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LAGRANGIAN H-UMBILICAL SUBMANIFOLDS IN QUATERNION EUCLIDEAN SPACES

By

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Abstract. In [2], B. Y. Chen proved that the Lagrangian H-umbilical submanifolds in complex Euclidean space $\mathbf{C}^n$ are Lagrangian pseudo-spheres and complex extensors of the unit hypersphere of $\mathbf{E}^n$, except the flat ones. Similar to this, we can define the Lagrangian H-umbilical submanifold in quaternion space forms. The main purpose of this paper is to classify the Lagrangian H-umbilical submanifolds in quaternion Euclidean space $\mathbf{H}^n$.

1. Introduction

It has been known that there is no totally umbilical Lagrangian submanifolds in complex-space-forms except the totally geodesic ones. It was natural to look for the simplest Lagrangian submanifold next to the totally geodesic ones in complex-space-form. To do so, B. Y. Chen introduced the notion of the Lagrangian H-umbilical submanifold [3] and also, in [2], he obtained the classification theorems for Lagrangian H-umbilical submanifolds in complex Euclidean space $\mathbf{C}^n$.

Similar to the above case, it also has been known that there exist no totally umbilical Lagrangian submanifold in quaternion-space-form $\tilde{M}^n(4c)$ except the totally geodesic ones. In order to find the next simplest case in quaternion-space-form $\tilde{M}^n(4c)$, we also introduce the notion of Lagrangian H-umbilical submanifold. By a Lagrangian H-umbilical submanifold M^n in a quaternion manifold $\tilde{M}^n$ we mean a non-totally geodesic Lagrangian submanifold whose second fundamental form is given by

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$$
(1)\quad
\begin{aligned}
&h(e_1,e_1) = \lambda_1 Ie_1 + \lambda_2 Je_1 + \lambda_3 Ke_1,\\
&h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu_1 Ie_1 + \mu_2 Je_1 + \mu_3 Ke_1,\\
&h(e_1,e_j) = \mu_1 Ie_j + \mu_2 Je_j + \mu_3 Ke_j, \quad j = 2,\dots,n\\
&h(e_j,e_k) = 0, \quad j \neq k, j, k = 2,\dots,n
\end{aligned}
$$

for some functions λ_1, λ_2, λ_3, μ_1, μ_2 and μ_3 with respect to some orthonormal local frame fields.

According to the above condition, the mean curvature vector H is given by $H = H_1 + H_2 + H_3$, where $H_1 = \gamma_1 Ie_1$, $H_2 = \gamma_2 Je_1$, $H_3 = \gamma_3 Ke_1$, and $\gamma_i = \frac{\lambda_i + (n-1)\mu_i}{n}$ $(i = 1,2,3)$. The condition (1) is equivalent to

$$
\begin{aligned}
h(X,Y) = \alpha_1(&\langle IX,H\rangle\langle IY,H\rangle H_1 + \alpha_2\langle JX,H\rangle\langle JY,H\rangle H_2\\
&+ \alpha_3\langle KX,H\rangle\langle KY,H\rangle H_3)\\
&+ \beta_1(\langle X,Y\rangle H_1 + \langle IY,H\rangle IX + \langle IX,H\rangle IY)\\
&+ \beta_2(\langle X,Y\rangle H_2 + \langle JY,H\rangle JX + \langle JX,H\rangle JY)\\
&+ \beta_3(\langle X,Y\rangle H_3 + \langle KY,H\rangle KX + \langle KX,H\rangle KY)
\end{aligned}
$$

for any tangent vectors X, Y, where $\alpha_i = \frac{\lambda_i - 3\mu_i}{\gamma_i^3}$ and $\beta_i = \frac{\mu_i}{\gamma_i}$ for $i = 1,2,3$. Clearly, a non-minimal Lagrangian H-umbilical submanifold has the shape operator A_H at H with two eigenvalues λ and μ, where $\lambda = \sum_{i=1}^n \lambda_i\gamma_i$ and $\mu = \sum_{i=1}^n \mu_i\gamma_i$ with respect to some orthonormal frame fields.

On the other hand, it also satisfies $\langle h(X,Y),\phi_i Z\rangle = \langle h(Z,Y),\phi_i X\rangle$, where ϕ_i is one of the element in $\{I,J,K\}$ and X, Y, Z are tangent vectors to M^n. Using this property, we can say that Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds next to totally geodesic submanifolds in quaternion Euclidean space.

The main purpose of this paper is to classify the Lagrangian H-umbilical submanifolds in quaternion Euclidean space.

2. Preliminaries [6]

Let $\tilde{M}^n$ be a $4n$-dimensional Riemannian manifold with metric g. $\tilde{M}^n$ is called a quaternion manifold if there exists a 3-dimensional vector space V of tensors of type $(1,1)$ with local basis of almost Hermitian structure I, J and K such that

(a) $IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J,\ I^2 = J^2 = K^2 = -1,$

(b) for any local cross-section φ of V, $\tilde{\nabla}_X \varphi$ is also a cross section of V, when X is an arbitrary vector field on $\tilde{M}^n$ and $\tilde{\nabla}$ the Riemannian connection on $\tilde{M}^n$.

Condition (b) is equivalent to the following condition:

(b') there exist local 1-forms p, q and r such that

$$\tilde{\nabla}_X I = r(X)J - q(X)K,$$

$$\tilde{\nabla}_X J = -r(X)I + p(X)K,$$

$$\tilde{\nabla}_X K = q(X)I - p(X)J$$

Let X be a unit vector on $\tilde{M}^n$. Then X, IX, JX, and KX form an orthonormal frame on $\tilde{M}^n$. We denote by $Q(X)$ the 4-plane spanned by them. For any two orthonormal vectors X, Y on $\tilde{M}^n$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X, Y is called a totally real plane. Any 2-plane in a $Q(X)$ is called a quaternion plane. The sectional curvature of a quaternion plane π is called the quaternion sectional curvature of π. A quaternion manifold is a quaternion-space-form if its quaternion sectional curvatures are equal to a constant $4c$. We denote such a $4n$-dimensional quaternion-space-form by $\tilde{M}^n(4c)$.

It is well known that a quaternion manifold $\tilde{M}^n$ is a quaternion-space-form if and only if its curvature tensor $\tilde{R}$ is of the following form:

$$\begin{aligned}\tilde{R}(X, Y)Z = c\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ \\ &+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\}\end{aligned} \tag{2}$$

for tangent vectors X, Y and Z on $\tilde{M}^n$.

Let M be an n-dimensional Riemannian manifold and $x : M \to \tilde{M}^n(4c)$ be an isometric immersion of M into a quaternion-space-form $\tilde{M}^n(4c)$. We call M a Lagrangian submanifold or a totally real submanifolds of $\tilde{M}^n(4c)$ if each 2-plane of M is mapped by x into a totally real plane in $\tilde{M}^n(4c)$. Consequently if M is a Lagrangian submanifold of $\tilde{M}^n(4c)$ then $\phi(TM) \subset T^{\perp}M$ for $\phi = I, J,$ or K, $T^{\perp}M$ is the normal bundle of M in $\tilde{M}^n(4c)$.

For any orthonormal vectors X, Y in TM, $\pi(X, Y)$ is totally real in $\tilde{M}^n(4c)$, $Q(X)$ and $Q(Y)$ are orthogonal and $g(X, \varphi Y) = g(\phi X, Y) = 0$ for $\varphi, \phi = I, J$ or K. By (2) we have

$$\tilde{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad \text{for } X, Y, Z \in TM$$

If we denote the Levi-Civita connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively, the formulas of Gauss and Weingarten are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X \zeta = -A_\zeta X + D_X \zeta,$$

for tangent vector fields X, Y and normal vector field ζ, where D is the normal connection. The second fundamental form h is related to the shape operator A_ζ by $\langle h(X,Y),\zeta\rangle = \langle A_\zeta X, Y\rangle$ for a normal vector ζ of M. The mean curvature vector H of M is defined by $H = \frac{1}{n}$ trace h. If we denote the curvature tensors of ∇ and D by R and R^D, then the equations of Gauss, Codazzi and Ricci are given by

$$\begin{aligned}
\langle R(X,Y)Z,W\rangle &= \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle \\
&\quad + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \\
(\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z), \\
\langle R^D(X,Y)\psi Z,\phi W\rangle &= \langle [A_{\psi Z},A_{\phi W}]X,Y\rangle \\
&\quad + c((\langle X,I\phi W\rangle\langle Y,I\psi Z\rangle - \langle X,I\psi Z\rangle\langle Y,I\phi W\rangle) \\
&\quad + (\langle X,J\phi W\rangle\langle Y,J\psi Z\rangle - \langle X,J\psi Z\rangle\langle Y,J\phi W\rangle) \\
&\quad + (\langle X,K\phi W\rangle\langle Y,K\psi Z\rangle - \langle X,K\psi Z\rangle\langle Y,K\phi W\rangle),
\end{aligned}$$

where X, Y, Z, W are tangent vector fields and $\phi,\psi = I,J,$ or K.

Finally, we recall a definition of warped product [1]. Let N_1, N_2 be two Riemannian manifolds with Riemannian metrics g_1, g_2, respectively and f a positive function on N_1. Then the metric $g = g_1 + f^2 g_2$ is called a warped product metric on $N_1 \times N_2$. The manifold $N_1 \times N_2$ with the warped product metric $g = g_1 + f^2 g_2$ is called a warped product manifold. The function f is called the warping function of the warped product manifold.

3. Quaternion Extensors

In this section we are going to investigate the geometry of quaternion extensors. First of all, we have the following definition.

Let $G : M^{n-1} \to \mathbf{E^m}$ be an isometric immersion of a Riemannian $(n-1)$-manifold into Euclidean m-space $\mathbf{E^m}$ and $F : I \to \mathbf{H}$ a unit speed curve in the quaternion plane. Consider the following extension ϕ given by

$$\phi = F \otimes G : I \times M^{n-1} \to \mathbf{H} \otimes \mathbf{E^m} = \mathbf{H^m},$$

where $\phi = F \otimes G$ is the tensor product immersion of F and G defined by

$$(F \otimes G)(s, p) = F(s) \otimes G(p); \quad s \in I,\ p \in M^{n-1}.$$

We call such an extension $\phi = F \otimes G$ a quaternion extensor of G via F.

An immersion $f : N \to \mathbf{E^m}$ is called spherical (respectively, unit spherical) if N is immersed into a hypersphere (respectively, unit hypersphere) of $\mathbf{E^m}$ centered at the origin. The quaternion extensor $\phi : F \otimes G : I \times M^{n-1} \to \mathbf{H^m}$ is called F-isometric if, for each $p \in M^{n-1}$, the immersion $F \otimes G(p) : I \to \mathbf{H^m} : s \mapsto F(s) \otimes G(p)$ is isometric. Similarly, the quaternion extensor is called G-isometric if, for each $s \in I$, the immersion $F(s) \otimes G : M^{n-1} \to \mathbf{H^m} : p \mapsto F(s) \otimes G(p)$ is isometric.

Lemma 3.1. *Let $G : M^{n-1} \to \mathbf{E^m}$ be an isometric immersion of a Riemannian $(n-1)$-manifold into Euclidean m-space $\mathbf{E^m}$ and $F : I \to \mathbf{H}$ a unit speed curve in the quaternion plane. Then*

(1) *the quaternion extensor $\phi = F \otimes G$ is F-isometric if and only if G is unit spherical,*

(2) *the quaternion extensor $\phi = F \otimes G$ is G-isometric if and only if F is unit spherical,*

(3) *the quaternion extensor $\phi = F \otimes G$ is totally real if and only if either G is spherical or $F(s) = cf(s)$ for some constant $c \in \mathbf{H}$ and real-valued function f.*

Proof. The statements (1) and (2) come from straightforward computations.

By a direct computation, the quaternion extensor is totally real if and only if, for any $s \in I$, $p \in M^{n-1}$ and $Y \in T_pM^{n-1}$, we have

$$Real(\varphi F(s)\bar{F}'(s)) \cdot \langle G(p), Y \rangle = 0,$$

where $\bar{F}'$ denotes the quaternionic conjugate of F' and $Real(\varphi F(s)\bar{F}'(s))$ the real part of $\varphi F(s)\bar{F}'(s)$ for $\varphi = i, j$ or k. Therefore, we have either $\langle G(p), Y \rangle = 0$ for all $p \in M^{n-1}$, $Y \in T_pM^{n-1}$ or $Real(iF(s)\bar{F}'(s)) = Real(jF(s)\bar{F}'(s)) = Real(kF(s)\bar{F}'(s)) = 0$ for all $s \in I$. If the first case occurs, then G is spherical. If F is given by $F(s) = a(s) + ib(s) + jc(s) + kd(s)$, where a, b, c, and d are real valued functions, and the second condition is true, then we have the following system of ODES:

$$ab' - a'b + cd' - c'd = 0$$
$$ac' - a'c + b'd - bd' = 0$$
$$ad' - a'd + bc' - b'c = 0$$

By solving this system, we can find that $F(s) = ca(s)$ for a constant $c \in \mathbf{H}$.

A submanifold M^{n-1} of E^m is said to be of essential codimension one if locally M^{n-1} is contained in an affine n-subspace of E^m.

PROPOSITION 3.2. *Let $G : M^{n-1} \to \mathbf{E}^m$ be an isometric immersion of a Riemannian $(n-1)$-manifold into Euclidean m-space $\mathbf{E}^m$ and $F : I \to \mathbf{H}$ a unit speed curve. Then the quaternion extensor $\phi = F \otimes G : I \times M^{n-1} \to \mathbf{H}^m$ is totally geodesic (with respect to the induced metric) if and only if one of the following two cases occurs:*

(1) $G : M^{n-1} \to \mathbf{E}^m$ is of essential codimension one and $F(s) = (s+a)c$ for some real number a and some unit quaternion number c.

(2) $n = 2$ and G is a line in $\mathbf{E}^m$.

PROOF. Since ϕ is totally geodesic, ϕ_{ss}, $YZ\phi$, $Y\phi_s$ are tangent vector fields for Y, Z vector fields tangent to the second component of $I \times M^{n-1}$. By using the fact $F''(s) \otimes \xi$ is normal to $I \times M^{n-1}$ in $\mathbf{H}^m$ (via ϕ) for any unit normal vector field ξ of M^{n-1} in $\mathbf{E}^m$, we get the following two equations.

$$\langle F''(s), F''(s)\rangle\langle \xi, G(p)\rangle = 0 \tag{3}$$

$$\langle F''(s), F(s)\rangle\langle \xi, h_G(Y,Z)\rangle = 0, \tag{4}$$

for any vector fields Y, Z tangent to M^{n-1} and for any $s \in I$ and point $p \in M^{n-1}$, where h_G is the second fundamental form of $G : M^{n-1} \to \mathbf{E}^m$.

We can divide our case as follows:

Case (1) $F'' = 0$

This case follows from the case (i) in proposition 2.2 in [2] so that we can deduce statement (1).

Case (2) $F'' \neq 0$

By (3), we get $\langle \xi, G(p)\rangle = 0$ for any normal vector field ξ to M^{n-1} in $\mathbf{E}^m$ and any point $p \in M^{n-1}$. Since ϕ is totally geodesic, $YZ\phi$ is a tangent vector field for Y, Z tangent to M^{n-1} in $\mathbf{E}^m$ which yields

$$0 = \langle F', F\rangle\langle \xi, h_G(Y,Z)\rangle \tag{5}$$

Suppose G is non totally geodesic. Then (5) gives $\langle F', F\rangle = 0$ and thus $\|F\|^2$ is a constant. Also, we get $\langle F'', F\rangle = 0$ because of (4). Combining these conditions for F implies $F' = 0$ which is impossible. Therefore, G must be totally geodesic which implies that $G(M^{n-1})$ is an open portion of an affine $(n-1)$-subspace, say E, of $\mathbf{E}^{\mathbf{m}}$.

Now, we can consider two cases below.

Case (2-i): $n \geq 3$. In this case, for each $p \in G(M^{n-1})$ with $G(p) \neq o$, there exist a nonzero vector $Y \in T_pM^{n-1}$ which is not parallel to $G(p)$. For such Y, we can say that $Y\phi_s = \alpha(s)F'(s) \otimes G + \beta(s)F \otimes Z$ for some real valued functions α, β and some tangent vector Z. It implies that F and F' are parallel which is impossible.

Case (2-ii): $n = 2$. In this case, G is an open portion of a line, say L in $\mathbf{E}^{\mathbf{m}}$.

The converse can be proved easily.

Proposition 3.3. *let $\imath : S^{n-1} \to \mathbf{E}^{\mathbf{n}}$ be the inclusion of the unit hypersphere of $\mathbf{E}^{\mathbf{n}}$ (centered at the origin). Then every quaternion extensor of $\imath$ via a unit speed curve F in $\mathbf{H}$ is a Lagrangian H-umbilical submanifold of $\mathbf{H}^{\mathbf{n}}$ unless $F(s) = (s+a)c$ for some real number a and some unit quaternion number c.*

Proof. By a direct computation, we can easily see that $\phi = F \otimes \imath$ is a Lagrangian H-umbilical submanifold satisfying

$$h(e_1, e_1) = \lambda_I Ie_1 + \lambda_J Je_1 + \lambda_K Ke_1,$$

$$h(e_1, e_j) = \mu_I Ie_j + \mu_J Je_j + \mu_K Ke_j, \quad \text{for } j = 2, \dots, n$$

$$h(e_j, e_j) = \mu_I Ie_1 + \mu_J Je_1 + \mu_K Ke_1, \quad \text{for } j = 2, \dots, n$$

$$h(e_j, e_k) = 0, \quad \text{for } j \neq k = 2, \dots, n,$$

where $\lambda_\varphi = \langle F'', \varphi F'\rangle$ and $\mu_\varphi = \left\langle \left(\frac{F}{\|F\|}\right)', \varphi\left(\frac{F}{\|F\|}\right)\right\rangle$ for $\varphi = I, J$ or K and $\{e_1, e_2, \dots, e_n\}$ is an orthonormal local frame field. Without difficulty, we can get that ϕ is totally geodesic if $F(s) = (s+a)c$ for some real number a and some unit quaternion number c.

4. Main Theorem

The main result of this section is to classify Lagrangian H-umbilical submanifolds of quaternion Euclidean space. To do this, we need to review the

Lagrangian pseudo-sphere in $\mathbf{C}^n$ ([2]). For a real number $b > 0$, let $F : \mathbf{R} \to \mathbf{C}$ be the unit speed curve given by

$$F(s) = \frac{e^{2bsi} + 1}{2bi}$$

With respect to the induced metric, the complex extensor $\phi = F \otimes \imath$ of the unit hypersphere of E^n via F is a Lagrangian isometric immersion of an open portion of an n-sphere $S^n(b^2)$ of sectional curvature b^2 into $\mathbf{C}^n$. It is called a Lagrangian pseudo-sphere. It has been shown that it is a Lagrangian H-umbilical submanifold in $\mathbf{C}^n$ satisfying the following second fundamental form:

$$\begin{aligned} &h(e_1, e_1) = 2bJe_1, \quad h(e_i, e_i) = bJe_1, \quad i \geq 2 \\ &h(e_1, e_j) = bJe_j, \quad h(e_j, e_k) = 0, \quad \text{for } j \neq k = 2, \dots, n, \end{aligned} \tag{6}$$

for some nontrivial function b with respect to some suitable orthonormal local frame field. Up to rigid motions in $\mathbf{C}^n$, it is unique.

THEOREM 4.1. *Let $n \geq 3$ and $L : M \to \mathbf{H}^n$ be a Lagrangian H-umbilical isometric immersion.*

We have one of these three cases:

(A) *M is flat or,*

(B) *up to rigid motions of $\mathbf{H}^n$, L is a Lagrangian pseudo-sphere in $\mathbf{C}^n$, or*

(C) *up to rigid motions of $\mathbf{H}^n$, L is a quaternion extensor of the unit hypersphere of $\mathbf{E}^n$.*

PROOF. Let $n \geq 3$ and $L : M \to \mathbf{H}^n$ be a Lagrangian H-umbilical isometric immersion whose second fundamental form is given by

$$\begin{aligned} &h(e_1, e_1) = \lambda_1 Ie_1 + \lambda_2 Je_1 + \lambda_3 Ke_1, \\ &h(e_1, e_j) = \mu_1 Ie_j + \mu_2 Je_j + \mu_3 Ke_j, \quad \text{for } j = 2, \dots, n \\ &h(e_j, e_j) = \mu_1 Ie_1 + \mu_2 Je_1 + \mu_3 Ke_1, \quad \text{for } j = 2, \dots, n \\ &h(e_j, e_k) = 0, \quad \text{for } j \neq k = 2, \dots, n, \end{aligned} \tag{7}$$

for some functions λ_i, μ_i $(i = 1, 2, 3)$ with respect to some suitable orthonormal local frame fields $\{e_1, e_2, \dots, e_n\}$ with the dual 1-forms $\omega^1, \dots, \omega^n$. Let (ω_A^B), $A, B = 1, \dots, n$ be the connection form on M defined by $\omega_i^j(e_k) = \langle \tilde{\nabla}_{e_k} e_i, e_j \rangle$ for $i, j, k = 1, \dots, n$. By (7) and Codazzi equation, we have

$$(8)\qquad \begin{aligned} e_1(\mu_1) &= (\lambda_1 - 2\mu_1)\omega_1^j(e_j) + \lambda_2\mu_3 - \lambda_3\mu_2 \\ e_1(\mu_2) &= (\lambda_2 - 2\mu_2)\omega_1^j(e_j) + \lambda_3\mu_1 - \lambda_1\mu_3 \\ e_1(\mu_3) &= (\lambda_3 - 2\mu_3)\omega_1^j(e_j) + \lambda_1\mu_2 - \lambda_2\mu_1 \end{aligned}$$

$$(9)\qquad \begin{aligned} e_j(\lambda_1) &= (\lambda_1 - 2\mu_1)\omega_1^j(e_1) \\ e_j(\lambda_2) &= (\lambda_2 - 2\mu_2)\omega_1^j(e_1) \\ e_j(\lambda_3) &= (\lambda_3 - 2\mu_3)\omega_1^j(e_1) \quad \text{for } j = 2,\dots,n \end{aligned}$$

$$(10)\qquad \begin{aligned} &(\lambda_1 - 2\mu_1)\omega_1^k(e_j) = 0 \\ &(\lambda_2 - 2\mu_2)\omega_1^k(e_j) = 0 \\ &(\lambda_3 - 2\mu_3)\omega_1^k(e_j) = 0 \quad \text{for } k \neq j = 2,\dots,n \end{aligned}$$

$$(11)\qquad \begin{aligned} e_j(\mu_1) &= 3\mu_1\omega_1^j(e_1) \\ e_j(\mu_2) &= 3\mu_2\omega_1^j(e_1) \\ e_j(\mu_3) &= 3\mu_3\omega_1^j(e_1) \quad \text{for } j = 2,\dots,n \end{aligned}$$

$$(12)\qquad \mu_1\omega_1^k(e_1) = \mu_2\omega_1^k(e_1) = \mu_3\omega_1^k(e_1) = 0, \quad k = 2,\dots,n$$

Here, (10) and (12) hold only for $n \geq 3$.

Let's first consider the case if M is of constant sectional curvature, then (7) implies that $\mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3) = 0$. Furthermore, the equations in (10) provide the following two cases:

(a) $\lambda_i = 2\mu_i$ for $i = 1, 2, 3$

If $\mu_1 = \mu_2 = \mu_3 = 0$, then M is flat.

From now on, we assume that there exists one i such that $\mu_i \neq 0$. Then the subset $V = \{p \in M \mid \mu_1(p) \neq 0 \text{ or } \mu_2(p) \neq 0 \text{ or } \mu_3(p) \neq 0\}$ is a nonempty open subset of M. The assumption and the equations in (8), (11) and (12) imply that μ_1, μ_2 and μ_3 are constants on V. Then Gauss equation shows V is a real-space-form of constant sectional curvature $\mu_1^2 + \mu_2^2 + \mu_3^2$, say $b^2 \neq 0$. By continuity, $V = M$. By making a proper translation and rescaling, we can say that M satisfies the second fundamental form given in (6). Moreover, we also can check that the first normal space is parallel with respect to the normal connection so that by applying the result of Erbacher [5], M can be immersed into complex Euclidean space $\mathbf{C}^n$ which implies that M is a Lagrangian pseudo-sphere in $\mathbf{C}^n$.

(b) There exists one i such that $\lambda_i \neq 2\mu_i$, and still we have $\mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3) = 0$.

By (10) and our assumption, we know that $\omega_1^j(e_k) = 0$ for $k \neq j = 2, \ldots, n$, and using (8) and (12), we get

$$\omega_1^j = \frac{e_1(\mu_2) - \lambda_3\mu_1 + \lambda_1\mu_3}{\lambda_2 - 2\mu_2}\omega^j \tag{13}$$

Suppose we define $f = \frac{e_1(\mu_2) - \lambda_3\mu_1 + \lambda_1\mu_3}{\lambda_2 - 2\mu_2}$. Let D be the distribution spanned by e_1 and $D^{\perp}$ be the distribution spanned by $\{e_2, e_3, \ldots, e_n\}$. Since $\omega_1^j(e_k) = 0$ for $k \neq j = 2, \ldots, n$, the distribution $D^{\perp}$ is integrable. Also, the distribution D is integrable since it is 1-dimensional. Therefore there exists local coordinates $\{x_1, x_2, \ldots, x_n\}$ such that $e_1 = \frac{\partial}{\partial x_1}$ and $D^{\perp}$ is spanned by $\left\{\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\}$. Using (13), we obtain

$$\langle \nabla_X Y, e_1 \rangle = -f\langle X, Y \rangle, \quad X, Y \in D^{\perp} \tag{14}$$

It implies that $D^{\perp}$ is a spherical distribution and furthermore, each leaf of $D^{\perp}$ is of constant sectional curvature $\mu_1^2 + \mu_2^2 + \mu_3^2 + f^2$. Now, by applying a result of Hiepko [8], M is isometric to a warped product $I \times_{\omega(s)} S^{n-1}$, where S^{n-1} is the unit $(n-1)$ sphere and $\omega(s)$ is a warping function.

Using the spherical coordinates $\{u_2, \ldots, u_n\}$ on the unit sphere, we can choose the metric

$$g = ds^2 + \omega^2(s)\{du_2^2 + \cos^2 u_2\, du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1}\, du_n^2\}$$

on $I \times_{\omega(s)} S^{n-1}$. By using this metric g, we have

$$\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \quad \nabla_{\partial/\partial s}\frac{\partial}{\partial u_k} = \frac{\omega'}{\omega}\frac{\partial}{\partial u_k}, \quad \nabla_{\partial/\partial u_2}\frac{\partial}{\partial u_2} = -\omega\omega'\frac{\partial}{\partial s},$$

$$\tag{15} \begin{aligned} &\nabla_{\partial/\partial u_i}\frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \leq i < j, \\ &\nabla_{\partial/\partial uj}\frac{\partial}{\partial u_j} = -\omega\omega' \cos^2 u_2 \cdots \cos^2 u_{j-1}\frac{\partial}{\partial s} \\ &\qquad + \sum_{k=2}^{n-1}\frac{\sin 2u_k}{2}\cos^2 u_{k+1} \cdots \cos^2 u_{j-1}\frac{\partial}{\partial u_k}, \quad j \geq 2 \end{aligned}$$

By substituting $X = Y = \frac{\partial}{\partial u_2}$ into (14) and using (15), we get

$$\frac{\omega'}{\omega} = f$$

Furthermore, computing the sectional curvatures spanned by $\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}$ and $\frac{\partial}{\partial s}, \frac{\partial}{\partial u_2}$ derive the following condition for the warping function:

$$\frac{1}{\omega^2} - f^2 = -\frac{\omega''}{\omega} = \mu_1^2 + \mu_2^2 + \mu_3^2 = \bar{\mu}^2$$

Note here that $\bar{\mu}$ is a constant by our assumption. These conditions provide a differential equation

$$f^2 + f' + \bar{\mu}^2 = 0$$

and then

$$\omega(s) = \cos \bar{\mu}s, \quad f(s) = -\bar{\mu} \tan \bar{\mu}s$$

By doing the same procedure in theorem 4.1 [2], we obtain that L is a quaternion extensor of the unit hypersphere of $\mathbf{E}^{\mathbf{n}}$.

Now, we assume that M does not contain open subset of constant sectional curvature. Then

$$U := \{p \in M : \mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3) \neq 0 \text{ at } p\}$$

is an open dense subset of M.

By (8), on U, we obtain

$$\omega_1^j(e_j) = \frac{\mu_1 e_1(\mu_1) + \mu_2 e_1(\mu_2) + \mu_3 e_1(\mu_3)}{\mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3)}$$

Since $n \geq 3$, $\omega_1^j(e_k) = 0$ for $k \neq j = 2, \dots, n$ on U. Therefore,

$$\omega_1^j = \frac{\mu_1 e_1(\mu_1) + \mu_2 e_1(\mu_2) + \mu_3 e_1(\mu_3)}{\mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3)} \omega^j$$

Let's define $\bar{f} = \frac{\mu_1 e_1(\mu_1) + \mu_2 e_1(\mu_2) + \mu_3 e_1(\mu_3)}{\mu_1(\lambda_1 - 2\mu_1) + \mu_2(\lambda_2 - 2\mu_2) + \mu_3(\lambda_3 - 2\mu_3)}$. Using this $\bar{f}$, the conclusion follows in the same way we have seen the above case.

Similar to the complex case, we have the following same result for Lagrangian H-umbilical surface in quaternion Euclidean space $\mathbf{H}^2$.

Theorem 4.2. *Let* $L : M \to \mathbf{H}^2$ *be a Lagrangian H-umbilical surface satisfying*

$$h(e_1, e_1) = \lambda_1 Ie_1 + \lambda_2 Je_1 + \lambda_3 Ke_1,$$

$$h(e_2, e_2) = \mu_1 Ie_1 + \mu_2 Je_1 + \mu_3 Ke_1,$$

$$h(e_1, e_2) = \mu_1 Ie_2 + \mu_2 Je_2 + \mu_3 Ke_2,$$

such that the integral curves of e_1 *are geodesics in* M*. Then*

(1) M *is flat or,*

(2) *up to rigid motions of* $\mathbf{H}^2$, L *is a Lagrangian pseudo-sphere in* $\mathbf{C}^2$ *or*

(3) *up to rigid motions of* $\mathbf{H}^2$, L *is a quaternion extensor of the unit circle of* $\mathbf{E}^2$.

PROOF. By Codazzi equation, we get

$$e_1(\mu_1) = (\lambda_1 - 2\mu_1)\omega_1^2(e_2) + \lambda_2\mu_3 - \lambda_3\mu_2 \tag{16}$$

$$e_1(\mu_2) = (\lambda_2 - 2\mu_2)\omega_1^2(e_2) + \lambda_3\mu_1 - \lambda_1\mu_3$$

$$e_1(\mu_3) = (\lambda_3 - 2\mu_3)\omega_1^2(e_2) + \lambda_1\mu_2 - \lambda_2\mu_1$$

The assumption that the integral curves of e_1 are geodesics in M yields

$$e_2(\mu_i) = e_2(\lambda_i) = 0, \quad i = 1, 2, 3 \tag{17}$$

We note here that this assumption is needed to replace the equation (13) obtained because $n \geq 3$. If we have $\lambda_i = 2\mu_i$ for $i = 1, 2, 3$ in (16), and using (17), the sectional curvature of the surface $\mu_1^2 + \mu_2^2 + \mu_3^2$ becomes a constant. If all μ_i's are identically zero, then it is a flat surface. Otherwise, doing the same work in the case (a) of theorem 4.1, we can say that it is a Lagrangian pseudo-sphere in $\mathbf{C}^2$. If there exists one i, saying $i = 1$ such that $\lambda_1 \neq 2\mu_1$, then we have

$$\omega_1^2 = \frac{e_1(\mu_1) - \lambda_2\mu_3 + \lambda_3\mu_2}{\lambda_1 - 2\mu_1}\omega^1 \tag{18}$$

and then the rest of the proof is exactly identical by taking into account $f = \frac{e_1(\mu_1) - \lambda_2\mu_3 + \lambda_3\mu_2}{\lambda_1 - 2\mu_1}$ in Theorem 4.1.

REMARK. The explicit description of flat Lagrangian H-umbilical submanifolds in a quaternion Euclidean space will be discussed in [7].

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