigma^{-1}Y_\mu) \\ &= -g^\sigma(Y_\lambda, D_V^\circ Y_\mu) + 2p^\sigma(V)g^\sigma(Y_\lambda, Y_\mu).\end{aligned}$$

Therefore, for local frame $\{Y_\lambda\}$ of $\mathscr{D}$, we have

$$(D_V^\circ g^\sigma)(Y_\lambda, Y_\mu) = -2p^\sigma(V)g^\sigma(Y_\lambda, Y_\mu) \tag{4.34}$$

from which we obtain $D_V^\circ g^\sigma = -2p^\sigma(V)g^\sigma$. □

Now assume that the torsion tensor $T$ of $D$ satisfies

(4.35) $$T(X, Y) \in \mathscr{L},$$

(4.36) $$T(L, X) \in \mathscr{D}, \quad T(L, JX) = -JT(L, X),$$

(4.37) $$(D_U T)(X, Y) = 0$$

for $U \in TM$, $X, Y \in \mathscr{D}$ and $L \in \mathscr{L}$. Then we have

$$\theta^\sigma(T(X, Y)) = -\omega^\sigma(X, Y), \quad \theta^\sigma(\xi_{p^\sigma}) = 1$$

because of (4.32) and (4.35). Therefore we obtain

(4.38) $$T(X, Y) = -\omega^\sigma(X, Y)\xi_{p^\sigma}$$

for $X, Y \in \mathscr{D}$. We define $(1,1)$ tensor $\phi^\sigma$ by $\phi^\sigma \xi^\sigma = 0$ and $\phi^\sigma X = JX$ for $X \in \mathscr{D}$, and moreover $\phi_{p^\sigma}$ by

(4.39) $$\phi_{p^\sigma} = \phi^\sigma - 2\theta \otimes \left(P^\sigma + \frac{1}{2} JB^\sigma\right).$$

It is easy to show that

(4.40) $$\phi_{p^\sigma} X = JX, \quad \phi_{p^\sigma} = \phi_{p^\tau}, \quad \theta^\sigma \circ \phi_{p^\sigma} = 0, \quad \phi_{p^\sigma}\xi_{p^\sigma} = 0.$$

Since $\theta^\sigma(D_V \xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma}) = 0$, we have

$$\begin{aligned} g^\sigma(D_V \xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma}, Y) &= \omega^\sigma(J(D_V \xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma}), Y) \\ &= \omega^\sigma(\phi_{p^\sigma}(D_V \xi_{p^\sigma}) - 2p^\sigma(V)\phi_{p^\sigma}\xi_{p^\sigma}, Y) \\ &= \omega^\sigma(\phi_{p^\sigma}(D_V \xi_{p^\sigma}), Y) \\ &= \omega^\sigma(-(D_V \phi_{p^\sigma})\xi_{p^\sigma}, Y) \\ &= g^\sigma(J(D_V \phi_{p^\sigma})\xi_{p^\sigma}, Y). \end{aligned}$$

Therefore we obtain

(4.41) $$D_V \xi_{p^\sigma} - 2p^\sigma(V)\xi_{p^\sigma} = J(D_V \phi_{p^\sigma})\xi_{p^\sigma}$$

for $V \in \mathfrak{X}(U^\sigma)$. From (4.37) and (4.33), we have

$$\begin{aligned} 0 &= D_V(T(X, Y)) - T(D_V X, Y) - T(X, D_V Y) \\ &= -D_V(\omega^\sigma(X, Y)\xi_{p^\sigma}) + \omega^\sigma(D_V X, Y)\xi_{p^\sigma} + \omega^\sigma(X, D_V Y)\xi_{p^\sigma} \\ &= 2p^\sigma(V)\omega^\sigma(X, Y)\xi_{p^\sigma} - \omega^\sigma(X, Y) D_V \xi_{p^\sigma}. \end{aligned}$$

Combining this equation with (4.41), we obtain

$$D_V\xi_{p^\sigma} = 2p^\sigma(V)\xi_{p^\sigma}, \quad D_V\phi_{p^\sigma} = 0. \tag{4.42}$$

In particular, $D_{\xi_{p^\sigma}}\phi_{p^\sigma} = 0$ and hence

$$\begin{aligned} 0 &= D_{\xi_{p^\sigma}}(\phi_{p^\sigma}X) - \phi_{p^\sigma}(D_{\xi_{p^\sigma}}X) \\ &= F_{p^\sigma}\phi_{p^\sigma}X + D_{\phi_{p^\sigma}X}\xi_{p^\sigma} + [\xi_{p^\sigma}, \phi_{p^\sigma}X] - \phi_{p^\sigma}(F_{p^\sigma}X + D_X\xi_{p^\sigma} + [\xi_{p^\sigma}, X]) \\ &= [F_{p^\sigma}, J]X + (\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X + 2p^\sigma(JX)\xi_{p^\sigma}, \end{aligned}$$

where we have put $F_{p^\sigma}X = T(\xi_{p^\sigma}, X)$. Equation (4.36) implies that

$$T(\xi_{p^\sigma}, X) = -\frac{1}{2}\phi_{p^\sigma}(\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X. \tag{4.43}$$

Finally, if $\sigma$ satisfies $\omega^\sigma(X, \xi^\sigma) = 0$, then we obtain, from Lemma 3.4,

$$dp^\sigma(JX, JY) + dp^\sigma(X, Y) = 0 \tag{4.44}$$

for $X, Y \in \mathscr{D}$.

PROPOSITION 4.3. *Let $\mathfrak{P}(M, G)$ be the subbundle determined by the CR structure and D a linear connection reducible to $\mathfrak{P}(M, G)$. Then there is a 1-dimensional distribution $\mathscr{L}$ on M transversal to $\mathscr{D}$. For a local cross section $\sigma$ of $\mathfrak{P}(M, G)$, D satisfies*

$$D_V\theta^\sigma = -2p^\sigma(V)\theta^\sigma, \quad D^\circ_V J = 0, \quad D^\circ_V g^\sigma = -2p^\sigma(V)g^\sigma$$

*for $V \in \mathfrak{X}(M)$. Moreover, if the torsion tensor T of D satisfies*

$$T(X, Y) \in \mathscr{L}, \quad T(L, X) \in \mathscr{D}, \quad T(L, JX) = -JT(L, X), \quad (D_V T)(X, Y) = 0$$

*for $X, Y \in \mathscr{D}$ and $L \in \mathscr{L}$, then D satisfies*

$$T(X, Y) = -\omega^\sigma(X, Y)\xi_{p^\sigma}, \quad T(\xi_{p^\sigma}, X) = -\frac{1}{2}\phi_{p^\sigma}(\mathscr{L}_{\xi_{p^\sigma}}\phi_{p^\sigma})X,$$

$$D_V\xi_{p^\sigma} = 2p^\sigma(V)\xi_{p^\sigma}, \quad D_V\phi_{p^\sigma} = 0$$

*and if $\omega^\sigma(X, \xi^\sigma) = 0$ holds for $X \in \mathscr{D}$, then $p^\sigma$ satisfies*

$$dp^\sigma(JX, JY) + dp^\sigma(X, Y) = 0.$$

REMARK. We assume that $M$ is orientable. Then we have a nonvanishing globally defined vector field $\xi$ transversal to $\mathscr{D}$. Then, for the local cross section $\sigma$ and $\tau$ of the form (4.15), $h$ reduces to a matrix that

$$\begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{b} & C \end{pmatrix},$$

where ${}^tC\widetilde{E_\gamma}C = \widetilde{E_\gamma}$. It follows from (4.11), (4.13) and (4.24) that $\theta$, $\phi$, $\omega$, $g$ and $p$ are globally defined on $M$, and $\xi_p$ is a global section of $\mathscr{L}$. Moreover, if we take $\xi$ such that $\omega(X,\xi) = 0$ for every $X \in \mathscr{D}$, then $p$ satisfies (3.16).

## 5. Curvature of CR Weyl Connection

In this section, we investigate the property of the curvature of a CR Weyl connection. Let $D$ be a CR Weyl connection of the CR structure $(\mathscr{D}, J)$. Let $R$ be the curvature tensor field of $D$ defined by

$$R(U,V)W = D_U D_V W - D_V D_U W - D_{[U,V]} W$$

for $U, V, W \in \mathfrak{X}(M)$. We fix a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ and let $p$ be the 1-form of $D$ corresponding to $(\phi, \xi, \theta)^*$. Since $D\xi_p = 2p \otimes \xi_p$, we see easily that

(5.1) $$R(U,V)\xi_p = 4\, dp(U,V)\xi_p, \quad U, V \in TM.$$

The property $D_U \Gamma(\mathscr{D}) \subset \Gamma(\mathscr{D})$ implies that

(5.2) $$R(U,V)\mathscr{D} \subset \mathscr{D}, \quad U, V \in TM.$$

Since $D\phi_p = 0$, we have

(5.3) $$R(U,V)\phi_p = \phi_p R(U,V), \quad U, V \in TM.$$

If we put $R(U,V,X,Y) = g(R(U,V)X,Y)$ for $U, V \in TM$ and $X, Y \in \mathscr{D}$, then we have the equation

(5.4) $$R(U,V,X,Y) = -R(U,V,Y,X) + 4\, dp(U,V) g(X,Y).$$

The first Bianchi identity is the formula (cf. [4]):

$$\mathfrak{S}\{R(U,V)W\} = \mathfrak{S}\{T(T(U,V),W) + (D_U T)(V,W)\},$$

where $U, V, W \in TM$ and $\mathfrak{S}$ denotes the cyclic sum with respect to $U$, $V$ and $W$. Replacing $U$, $V$, $W$ with $X, Y, Z \in \mathscr{D}$ respectively in the first Bianchi identity above, we have, from (3.2),

$$\mathfrak{S}\{R(X,Y)Z\} = \mathfrak{S}\{T(T(X,Y),Z)\}.$$

Moreover, applying the condition (e) in the definition of a CR Weyl connection to the above equation, we obtain

$$\mathfrak{S}\{R(X,Y)Z\} = -\mathfrak{S}\{\omega(X,Y)F_pZ\} \tag{5.5}$$

for every $X, Y, Z \in \mathscr{D}$. Putting $U = \xi_p$ and replacing $V$, $W$ with $Y, Z \in \mathscr{D}$ in the first Bianchi identity, we have

$$\begin{aligned}
R(\xi_p, Y)Z &+ R(Y,Z)\xi_p + R(Z,\xi_p)Y \\
&= T(T(\xi_p, Y), Z) + T(T(Y,Z),\xi_p) + T(T(Z,\xi_p), Y) \\
&\quad + (D_{\xi_p}T)(Y,Z) + (D_YT)(Z,\xi_p) + (D_ZT)(\xi_p, Y) \\
&= T(F_pY,Z) + T(-\omega(Y,Z)\xi_p,\xi_p) - T(T(\xi_p,Z), Y) \\
&\quad - (D_YT)(\xi_p, Z) + (D_ZT)(\xi_p, Y) \\
&= -\omega(F_pY,Z)\xi_p + \omega(F_pZ, Y)\xi_p - D_Y(T(\xi_p, Z)) \\
&\quad + T(D_Y\xi_p, Z) + T(\xi_p, D_YZ) \\
&\quad + D_Z(T(\xi_p, Y)) - T(D_Z\xi_p, Y) - T(\xi_p, D_ZY) \\
&= -\omega(F_pY,Z)\xi_p + \omega(F_pZ, Y)\xi_p - (D_YF_p)Z \\
&\quad + 2p(Y)F_pZ + (D_ZF_p)Y - 2p(Z)F_pY,
\end{aligned}$$

where we have used (3.2) and (b), (e) in the definition of a CR Weyl connection. In addition, when we rewrite (3.17) with $\omega$, we have

$$\omega(F_pX, Y) + \omega(X,F_pY) = -4\, dp(X, Y). \tag{5.6}$$

Substituting (5.1) and (5.6) into the first Bianchi identity including $\xi_p$ above, we obtain

$$R(\xi_p, Y)Z - R(\xi_p,Z)Y = -\{(D_YF_p)Z - 2p(Y)F_pZ\} + \{(D_ZF_p)Y - 2p(Z)F_pY\} \tag{5.7}$$

for $Y, Z \in \mathscr{D}$. Since the second Bianchi identity is the formula:

$$\mathfrak{S}\{(D_UR)(V,W)\} = -\mathfrak{S}\{R(T(U,V),W)\}$$

for $U, V, W \in TM$, we have immediately

$$\mathfrak{S}\{(D_XR)(Y,Z)\} = \mathfrak{S}\{\omega(X,Y)R(\xi_p,Z)\} \tag{5.8}$$

for $X, Y, Z \in \mathscr{D}$. Furthermore, if we put $U = \xi_p$ and replace $V$, $W$ with $Y, Z \in \mathscr{D}$ respectively in the second Bianchi identity, then

(5.9)
$$(D_{\xi_p}R)(Y,Z) - (D_Y R)(\xi_p, Z) + (D_Z R)(\xi_p, Y) = -R(F_p Y, Z) + R(F_p Z, Y).$$

We shall prove the following formula:

$$\begin{aligned}(5.10)\quad & R(X,Y,Z,W) - 2g(JX,Y)\,dp(JZ,W) + 2g(X,Y)\,dp(Z,W) \\ &\quad - R(Z,W,X,Y) + 2g(JZ,W)\,dp(JX,Y) - 2g(Z,W)\,dp(X,Y) \\ &= -g(JX,Z)g(F_p Y,W) + 2g(JX,Z)\,dp(JY,W) + 2g(X,Z)\,dp(Y,W) \\ &\quad + g(JY,Z)g(F_p X,W) - 2g(JY,Z)\,dp(JX,W) - 2g(Y,Z)\,dp(X,W) \\ &\quad - g(JY,W)g(F_p X,Z) + 2g(JY,W)\,dp(JX,Z) + 2g(Y,W)\,dp(X,Z) \\ &\quad + g(JX,W)g(F_p Y,Z) - 2g(JX,W)\,dp(JY,Z) - 2g(X,W)\,dp(Y,Z),\end{aligned}$$

where $X, Y, Z, W \in \mathscr{D}$. If we put

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W),$$

then we have

$$\begin{aligned}&\tilde{R}(X,Y,Z,W) - \tilde{R}(Y,Z,W,X) - \tilde{R}(Z,W,X,Y) + \tilde{R}(W,X,Y,Z) \\ &\quad = 2\{R(Y,Z,X,W) - R(W,X,Z,Y)\} \\ &\qquad + 4\,dp(X,Y)g(Z,W) - 4\,dp(Y,Z)g(X,W) + 4\,dp(Z,X)g(Y,W) \\ &\qquad - 4\,dp(Z,W)g(Y,X) + 4\,dp(Y,W)g(Z,X) + 4\,dp(W,X)g(Y,Z)\end{aligned}$$

because of (5.4). The equation (5.5) shows

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = -\{&\omega(X,Y)g(F_p Z,W) + \omega(Y,Z)g(F_p X,W) \\ &+ \omega(Z,X)g(F_p Y,W)\}.\end{aligned}$$

Combining the two equations above, applying (3.16) and (3.17) to the obtained equation and changing $Y$ for $X$, $Z$ for $Y$ and $X$ for $Z$, we have (5.10). From (5.7) we have

$$\begin{aligned}R(\xi_p, Y, Z, W) - R(\xi_p, Z, Y, W) = &-g((D_Y F_p)Z, W) + 2p(Y)g(F_p Z, W) \\ &+ g((D_Z F_p)Y, W) - 2p(Z)g(F_p Y, W),\end{aligned}$$

in which we permute the letters $Y$, $Z$ and $W$ cyclically and subtract one from the sum of the other two. Then we have

$$(5.11)\quad \begin{aligned} &2R(\xi_p, Z, W, Y) \\ &\quad + 4\,dp(\xi_p, Y)g(Z, W) - 4\,dp(\xi_p, Z)g(W, Y) - 4\,dp(\xi_p, W)g(Z, Y) \\ &= -g((D_Y F_p)Z, W) + 2p(Y)g(F_p Z, W) \\ &\quad + g((D_Z F_p)Y, W) - 2p(Z)g(F_p Y, W) \\ &\quad + g((D_W F_p)Y, Z) - 2p(W)g(F_p Y, Z) \\ &\quad - g((D_Y F_p)W, Z) + 2p(Y)g(F_p W, Z) \\ &\quad - g((D_Z F_p)W, Y) + 2p(Z)g(F_p W, Y) \\ &\quad + g((D_W F_p)Z, Y) - 2p(W)g(F_p Z, Y) \end{aligned}$$

because of (5.4). Note that $D_V F_p$ satisfies the following equation

$$(5.12)\quad g((D_V F_p)X, Y) = g(X, (D_V F_p)Y) + 4(D_V\, dp)(JX, Y) + 8p(V)\, dp(JX, Y)$$

for $V \in TM$ and $X, Y \in \mathscr{D}$, which is obtained from (3.17). Moreover note that (3.16) shows that

$$(5.13)\qquad (D_V\, dp)(JX, JY) = -(D_V\, dp)(X, Y)$$

for $V \in TM$ and $X, Y \in \mathscr{D}$. Applying (5.12) and (5.13) to (5.11), we obtain

$$(5.14)\quad \begin{aligned} &R(\xi_p, Y, Z, W) \\ &= g(Y, (D_Z F_p)W - (D_W F_p)Z) + g(Y, 2p(W)F_p Z - 2p(Z)F_p W) \\ &\quad - 2(D_W\, dp)(JY, Z) + 2(D_Y\, dp)(JW, Z) + 2(D_Z\, dp)(JY, W) \\ &\quad - 2\, dp(\xi_p, W)g(Y, Z) + 2\, dp(\xi_p, Y)g(Z, W) + 2\, dp(\xi_p, Z)g(Y, W) \end{aligned}$$

for every $Y, Z, W \in \mathscr{D}$.

Next we get the following formula for the difference of $R(JX, JY)$ and $R(X, Y)$:

$$(5.15)\quad \begin{aligned} &R(JX, JY)Z - R(X, Y)Z \\ &= g(JX, Z)F_p Y - g(JY, Z)F_p X + g(X, Z)F_p JY - g(Y, Z)F_p JX \\ &\quad + f_p(X, Z)JY - f_p(Y, Z)JX + f_p(JX, Z)Y - f_p(JY, Z)X \\ &\quad - 4\, dp(X, Y)Z + 4\, dp(JX, Y)JZ, \end{aligned}$$

where $X, Y, Z \in \mathscr{D}$ and we have defined $f_p$ by

$$f_p(X, Y) = g(F_pX, Y), \quad X, Y \in \mathscr{D}.$$

This formula can be proved by using equations (3.16), (5.3) and (5.10). In fact we see that

$$\begin{aligned}
&R(JX,JY,Z,W) - 2g(J^2X,JY)\,dp(JZ,W) + 2g(JX,JY)\,dp(Z,W)\\
&\quad = R(Z,W,JX,JY) - 2g(JZ,W)\,dp(J^2X,JY) + 2g(Z,W)\,dp(JX,JY)\\
&\qquad - g(J^2X,Z)g(F_pJY,W) + 2g(J^2X,Z)\,dp(J^2Y,W) + 2g(JX,Z)\,dp(JY,W)\\
&\qquad + g(J^2Y,Z)g(F_pJX,W) - 2g(J^2Y,Z)\,dp(J^2X,W) - 2g(JY,Z)\,dp(JX,W)\\
&\qquad - g(J^2Y,W)g(F_pJX,Z) + 2g(J^2Y,W)\,dp(J^2X,Z) + 2g(JY,W)\,dp(JX,Z)\\
&\qquad + g(J^2X,W)g(F_pJY,Z) - 2g(J^2X,W)\,dp(J^2Y,Z) - 2g(JX,W)\,dp(JY,Z)\\
&\quad = \{R(Z,W,X,Y) - 2g(JZ,W)\,dp(JX,Y) + 2g(Z,W)\,dp(X,Y)\}\\
&\qquad + 4g(JZ,W)\,dp(JX,Y) - 4g(Z,W)\,dp(X,Y)\\
&\qquad + g(X,Z)g(F_pJY,W) + 2g(X,Z)\,dp(Y,W) + 2g(JX,Z)\,dp(JY,W)\\
&\qquad - g(Y,Z)g(F_pJX,W) - 2g(Y,Z)\,dp(X,W) - 2g(JY,Z)\,dp(JX,W)\\
&\qquad + g(Y,W)g(F_pJX,Z) + 2g(Y,W)\,dp(X,Z) + 2g(JY,W)\,dp(JX,Z)\\
&\qquad - g(X,W)g(F_pJY,Z) - 2g(X,W)\,dp(Y,Z) - 2g(JX,W)\,dp(JY,Z)\\
&\quad = \{R(X,Y,Z,W) - 2g(JX,Y)\,dp(JZ,W) + 2g(X,Y)\,dp(Z,W)\\
&\qquad\ + g(JX,Z)g(F_pY,W) - 2g(JX,Z)\,dp(JY,W) - 2g(X,Z)\,dp(Y,W)\\
&\qquad\ - g(JY,Z)g(F_pX,W) + 2g(JY,Z)\,dp(JX,W) + 2g(Y,Z)\,dp(X,W)\\
&\qquad\ + g(JY,W)g(F_pX,Z) - 2g(JY,W)\,dp(JX,Z) - 2g(Y,W)\,dp(X,Z)\\
&\qquad\ - g(JX,W)g(F_pY,Z) + 2g(JX,W)\,dp(JY,Z) + 2g(X,W)\,dp(Y,Z)\}\\
&\qquad + 4g(JZ,W)\,dp(JX,Y) - 4g(Z,W)\,dp(X,Y)\\
&\qquad + g(X,Z)g(F_pJY,W) + 2g(X,Z)\,dp(Y,W) + 2g(JX,Z)\,dp(JY,W)\\
&\qquad - g(Y,Z)g(F_pJX,W) - 2g(Y,Z)\,dp(X,W) - 2g(JY,Z)\,dp(JX,W)\\
&\qquad + g(Y,W)g(F_pJX,Z) + 2g(Y,W)\,dp(X,Z) + 2g(JY,W)\,dp(JX,Z)\\
&\qquad - g(X,W)g(F_pJY,Z) - 2g(X,W)\,dp(Y,Z) - 2g(JX,W)\,dp(JY,Z).
\end{aligned}$$

We turn to the study of the Ricci tensor field of a CR Weyl connection. We shall define two kinds of Ricci tensors. In general, Ricci tensor field $s$ is defined by

$$s(V, W) = \text{trace of } (U \to R(U, V)W) \tag{5.16}$$

for $V, W \in TM$. We define another Ricci tensor field $k$ by

$$k(V, W) = \frac{1}{2}\,\text{trace}(\phi_p R(V, \phi_p W)) \tag{5.17}$$

for $V, W \in TM$. Restricting $s$ to $\mathscr{D}$, we obtain the following equation

$$s(X, Y) - s(Y, X) = -4(n+1)\, dp(X, Y) \tag{5.18}$$

for every $X, Y \in \mathscr{D}$. The proof of (5.18) is as follows: Noting that $R$ satisfies (5.2), we may consider the contraction in only $\mathscr{D}$. Since

$$\text{trace}_{\mathscr{D}}(R(V, W)) = 4n\, dp(V, W), \quad V, W \in TM, \tag{5.19}$$

where we have used (5.4) and $\text{trace}_{\mathscr{D}}$ denotes the trace in only $\mathscr{D}$, we have

$$s(X, Y) - s(Y, X) = \text{trace}_{\mathscr{D}}(Z \to \mathfrak{S}\{R(Z, X)Y\}) - 4n\, dp(X, Y).$$

Therefore, from (5.5), (3.17) and the fact that $\text{trace}_{\mathscr{D}} F_p = 0$,

$$\begin{aligned} s(X, Y) - s(Y, X) &= -\text{trace}_{\mathscr{D}}(Z \to \mathfrak{S}\{\omega(Z, X)F_p Y\}) - 4n\, dp(X, Y) \\ &= g(F_p X, JY) - g(F_p JY, X) - \omega(X, Y)\,\text{trace}_{\mathscr{D}} F_p - 4n\, dp(X, Y) \\ &= -4(n+1)\, dp(X, Y). \end{aligned}$$

Next we obtain the relation between $s$ and $k$:

$$k(X, Y) = s(X, Y) - (n-1) f_p(JX, Y) - 2n\, dp(X, Y), \quad X, Y \in \mathscr{D}. \tag{5.20}$$

The equation (5.20) can be shown as follows:

$$\begin{aligned} s(X, Y) &= \text{trace}_{\mathscr{D}}(Z \to -JR(Z, X)JY) \\ &= \text{trace}_{\mathscr{D}}(Z \to JR(X, JY)Z + JR(JY, Z)X \\ &\qquad + \omega(X, JY)JF_p Z + \omega(JY, Z)JF_p X + \omega(Z, X)F_p Y) \\ &= 2k(X, Y) + \text{trace}_{\mathscr{D}}(Z \to JR(JY, Z)X) \\ &\qquad - \omega(X, JY)\,\text{trace}_{\mathscr{D}}(F_p J) + g(JF_p X, Y) + g(F_p Y, JX) \\ &= 2k(X, Y) + \text{trace}_{\mathscr{D}}(Z \to JR(JY, Z)X) + 4\, dp(X, Y), \end{aligned}$$

where we have used (5.5), (3.17) and the fact that $F_p$ anticommutes with $J$, and using (5.15) and (3.17) again, we have

$$\begin{aligned}
&\mathrm{trace}_{\mathscr{D}}(Z \to JR(JY,Z)X)\\
&\quad = \mathrm{trace}_{\mathscr{D}}(JZ \to JR(JY,JZ)X)\\
&\quad = \mathrm{trace}_{\mathscr{D}}(Z \to R(JY,JZ)X)\\
&\quad = \mathrm{trace}_{\mathscr{D}}(Z \to R(Y,Z)X + g(JY,X)F_pZ - g(JZ,X)F_pY + g(Y,X)F_pJZ\\
&\qquad - g(Z,X)F_pJY + f_p(Y,X)JZ - f_p(Z,X)JY + f_p(JY,X)Z - f_p(JZ,X)Y\\
&\qquad - 4\,dp(Y,Z)X + 4\,dp(JY,Z)JX)\\
&\quad = -s(Y,X) + g(JY,X)\,\mathrm{trace}_{\mathscr{D}}\,F_p + g(F_pY,JX) + g(Y,X)\,\mathrm{trace}_{\mathscr{D}}(F_pJ)\\
&\qquad - g(F_pJY,X) + f_p(Y,X)\,\mathrm{trace}_{\mathscr{D}}\,J - g(F_pJY,X) + 2ng(F_pJY,X)\\
&\qquad - g(F_pJY,X) - 4\,dp(Y,X) + 4\,dp(X,Y)\\
&\quad = -s(Y,X) + 2(n-1)f_p(JX,Y) + 8n\,dp(X,Y),
\end{aligned}$$

which shows (5.20). From equations (5.18) and (5.20) we obtain

$$k(X,Y) - k(Y,X) = -4(n+2)\,dp(X,Y) \tag{5.21}$$

for every $X, Y \in \mathscr{D}$. The defining equation (5.17) of $k$ shows the following property

$$k(JX,JY) - k(X,Y) = 4(n+2)\,dp(X,Y) \tag{5.22}$$

for every $X, Y \in \mathscr{D}$. It follows that

$$s(JX,JY) - s(X,Y) = -2(n-1)f_p(JX,Y) + 8\,dp(X,Y) \tag{5.23}$$

for every $X, Y \in \mathscr{D}$. It is easy to show

$$s(X,\xi_p) = -4\,dp(X,\xi_p), \quad X \in \mathscr{D}. \tag{5.24}$$

Furthermore, by making use of (5.7) and (5.19) we obtain

$$\begin{aligned}
s(\xi_p,X) &= \mathrm{trace}_{\mathscr{D}}(Z \to (D_ZF_p)X) - 2p(F_pX) - 4n\,dp(\xi_p,X)\\
&= \mathrm{trace}_{\mathscr{D}}(Z \to (D_ZT)(\xi_p,X)) - 4n\,dp(\xi_p,X), \quad X \in \mathscr{D}.
\end{aligned} \tag{5.25}$$

We introduce two notations for later use. Define $S \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$g(SX,Y) = s(X,Y), \quad X, Y \in \mathscr{D} \tag{5.26}$$

and $\rho$ by

(5.27) $$\rho = \mathrm{trace}_{\mathscr{D}}\, S$$

which is a smooth function on $M$ and will be called scalar curvature.

Finally we state the following lemma and conclude this section.

PROPOSITION 5.1. *The Ricci tensor field s satisfies*

(5.28) $$\sum_{i=1}^{2n} \varepsilon_i (D_{e_i} s)(X, e_i) = \frac{1}{2}(d\rho - 2\rho p)(X)$$

*for* $X \in \mathscr{D}$, *where* $\{e_i\}$ *denotes an orthonormal frame of* $\mathscr{D}$ *with respect to the pseudo metric* $g$ *and* $\varepsilon_i = g(e_i, e_i) = \pm 1$.

PROOF. From the second Bianchi identity (5.8) we have

$$\mathfrak{S}\{(D_X R)(Y, Z)W\} = \mathfrak{S}\{\omega(X, Y)R(\xi_p, Z)W\}$$

for $W \in \mathscr{D}$. Therefore, if, in the above equation, we replace $Y$ with $e_i$ and take the inner product with $e_i$, we have

(5.29) $$(D_X s)(Z, W) + \sum_{i=1}^{2n} \varepsilon_i g((D_{e_i} R)(Z, X)W, e_i) - (D_Z s)(X, W)$$
$$= -g(JX, R(\xi_p, Z)W) - g(JR(\xi_p, X)W, Z) + g(JZ, X)s(\xi_p, W),$$

where we have used the following equation

(5.30) $$D_X e_i = -\sum_{j=1}^{2n} \varepsilon_j g(D_X e_j, e_i) e_j + 2p(X) e_i.$$

Moreover, replace both $Z$ and $W$ with $e_j$ and sum with respect to $j$. Then we have

(5.31) $$\sum_{j=1}^{2n} \varepsilon_j (D_X s)(e_j, e_j) + \sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j g((D_{e_i} R)(e_j, X)e_j, e_i) - \sum_{j=1}^{2n} \varepsilon_j (D_{e_j} s)(X, e_j)$$
$$= -\sum_{j=1}^{2n} \varepsilon_j g(JX, R(\xi_p, e_j)e_j) - \sum_{j=1}^{2n} \varepsilon_j g(JR(\xi_p, X)e_j, e_j)$$
$$+ \sum_{j=1}^{2n} \varepsilon_j g(Je_j, X)s(\xi_p, e_j).$$

We calculate the each term of the equation (5.31). Applying (5.30) to the first term of the left hand side of (5.31), we have

$$\sum_{j=1}^{2n} \varepsilon_j (D_X s)(e_j, e_j) = (d\rho - 2\rho p)(X). \tag{5.32}$$

Applying (5.4) and (5.30) to the second term of the left hand side of (5.31), we have

$$\sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j g((D_{e_i} R)(e_j, X)e_j, e_i) = -\sum_i \varepsilon_i (D_{e_i} s)(X, e_i) + 4 \sum_i (D_{e_i}\, dp)(e_i, X). \tag{5.33}$$

For the first term of the right hand side of (5.31), we have, from (5.4),

$$-\sum_{j=1}^{2n} \varepsilon_j g(JX, R(\xi_p, e_j)e_j) = -s(\xi_p, JX) - 4\, dp(\xi_p, JX). \tag{5.34}$$

To compute the second term of the right hand side of (5.31), we prepare the following equation

$$\sum_i \varepsilon_i g((D_X F_p)e_i, Je_i) = \sum_i \varepsilon_i g((D_X F_p J)e_i, e_i) = \operatorname{trace}_{\mathscr{D}} D_X(F_p J) = 0.$$

By using (5.3), (5.7) and the equation $\operatorname{trace}_{\mathscr{D}}(F_p J) = 0$, we have

$$-\sum_{i=1}^{2n} \varepsilon_i g(JR(\xi_p, X)e_i, e_i) = \operatorname{trace}_{\mathscr{D}}(Z \to (D_Z F_p)JX) - 2p(F_p JX) + s(\xi_p, JX). \tag{5.35}$$

For the third term of the right hand side of (5.31), we have

$$\sum_{j=1}^{2n} \varepsilon_j g(Je_j, X) s(\xi_p, e_j) = -s(\xi_p, JX). \tag{5.36}$$

We see from (5.34), (5.35), (5.36) and (5.25) that the right hand side of (5.31) becomes $4(n-1)\, dp(\xi_p, JX)$. Substituting (5.32) and (5.33) into (5.31), we have

$$\begin{aligned} &-2\sum_i \varepsilon_i (D_{e_i} s)(X, e_i) + 4 \sum_i \varepsilon_i (D_{e_i}\, dp)(e_i, X) + (d\rho - 2\rho p)(X) \\ &\quad = 4(n-1)\, dp(\xi_p, JX). \end{aligned} \tag{5.37}$$

If we prove

$$2\sum_i \varepsilon_i (D_{e_i}\, dp)(JX, e_i) = 2(n-1)\, dp(\xi_p, X), \tag{5.38}$$

then we conclude (5.28). The proof of (5.38) is as follows. We calculate the exterior derivative of $dp$.

$$3\, d(dp)(Y, Z, W) = \mathfrak{S}\{(D_Y\, dp)(Z, W) - \omega(Y, Z)\, dp(\xi_p, W)\}$$

for $Y, Z, W \in \mathscr{D}$, where we have used (e) in the definition of the torsion tensor of CR Weyl connection. Replacing $Y$ with $e_i$, $Z$ with $Je_i$ and $W$ with $X$ in the above equation, and summing with respect to $i$, we have

$$\sum_i \varepsilon_i (D_{e_i}\, dp)(Je_i, X) + \sum_i \varepsilon_i (D_{Je_i}\, dp)(X, e_i) = -(2n-1)\, dp(\xi_p, X). \tag{5.39}$$

For the first term of the left hand side of (5.39), we have

$$-\sum_i \varepsilon_i (D_{e_i}\, dp)(Je_i, X) = \sum_i \varepsilon_i (D_{e_i}\, dp)(JX, e_i). \tag{5.40}$$

For the second term of the left hand side of (5.39), we also have

$$\begin{aligned} -\sum_i \varepsilon_i (D_{Je_i}\, dp)(X, e_i) &= \sum_i \varepsilon_i (D_{Je_i}\, dp)(JX, Je_i) \\ &= \sum_i \varepsilon_i (D_{e_i}\, dp)(JX, e_i), \end{aligned} \tag{5.41}$$

where we have used (3.16). Substituting (5.40) and (5.41) into (5.39), we obtain (5.38). □

## 6. CR Einstein-Weyl Structures

Let $D$ be a CR Weyl connection on a CR manifold $(M, \mathscr{D}, J)$. Fixing a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ belonging to the CR structure $(\mathscr{D}, J)$, we know that there exists uniquely a Tanaka connection $\nabla$ associated with the almost contact structure $(\phi, \xi, \theta)^*$ (cf. [9], [12]). Then the difference tensor $H$ between $D$ and $\nabla$ is given in Proposition 3.5. Thus we may calculate the difference $R(X, Y)Z - R^{\nabla}(X, Y)Z$ for $X, Y, Z \in \mathscr{D}$, where $R^{\nabla}$ denotes the curvature tensor of $\nabla$. We introduce suitable 2-forms and rewrite the resulting long equation comfortably. Next we shall calculate $k - k^{\nabla}$ and $\rho - \rho^{\nabla}$. In this way, the curvature tensor $R$ will be expressed as the equation including Bochner curvature tensor. Making use of this equation, we can define a CR Einstein-Weyl structure on a CR manifold.

To begin with, we calculate the difference $R - R^{\nabla}$. Since

$$
\begin{aligned}
D_X D_Y Z &= D_X(\nabla_Y Z + H(Y,Z)) \\
&= \nabla_X \nabla_Y Z + H(X, \nabla_Y Z) + (\nabla_X H)(Y,Z) + H(\nabla_X Y, Z) \\
&\quad + H(Y, \nabla_X Z) + H(X, H(Y,Z)),
\end{aligned}
$$

$$[X, Y] = \nabla_X Y - \nabla_Y X - T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X + \omega(X, Y)\xi$$

for $X, Y, Z \in \Gamma(\mathscr{D})$, where we have used the equation $T^{\nabla}(X, Y) = -\omega(X, Y)\xi$, we have

$$
\begin{aligned}
(6.1) \qquad & R(X, Y)Z - R^{\nabla}(X, Y)Z \\
&= (\nabla_X H)(Y,Z) - (\nabla_Y H)(X,Z) \\
&\quad + H(X, H(Y,Z)) - H(Y, H(X,Z)) - \omega(X, Y)H(\xi, Z).
\end{aligned}
$$

We substitute (3.18) and (3.19) into (6.1). The calculation is long but routine and hence we omit the proof. The result is as follows (cf. [9]):

$$
\begin{aligned}
(6.2) \qquad & R(X, Y)Z - R^{\nabla}(X, Y)Z \\
&= -\{(\nabla_Y p)(Z) - p(Y)p(Z) + q(Y)q(Z) + p(P)g(Y,Z)\}X \\
&\quad + \{(\nabla_X p)(Z) - p(X)p(Z) + q(X)q(Z) + p(P)g(X,Z)\}Y \\
&\quad - \{(\nabla_Y q)(Z) - q(Y)p(Z) - p(Y)q(Z) + p(P)g(JY,Z)\}JX \\
&\quad + \{(\nabla_X q)(Z) - q(X)p(Z) - p(X)q(Z) + p(P)g(JX,Z)\}JY \\
&\quad - g(Y,Z)\{\nabla_X P - p(X)P + q(X)Q\} \\
&\quad + g(X,Z)\{\nabla_Y P - p(Y)P + q(Y)Q\} \\
&\quad - g(JY,Z)\{\nabla_X Q - q(X)P - p(X)Q\} \\
&\quad + g(JX,Z)\{\nabla_Y Q - q(Y)P - p(Y)Q\} \\
&\quad + \{(\nabla_X p)(Y) - (\nabla_Y p)(X)\}Z + \{(\nabla_X q)(Y) - (\nabla_Y q)(X)\}JZ \\
&\quad + g(JX, Y)\{\nabla_{JZ} P + \nabla_Z Q + 2p(P)JZ\}.
\end{aligned}
$$

Now we define $\alpha \in \Gamma(\mathscr{D}^* \otimes \mathscr{D}^*)$ by

(6.3)
$$\alpha(Y,Z) = (\nabla_Y p)(Z) - p(Y)p(Z) + q(Y)q(Z) + \frac{1}{2}p(P)g(Y,Z) + \frac{1}{2}p(\xi)g(JY,Z)$$

and $\gamma \in \Gamma(\mathscr{D}^* \otimes \mathscr{D}^*)$ by

(6.4)
$$\gamma(Y,Z) = (\nabla_Y q)(Z) - q(Y)p(Z) - p(Y)q(Z) + \frac{1}{2}p(P)g(JY,Z) - \frac{1}{2}p(\xi)g(Y,Z).$$

Then they are related as

$$\alpha(Y,Z) = \gamma(Y,JZ). \tag{6.5}$$

Rewriting the exterior differentiation $dp$ and $dq$ of the 1-form $p$ and $q$ in terms of the Tanaka connection respectively, we obtain

$$2\,dp(Y,Z) = (\nabla_Y p)(Z) - (\nabla_Z p)(Y) - p(\xi)\omega(Y,Z), \tag{6.6}$$

$$2\,dq(Y,Z) = (\nabla_Y q)(Z) - (\nabla_Z q)(Y) \tag{6.7}$$

for $Y, Z \in \mathscr{D}$, where we have used $q(\xi) = 0$. From (6.3) and (6.6), we have

$$\alpha(Y,Z) - \alpha(Z,Y) = 2\,dp(Y,Z). \tag{6.8}$$

We also have, from (6.4) and (6.7),

$$\gamma(Y,Z) - \gamma(Z,Y) = 2\,dq(Y,Z) + p(P)g(JY,Z). \tag{6.9}$$

Furthermore, define $A, C \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$AY = \nabla_Y P - p(Y)P + q(Y)Q + \frac{1}{2}p(P)Y + \frac{1}{2}p(\xi)JY, \tag{6.10}$$

$$CY = \nabla_Y Q - q(Y)P - p(Y)Q + \frac{1}{2}p(P)JY - \frac{1}{2}p(\xi)Y. \tag{6.11}$$

Then we have

$$g(AY,Z) = \alpha(Y,Z), \quad g(CY,Z) = \gamma(Y,Z), \tag{6.12}$$

and from (6.5)

$$JA = C. \tag{6.13}$$

Substituting (6.3), (6.4), (6.10) and (6.11) into (6.2), we easily obtain the following equation and we omit the proof (cf. [10]).

LEMMA 6.1. *$R - R^{\nabla}$ is given by*

$$R(X,Y)Z - R^{\nabla}(X,Y)Z \tag{6.14}$$
$$= -\alpha(Y,Z)X + \alpha(X,Z)Y - \gamma(Y,Z)JX + \gamma(X,Z)JY$$
$$- g(Y,Z)AX + g(X,Z)AY - g(JY,Z)CX + g(JX,Z)CY$$
$$+ \{\alpha(X,Y) - \alpha(Y,X)\}Z + \{\gamma(X,Y) - \gamma(Y,X)\}JZ$$
$$+ g(JX,Y)(AJZ + CZ).$$

REMARK. We can represent the equation (6.14) in the form similar to [10]:

$$R(X,Y)Z - R^{\nabla}(X,Y)Z$$
$$= -\alpha(Y,Z)X + \alpha(X,Z)Y - \gamma(Y,Z)JX + \gamma(X,Z)JY$$
$$- g(Y,Z)AX + g(X,Z)AY - g(JY,Z)CX + g(JX,Z)CY$$
$$+ \{\gamma(X,Y) - \gamma(Y,X)\}JZ + g(JX,Y)\{CZ - {}^{t}CZ\}$$
$$+ 2\,dp(X,Y)Z + 2g(JX,Y)\,dp^{\#}(JZ),$$

where ${}^{t}C$ denotes the transpose of the linear transformation $C$ of $\mathscr{D}$ with respect to $g$ and $dp^{\#}$ is the linear transformation of $\mathscr{D}$ defined by $g(dp^{\#}X, Y) = dp(X, Y)$.

Next we shall compute $k(Y,Z) - k^{\nabla}(Y,Z)$ for $Y, Z \in \mathscr{D}$, where $k^{\nabla}$ is the Ricci tensor of the fixed Tanaka connection $\nabla$. Before contracting the equation (6.14), we consider the symmetric part of $\gamma$. For $f_p(Y,Z) - f(Y,Z)$, we obtain

$$\gamma(Y,Z) + \gamma(Z,Y) = -f_p(Y,Z) + f(Y,Z) + 2\,dp(JY,Z), \tag{6.15}$$

where $f(Y,Z) = g(FY,Z)$. In fact, since

$$f_p(Y,Z) - f(Y,Z) = (\nabla_{JY}p)(Z) - (\nabla_Y q)(Z) + 2p(Y)q(Z) + 2q(Y)p(Z)$$

because of (3.24), the bilinear form $\alpha$ satisfies

$$\alpha(JY,Z) + \alpha(Y,JZ) = f_p(Y,Z) - f(Y,Z), \tag{6.16}$$

which implies (6.15).

Well we compute $s(Y,Z) - s^{\nabla}(Y,Z)$, where $s^{\nabla}$ is the Ricci tensor of $\nabla$. Contracting (6.14), we see that

$$\begin{aligned} s(Y,Z) - s^{\nabla}(Y,Z) \\ &= -2n\alpha(Y,Z) + \alpha(Y,Z) - \gamma(Y,Z)\,\mathrm{trace}_{\mathscr{D}}\,J + \gamma(JY,Z) \\ &\quad - g(Y,Z)\,\mathrm{trace}_{\mathscr{D}}\,A + \alpha(Y,Z) - g(JY,Z)\,\mathrm{trace}_{\mathscr{D}}\,C - \gamma(Y,JZ) \\ &\quad + \{\alpha(Z,Y) - \alpha(Y,Z)\} - \{\gamma(JZ,Y) - \gamma(Y,JZ)\} \\ &\quad - g(AJZ,JY) - g(CZ,JY). \end{aligned}$$

Since $\mathrm{trace}_{\mathscr{D}}\,F_p = \mathrm{trace}_{\mathscr{D}}\,F = 0$, we obtain $\mathrm{trace}_{\mathscr{D}}\,C = 0$ by virtue of the equation (6.15). Making use of (6.5), (6.8), (6.15) and (6.16), we have

$$\begin{aligned} s(Y,Z) - s^{\nabla}(Y,Z) &= -2(n+2)\alpha(Y,Z) - 3f_p(JY,Z) \\ &\quad + f(JY,Z) - g(Y,Z)\,\mathrm{trace}_{\mathscr{D}}\,A - 4\,dp(Y,Z). \end{aligned} \tag{6.17}$$

Therefore, by the equation (5.20), we get

LEMMA 6.2. *The difference $k(Y,Z) - k^{\nabla}(Y,Z)$ is given by*

$$\begin{aligned} k(Y,Z) - k^{\nabla}(Y,Z) &= -(n+2)\{\alpha(Y,Z) + \alpha(JY,JZ)\} \\ &\quad - g(Y,Z)\,\mathrm{trace}_{\mathscr{D}}\,A - 2(n+2)\,dp(Y,Z) \end{aligned} \tag{6.18}$$

*for every $Y, Z \in \mathscr{D}$.*

Using the equation (6.17), we have

$$S - S^{\nabla} = -2(n+2)A - 3F_pJ + 3FJ - (\mathrm{trace}_{\mathscr{D}}\,A)I_{\mathscr{D}} - 4\,dp^{\#}, \tag{6.19}$$

where $S^{\nabla}$ denotes the linear transformation of $\mathscr{D}$ defined by $g(S^{\nabla}Y,Z) = s^{\nabla}(Y,Z)$ and $I_{\mathscr{D}}$ denotes the identity transformation of $\mathscr{D}$. We obtain, from (6.19),

LEMMA 6.3. *The difference $\rho - \rho^{\nabla}$ is given by*

$$\rho - \rho^{\nabla} = -4(n+1)\,\mathrm{trace}_{\mathscr{D}}\,A, \tag{6.20}$$

*where $\rho^{\nabla}$ denotes the scalar curvature of $\nabla$.*

Let us define $l$ and $m$ by

$$l(Y,Z) = -\frac{1}{2(n+2)}k(Y,Z) + \frac{1}{8(n+1)(n+2)}\rho g(Y,Z) \tag{6.21}$$

and

$$m(Y,Z) = -\frac{1}{2(n+2)}k(JY,Z) + \frac{1}{8(n+1)(n+2)}\rho g(JY,Z) \tag{6.22}$$

respectively, where $Y, Z \in \mathscr{D}$. From the equation (5.21) and (5.22) we obtain

$$l(Y,Z) - l(Z,Y) = 2\,dp(Y,Z), \tag{6.23}$$

$$l(JY,JZ) - l(Y,Z) = -2\,dp(Y,Z). \tag{6.24}$$

Also we similarly obtain

$$m(Y,Z) = -m(Z,Y), \tag{6.25}$$

$$m(JY,JZ) - m(Y,Z) = -2\,dp(JY,Z). \tag{6.26}$$

The forms $l$ and $m$ are related as

$$m(Y,Z) = l(JY,Z). \tag{6.27}$$

We define $L \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ and $M \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$g(LY,Z) = l(Y,Z), \tag{6.28}$$

$$g(MY,Z) = m(Y,Z) \tag{6.29}$$

for every $Y, Z \in \mathscr{D}$ respectively.

We express $\alpha$, $\gamma$, $A$ and $C$ by the above notations:

LEMMA 6.4. *The bilinear form $\alpha$ on $\mathscr{D}$ is given by*

$$\alpha(Y,Z) = l(Y,Z) - l^{\nabla}(Y,Z) - \frac{1}{2}\{f_p(JY,Z) - f(JY,Z)\} - dp(Y,Z), \tag{6.30}$$

*so that we have*

$$A = L - L^{\nabla} - \frac{1}{2}(F_pJ - FJ) - dp^{\#}, \tag{6.31}$$

*and the bilinear form $\gamma$ is given by*

$$\gamma(Y,Z) = m(Y,Z) - m^{\nabla}(Y,Z) - \frac{1}{2}\{f_p(Y,Z) - f(Y,Z)\} - dp(JY,Z), \tag{6.32}$$

*so that we have*

$$C = M - M^{\nabla} - \frac{1}{2}(F_p - F) - dp^{\#}J, \tag{6.33}$$

*where* $l^\nabla$, $m^\nabla$, $L^\nabla$ *and* $M^\nabla$ *denote the tensors similarly defined by* (6.21), (6.22), (6.28) *and* (6.29) *with respect to* $\nabla$ *respectively.*

REMARK. In [10], the following equations are easily verified:

$$l^\nabla(Y,Z) = l^\nabla(Z,Y), \quad m^\nabla(Y,Z) = -m^\nabla(Z,Y)$$

$$l^\nabla(JY,JZ) = l(Y,Z), \quad m^\nabla(JY,JZ) = m^\nabla(Y,Z)$$

for $Y, Z \in \mathscr{D}$. These are derived from the fact that $k^\nabla$ is symmetric on $\mathscr{D}$ and satisfies $k^\nabla(JY,JZ) = k(Y,Z)$ for $Y, Z \in \mathscr{D}$.

PROOF. It suffices to prove the equation (6.30) from which the others are trivially derived from the above remark. From the defining equation (6.21), we have

$$\begin{aligned} l(Y,Z) - l^\nabla(Y,Z) = &-\frac{1}{2(n+2)}\{k(Y,Z) - k^\nabla(Y,Z)\} \\ &+ \frac{1}{8(n+1)(n+2)}(\rho - \rho^\nabla)g(Y,Z). \end{aligned}$$

We substitute the equation (6.18) and (6.20) into the above equation. Then we have

$$l(Y,Z) - l^\nabla(Y,Z) = \frac{1}{2}\{\alpha(Y,Z) + \alpha(JY,JZ)\} + dp(Y,Z),$$

and hence, we obtain (6.30) from (6.16). □

Next we shall rewrite the equation (6.14) by making use of Lemma 6.4. Before we do so, we need to state the Bochner curvature tensor which is invariant under the change (2.8).

Sakamoto and Takemura (cf. [10]) state the Bochner curvature tensor in the following form.

LEMMA 6.5. *Let* $B_0, B_1 \in \Gamma(\mathscr{D}^{*3} \otimes \mathscr{D})$ *be defined by*

(6.34) $B_0(X,Y)Z$

$$\begin{aligned} &= R^\nabla(X,Y)Z + l^\nabla(Y,Z)X - l^\nabla(X,Z)Y + m^\nabla(Y,Z)JX - m^\nabla(X,Z)JY \\ &\quad + g(Y,Z)L^\nabla X - g(X,Z)L^\nabla Y + g(JY,Z)M^\nabla X - g(JX,Z)M^\nabla Y \\ &\quad - 2\{m^\nabla(X,Y)JZ + g(JX,Y)M^\nabla Z\}, \end{aligned}$$

$$B_1(X, Y)Z = \frac{1}{2}\{R^\nabla(JX, JY)Z - R^\nabla(X, Y)Z\}. \tag{6.35}$$

*Then $B = B_0 + B_1$ is invariant under the change* (2.8). (*The tensor field $B$ on $\mathscr{D}$ is called Bochner curvature tensor.*)

The right hand side of the definition of $B_1$ is given by

$$\begin{aligned} R^\nabla(JX, JY)Z - R^\nabla(X, Y)Z &= g(JX, Z)FY - g(JY, Z)FX + g(X, Z)FJY - g(Y, Z)FJX \\ &\quad + f(X, Z)JY - f(Y, Z)JX + f(JX, Z)Y - f(JY, Z)X \end{aligned} \tag{6.36}$$

for $X, Y, Z \in \mathscr{D}$ (cf. [10]).

We introduce the important notations for a CR Einstein-Weyl structure by which we rewrite the equation (6.14). We define $ric^D$ by

$$ric^D(Y, Z) = l(Y, Z) - dp(Y, Z) \tag{6.37}$$

for $Y, Z \in \mathscr{D}$. From the equation (6.23) we see that the tensor $ric^D$ is symmetric and hence $ric^D$ is the symmetric part of $l$. We obtain, from (6.27),

$$ric^D(JY, Z) = m(Y, Z) - dp(JY, Z). \tag{6.38}$$

Furthermore we define $Ric^D \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ by

$$g(Ric^D\, Y, Z) = ric^D(Y, Z) \tag{6.39}$$

for $Y, Z \in \mathscr{D}$. It follows that

$$Ric^D = L - dp^\#, \quad Ric^D\, J = M - dp^\# J. \tag{6.40}$$

We obtain, from Lemma 6.1,

THEOREM 6.6. *Let $(\mathscr{D}, J)$ be a nodegenerate CR structure on $M^{2n+1}$ and $(\phi, \xi, \theta)^*$ a $\mathscr{D}$-preserving almost contact structure belonging to $(\mathscr{D}, J)$. Let $D$ be a CR Weyl connection. Then the curvature tensor $R$ of $D$ satisfies*

$$\begin{aligned} \frac{1}{2}\{R(JX, JY)Z + R(X, Y)Z\} &= -ric^D(Y, Z)X + ric^D(X, Z)Y - ric^D(JY, Z)JX + ric^D(JX, Z)JY \\ &\quad - g(Y, Z)\, Ric^D\, X + g(X, Z)\, Ric^D\, Y - g(JY, Z)\, Ric^D\, JX \\ &\quad + g(JX, Z)\, Ric^D\, JY + 2\{ric^D(JX, Y)JZ + g(JX, Y)\, Ric^D\, JZ\} \\ &\quad + B(X, Y)Z \end{aligned} \tag{6.41}$$

*for every $X, Y, Z \in \mathscr{D}$.*

PROOF. Substitute the equations from (6.30) to (6.40) into (6.14). Then we obtain (6.41). □

REMARK. For $X, Y \in \mathscr{D}$ we define the transformation $X \wedge Y$ on $\mathscr{D}$ by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{6.42}$$

for $Z \in \mathscr{D}$. Furthermore, for $Ric^D, I_{\mathscr{D}} \in \Gamma(\mathscr{D}^* \otimes \mathscr{D})$ we define $Ric^D \wedge I_{\mathscr{D}}$ by

$$(Ric^D \wedge I_{\mathscr{D}})_{X,Y}Z = Ric^D\ Y \wedge I_{\mathscr{D}}X - Ric^D\ X \wedge I_{\mathscr{D}}Y \tag{6.43}$$

for $X, Y, Z \in \mathscr{D}$. Using such notations as (6.42) and (6.43) and rewriting the equation (6.41) and (5.15), we obtain

$$\begin{aligned} \frac{1}{2}\{R(JX,JY)Z + R(X,Y)Z\} = {} & \{Ric^D \wedge I_{\mathscr{D}} + Ric^D\ J \wedge J\}_{X,Y}Z \\ & + 2\{ric^D(JX,Y)JZ + g(JX,Y)\ Ric^D\ JZ\} \\ & + B(X,Y)Z, \end{aligned} \tag{6.44}$$

$$\begin{aligned} \frac{1}{2}\{R(JX,JY)Z - R(X,Y)Z\} = {} & \frac{1}{2}\{F_p \wedge J + F_pJ \wedge I_{\mathscr{D}}\}_{X,Y}Z \\ & - 2\{dp(X,Y)Z - dp(JX,Y)JZ\}. \end{aligned} \tag{6.45}$$

We find that the equations (6.44) and (6.45) are similar to the equation in [2] which describes the relation between the curvature $R$ of a Weyl connection and the Weyl conformal curvatrue tensor $W$. The definition of an Einstein-Weyl connection is that the symmetric part of $h^D$ in [2] is proportional to $g$ pointwise. Therefore it will be appropriate that we define a CR Einstein-Weyl connection as follows:

DEFINITION. A pair of a nondegenerate CR structure $(\mathscr{D}, J)$ and a CR Weyl connection $D$ is *CR Einstein-Weyl* if the bilinear form $ric^D$ is proportional to $g$ pointwise, where $g$ is the Levi metric of arbitrary $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathscr{D}, J)$. And a CR manifold $M$ furnished with a CR Einstein-Weyl structure is called a *CR Einstein-Weyl manifold.*

REMARK. The bilinear form $\rho g$ does not depend on the choice of $(\phi, \xi, \theta)^*$ and so does $ric^D$. Therefore the definition that the CR Weyl connection is CR Einstein-Weyl is independent of the choice of $(\phi, \xi, \theta)^*$.

By the following proposition, we may state that a certain pair of a 1-form $p$ and $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ determines a CR Einstein-Weyl structure as in the case of Einstein-Weyl structure.

PROPOSITION 6.7. *The CR structure $(\mathscr{D}, J)$ admits a CR Einstein-Weyl connection $D$ if and only if $D$ is determined by a pair of a 1-from $p$ satisfying (3.16) and a $\mathscr{D}$-preserving almost contact structure $(\phi, \xi, \theta)^*$ which satisfy*

$$k^{\nabla}(Y,Z) - (n+2)\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY,Z)\} = \Lambda g(Y,Z) \tag{6.46}$$

*for every $Y, Z \in \mathscr{D}$.*

PROOF. First we assume that $ric^D(Y,Z)$ is proportional to $g(Y,Z)$ pointwise. Then by the definition of $ric^D$ and (6.30), we have

$$ric^D(Y,Z) = l^{\nabla}(Y,Z) + \alpha(Y,Z) + \frac{1}{2}\{f_p(JY,Z) - f(JY,Z)\}. \tag{6.47}$$

Moreover, applying (3.24) to (6.47), we have

$$\begin{aligned} ric^D(Y,Z) = l^{\nabla}(Y,Z) &+ \frac{1}{2}\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY,Z)\} \\ &+ \frac{1}{2}p(P)g(Y,Z). \end{aligned} \tag{6.48}$$

Substituting the definition of $l^{\nabla}$ into (6.48), we have

$$\begin{aligned} ric^D(Y,Z) = &-\frac{1}{2(n+2)}k^{\nabla}(Y,Z) \\ &+ \frac{1}{2}\{(\nabla_Y p)(Z) - (\nabla_{JY} q)(Z) + p(\xi)g(JY,Z)\} \\ &+ \frac{1}{8(n+1)(n+2)}\{\rho^{\nabla} + 4(n+1)(n+2)p(P)\}g(Y,Z), \end{aligned} \tag{6.49}$$

which implies (6.46).

Conversely, we assume that there exist $p$ and $(\phi, \xi, \theta)^*$ which satisfy (6.46). By Proposition 3.6, we have a CR Weyl connection $D$. Then we define the tensor $ric^D$ of the CR Weyl connection $D$. Substituting (6.46) into (6.49), we see that $ric^D$ is proportional to $g$ pointwise. □

Next we state the main theorem in terms of a holomorphic 1-form. If $(\phi, \xi, \theta)^*$ is a $\mathscr{D}$-preserving almost contact structure such that the Ricci tensor $k^{\nabla}$ of the Tanaka connection $\nabla$ associated with $(\phi, \xi, \theta)^*$ is proportional to $g$ pointwise, that is, (6.50) $$k^{\nabla}(Y,Z) = cg(Y,Z)$$

for $Y, Z \in \mathscr{D}$, where $c$ is a smooth function on $M$ and $g$ is the Levi metric of $(\phi, \xi, \theta)^*$, then $(\phi, \xi, \theta)^*$ is said to be *pseudo-Einstein* (cf. [6]).

THEOREM 6.8. *Let $(\mathscr{D}, J)$ be a nondegenerate CR structure on a $(2n+1)$-dimensional manifold $M$. Assume that there exists a $\mathscr{D}$-preserving pseudo-Einstein almost contact structure $(\phi, \xi, \theta)^*$ belonging to $(\mathscr{D}, J)$. If there exists a holomorphic 1-form $p + \sqrt{-1}q$, where $p$ is a real 1-form and $q = -p \circ \phi$, then the CR Weyl connection $D$ determined by $p$ and $\nabla$ (Tanaka connection associated with $(\phi, \xi, \theta)^*$) is CR Einstein-Weyl.*

PROOF. First we put $u = p + \sqrt{-1}q$. We see from (2.19) that $d''u = 0$ if and only if $u$ satisfies the following equations:

(6.51) $$(\nabla_{Z+\sqrt{-1}JZ}u)(Y - \sqrt{-1}JY) - u(T^{\nabla}(Y - \sqrt{-1}JY, Z + \sqrt{-1}JZ)) = 0$$

(6.52) $$(\nabla_{Z+\sqrt{-1}JZ}u)(\xi) - u(T^{\nabla}(\xi, Z + \sqrt{-1}JZ)) = 0$$

for $Y, Z \in \mathscr{D}$. We have, from (6.51),

(6.53) $$\begin{aligned}&(\nabla_Z p)(Y) - (\nabla_{JZ} q)(Y) + p(\xi)g(JZ, Y)\\ &\quad + \sqrt{-1}\{(\nabla_{JZ} p)(Y) + (\nabla_Z q)(Y) - p(\xi)g(Z, Y)\} = 0.\end{aligned}$$

Combining (6.53) with the assumption that $(\phi, \xi, \theta)^*$ is pseudo-Einstein, we see that (6.46) is satisfied. Since

$$\begin{aligned}2\{dp(X, Y) + dp(JX, JY)\} &= (\nabla_X p)(Y) - (\nabla_Y p)(X) + p(T(X, Y))\\ &\quad + (\nabla_{JX} p)(JY) - (\nabla_{JY} p)(JX) + p(T(JX, JY))\\ &= (\nabla_X p)(Y) - (\nabla_{JX} q)(Y) + p(\xi)g(JX, Y)\\ &\quad - \{(\nabla_Y p)(X) - (\nabla_{JY} q)(X) + p(\xi)g(JY, X)\}\end{aligned}$$

for $X, Y \in \mathscr{D}$, we also obtain (3.16). Therefore, by Theorem 6.7, the CR Weyl connection $D$ determined by $p$ and $\nabla$ is CR Einstein-Weyl. □

## 7. Example of CR Einstein-Weyl Manifolds

We shall explain an example of a CR Einstein-Weyl manifold. We shall show that the total space of $SO(3)$-principal bundle over a quaternion Kähler manifold has a CR Einstein-Weyl structure.

Let $M$ be a Riemannian manifold of dimension $4m$ $(m \geq 2)$. The manifold $M$ is a quaternion Kähler manifold if the holonomy group of the Levi-Civita connection is contained in $Sp(m) \cdot Sp(1)$, where $Sp(m)$ acts on $\mathbf{H}^m$ on the left and $Sp(1)$ acts on $\mathbf{H}^m$ as $\vec{q} \mapsto \vec{q} \cdot \bar{u}$ on the right for $\vec{q} \in \mathbf{H}^m$. Thus $Sp(m) \cdot Sp(1)$ is a subgroup of $SO(4m)$, which is isomorphic to $Sp(m) \times Sp(1)/\{\pm 1\}$ (cf. [1]).

A Riemannian manifold $(M, g)$ is a quaternion Kähler manifold if and only if there are an open covering $\{U\}$ of $M$ and $(1,1)$ tensor fields $J_1$, $J_2$, $J_3$ (defined on $U$) satisfying

$$J_1^2 = -I, \quad J_2^2 = -I, \quad J_3^2 = -I$$

$$J_1 J_2 = -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2,$$

$$g(J_i X, J_i Y) = g(X, Y) \quad (i = 1, 2, 3)$$

and

$$\begin{cases} \nabla_X^g J_1 = & & 2\theta_3(X) J_2 & -2\theta_2(X) J_3 \\ \nabla_X^g J_2 = -2\theta_3(X) J_1 & & & +2\theta_1(X) J_3 \\ \nabla_X^g J_3 = & 2\theta_2(X) J_1 & -2\theta_1(X) J_2 & \end{cases} \tag{7.1}$$

for $X, Y \in TU$, where $\nabla^g$ is the Levi-Civita connection of $g$. The tensors $J_1$, $J_2$ and $J_3$ form a local basis of a vector bundle $V(M)$ over $M$. For another local basis $J_1'$, $J_2'$ and $J_3'$ on $U'$, we have, on $U \cap U'$,

$$(J_1', J_2', J_3') = (J_1, J_2, J_3) s_{UU'}, \quad s_{UU'} \in SO(3), \tag{7.2}$$

where the product of the right hand side is the matrix multiplication.

Let $\mathscr{P}$ be the principal $SO(3)$-bundle associated with $V(M)$, that is, $\mathscr{P}$ is the principal bundle consisting of frames of $V(M)$. The dimension of the total space of $\mathscr{P}$ is equal to $4m + 3$. We shall show that the total space $\mathscr{P}$ admits a CR Einstein-Weyl structure. We take a basis of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ as follows:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

Then the basis satisfies

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2. \tag{7.4}$$

By using 1-forms $\theta_1$, $\theta_2$ and $\theta_3$ appearing in (7.1), we define $\omega_U$ by

(7.5) $$\omega_U = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3$$

on the each $U$. Hence $\omega_U$ is a $\mathfrak{so}(3)$-valued 1-form. We have, from (7.1) and (7.2),

(7.6) $$\omega_{U'} = s_{UU'}^{-1}\, ds_{UU'} + s_{UU'}^{-1}\omega_U s_{UU'}$$

on $U \cap U'$. Therefore the family $\mathfrak{so}(3)$-valued 1-form $\{\omega_U\}$ determines a connection $\omega$ in the principal bundle $\mathscr{P}$. If we consider $\sigma = \{J_1, J_2, J_3\}$ as a cross section of $\mathscr{P}$ on $U$, then $\sigma^*\omega = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3$. We put

(7.7) $$\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3,$$

where $\omega_1$, $\omega_2$ and $\omega_3$ are 1-forms on $\mathscr{P}$. The curvature form $\Omega$ of $\omega$ is given by

$$\Omega_1 = d\omega_1 + \omega_2 \wedge \omega_3, \quad \Omega_2 = d\omega_2 + \omega_3 \wedge \omega_1, \quad \Omega_3 = d\omega_3 + \omega_1 \wedge \omega_2,$$

where $\Omega = \Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3$. Let $\zeta_i$ be the fundamental vector field corresponding to $e_i$ $(i = 1, 2, 3)$. Then we have from (7.4)

(7.8) $$[\zeta_1, \zeta_2] = 2\zeta_3, \quad [\zeta_2, \zeta_3] = 2\zeta_1, \quad [\zeta_3, \zeta_1] = 2\zeta_2$$

and

(7.9) $$\omega_i(\zeta_j) = \delta_{ij}.$$

The Ricci identity for $J_i$ and (7.1) imply that

(7.10)
$$\begin{cases} [R^g(X, Y), J_1] = & & 4\sigma^*\Omega_3(X, Y)J_2 & -4\sigma^*\Omega_2(X, Y)J_3 \\ [R^g(X, Y), J_2] = & -4\sigma^*\Omega_3(X, Y)J_1 & & +4\sigma^*\Omega_1(X, Y)J_3 \\ [R^g(X, Y), J_3] = & 4\sigma^*\Omega_2(X, Y)J_1 & -4\sigma^*\Omega_1(X, Y)J_2 & \end{cases}$$

for $X, Y \in TU$. If $m \geq 2$, then it can be shown that $(M, g)$ is Einstein. For the proof of this fact, [p. 403, 2] or [3] where (7.10) is used as a key equation. Let $X^H$ denote the horizontal lift of $X \in TM$. Then we have

(7.11) $$\Omega_i(X^H, Y^H)_\sigma = -\frac{c}{2} g(J_i X, Y) \quad (i = 1, 2, 3),$$

where $c = \rho^g/\{8m(m+2)\}$ and $\rho^g$ is the scalar curvature of $(M, g)$. In the sequel, we assume that the constant $\rho^g$ does not vanish. We put $\theta = -\omega_1/c$ and $\xi = -c\zeta_1$. Let $\mathscr{D}$ be the hyperdistribution spanned by the horizontal distribution $\mathscr{H}$ of $\omega$, $\zeta_2$ and $\zeta_3$ at each point of $\mathscr{P}$. Then we have $\theta(\xi) = 1$ and $\theta(\mathscr{D}) = 0$. Moreover

$$(7.12)\qquad -2\,d\theta(X^H, Y^H)_\sigma = \frac{2}{c}\{\Omega_1(X^H, Y^H) - 2(\omega_2 \wedge \omega_3)(X^H, Y^H)\}_\sigma$$
$$= -g(J_1 X, Y)$$

for $X, Y \in TU$. We define $J_u : \mathscr{D}_u \to \mathscr{D}_u$ at $u = \{J_1, J_2, J_3\} \in \mathscr{P}$ by

$$(7.13)\qquad J_u V = (J_1 X)^H - \omega_3(V_3)\zeta_2 + \omega_2(V_2)\zeta_3$$

for $V = X^H + V_2 + V_3 \in \mathscr{H}_u \oplus \operatorname{span}\{\zeta_2\} \oplus \operatorname{span}\{\zeta_3\}$. It is easily verified that $J\zeta_2 = \zeta_3$, $J\zeta_3 = -\zeta_2$ and hence $J$ is a complex structure on $\mathscr{D}$. We also define $\omega_L$ and $g_L$ by $\omega_L = -2\,d\theta$ and $g_L(\cdot\,,\cdot) = \omega_L(J\cdot\,,\cdot)$, respectively. Then

$$(7.14)\qquad \omega_L(X^H, Y^H)_u = -g(J_1 X, Y), \quad \omega_L(X^H, \zeta_2) = \omega_L(X^H, \zeta_3) = 0,$$
$$\omega_L(\zeta_2, \zeta_3) = -\frac{2}{c},$$

since $[\zeta_2, \zeta_3] = 2\zeta_1$ and $[X^H, \zeta_i] = 0$. Putting $\xi_j = \sqrt{|c|/2}\,\varepsilon\zeta_j$ $(j = 2, 3)$, we have

$$(7.15)\qquad g_L(X^H, Y^H) = g(X, Y), \quad g_L(X^H, \xi_j) = 0, \quad g_L(\xi_j, \xi_j) = \varepsilon$$

for $j = 2, 3$, where $\varepsilon$ is the signature of $c$. It follows that $g_L$ is nondegenerate and positive definite (resp. pseudo-metric with $\gamma = 2$) if the scalar curvature $\rho^g$ is positive (resp. negative). It is easy to show that $g_L$ is Hermitian, that is, $g_L(JV, JW) = g_L(V, W)$ for $V, W \in \mathscr{D}$. Thus we see that the nondegenerate pair $(\mathscr{D}, J)$ satisfies (C.1) in Section 1. To prove (C.2), we first show

$$(7.16)\qquad [JX^H, JY^H] - [X^H, Y^H] - J([X^H, JY^H] + [JX^H, Y^H]) = 0$$

for $X, Y \in \mathfrak{X}(U)$. For an arbitrarily fixed $u \in \mathscr{P}$, we can take a cross section $\sigma = \{J_1, J_2, J_3\}$ on $U$ such that $\sigma(x) = u$ and $d\sigma(T_x M) = \mathscr{H}_u$, where $\pi(u) = x$, $\pi$ being the projection $\mathscr{P} \to M$. Then the left hand side of the above equation is equal to

$$(7.17)\quad [(J_1 X)^H, (J_1 Y)^H] - [X^H, Y^H] - J([X^H, (J_1 Y)^H] + [(J_1 X)^H, Y^H])$$

at $u$, since $JX^H = (J_1 X)^H$ along $\sigma$. The horizontal component of (7.17) is the horizontal lift of

$$(7.18)\qquad [J_1 X, J_1 Y] - [X, Y] - J_1([X, J_1 Y] + [J_1 X, Y]).$$

Since $\theta_i = 0$ at $x$ $(i = 1, 2, 3)$, we see from (7.1) that (7.18) vanishes at $x$. The vertical component of (7.17) also vanishes at $u$ since

$$\begin{aligned}\omega_j([(J_1X)^H, (J_1Y)^H] - [X^H, Y^H])\\ &= 2\{-\Omega_j((J_1X)^H, (J_1Y)^H) + \Omega_j(X^H, Y^H)\}\\ &= -2cg(J_jX, Y)\end{aligned}$$

and

$$\begin{aligned}&\omega_j(J[X^H, (J_1Y)^H] + J[(J_1X)^H, Y^H])\\ &\quad = \omega_j((J_1[X, J_1Y])^H - \omega_3([X^H, (J_1Y)^H])\zeta_2 + \omega_2([X^H, (J_1Y)^H])\zeta_3)\\ &\qquad + \omega_j((J_1[J_1X, Y])^H - \omega_3([(J_1X)^H, Y^H])\zeta_2 + \omega_2([(J_1X)^H, Y^H])\zeta_3)\\ &\quad = -2cg(J_jX, Y)\end{aligned}$$

at $u$ for $j = 2, 3$. Secondly we show, for $j = 2, 3$,

$$[JX^H, J\zeta_j] - [X^H, \zeta_j] - J([X^H, J\zeta_j] + [JX^H, \zeta_j]) = 0. \tag{7.19}$$

Note that $[X^H, \zeta_j] = [X^H, J\zeta_j] = 0$. Thus it suffices to show that

$$[JX^H, J\zeta_j] - J[JX^H, \zeta_j] = 0 \tag{7.20}$$

at $u$. Since

$$\begin{aligned}[JX^H, \zeta_j]_u &= \lim_{t\to 0} \frac{1}{t}\{(d\varphi_t(JX^H))_u - (JX^H)_u\}\\ &= \lim_{t\to 0} \frac{1}{t}\{(J_1(-t) - J_1)X\}_u^H\\ &= -((ue_j)_1X)_u^H \quad (\varphi_t = R\exp(te_j)),\end{aligned}$$

where $(J_1(t), J_2(t), J_3(t)) = (J_1, J_2, J_3)\exp(te_j)$ and $((ue_j)_1, (ue_j)_2, (ue_j)_3) = (J_1, J_2, J_3)e_j$. If $j = 2$, then $[JX^H, \zeta_2]_u = 2(J_3X)_u^H$ and hence $J[JX^H, \zeta_2]_u = 2(J_1J_3X)_u^H = -2(J_2X)_u^H$. Since $[JX^H, \zeta_3]_u = -2(J_2X)_u^H$, we see that

$$[JX^H, J\zeta_2] - J[JX^H, \zeta_2] = 0$$

at $u$. We have (7.20) for $j = 3$ in the similar way. Thirdly it is easy to show

$$[J\zeta_2, J\zeta_3] - [\zeta_2, \zeta_3] - J([\zeta_2, J\zeta_3] + [J\zeta_2, \zeta_3]) = 0.$$

We have proved that the condition (C.2) is satisfied. The pair $(\mathscr{D}, J)$ is a nondegenerate CR structure on $\mathscr{P}$.

Let $\phi$ be defined by $\phi\xi = 0$ and $\phi|_{\mathscr{D}} = J$. Then $(\phi, \xi, \theta)$ is an almost contact structure belonging to $(\mathscr{D}, J)$. For $fX^H + g\zeta_2 + h\zeta_3 \in \Gamma(\mathscr{D})$, we have

$$[fX^H + g\zeta_2 + h\zeta_3, \xi] \equiv 2cg\zeta_3 - 2ch\zeta_2 \quad (\text{mod } \mathscr{D}).$$

Therefore $(\phi, \xi, \theta)$ is a $\mathscr{D}$-preserving almost contact structure.

Next we compute the curvature tensor of the Tanaka connection $\nabla$ associated with $(\phi, \xi, \theta)$. Since $F = -1/2\phi(\mathscr{L}_\xi \phi)$ on $\mathscr{D}$, we easily have $F\xi_j = 0$ $(j = 2, 3)$. Moreover, $\phi(\mathscr{L}_\xi \phi)X^H = J[\xi, JX^H]$ and hence $FX^H = 0$ by the same method as the proof of (7.20). Thus we have $F = 0$. It follows that

$$\nabla_\xi \xi_2 = -2c\xi_3, \quad \nabla_\xi \xi_3 = 2c\xi_2. \tag{7.21}$$

Since

$$\begin{aligned} -\omega_L(\xi_2, \xi_3)\xi = T(\xi_2, \xi_3) &= \nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2 - [\xi_2, \xi_3] \\ &= \nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2 + \varepsilon\xi \end{aligned}$$

and $\omega_L(\xi_2, \xi_3) = -\varepsilon$, we see that $\nabla_{\xi_2}\xi_3 = \nabla_{\xi_3}\xi_2$. Note that

$$\begin{aligned} \nabla_{\xi_2}\xi_3 &= \nabla_{\xi_2}(\phi\xi_2) = \phi\nabla_{\xi_2}\xi_2, \\ \nabla_{\xi_3}\xi_3 &= \phi\nabla_{\xi_3}\xi_2 = \phi\nabla_{\xi_2}\xi_3 = \phi^2\nabla_{\xi_2}\xi_2. \end{aligned}$$

To prove

$$\nabla_{\xi_j}\xi_k = 0 \quad (j, k = 2, 3), \tag{7.22}$$

we have only to show $\nabla_{\xi_2}\xi_2 = 0$. Since $\nabla^\circ g_L = 0$ and $T_{\mathscr{D}_\xi} = 0$ on $\mathscr{D}$, we have

$$\begin{aligned} 2g_L(\nabla_{\xi_2}\xi_2, W) &= \xi_2 \cdot g_L(\xi_2, W) + \xi_2 \cdot g_L(\xi_2, W) - W \cdot g_L(\xi_2, \xi_2) \\ &\quad - g_L(\xi_2, [\xi_2, W]_{\mathscr{D}_\xi}) - g_L(\xi_2, [\xi_2, W]_{\mathscr{D}_\xi}) + g_L(W, [\xi_2, \xi_2]_{\mathscr{D}_\xi}) \end{aligned}$$

for $W \in \Gamma(\mathscr{D})$. If $W = \xi_j$ $(j = 2, 3)$, then the right hand side vanishes. If $W = X^H$, then the right hand side also vanishes because of the equation $[\xi_2, X^H] = 0$. By the equation $\nabla_\xi X^H = FX^H + [\xi, X^H]$, we obtain

$$\nabla_\xi X^H = 0 \tag{7.23}$$

for every $X \in \mathfrak{X}(U)$. Note that

$$\begin{aligned} 2g_L(\nabla_{\xi_j}X^H, W) &= \xi_j \cdot g_L(X^H, W) + X^H \cdot g_L(\xi_j, W) - W \cdot g_L(\xi_j, X^H) \\ &\quad - g_L(\xi_j, [X^H, W]_{\mathscr{D}_\xi}) - g_L(X^H, [\xi_j, W]_{\mathscr{D}_\xi}) + g_L(W, [\xi_j, X^H]_{\mathscr{D}_\xi}). \end{aligned}$$

If $W = \xi_k$ $(k = 2, 3)$, then the right hand side vanishes and if $W = Y^H$, then

$$
\begin{aligned}
2g_L(\nabla_{\xi_j} X^H, Y^H) &= -g_L(\xi_j, [X^H, Y^H]_{\mathscr{D}_\xi}) \\
&= -g_L(\xi_j, \omega_2([X^H, Y^H])\zeta_2 + \omega_3([X^H, Y^H])\zeta_3) \\
&= -\frac{1}{a}\omega_j([X^H, Y^H]) \\
&= \frac{2}{a}\Omega_j(X^H, Y^H),
\end{aligned}
$$

where $a = \sqrt{|c|/2}$. We define $K_j$ $(j = 2, 3)$ by $(K_j)_u X^H = (J_j X)_u^H$ and $K_j\xi_2 = K_j\xi_3 = 0$ at $u = \{J_1, J_2, J_3\} \in \mathscr{P}$. Then $K_j$ is a linear transformation of $\mathscr{D}$ such that

$$\Omega_j(V, W) = -\frac{c}{2} g_L(K_j V, W) \tag{7.24}$$

for every $V, W \in \mathscr{D}$. With this notation, we have

$$\nabla_{\xi_j} X^H = -\varepsilon a K_j X^H. \tag{7.25}$$

Since $[X^H, \xi_j] = 0$ and $\omega_L(X^H, \xi_j) = 0$, we obtain

$$\nabla_{X^H} \xi_j = -\varepsilon a K_j X^H. \tag{7.26}$$

We also use

$$
\begin{aligned}
&2g_L(\nabla_{X^H} Y^H, Z^H) \\
&= X^H \cdot g_L(Y^H, Z^H) + Y^H \cdot g_L(X^H, Z^H) - Z^H \cdot g_L(X^H, Y^H) \\
&\quad - g_L(X^H, [Y^H, Z^H]_{\mathscr{H}}) - g_L(Y^H, [X^H, Z^H]_{\mathscr{H}}) + g_L(Z^H, [X^H, Y^H]_{\mathscr{H}}) \\
&= X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) \\
&\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \\
&= 2g(\nabla^g_X Y, Z) = 2g_L((\nabla^g_X Y)^H, Z^H),
\end{aligned}
$$

from which

$$\nabla_{X^H} Y^H = a\{g_L(K_2 X^H, Y^H)\xi_2 + g_L(K_3 X^H, Y^H)\xi_3\} + (\nabla^g_X Y)^H. \tag{7.27}$$

To calculate the curvature tensor easily, we assume that $\nabla^g X = \nabla^g Y = \nabla^g Z = 0$ at $x$ and $\sigma = \{J_1, J_2, J_3\}$ is a cross section of $\mathscr{P}|_U$ such that $\sigma(x) = u$ and $d\sigma(T_x M) = \mathscr{H}_u$ for an arbitrarily fixed $u \in \mathscr{P}|_U$. So the calculation is always evaluated at $u$. Using (7.27), we have

$$\begin{aligned}(7.28)\quad & R^\nabla(X^H, Y^H)Z^H \\ &= a(X^H \cdot g_L(K_2 Y^H, Z^H))\xi_2 + a g_L(K_2 Y^H, Z^H)\nabla_{X^H}\xi_2 \\ &\quad + a(X^H \cdot g_L(K_3 Y^H, Z^H))\xi_3 + a g_L(K_3 Y^H, Z^H)\nabla_{X^H}\xi_3 \\ &\quad - a(Y^H \cdot g_L(K_2 X^H, Z^H))\xi_2 - a g_L(K_2 X^H, Z^H)\nabla_{Y^H}\xi_2 \\ &\quad - a(Y^H \cdot g_L(K_3 X^H, Z^H))\xi_3 - a g_L(K_3 X^H, Z^H)\nabla_{Y^H}\xi_3 \\ &\quad + (R^g(X, Y)Z)^H - 2\Omega_2(X^H, Y^H)K_2 Z^H - 2\Omega_3(X^H, Y^H)K_3 Z^H,\end{aligned}$$

from which and (7.26),

$$\begin{aligned}(7.29)\quad & g_L(R^\nabla(X^H, Y^H)Z^H, X^H) \\ &= -\frac{3}{2}c\{g(X, J_2 Z)g(J_2 Y, X) + g(X, J_3 Z)g(J_3 Y, X)\} \\ &\quad + g(R^g(X, Y)Z, X).\end{aligned}$$

Similarly we have

$$\varepsilon g_L(R^\nabla(\xi_j, Y^H)Z^H, \xi_j) = a^2 \varepsilon \zeta_j \cdot g_L(K_j Y^H, Z^H) + \frac{c}{2} g(Y, Z)$$

for $j = 2, 3$. Since

$$\begin{aligned}\zeta_j \cdot g_L(K_j Y^H, Z^H) &= \frac{d}{dt} g_L(K_j Y^H, Z^H)_{u \exp te_j}|_{t=0} \\ &= \frac{d}{dt} g(J_j(t) Y, Z)|_{t=0} \\ &= g((ue_j)_j Y, Z) \\ &= 0,\end{aligned}$$

where the notation $J_j(t)$ and $(ue_j)_j$ are defined in the proof of (7.20), we have

$$(7.30)\qquad \varepsilon g_L(R^\nabla(\xi_j, Y^H)Z^H, \xi_j) = \frac{c}{2} g(Y, Z).$$

It follows from (7.29) and (7.30) that

$$(7.31)\qquad s^\nabla(Y^H, Z^H) = \frac{m+1}{4m(m+2)} \rho^g g_L(Y^H, Z^H).$$

The calculation of $g_L(R^\nabla(X^H, \xi_j)\xi_j, X^H)$ and $\varepsilon g_L(R^\nabla(\xi_k, \xi_j)\xi_j, \xi_k)$ $(j, k = 2, 3, j \neq k)$ is easy. The results are

$$g_L(R^\nabla(X^H, \xi_j)\xi_j, X^H) = \frac{1}{2}c\varepsilon g(X, X) \tag{7.32}$$

and

$$\varepsilon g_L(R^\nabla(\xi_k, \xi_j)\xi_j, \xi_k) = 2c\varepsilon. \tag{7.33}$$

It follows from (7.32) and (7.33) that

$$s^\nabla(\xi_j, \xi_j) = \frac{m+1}{4m(m+2)}\rho^g g_L(\xi_j, \xi_j) \tag{7.34}$$

for $j = 2, 3$. The equation (7.28) implies that $g_L(R^\nabla(X^H, Y^H)\xi_j, X^H) = 0$. We have, from $R^\nabla(\xi_k, \xi_j)\xi_j = 2c\varepsilon\xi_j$,

$$\begin{aligned} g_L(R^\nabla(\xi_k, Y^H)\xi_j, \xi_k) &= g_L(R^\nabla(\xi_j, \xi_k)\xi_k, Y^H) \\ &= 0, \end{aligned} \tag{7.35}$$

where we note that the first equality is derived from $F = 0$. Therefore we obtain

$$s^\nabla(Y^H, \xi_j) = 0. \tag{7.36}$$

Similarly we have

$$g_L(R^\nabla(X^H, \xi_2)\xi_3, X^H) = 0, \quad g_L(R^\nabla(\xi_j, \xi_2)\xi_3, \xi_j) = 0$$

and hence

$$s^\nabla(\xi_2, \xi_3) = 0. \tag{7.37}$$

The two Ricci tensors $s^\nabla$ and $k^\nabla$ coincide when $F = 0$ (cf. [10]). Therefore we conclude that $(\phi, \xi, \theta)$ is pseudo-Einstein.

Finally we show that $p = \alpha\omega_2 + \beta\omega_3$ ($\alpha, \beta$: constant) satisfies (3.16) and $(\nabla_V p)(W) - (\nabla_{JV} q)(W) + p(\xi)g_L(JV, W) = 0$ for $V, W \in \mathscr{D}$. It is easy to show that $p$ satisfies (3.16) by virtue of the structure equation of the connection $\omega$. Since

$$(\nabla_V p)(W) - (\nabla_{JV} q)(W) = V \cdot p(W) - p(\nabla_V W) + JV \cdot p(JW) - p(J\nabla_{JV} W),$$

we easily see that $(\nabla_V p)(W) - (\nabla_{JV} q)(W) = 0$ in the cases where ($V = X^H$, $W = Y^H$), ($V = X^H$, $W = \xi_j$), ($V = \xi_j$, $W = X^H$) and ($V = \xi_j$, $W = \xi_k$). Noting that $p(\xi) = 0$, we obtain the assertion.

In this way, we have shown that the total space of the $SO(3)$-bundle associated with a quaternion Kähler manifold of dimension $4m$ $(m \geq 2)$ with non vanishing scalar curvature admits a CR Einstein-Weyl structure.

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