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ON THE REGULARITY OF WEAK SUBELLIPTIC *F*-HARMONIC MAPS

By

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Abstract. Building on work by L. Capogna & D. Danielli & N. Garofalo (cf. [7]–[8]), G. Citti & N. Garofalo & E. Lanconelli (cf. [10]) and P. Hájlasz & P. Strzelecki (cf. [16]) we study local properties of weak subelliptic F-harmonic maps (cf. [4]) of a nondegenerate CR manifold into a sphere S^m, where $\rho(t) = F'(t/2)$, $F \in C^2$, $F(t) \geq 0$, $F'(t) > 0$, $t \geq 0$. If $\Omega \subset \boldsymbol{R}^n$ is a bounded domain and X is a Hörmander system on $\boldsymbol{R}^n$, we show that any weak solution $\phi \in W_X^{1,D}(\Omega, S^m)$ to the nonlinear subelliptic system $-X^* \cdot (\rho(|X\phi|^2)X\phi) = \rho(|X\phi|^2)\phi|X\phi|^2$ is locally Hölder continuous, where D is a homogeneous dimension of Ω with respect to X, provided that $t^p/K \leq \rho(t) \leq Kt^p$ for some $0 < p < (D-2)/2$.

1 Subelliptic Harmonic Maps

Let Ω be a domain in $\boldsymbol{R}^n$, $n \geq 2$, and $X := \{X_1, \dots, X_k\}$ a system of vector fields with smooth real coefficients defined on some open set $U \subseteq \boldsymbol{R}^n$ with $\Omega \subset\subset U$. Let us assume that X is a *Hörmander system* on U, i.e. $X_1, \dots, X_k$ together with their commutators up to a certain fixed length span the tangent space $T_x(U)$, for any $x \in U$. The adjoint of $X_a = b_a^j \partial/\partial x^j$ is given by $X_a^*(f) = -\partial(b_a^j f)/\partial x^j$ for $f \in C_0^1(U)$ and the *Hörmander operator* is

$$Hu \equiv \sum_{a=1}^{k} X_a^* X_a u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial u}{\partial x^j}\right),$$

where $a^{ij} = \sum_{a=1}^{k} b_a^i b_a^j$. Then $a^{ij} \in C^\infty(U)$ and a^{ij} is symmetric and positive semidefinite, yet a^{ij} might fail to be positive definite, hence H is a degenerate

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elliptic operator (in the sense of J. M. Bony, [6]). Given a Riemannian manifold (N, h), covered by a single coordinate system $(y^1, \dots, y^m) : N \to \boldsymbol{R}^m$, a smooth map $\phi : \bar{\Omega} \to N$ is a *subelliptic harmonic map* if it is a critical point of the functional

$$E(\phi) = \frac{1}{2}\int_{\Omega} \sum_{a=1}^{k} (X_a\phi^i)(X_a\phi^j)(h_{ij} \circ \phi)\, dx, \tag{1}$$

where $\phi^i := y^i \circ \phi$ and h_{ij} are the coefficients of h with respect to (y^i). The notion is due to J. Jost & C-J. Xu, [21]. The Euler-Lagrange equations of the variational principle $\delta E(\phi) = 0$ are

$$H_N\phi^i \equiv H\phi^i + \sum_{a=1}^{k} (\Gamma^i_{j\ell} \circ \phi)(X_a\phi^j)(X_a\phi^\ell) = 0, \quad 1 \le i \le m.$$

H is a subelliptic operator (in the sense of D. Jerison & A. Sánchez-Calle, [19], p. 46) hence $H_N\phi = 0$ is a nonlinear subelliptic system of PDEs, thus motivating the terminology in [21]. By a classical result of L. Hörmander (cf. [18]) H is hypoelliptic and it is indeed a natural problem to extend existence and regularity results known for elliptic PDEs (of a variational origin) to the hypoelliptic case.

The reason for studying subelliptic operators is provided by the function theory in several complex variables. Indeed, let $(M, T_{1,0}(M))$ be a CR submanifold of $\boldsymbol{C}^{n+1}$ and $\bar{\partial}_M f = 0$ the tangential Cauchy-Riemann equations. In order to develop a Hodge theory for the $\bar{\partial}_M$-complex one needs (cf. e.g. J. J. Kohn, [22]) to consider the Kohn-Rossi laplacian $\Box_M = \bar{\partial}_M \bar{\partial}_M^* + \bar{\partial}_M^* \bar{\partial}_M$ and then the principal part of $-\Box_M$ may be shown to be a subelliptic operator. To see that J. Jost & C-J. Xu's subelliptic harmonic maps are tied to the same circle of ideas, assume M to be strictly pseudoconvex and consider the canonical S^1-bundle $\pi : C(M) \to M$, cf. e.g. [11], p. 104. Given a contact form θ on M, the total space $C(M)$ carries a Lorentzian metric F_θ, the *Fefferman metric* of (M, θ) [cf. C. Fefferman, [12], for the case of a real hypersurface in $\boldsymbol{C}^{n+1}$, and J. M. Lee, [23], for an abstract (i.e. not necessarily embedded) strictly pseudoconvex CR manifold]. Let $\{X_1, \dots, X_{2n}\}$ be a local orthonormal (with respect to the Levi form G_θ) frame in the Levi, or maximally complex, distribution $H(M) := Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$, defined on a local coordinate neighborhood (U, φ) of M. Then $X = \{(d\varphi)X_a : 1 \le a \le 2n\}$ is a Hörmander system on $\varphi(U) \subseteq \boldsymbol{R}^{2n+1}$. By a result of H. Urakawa *et alt.*, [5], if $\phi : M \to N$ is a smooth map into a Riemannian manifold such that its vertical lift $\phi \circ \pi$ is an ordinary harmonic map (in the sense of B. Fuglede, [14]) of $(C(M), F_\theta)$ into (N, h), then $\phi \circ \varphi^{-1}$ is a subelliptic harmonic map (with respect to X). Moreover, this class of maps consists (cf. [5]) precisely of the critical points (referred to as *pseudoharmonic maps* in [5]) of the functional

$$E(\phi) = \frac{1}{2} \int_M trace_{G_\theta}(\pi_H \phi^* h) \theta \wedge (d\theta)^n. \qquad (2)$$

Here we assume (for simplicity) that M is compact. Also $\pi_H \phi^* h$ denotes the restriction of the bilinear form $\phi^* h$ to $H(M) \otimes H(M)$. See [11] for a brief introduction to CR and pseudohermitian geometry.

The function spaces suited for the study of solutions to $H_N \phi = 0$ are the *Folland-Stein spaces* (cf. G. B. Folland & E. M. Stein, [13])

$$W_X^{1,p}(U) = \{f \in L^p(U) : X_a f \in L^p(U), 1 \leq a \leq k\},$$

where $X_a f$ is meant in distributional sense. As N is covered by a single chart the space $W_X^{1,p}(U, N)$ is also unambiguously defined (it consists of all $\phi : U \to N$ such that $\phi^i \in W_X^{1,p}(U)$, $1 \leq i \leq m$). Although the equations $H_N \phi = 0$ are nonlinear there is a naturally associated concept of weak solution, that is a map $\phi : U \to N$ such that $\phi \in W_X^{1,2}(U, N)$, and

$$\int_U (X\phi^i) \cdot (X\varphi) \, dx = \int_U (\Gamma^i_{j\ell} \circ \phi)(X\phi^j) \cdot (X\phi^\ell) \varphi \, dx, \quad 1 \leq i \leq m,$$

for any $\varphi \in C_0^\infty(U)$. Here, for two vector fields $E = (E_1, \ldots, E_k)$ and $F = (F_1, \ldots, F_k)$ we set $E \cdot F = \delta^{ab} E_a F_b$.

Given $f \in C^0(\overline{\Omega}, N) \cap W_X^{1,2}(\Omega, N)$ such that $f(\overline{\Omega})$ is contained in a regular ball[1] $B(p, r) \subset N$, J. Jost & C-J. Xu have solved (cf. *op. cit.*) the Dirichlet problem $H_N \phi = 0$, $\phi|_{\partial\Omega} = f$. It possesses a unique solution $\phi \in W_X^{1,2}(\Omega, N) \cap L^\infty(\Omega, N)$ such that $\phi(\overline{\Omega}) \subset B(p, r)$. Moreover, if $\partial\Omega$ is smooth and non-characteristic[2] for X and f is smooth then ϕ is continuous up to the boundary. Z-R. Zhou has redefined (cf. [29]) subelliptic harmonic maps as the smooth solutions $\phi : U \to N$ to

$$\frac{1}{\sqrt{\gamma}} X_b^* (\sqrt{\gamma} \gamma^{ab} X_a \phi^i) - \gamma^{ab} (\Gamma^i_{j\ell} \circ \phi)(X_a \phi^j)(X_b \phi^\ell) = 0, \quad 1 \leq i \leq m, \qquad (3)$$

[1] A ball $B(p, r) \subset N$ is *regular* if $r < \min\{\pi/(2\kappa), i(p)\}$, where κ^2 is an upper bound for the sectional curvature of N and $i(p)$ is the injectivity radius of $p \in N$.

[2] The boundary $\partial\Omega$ is *noncharacteristic* for X if for any $x \in \partial\Omega$ there is $a \in \{1, \ldots, k\}$ such that $X_a(x) \notin T_x(\partial\Omega)$.

where γ_{ab} is a positive definite symmetric matrix of smooth functions on U, $\gamma = \det(\gamma_{ab})$ and $\gamma^{ac}\gamma_{cb} = \delta^a_b$. Existence and continuity up to the boundary of the solution to the Dirichlet problem for (3) may be treated as in [21]. Combining this result with Theorem 1.1 of C-J. Xu & C. Zuily, [28], (dealing with higher interior regularity for a class of quasilinear subelliptic systems which includes (3)) shows that solutions are actually smooth. Z-R. Zhou proves (cf. *op. cit.*) that two solutions $\phi_1, \phi_2 : \bar{\Omega} \to B(p,r) \subset N$ to (3) with the same boundary values ($\phi_1|_{\partial\Omega} = \phi_1|_{\partial\Omega}$) actually coincide ($\phi_1 = \phi_2$). A moment's thought shows that, while Z-R. Zhou's concept of a subelliptic harmonic map is more general than that of J. Jost & C-J. Xu (as $\gamma_{ab} = \delta_{ab}$ in [21]), both are but local manifestations of the same global notion, that of a pseudoharmonic map. Indeed, given a nondegenerate CR manifold M and a local frame (not necessarily orthonormal) $\{X_1, \dots, X_{2n}\}$ of $H(M)$ (defined on a coordinate neighborhood of M) the Euler-Lagrange equations (locally written, with respect to $\{X_a\}$) associated to the functional (2) are precisely the equations (3) (with $\gamma_{ab} := G_\theta(X_a, X_b)$).

When $X_a = \partial/\partial x^a$, $1 \le a \le n$, a critical point of (1) is an ordinary harmonic map (cf. e.g. J. Jost, [20], p. 389) $\phi : U \to N$. Then, in analogy with p-harmonic maps (cf. e.g. P. Baird & S. Gudmundson, [2]), P. Hájlasz & P. Strzelecki's notion (cf. [16]) of a *subelliptic p-harmonic map* $\phi : \Omega \to S^m$, that is a critical point of the functional

$$E_p(\phi) = \int_\Omega |X\phi|^p \, dx,$$

appears as quite natural (here $\Omega \subset \boldsymbol{R}^n$ is a bounded domain and $|X\phi|^2 = \sum_{a=1}^k |X_a\phi|^2$, for a given Hörmander system $X = \{X_a\}$ on $\boldsymbol{R}^n$). P. Hájlasz & P. Strzelecki prove (cf. *op. cit.*) the local Hölder continuity of every subelliptic D-harmonic map $\phi \in W^{1,D}_X(\Omega, S^m)$, where D is a homogeneous dimension of Ω (with respect to X). The subelliptic analog to the exponential harmonic maps (cf. e.g. M. C. Hong, [17]) has not been studied, so far.

Building on ideas due to M. Ara, [1], and K. Uhlenbeck, [27], the first named author has considered (cf. [4]) *F-pseudoharmonic maps* $\phi : M \to N$ of a (compact) strictly pseudoconvex CR manifold M into a Riemannian manifold N, defined as critical points of the functional

$$E_F(\phi) = \int_M F\left(\frac{1}{2}\, trace_{G_\theta}(\pi_H \phi^* h)\right) \theta \wedge (d\theta)^n, \tag{4}$$

where $F : [0, \infty) \to [0, \infty)$ is a C^2 function such that $F'(t) > 0$. The Euler-Lagrange equations of the variational principle $\delta E_F(\phi) = 0$ are (cf. [4])

$$div(\rho(Q)\nabla^H\phi^i) + \sum_{a=1}^{2n}\rho(Q)(\Gamma^i_{j\ell}\circ\phi)(X_a\phi^j)(X_a\phi^\ell) = 0, \tag{5}$$

$$\rho(t) := F'(t/2), \quad Q := trace_{G_\theta}(\pi_H\phi^*h),$$

where $\{X_a\}$ is a local G_θ-orthonormal frame of $H(M)$. The divergence operator is defined with respect to the volume form $\Psi := \theta\wedge(d\theta)^n$, that is $\mathscr{L}_X\Psi = div(X)\Psi$, for any C^1 vector field X on M, and $\nabla^H u := \pi_H\nabla u$, for any $u\in C^1(M)$, where ∇u is the gradient of u with respect to the Webster metric g_θ (cf. e.g. [11])

$$g_\theta(X,Y) := G_\theta(\pi_H X, \pi_H Y) + \theta(X)\theta(Y), \quad X,Y\in T(M),$$

and $\pi_H : T(M)\to H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M)\oplus \boldsymbol{R}T$. Here T is the *characteristic direction* of $d\theta$, i.e. the unique (globally defined) vector field on M determined by $\theta(T) = 1$ and $T\rfloor d\theta = 0$. When $N = S^m$ the equations (5) become (by also taking into account the (local) expression of *div* with respect to $\{X_a\}$)

$$-\sum_{a=1}^{2n} X_a^*(\rho(Q)X_a\phi^i) = \rho(Q)\phi^i|X\phi|^2, \quad 1\le i\le m, \tag{6}$$

$$|X\phi|^2 := \sum_{a=1}^{2n}\sum_{A=1}^{m+1} X_a(\phi^A)^2, \quad \phi^A := y^A\circ\phi,$$

where (y^A) are the Cartesian coordinates on $\boldsymbol{R}^{m+1}$. The adjoint X_a^* is with respect to Ψ, i.e. $\int uX_a^*v\Psi = -\int(X_a u)v\Psi$, for any $u\in C_0^\infty$ and $v\in C^\infty$. Taking into account the constraint $\sum_{A=1}^{m+1}\phi_A^2 = 1$ (where $\phi_A = \phi^A$) it follows that ϕ_{m+1} satisfies (6) as well.

Our purpose in the present paper is to start a study of the regularity of weak solutions to (6). In the spirit of P. Hájlasz & P. Strzelecki, [16], we first deal with the problem where $\phi:\Omega\to N$, for some bounded domain $\Omega\subset\boldsymbol{R}^n$. Then, corresponding to (4) and (6) we have

$$E_F(\phi) = \int_\Omega F\left(\frac{1}{2}|X\phi|^2\right)dx,$$

$$-X^*\cdot(\rho(|X\phi|^2)X\phi) = \rho(|X\phi|^2)\phi|X\phi|^2. \tag{7}$$

Compare our (7) to (0.1) in K. Uhlenbeck, [27]. Note that when $F(t) := (2t)^{p/2}$, $t\ge 0$, and $m=1$ the left hand side of (7) becomes $\mathscr{L}_p\phi$, where $\mathscr{L}_p u \equiv -X^*\cdot(|Xu|^{p-2})Xu)$ is the *subelliptic p-Laplacian* in [7]. L. Capogna & D. Danielli

& N. Garofalo were (cf. *op. cit.*) the first to study regularity properties of weak solutions to (a single equation) $\mathscr{L}_p u = 0$.

We shall need the *Carnot-Carathéodory* distance $d_C(x,y)$ defined as the infimum of $T > 0$ for which there is an absolutely continuous curve $C : [0,T] \to \boldsymbol{R}^n$ such that $\dot{C}(t) = \sum_{a=1}^k f_a(t) X_a(C(t))$, for some functions $f_j(t)$ satisfying $\sum_{a=1}^k f_a(t)^2 \le 1$, and $C(0) = x$, $C(T) = y$, $x, y \in \boldsymbol{R}^n$. Also, for a bounded open set $\Omega \subset \boldsymbol{R}^n$ we recall that a number D is a *homogeneous dimension* with respect to X if there is a constant $C > 0$ such that

$$\frac{|B(x,r)|}{|B(x_0,r_0)|} \ge C\left(\frac{r}{r_0}\right)^D, \tag{8}$$

for any ball $B_0 = B(x_0,r_0)$ of center $x_0 \in \Omega$ and radius $0 < r_0 \le diam(\Omega)$, and any ball $B = B(x,r)$ of center $x \in B_0$ and radius $0 < r \le r_0$. Here $B = \{y \in \boldsymbol{R}^n : d_C(x,y) < r\}$ is a metric ball and $diam(\Omega)$ is the diameter of Ω with respect to d_C. Also $|A|$ denotes the Lebesgue measure of the set A. Clearly, any $D' \ge D$ is a homogeneous dimension of Ω, as well. Our result is

THEOREM 1. *Let $X = \{X_1, \dots, X_k\}$ be a Hörmander system on $\boldsymbol{R}^n$ such that each $X_a = b_a^A \partial/\partial x^A$ has components $b_a^A(x)$ which are globally Lipschitz on $\boldsymbol{R}^n$. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain and D a homogeneous dimension of Ω relative to X. Assume that $t^p/K \le \rho(t) \le K t^p$, for some constant $K \ge 1$ and some $0 < p < (D-2)/2$. Let $\phi \in W_X^{1,D}(\Omega, S^m)$ be a weak solution to the nonlinear subelliptic system (7). Let $R_0 > 0$ and $\Omega_1 \subset\subset \Omega$ such that $B(x, 2R_0/\tau) \subset \Omega$ for any $x \in \Omega_1$, $\tau := 1/200$. There is $\lambda \in [1/2, 1)$ and $0 < r_0 \le R_0$ such that*

$$I_p(r) := \int_{B(x,r)} |X\phi|^{2(p+1)}(y)\, dy \le C r^\gamma, \quad \gamma := (\log \lambda)/(\log \tau),$$

for any $x \in \Omega_1$ and any $0 < r \le r_0$. Consequently, if $\tau^D < \lambda < \tau^{D-2(p+1)}$ then $\phi \in S_{loc}^{0,\alpha}(\Omega)$ with $\alpha := 1 + (\gamma - D)/(2p+2)$, hence ϕ is locally Hölder continuous.

The assumption that X_a have globally Lipschitz coefficients guarantees (by Prop. 2.8 in N. Garofalo & D. M. Nhieu, [15]) that a subset in $\boldsymbol{R}^n$ is bounded with respect to d_C if and only if it is bounded with respect to the Euclidean metric[3]. The Hölder like spaces (associated to the given Hörmander system) in Theorem 1 are given by

[3] If a set A is bounded with respect to the Euclidean metric then A is d_C-bounded, yet the converse fails, in general.

$$S^{0,\alpha}(\Omega) = \left\{ f \in L^\infty(\Omega) : \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{d_C(x,y)^\alpha} < \infty \right\}, \quad 0 < \alpha \le 1.$$

Folland-Stein spaces may be defined on a (not necessarily compact) strictly pseudoconvex CR manifold, as well. For instance, let $W_H^{1,p}(M)$ be the completion of $\{u \in C^\infty(M) \cap L^p(M) : \nabla^H u \in L^p(H(M))\}$ with respect to the norm

$$\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|\nabla^H u\|_{L^p},$$

where

$$\|u\|_{L^p} = \left(\int_M |u|^p \theta \wedge (d\theta)^n\right)^{1/p}, \quad \|X\|_{L^p} = \left(\int_M G_\theta(X,X)^{p/2} \theta \wedge (d\theta)^n\right)^{1/p}.$$

Of course $L^p(M)$ (respectively $L^p(H(M))$) is the completion of $C_0^\infty(M)$ (respectively of $\Gamma_0^\infty(H(M))$) with respect to the norm $\|\cdot\|_{L^p}$. Also, an analog of the Carnot-Carathéodory metric is available on any strictly pseudoconvex CR manifold M. Let us briefly recall its construction (under the conventions of R. S. Strichartz, [26]). The Levi form

$$G_\theta(X,Y) := (d\theta)(X,JY), \quad X,Y \in H(M),$$
$$J : H(M) \to H(M), \quad J(Z+\bar{Z}) := \sqrt{-1}(Z-\bar{Z}), \quad Z \in T_{1,0}(M),$$

is a *sub-Riemannian metric* on $H(M)$ (cf. [26], p. 225) and $H(M)$ satisfies the *strong bracket generating hypothesis* (as θ is a contact form, cf. [26], p. 229) hence the study of $(M, H(M), G_\theta)$ lies within *sub-Riemannian geometry*. A piecewise C^1 curve $\gamma : I \to M$, where $I \subseteq \boldsymbol{R}$ is an interval, is *lengthy* if $\dot{\gamma}(t) \in H(M)_{\gamma(t)}$, for every $t \in I$ where $\dot{\gamma}(t)$ is defined. Let $g(x) : T_x^*(M) \to H(M)_x$, $x \in M$, be the $\boldsymbol{R}$-linear map defined by $G_{\theta,x}(g(x)\alpha, X) = \alpha(X)$, for any $\alpha \in T_x^*(M)$ and $X \in H(M)_x$. Then $Ker(g(x)) = H(M)_x^\perp$, $x \in M$. A piecewise C^0 curve $\xi : I \to T^*(M)$ is a *cotangent lift* of (the lengthy curve) γ if $\xi(t) \in T_{\gamma(t)}^*(M)$ and $g(\gamma(t))\xi(t) = \dot{\gamma}(t)$, for every t (where defined). Any such ξ descends to a (well defined) map $I \to T^*(M)/H(M)^\perp$ (the *uniqueness of piecewise C^0 cotangent lifts* modulo sections of $H(M)^\perp$). The *length* of a lengthy curve γ is

$$\ell(\gamma) = \int_I \langle \xi(t), g(\gamma(t))\xi(t)\rangle^{1/2}\, dt,$$

where $\langle \alpha, v\rangle = \alpha(v)$, $\alpha \in T_x^*(M)$, $v \in T_x(M)$. The definition of $\ell(\gamma)$ does not depend upon the choice of a cotangent lift of γ. The distance $d_S(x,y)$ between two points $x, y \in M$ is the infimum of the lengths of all lengthy curves joining x

and y (that such curves exist is a classical theorem of W. L. Chow, [9]). We refer to d_S as the *Strichartz distance* on (M, θ).

While we deliberately made use of the language of sub-Riemannian geometry (to emphasize that CR geometry embeds there) we must nevertheless observe that the Webster metric g_θ is a *contraction* of the sub-Riemannian metric G_θ (G_θ is an *expansion* of g_θ) hence a lengthy curve has the same length in (M, g_θ). In particular, if d_R is the Riemannian distance (associated to (M, g_θ)) then $d_R(x, y) \leq d_S(x, y)$, for any $x, y \in M$. The metrics d_R and d_S define the same topology on M (cf. [26], p. 230) yet they are not equivalent metrics.

A number D is called a *homogeneous dimension* of (M, θ) if there is a constant $C > 0$ such that (8) holds for any d_S-balls B_0 of radius r_0 and B of center $x \in B_0$ and radius $r \leq r_0$. The Lebesgue measure of the sets appearing in (8) is replaced by their Riemannian volume in (M, g_θ). Let $D(M)$ be the smallest such D (*the* homogeneous dimension of (M, θ)). If M has finite diameter (with respect to the Strichartz distance d_S) and the Riemannian measure (associated to g_θ) has the doubling property then (by Lemma 2.7 in [16]) a homogeneous dimension of (M, θ) exists. It is unknown whether $D(M)$ is a CR invariant. The (3-dimensional) *Heisenberg group* $\boldsymbol{H}_1 = \boldsymbol{C} \times \boldsymbol{R}$, with the CR structure spanned by $\partial/\partial z + i\bar{z}\partial/\partial t$ (the *Lewy operator*) and the contact form $\theta = dt + (i/2)(z\,d\bar{z} - \bar{z}\,dz)$, has homogeneous dimension $D(\boldsymbol{H}_1) = 4$ (cf. e.g. [16], p. 349). We conjecture that each weak solution $\phi \in W_H^{1,D}(M, S^m)$ to the nonlinear subelliptic system (6) with $\rho(t) = Dt^{(D-2)/2}$, $t \geq 0$ (where D is a homogeneous dimension of M) is smooth.

2 A Caccioppoli Type Estimate

$C > 0$ denotes a generic constant (which may change even within the current computation). If $C > 0$ is a constant then $CB(x_0, r)$ is the ball $B(x_0, Cr)$. Also, by $r \approx s$ we mean $r/C \leq s \leq Cr$, for some $C \geq 1$. To start the study of weak solutions ϕ to (7) with the constraint $\sum_{i=1}^{m+1} \phi_i^2 = 1$, we set $V_{i,a} := \rho(Q) X_a \phi_i$, $1 \leq a \leq k$, and $V_i = (V_{i,1}, \dots, V_{i,k})$. Then

$$V_i = \sum_{j=1}^{m+1} \phi_j(\phi_j V_i - \phi_i V_j),$$

merely as a consequence of the constraint. Next, we set $E_{i,j} := \phi_j V_i - \phi_i V_j$ and then (7) implies

$$X^* \cdot E_{i,j} = 0, \quad 1 \leq i, j \leq m+1. \tag{9}$$

Indeed, for any $\psi \in C_0^\infty$

$$\begin{aligned}\int_\Omega (X^* \cdot (\phi_i V_j))\psi \, dx &= \sum_{a=1}^{k} \int_\Omega X_a^*(\phi_i V_{j,a})\psi \, dx = -\sum_a \int_\Omega \phi_i V_{j,a} X_a \psi \, dx \\ &= -\sum_a \int \rho(Q)(X_a\phi_j)[X_a(\phi_i\psi) - \psi X_a\phi_i] \, dx \\ &= \sum_a \int X_a^*(\rho(Q)X_a\phi_j)\phi_i\psi \, dx + \sum_a \int \psi\rho(Q)(X_a\phi_i)(X_a\phi_j) \, dx \\ (by\ (7)) \quad &= \int \rho(Q)\left[-Q\phi_i\phi_j + \sum_a (X_a\phi_i)(X_a\phi_j)\right]\psi \, dx\end{aligned}$$

hence $X^* \cdot (\phi_i V_j)$ is symmetric in i, j, which yields (9). The identity (9) implies the following

Lemma 1 (The duality inequality). *Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain and $X_a = b_a^A \partial/\partial x^A$ a Hörmander system on $\boldsymbol{R}^n$ with $b_a^A(x)$ globally Lipschitz. Let $R_0 > 0$ and $\Omega_1 \subset\subset \Omega$ such that $B(x, 400R_0) \subset \Omega$, for any $x \in \Omega_1$. Let $B = B(x_0, r)$, $x_0 \in \Omega_1$, be a ball such that $0 < r \le R_0$ and $\varphi \in W_X^{1,D}(B)$ a function of compact support. Then*

$$\left| \int_B X^* \cdot (\phi_j E_{i,j})\varphi \, dx \right| \le CK \|X\varphi\|_{L^D(B)} (\|X\phi\|_{L^{2p+2}(100B)})^{2p+2}, \tag{10}$$

for some constant $C = C(\Omega_1, D, C_d, R_0) > 0$, provided that $\rho(t) \le Kt^p$, $t \ge 0$, for some $K > 0$ and $0 < p < (D-2)/2$.

Aside from some additional technical difficulties (e.g. one applies twice the fractional integration theorem), the proof of Lemma 1 is similar to that of Lemma 3.2 in P. Hájlasz & P. Strzelecki, [16], p. 354. We give a proof of Lemma 1 in Section 3. To prove Theorem 1 we fix $\Omega_1 \subset\subset \Omega$ and $R_0 > 0$ as in Lemma 1. Taking the dot product of $V_i = \sum_{j=1}^{m+1} \phi_j E_{i,j}$ with X^* we get

$$X^* \cdot (\rho(Q)X\phi_i) = \sum_{j=1}^{m+1} X^* \cdot (\phi_j E_{i,j}),$$

(a consequence of the constraint alone) and integrating over $2B$ [where $B = B(x, r)$, $x \in \Omega_1$, $0 < r < R_0$] against $\psi_i := \eta(\phi_i - (\phi_i)_{2B})$, where $0 \le \eta \le 1$ is a smooth cut-off function such that $\eta = 1$ on B, $\eta = 0$ on $\Omega \backslash 2B$, and $|X\eta| \le C/r$

$$\int X^* \cdot (\rho(Q)X\phi_i)\psi_i \, dx = \sum_{j=1}^{m+1} \int X^* \cdot (\phi_j E_{i,j})\psi_i \, dx. \tag{11}$$

The left hand side may be also written

$$\int X^* \cdot (\rho(Q)X\phi_i)\psi_i \, dx = -\int \rho(Q)(X\phi_i) \cdot (X\psi_i) \, dx$$
$$= -\int \rho(Q)(X\phi_i) \cdot [(X\eta)(\phi_i - (\phi_i)_{2B}) + \eta(X\phi_i)] \, dx$$

hence (11) becomes (by summing over $1 \le i \le m+1$)

$$\int_{2B} \eta Q\rho(Q) \, dx + \sum_{i=1}^{m+1} \int_{2B} \rho(Q)(\phi_i - (\phi_i)_{2B})(X\phi_i) \cdot (X\eta) \, dx$$
$$= -\sum_{i,j} \int_{2B} X^* \cdot (\phi_j E_{i,j})\psi_i \, dx.$$

Consequently

$$\int_B Q\rho(Q) \, dx \le \int_{2B} \eta Q\rho(Q) \, dx$$
$$\le \sum_i \int_{2B} \rho(Q)|\phi_i - (\phi_i)_{2B}| \, |X\phi_i| \, |X\eta| \, dx + \sum_{i,j} |I_{i,j}|,$$

where

$$I_{i,j} := \int_{2B} X^* \cdot (\phi_j E_{i,j})\psi_i \, dx.$$

Moreover, by $|X\phi_i| \le |X\phi| = Q^{1/2}$ and by the Hölder inequality (with $1/[2(p+1)] + 1/q = 1$)

$$\int_B Q\rho(Q) \, dx \le \sum_i \int_{2B} Q^{1/2}\rho(Q)|X\eta| \, |\phi_i - (\phi_i)_{2B}| \, dx + \sum_{i,j} |I_{i,j}|$$
$$\le \sum_{i,j} |I_{i,j}| + \sum_i \left(\int_{2B} |\phi_i - (\phi_i)_{2B}|^{2(p+1)} \right)^{1/[2(p+1)]}$$
$$\times \left(\int_{2B \setminus B} (Q^{1/2}\rho(Q)|X\eta|)^{2(p+1)/(2p+1)} \right)^{(2p+1)/[2(p+1)]}.$$

At this point, we may apply the *Poincaré inequality*

$$\left(\int_{2B} |u - u_{2B}|^s \, dx\right)^{1/s} \le Cr\left(\int_{2B} |Xu|^s \, dx\right)^{1/s}, \quad 1 \le s < \infty,$$

and Lemma 1 (with φ replaced by ψ_i) so that to get

$$\begin{aligned}\int_B Q\rho(Q) \, dx &\le C\sum_i \left(\int_{2B} |X\phi_i|^{2(p+1)} \, dx\right)^{1/[2(p+1)]} \\ &\quad \times \left(\int_{2B\setminus B} (Q^{1/2}\rho(Q))^{2(p+1)/(2p+1)} \, dx\right)^{(2p+1)/[2(p+1)]} + \sum_{i,j} |I_{i,j}| \\ &\le C\left(\int_{2B} Q^{p+1} \, dx\right)^{1/[2(p+1)]} \left(\int_{2B\setminus B} (Q\rho(Q)^2)^{(p+1)/(2p+1)} \, dx\right)^{(2p+1)/[2(p+1)]} \\ &\quad + C\sum_i \|X\psi_i\|_{L^D(2B)} (\|X\phi\|_{L^{2p+2}(200B)})^{2p+2}.\end{aligned}$$

By $\rho(t) \le Kt^p$ we have

$$\begin{aligned}&\left(\int_{2B\setminus B} (Q\rho(Q)^2)^{(p+1)/(2p+1)} \, dx\right)^{(2p+1)/[2(p+1)]} \\ &\quad \le K\left(\int_{2B\setminus B} |X\phi|^{2(p+1)} \, dx\right)^{(2p+1)/[2(p+1)]}.\end{aligned}$$

Now we may use $t^p/K \le \rho(t)$ to estimate $\int Q\rho(Q) \, dx$ by below, and the inequality

$$\sum_i \|X\psi_i\|_{L^D(2B)} \le C\|X\phi\|_{L^D(2B)} \tag{12}$$

to obtain

$$\begin{aligned}I_p(r) &\le C[I_p(2r)^{1/(2p+2)}(I_p(2r) - I_p(r))^{(2p+1)/(2p+2)} \\ &\quad + \|X\phi\|_{L^D(2B)} I_p(200r)],\end{aligned} \tag{13}$$

where

$$I_p(r) := \int_{B(x,r)} |X\phi|^{2p+2} \, dx.$$

As to (12), it follows from

$$\sum_i \|X\psi_i\|_{L^D(2B)} \le \sum_i (\|(X\eta)(\phi_i - (\phi_i)_{2B})\|_{L^D(2B)} + \|\eta X\phi_i\|_{L^D(2B)})$$

$$= \sum_i \left(\int_{2B} |X\eta|^D |\phi_i - (\phi_i)_{2B}|^D \, dx\right)^{1/D} + \sum_i \left(\int_{2B} |\eta|^D |X\phi_i|^D \, dx\right)^{1/D} dx$$

$$\le \frac{C}{r} \sum_i \left(\int_{2B} |\phi_i - (\phi_i)_{2B}|^D \, dx\right)^{1/D} + \sum_i \left(\int_{2B} |X\phi_i|^D \, dx\right)^{1/D} dx$$

(*by the Poincaré inequality*)

$$\le C\left(\int_{2B} |X\phi|^D \, dx\right)^{1/D} dx.$$

Using (13) we may establish

LEMMA 2. *There are* $r_0 > 0$ *and* $\lambda \in [1/2, 1)$ *such that*

$$I_p(r) \le \lambda I_p(200r), \tag{14}$$

for any $0 < r \le r_0$.

The proof is by contradiction. Assume that for any $r_0 > 0$ and any $\lambda \in [1/2, 1)$ there is $0 < r \le r_0$ such that $\lambda I_p(200r) < I_p(r)$. Then (by (13))

$$\begin{aligned} \lambda I_p(200r) &< I_p(r) \\ &\le C[I_p(2r)^{1/(2p+2)}(I_p(2r) - I_p(r))^{(2p+1)/(2p+2)} + \|X\phi\|_{L^D(2B)} I_p(200r)] \\ &\le C[I_p(200r)(1-\lambda)^{(2p+1)/(2p+2)} + \|X\phi\|_{L^D(2B)} I_p(200r)] \end{aligned}$$

That is

$$\frac{1}{2} \le \lambda < C[(1-\lambda)^{(2p+1)/(2p+2)} + \|X\phi\|_{L^D(2B)}].$$

Consequently, for any $r_0 > 0$ there is $0 < r \le r_0$ such that

$$\left(\frac{1}{2C}\right)^D \le \int_{2B} |X\phi|^D \, dx.$$

Indeed, let $\lambda_\nu \in [1/2, 1)$, $\lambda_\nu \to 1$ as $\nu \to \infty$, and $0 < r_\nu \le r_0$ correspondingly. By eventually passing to a subsequence, we may assume $r_\nu \to r_\infty$ as $\nu \to \infty$, for some $0 \le r_\infty \le r_0$. Let us take $\nu \to \infty$ in $1/2 < C[(1-\lambda_\nu)^{(2p+1)/(2p+2)} +$

$(\int_{B(x,2r_\nu)} |X\phi|^D\, dy)^{1/D}]$. Then we may use the Vitali absolute continuity of the integral to conclude that either $r_\infty > 0$ and then we get the desired inequality, or $r_\infty = 0$ and then $1/2 \le 0$, a contradiction. In particular, for $r_0 = 1/k$ there is $0 < r \le 1/k$ such that

$$\left(\frac{1}{2C}\right)^D \le \int_{B(x,2r)} |X\phi|^D\, dy \le \int_{B(x,2/k)} |X\phi|^D\, dy$$

and (again using absolute continuity) the last integral goes to 0 as $k \to \infty$, a contradiction. Lemma 2 is proved.

The inequality (14) may be written $I_p(\tau r) \le \lambda I_p(r)$, where $\tau = 1/200$. Therefore $I_p(\tau^m r) \le \lambda^m I_p(r)$, for any integer $m \ge 1$. The following argument (leading to the estimate (15)) is standard. Details are for the sake of completeness. $\{(\tau^m, \tau^{m-1}] : m \ge 1\}$ is a cover of $(0,1]$ hence $\tau^m < r/r_0 \le \tau^{m-1}$, for some $m \ge 1$. Now $r \le \tau^{m-1} r_0$ yields

$$I_p(r) \le I_p(\tau^{m-1} r_0) \le \lambda^{m-1} I_p(r_0).$$

Let us set $\gamma := (\log \lambda)/(\log \tau)$ (then $0 < \gamma < 1$, because of $\lambda \ge 1/2 > \tau$). On the other hand $r/r_0 \ge \tau^m$ yields

$$\left(\frac{r}{r_0}\right)^\gamma > \tau^{m\gamma} = \tau^{(\log \lambda^m)/(\log \tau)} = \lambda^m.$$

Then $\lambda^{m-1} < (r/r_0)^\gamma/\lambda$ where from

$$I_p(r) \le \lambda^{m-1} I_p(r_0) < \frac{1}{\lambda}\left(\frac{r}{r_0}\right)^\gamma I_p(r_0) = Cr^\gamma,$$

(where $C = I_p(r_0)/(\lambda r_0^\gamma)$). We have obtained

$$\int_{B(x,r)} |X\phi|^{2(p+1)}(y)\, dy \le Cr^\gamma, \tag{15}$$

which is the (Caccioppoli type) estimate sought after. To end the proof of Theorem 1 we need to recall (cf. Prop. 2.1 in C-J. Xu & C. Zuily, [28], p. 326) the following result. Let $u \in L^2(\Omega)$. Then the following two conditions are equivalent i) $u \in S^{0,\alpha}_{loc}(\Omega)$, and ii) there are constants $r_0 > 0$ and $C > 0$ such that for any $0 < r \le r_0$ and any $x \in \Omega$ such that $B(x,2r) \subset \Omega$ one has

$$\int_{B(x,r)} |u(y) - u_{B(x,r)}|^2\, dy \le C|B(x,r)|r^{2\alpha}.$$

By the Poincaré inequality

$$\int_{B(x,r)} |\phi_i(y) - (\phi_i)_{B(x,r)}|^2 \, dy \le Cr^2 \int_{B(x,r)} |X\phi|^2 \, dy$$

by the Hölder inequality (with $1/(p+1) + 1/q = 1$)

$$\le Cr^2 \left(\int_{B(x,r)} |X\phi|^{2p+2} \, dy \right)^{1/(p+1)} |B(x,r)|^{p/(p+1)}$$

by (15) and by the definition of homogeneous dimension

$$\le Cr^2 |B(x,r)|^{p/(p+1)} r^{\gamma/(p+1)} \le C|B(x,r)| r^{2\alpha},$$

where $\alpha := 1 + (\gamma - D)/(2p + 2)$. Now $\alpha > 0$ provided that $p > (D - 2)/2 - \gamma/2$, and $\alpha \le 1$ when $D > \gamma$, which holds as D is tacitly assumed to be large (usually D is larger than the Euclidean dimension). Theorem 1 is proved.

3 The Duality Inequality

It suffices to prove Lemma 1 for $\varphi \in C_0^\infty(B)$. Since the proof is fairly long, we organize it in several steps, as follows. For any bounded domain $\Omega \subset \boldsymbol{R}^n$ and any $u \in C_0^\infty(\Omega)$ one has the representation formula

$$u(x) = \int_\Omega (X_y G(y,x)) \cdot (X_y u(y)) \, dy, \quad x \in \Omega, \tag{16}$$

where $G(x,y)$ is the Green function[4] of H on Ω. By a result of G. Citti & N. Garofalo & E. Lanconelli, [10], we may construct a smooth cut-off function η_0 such that $\eta_0 = 1$ on $2B$, $\eta_0 = 0$ on $\Omega \backslash 4B$ and $|X\eta_0| \le C/diam(B)$. The diameter of a set is meant with respect to d_C. Then, using (16) (with $u = \varphi$)

$$\int_B X^* \cdot (\phi_j E_{i,j})\varphi \, dx = \int_B X^* \cdot (\phi_j E_{i,j})\varphi\eta_0 \, dx$$

$$= \iint_{(x,y) \in B \times B} X^* \cdot (\phi_j E_{i,j})(x)\eta_0(x)(X_y G(y,x)) \cdot (X_y \varphi(y)) \, dxdy,$$

hence

$$\int_B X^* \cdot (\phi_j E_{i,j})\varphi \, dx = \int_B A \cdot (X\varphi) \, dy,$$

[4] The existence of the Green function is well known to follow from the hypoellipticity of H and Bony's maximum principle (cf. J. M. Bony, [6]).

where

$$A(y) := \int_B X^* \cdot (\phi_j E_{i,j})(x)\eta_0(x)X_y G(y,x)\,dx.$$

STEP 1. *A bound on* $|A_a(y)|$.

Let $y \in B$ and let $\{\theta_\alpha\}_{\alpha\in I}$ be a smooth partition of unity associated to a Whitney decomposition of $\Omega_y := \Omega\backslash\{y\}$. Precisely, for $x \in \Omega_y$ we set $r_x := d_C(x, \boldsymbol{R}^n\backslash\Omega_y)/1000$ and choose, among $\{B(x,r_x)\}_{x\in\Omega_y}$, a maximal family of mutually disjoint balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in I}$ (hence $\Omega_y = \bigcup_{\alpha\in I} B(x_\alpha, 3r_\alpha)$ and there is $N \geq 1$ such that each $x \in \Omega$ belongs to at most N balls $B(x_\alpha, 6r_\alpha)$). Then we may consider a family of smooth functions $\{\theta_\alpha\}_{\alpha\in I}$ such that $0 \leq \theta_\alpha \leq 1$, $\sum_{\alpha\in I}\theta_\alpha = 1$ on Ω_y, $supp(\theta_\alpha) \subset B_\alpha := B(x_\alpha, 6r_\alpha)$ and $|X\theta_\alpha| \leq C/r_\alpha$ (cf. e.g. R. A. Macías & C. Segovia, [24], for a general approach within the framework of metric spaces endowed with a Borel measure. To get the bounds on the gradients one also uses a result of G. Citti & N. Garofalo & E. Lanconelli, [10]). Then

$$\begin{aligned} A_a(y) &= \sum_{\alpha\in I}\int_{B_\alpha} X^* \cdot (\phi_j E_{i,j})(x)\eta_0(x)\theta_\alpha(x)X_{a,y}G(y,x)\,dx \\ (by\ (9)) \quad &= \sum_{\alpha\in I}\int_{B_\alpha} X^* \cdot [\phi_j - (\phi_j)_{B_\alpha}]E_{i,j}(x)\eta_0(x)\theta_\alpha(x)X_{a,y}G(y,x)\,dx, \end{aligned}$$

where $(\phi_j)_{B_\alpha} := (1/|B_\alpha|)\int_{B_\alpha}\phi_j(x)\,dx$. Next

$$A_a(y) = -\sum_{\alpha\in I}\int_{B_\alpha}[\phi_j - (\phi_j)_{B_\alpha}]E_{i,j}(x)\cdot X_x(\eta_0(x)\theta_\alpha(x)X_{a,y}G(y,x))\,dx.$$

By a result of A. Sánchez-Calle, [25] (for $n \geq 2$)

$$|X_a G(x,y)| \leq C\frac{d_C(x,y)}{|B(x,d_C(x,y))|}, \quad |X_a X_b G(x,y)| \leq \frac{C}{|B(x,d_C(x,y))|},$$

for any $x, y \in \Omega$ (and it is irrelevant whether differentiation is with respect to x or y). Using also $|X\eta_0(x)| \leq C d_C(x,y)^{-1}$ and $|X\theta_\alpha(x)| \leq C d_C(x,y)^{-1}$, $\alpha \in I$, we obtain

$$|X_{b,x}(\eta_0(x)\theta_\alpha(x)X_{a,y}G(y,x))| \leq \frac{C}{|B(y,d_C(x,y))|}$$

hence

$$|A_a(y)| \leq C\sum_{\alpha\in I}\int_{B_\alpha}\frac{|\phi_j(x) - (\phi_j)_{B_\alpha}|\,|E_{i,j}|}{|B(y,d_C(x,y))|}\,dx. \tag{17}$$

Note that

$$|B(y, d_C(x, y))| \geq C|B_\alpha|, \quad x \in B_\alpha. \tag{18}$$

Indeed, $d_C(x, x_\alpha) < 6r_\alpha$. On the other hand, $d_C(y, x_\alpha) \geq 1000r_\alpha$, from $y \in \boldsymbol{R}^n \backslash \Omega_y$ and the very definition of r_α. Hence $1000r_\alpha \leq d_C(y, x_\alpha) \leq d_C(x, y) + d_C(x, x_\alpha) \leq d_C(x, y) + 6r_\alpha$, that is $r_\alpha \leq d_C(x, y)/994$ or $6r_\alpha \leq d_C(x, y)$. This yields $|B(y, 6r_\alpha)| \leq |B(y, d_C(x, y))|$. Moreover $|B(y, 6r_\alpha)|/|B(x_\alpha, 6r_\alpha)| \geq C$, as a consequence of (8). Combining the last two inequalities leads to (18). Let us consider the set of indices $J := \{\alpha \in I : supp(\theta_\alpha) \cap 4B \neq \emptyset\}$. By (17)–(18) and the Hölder inequality (with $1/v^* + 1/\beta = 1$)

$$\begin{aligned} |A_a(y)| &\leq C \sum_{\alpha \in J} \frac{1}{|B_\alpha|} \int_{B_\alpha} |\phi_j(x) - (\phi_j)_{B_\alpha}| \, |E_{i,j}| \, dx \\ &\leq C \sum_{\alpha \in J} \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |\phi_j(x) - (\phi_j)_{B_\alpha}|^{v^*} dx \right)^{1/v^*} \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |E_{i,j}|^\beta dx \right)^{1/\beta}. \end{aligned}$$

Next, we need to apply the *Sobolev inequality*, in the form stated for instance by G. Capogna & D. Danielli & N. Garofalo, [8]. Precisely, given $1 \leq p < D$ there is a constant $C > 0$ such that for any ball $B = B(x, r)$ with $x \in \Omega$ and $0 < r \leq diam(\Omega)$ one has

$$\left(\frac{1}{B} \int_B |u - u_B|^{p^*} dx \right)^{1/p^*} \leq Cr \left(\frac{1}{B} \int_B |Xu|^p dx \right)^{1/p}, \quad p^* = \frac{Dp}{D - p},$$

(where D is a homogeneous dimension of Ω relative to X). Let us choose $v^* := Dv/(D - v)$ (hence $\beta = v^*/(v^* - 1) = Dv/[D(v - 1) + v]$) with $1 \leq v < D$. Then (as $X_a \phi_j \in L^v$)

$$|A_a(y)| \leq C \sum_{\alpha \in J} r_\alpha \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |X\phi_j|^v dx \right)^{1/v} \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |E_{i,j}|^\beta dx \right)^{1/\beta}.$$

Note that (by the very definition of $E_{i,j}$) one has $|E_{i,j}| \leq 2\rho(Q)|X\phi|$. Therefore, using also $\rho(Q) \leq KQ^p$

$$|A_a(y)| \leq CK \sum_{\alpha \in J} r_\alpha \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |X\phi|^v dx \right)^{1/v} \left(\frac{1}{|B_\alpha|} \int_{B_\alpha} |X\phi|^{\beta(2p+1)} dx \right)^{1/\beta}.$$

The second integral converges if $\beta \leq D/(2p + 1)$. Later on, we shall choose v (and this will produce a limitation on p). Given $\alpha \in J$, there is $k \in \boldsymbol{Z}$ such that $x_\alpha \in B(y, 2^{k-1}) \backslash B(y, 2^{k-2})$. Let us observe (together with P. Hájlasz & P. Strzelecki, [16], p. 356) that $B_\alpha = B(x_\alpha, 6r_\alpha) \subset B(y, 2^k)$. Moreover $r_\alpha \approx 2^k$ hence, by applying (8) with $x_0 = y$, $r_0 = 2^k$ and $x = x_\alpha$, $r = 6r_\alpha$

$$\frac{|B(x_\alpha, 6r_\alpha)|}{|B(y, 2^k)|} \geq C\left(\frac{6r_\alpha}{2^k}\right)^D$$

we get $|B_\alpha| \approx |B(y, 2^k)|$. In the end, when $2^{k-2} \geq diam(8B)$ the set $\{\alpha \in J : x_\alpha \in B(y, 2^{k-1}) \backslash B(y, 2^{k-2})\}$ is empty. Therefore

$$|A_a(y)| \leq C \sum_{2^k \leq 4\, diam(8B)} 2^k \left(\frac{1}{|B(y, 2^k)|} \int_{B(y,2^k)} |X\phi|^\nu \, dx\right)^{1/\nu}$$

$$\times \left(\frac{1}{|B(y, 2^k)|} \int_{B(y,2^k)} |X\phi|^{\beta(2p+1)} \, dx\right)^{1/\beta}. \tag{19}$$

STEP 2. *Rewriting the estimate* (19) *in terms of Riesz potentials.*

Let us consider the *abstract Riesz potentials*

$$J_{h,q}^{\sigma,A} g(x) = \sum_{2^k \leq 2\sigma\, diam(A)} 2^{kh} \left(\frac{1}{|B(x, 2^k)|} \int_{B(x,2^k)} |g(z)|^q \, dz\right)^{1/q},$$

for $h > 0$, $q > 0$, $\sigma \geq 1$ and a bounded open set $A \subset \boldsymbol{R}^n$, and recall the *fractional integration theorem* (due to P. Hájlasz & P. Koskela, cf. [16], p. 351), that is

$$\|J_{h,q}^{\sigma,A} g\|_{L^{s^*}(A)} \leq C\left(\frac{diam(A)}{|A|^{D_1}}\right)^h \|g\|_{L^s(V)}, \quad s^* = \frac{D_1 s}{D_1 - hs},$$

where $V = \{y \in \boldsymbol{R}^n : d_C(y, A) < 2\sigma\, diam(A)\}$, provided that $h > 0$, $0 < q < s < D_1/h$ and

$$|B(x,r)| \geq C\left(\frac{r}{diam(A)}\right)^{D_1} |A|, \quad x \in A,\ 0 < r \leq 2\sigma\, diam(A).$$

Then

$$|A_a(y)| \leq C(J_{1/2,\nu}^{2,8B} |X\phi|(y))(J_{1/2,\beta}^{2,8B} |X\phi|^{2p+1}(y)). \tag{20}$$

STEP 3. *End of proof of Lemma* 1.

By the Hölder inequality (with $1/D + 1/D' = 1$)

$$\left|\int_B X^* \cdot (\phi_j E_{i,j})\varphi \, dx\right| \leq \sum_a \|X\varphi\|_{L^D(B)} \left(\int_B |A_a(y)|^{D'} \, dy\right)^{1/D'}$$

(by (20) in Step 2)

$$\leq C\|X\varphi\|_{L^D(B)} \left(\int_B (J_\alpha |X\phi|)^{D'} (J_\beta |X\phi|^{2p+1})^{D'} \, dy\right)^{1/D'}$$

(again by the Hölder inequality, with $D'/s^* + 1/r' = 1$)

$$\leq C\|X\varphi\|_{L^D(B)}(\|(J_\nu|X\phi|)^{D'}\|_{L^{s^*/D'}(8B)} \cdot \|(J_\beta|X\phi|^{2p+1})^{D'}\|_{L^{r'}(8B)})^{1/D'}.$$

At this point we may apply (twice) the fractional integration theorem (with $A = 8B$, $D_1 = D$, $\sigma = 2$, $h = 1/2$ and $q = \nu$ (respectively $q = \beta$)). Let us set $J_q := J^{2,8B}_{1/2,q}$, for simplicity. Now, on one hand

$$\|J_\nu|X\phi|\,\|_{L^{s^*}(8B)} \leq C\left(\frac{diam(8B)}{|8B|^{1/D}}\right)^{1/2}\|X\phi\|_{L^s(V)}, \quad s^* = \frac{2Ds}{2D-s},$$

and on the other

$$\|J_\beta|X\phi|^{2p+1}\|_{L^{r^*}(8B)} \leq C\left(\frac{diam(8B)}{|8B|^{1/D}}\right)^{1/2}\|\,|X\phi|^{2p+1}\|_{L^r(V)}, \quad r^* = \frac{2Dr}{2D-r}.$$

We wish to have $r^* = s^*D'/(s^* - D') = 2Ds/[2s(D-1) - 2D + s]$ hence we must take $r := s/(s-1)$ and request that $0 < \nu < s$ and $0 < \beta < s/(s-1)$. Summing up (by $\|g^n\|_{L^{m/n}} = (\|g\|_{L^m})^n$)

$$\left|\int_B X^* \cdot (\phi_j E_{i,j})\varphi\, dx\right|$$

$$\leq C\frac{diam(8B)}{|8B|^{1/D}}\|X\varphi\|_{L^D(B)}\|X\phi\|_{L^s(V)}\|\,|X\phi|^{2p+1}\|_{L^{s/(s-1)}(V)},$$

and the integrals in the right hand member are convergent if

$$D/(D-2p-1) \leq s \leq D. \tag{21}$$

At this point, we choose $s := 2(p+1)$ (hence $s/(s-1) = 2(p+1)/(2p+1)$). The inequalities (21) are satisfied (because $0 < p < (D-2)/2$). With this choice of s we must have $\beta = D\nu/[D(\nu-1)+\nu] < 2(p+1)/(2p+1)$ hence

$$2D(p+1)/[D+2(p+1)] < \nu < 2(p+1),$$

(again, such a choice of ν is possible because $p < (D-2)/2$). Finally, note that $\|\,|X\phi|^{2p+1}\|_{L^{s/(s-1)}(V)} = (\|X\phi\|_{L^{2(p+1)}(V)})^{2p+1}$ and $V \subset 100B$, hence

$$\left|\int_B X^* \cdot (\phi_j E_{i,j})\varphi\, dx\right| \leq C\frac{diam(8B)}{|8B|^{1/D}}\|X\varphi\|_{L^D(B)}(\|X\phi\|_{L^{2(p+1)}(100B)})^{2(p+1)}.$$

To end the proof of Lemma 1, let $R_0 > 0$ and consider a relatively compact subset $\Omega_1 \subset\subset \Omega$ such that $B(x, 400R_0) \subset \Omega$, for any $x \in \Omega_1$. For any $0 < r \leq R_0$, from the definition of the homogeneous dimension[5]

[5] Cf. also Lemma 2.7 in P. Hájlasz & P. Strzelecki, [16], p. 350.

$$|8B| \geq C\left(\frac{8r}{400R_0}\right)^D |B(x, 400R_0)| \geq C(8r)^D (400R_0)^{s_d - D} \frac{|\Omega|}{(2\, diam(\Omega))^{s_d}}$$

where $s_d := \log_2 C_d$ and $C_d \geq 1$ is the doubling constant (on Ω, relative to the Lebesgue measure). In the end, $diam(8B)/|8B|^{1/D} \leq C$, for some constant $C = C(\Omega_1, D, C_d, R_0) > 0$. The inequality (10) is proved.

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