TSUKUBA J. MATH.
Vol. 28 No. 2 (2004), 401–416

# RIEMANNIAN MANIFOLDS STRUCTURED BY A LOCAL CONFORMAL SECTION

By

Filip DEFEVER and Radu ROSCA

**Abstract.** Geometrical and structural properties are proved for manifolds which are structured by the presence of a local conformal section.

## 1 Introduction

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $\mathcal{O} = \mathrm{vect}\{e_i \mid i = 1, \dots n\}$ be a local field of orthonormal frames over $M$ and let $\mathcal{O}^* = \mathrm{covect}\{\omega^i \mid i = 1, \dots n\}$ be its associated coframe. Let $\alpha = \sum_{i=1}^n t_i \omega^i$ be a globally defined 1-form and let $\mathcal{T} = \alpha^\#$ be its dual vector field. If the connection forms $\theta$ associated with $\mathcal{O}$ satisfy

$$\langle e_i \wedge e_j, \mathcal{T} \rangle = \theta_j^i,$$

we say that $M$ is structured by a local conformal section $\mathcal{T}$.

In the present paper, we prove that in this case $\mathcal{T}$ is a concurrent vector field [2] which satisfies

$$\nabla_Z \mathcal{T} = \rho Z, \quad Z \in \Xi(M), \rho \in C^\infty(M).$$

In consequence of this fact, $\mathcal{T}$ is both a conformal vector field (with $\rho$ as conformal factor) and an exterior concurrent vector field [12]. Moreover, in Section 3 the following properties are also proved:

- **(i)** the dual form, the connection forms, and the curvature forms associated with $\mathcal{O}$ are $d^{-\alpha}$-exact, $d^{-2\alpha}$-exact, and $d^{-4\rho\alpha}$-exact, respectively;
- **(ii)** $\mathcal{T}$ commutes with the dual vectors and the connection forms $\theta$;
- **(iii)** the divergences of $e_i$ constitute an $m$-dimensional eigenspace of $\Delta$, corresponding to the eigenvalue $-((n+1)/2)\rho$;

Received April 7, 2003.

**(iv)** the scalar curvature $S$ of $M$ is expressed by

$$S = -\frac{n^2 + n - 2}{2}\rho;$$

**(v)** if $\mathscr{U}$ is any parallel vector field, then through Weitzenbock's formula [10] one finds that $g(\mathscr{T}, \mathscr{U})$ is an eigenfunction of $\Delta$.

Next, in Section 4 we study some properties of the Lie algebra of infinitesimal transformations induced by $\mathscr{T}$ and prove:

**(i)** $\mathscr{T}$ defines an infinitesimal conformal transformation of $S$ and of the function $g(\mathscr{T}, Z)$, for $Z \in \Xi(M)$, which means that

$$\mathscr{L}_{\mathscr{T}} S = \rho S, \quad \mathscr{L}_{\mathscr{T}} g(\mathscr{T}, Z) = \frac{2n-3}{n-1}\rho g(\mathscr{T}, Z);$$

**(ii)** we denote by $V, \mu, \psi$, and $L$, the canonical vector field, the Liouville 1-form [4], the canonical symplectic form on $TM$, and the operator of Yano and Ishihara, respectively; then $\psi$ is a Finslerian form [4] which is invariant by $\mathscr{T}$;

**(iii)** the complete lift $\Omega^C$ of the symplectic form $\Omega$ of $M$ is also conformally symplectic on $TM$;

**(iv)** the complete lift $\alpha^C$ of $\alpha = \mathscr{T}^\flat$ is also an exact form.

## 2 Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and let $\nabla$ be the covariant differential operator defined by the metric tensor. We assume in the sequel that $M$ is oriented and that the connection $\nabla$ is symmetric.

Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle $TM$, and

$$\flat : TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \# : TM \xleftarrow{\#} T^*M$$

the classical isomorphisms defined by the metric tensor $g$ (i.e. $^\flat$ is the index lowering operator, and $^\#$ is the index raising operator).

Following [10], we denote by

$$A^q(M, TM) = \Gamma\ \mathrm{Hom}(\Lambda^q TM, TM),$$

the set of vector valued $q$-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to $\nabla$

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM).$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$.

Furthermore, we denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of $M$, which is also called the soldering form of $M$ [3]; since $\nabla$ is assumed to be symmetric, we recall that the identity $d^\nabla(dp) = 0$ is valid.

The operator

$$d^\omega = d + e(\omega), \tag{1}$$

acting on $\Lambda M$ is called the cohomology operator [5]. In (1), $e(\omega)$ means the exterior product by the closed 1-form $\omega$, i.e.

$$d^\omega u = du + \omega \wedge u,$$

with $u \in \Lambda M$. Clearly one has the identity

$$d^\omega \circ d^\omega = 0. \tag{2}$$

A form $u \in \Lambda M$ such that

$$d^\omega u = 0, \tag{3}$$

is said to be $d^\omega$-closed, and $\omega$ is called the cohomology form.

A vector field $X$ which satisfies

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M, \tag{4}$$

is defined to be an exterior concurrent vector field [12]. The 1-form $\pi$ in (2) is called the concurrence form and is defined by

$$\pi = \lambda X^\flat, \tag{5}$$

where $\lambda \in C^\infty(M)$ is a nonzero conformal scalar associated with $X$. If $\mathscr{T}$ is any conformal vector field on $M$, which means that

$$\mathscr{L}_\mathscr{T} g = \rho g \Leftrightarrow \langle \nabla_Z \mathscr{T}, Z' \rangle + \langle \nabla_{Z'} \mathscr{T}, Z \rangle = \rho \langle Z, Z' \rangle, \tag{6}$$

then it follows that

$$\rho = \frac{2}{n} \operatorname{div} \mathscr{T}. \tag{7}$$

Therefore, in application of Orsted's lemma [1] one can write

$$\mathscr{L}_\mathscr{T} Z^\flat = \rho Z^\flat + [\mathscr{T}, Z]^\flat, \tag{8}$$

where $[,]$ stands for the Lie bracket. If $S$ is the scalar curvature of $M$, then Yano's formula [15] reads

$$\mathscr{L}_\mathscr{T} S = (n-1)\Delta\rho - S\rho. \tag{9}$$

Let

$$(\mathrm{Hess}_\nabla\, \rho)(Z,Z') = g(Z, H_\rho Z'), \tag{10}$$

where

$$H_\rho Z' = \nabla_{Z'}\, \mathrm{grad}\, \rho, \tag{11}$$

then

$$2\mathscr{L}_{\mathscr{T}}\mathscr{R}(Z,Z') = (\Delta\rho)g(Z,Z') - (n-2)(\mathrm{Hess}_\nabla\, \rho)(Z,Z'). \tag{12}$$

In Section 5 we will rely on the following concepts concerning the tangent bundle manifold $TM$ having as basis manifold $M$. Denote by $V(V^i)$ $(i = 1,\dots n)$ the Liouville vector field (or the canonical vector field on $TM$ [6]). Accordingly, one may consider the sets

$$\mathscr{B} = \left\{ e_i, \frac{\partial}{\partial V^i} \,|\, i = 1,\dots n \right\}, \quad \text{and} \quad \mathscr{B}^* = \{\omega^i, dV^i \,|\, i = 1,\dots n\},$$

as an adapted vectorial basis, and an adapted cobasis in $TM$, respectively.

For application in the sequel, we remind that the vertical differential operator $d_V$ is an antiderivation of degree 1 on $\Lambda(TM)$, and is defined by [4]

$$d_V(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial V^i}\omega^i, \quad d_V(\omega^i) = 0, \quad d_V(dV^i) = 0; \tag{13}$$

the vertical operator $i_V$, which is a derivation of degree 0 on $\Lambda(TM)$, is defined by [4]

$$i_V(f) = 0, \quad i_V(\omega^i) = 0, \quad i_V(dV^i) = \omega^i. \tag{14}$$

Moreover, both operators $d_V$ and $i_V$ satisfy the following relation

$$[d_V, i_V] = d_V.$$

Next, with $V$ denoting the Liouville vector field $V$, which may be expressed as [6]

$$V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial V^i},$$

then, by definition, any 1-form $u$ such that

$$\mathscr{L}_V u = \gamma u \tag{15}$$

is said to be homogeneous of degree $\gamma$.

The vertical lift $Z^V$ [16] of any vector field $Z$ on $M$ with components $Z^i$ $(i = 1, \dots n)$ is expressed by

$$Z^V = \sum_{i=1}^{n} Z^i \frac{\partial}{\partial V^i} =: \begin{pmatrix} 0 \\ Z^i \end{pmatrix}; \tag{16}$$

and the complete lift $Z^C$ of $Z(Z^i)$ $(i = 1, \dots n)$ is given by

$$Z^C = \sum_{i=1}^{n} \left( Z^i e_i + \partial Z^i \frac{\partial}{\partial V^i} \right) =: \begin{pmatrix} Z^i \\ \partial Z^i \end{pmatrix}, \tag{17}$$

where $\partial Z^i = \sum_{\kappa=1}^{n} V^\kappa \partial_\kappa Z^i$, with $\partial_\kappa$ the pfaffian derivative.

Finally, the complete lift $\beta^C$ of a 1-form $\beta = \sum_{i=1}^{n} \beta_i \omega^i$ is defined by

$$\beta^C = \sum_{i=1}^{n} (\partial \beta^i \omega^i + \beta^i \, dV^i) =: (\partial \beta^i, \beta^i). \tag{18}$$

## 3 Manifolds with a Local Conformal Section

Considering an $n$-dimensional manifold $(M, g)$, then in terms of the local field of adapted vectorial frames $\mathcal{O} = \operatorname{vect}\{e_i \mid i = 1, \dots n\}$ and its associated coframe $\mathcal{O}^* = \operatorname{covect}\{\omega^i \mid i = 1, \dots n\}$, the soldering form $dp$ can be expressed as

$$dp = \sum_{i=1}^{n} \omega^i \otimes e_i; \tag{19}$$

and we recall that $E$. Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=1}^{n} \theta_A^B \otimes e_B, \tag{20}$$

$$d\omega^A = -\sum_{B=1}^{n} \theta_B^A \wedge \omega^B, \tag{21}$$

$$d\theta_B^A = -\sum_{C=1}^{n} \theta_B^C \wedge \theta_C^A + \Theta_B^A. \tag{22}$$

In the above equations $\theta$ (respectively $\Theta$) are the local connection forms in the tangent bundle $TM$ (respectively the curvature 2-forms on $M$).

Let

$$\alpha = \sum_{i=1}^{n} t_i \omega^i \tag{23}$$

be a globally defined 1-form on $M$ and let $\mathscr{T} = \alpha^{\#}$ be its dual vector field. If the connection forms satisfy

$$\langle e_i \wedge e_j, \mathscr{T} \rangle = \theta_i^j, \tag{24}$$

then one says that $M$ is structured by a local conformal section $\mathscr{T}$. From (23) and (24) one gets that

$$\theta_i^j = t_i \omega^j - t_j \omega^i. \tag{25}$$

This implies that

$$\theta_i^j(\mathscr{T}) = 0, \tag{26}$$

which shows that the forms $\theta$ are integral relations of invariance [8]. Now, in consequence of (24), and making use of (23), one finds that

$$d\omega^i = \alpha \wedge \omega^i \Rightarrow d\alpha = 0. \tag{27}$$

Hence, in terms of $d^{\omega}$-cohomology, and in view of (3), one may write that

$$d^{-\alpha}\omega^i = 0, \tag{28}$$

i.e. all covectors of $\mathcal{O}^*$ are $d^{-\alpha}$-closed.

Since $\mathscr{T} = \sum_{i=1}^n t_i e_i$, and taking into account (20) and (25), it follows that

$$\nabla e_i = t_i \, dp - \omega^i \otimes \mathscr{T}. \tag{29}$$

Recalling now that at each point $p \in M$, $\operatorname{div} Z = \operatorname{Tr}(\nabla Z) = \sum_{i=1}^n \omega^i(\nabla_{e_i} Z)$, one derives from (29) that

$$\operatorname{div} e_i = (n-1)t_i, \tag{30}$$

which provides a geometrical interpretation for the components of $\mathscr{T}$.

On the other hand, on behalf of (27) one gets

$$dt_i = t_i \alpha + a\omega^i, \quad a \in C^{\infty}(M), \tag{31}$$

and by exterior differentiation is can be seen that the scalar function $a$ must in fact be a constant. Setting $2t = \|\mathscr{T}\|^2$ for notational brevity, one finds by (31) and (32) that

$$dt = (2t + a)\alpha, \tag{32}$$

which shows that $\alpha$ is an exact form. If we put

$$\rho = 2(2t + a) \in C^{\infty}(M), \tag{33}$$

one derives that

$$d\rho = 2\rho\alpha, \tag{34}$$

and

$$\mathcal{L}_{\mathcal{T}}\omega^i = \frac{\rho}{2}\omega^i. \tag{35}$$

The above equation expresses that the vector field $\mathcal{T}$ defines an infinitesimal conformal transformation of all covectors of $\mathcal{O}^*$; according to a well know definition [8], we say that $\mathcal{T}$ defines a local conformal section of the manifold $M$.

Next, with the help of (29) and (31), we get

$$\nabla\mathcal{T} = \frac{\rho}{2}\, dp, \tag{36}$$

which shows that $\mathcal{T}$ is a concurrent vector field [2]. In turn, this implies the following two properties for $\mathcal{T}$:

**(a)** $\mathcal{T}$ is a conformal vector field on $M$, i.e.

$$\mathcal{L}_{\mathcal{T}}g = \rho g, \tag{37}$$

and

$$\operatorname{div}\mathcal{T} = \frac{n}{2}\rho; \tag{38}$$

**(b)** $\mathcal{T}$ is an exterior concurrent vector field [12], which by (34) satisfies

$$\nabla^2\mathcal{T} = \rho\alpha \wedge dp = \rho\mathcal{T}^\flat \wedge dp. \tag{39}$$

Further, invoking (25) and (31) yields

$$d\theta^i_j = 2\alpha \wedge \theta^i_j + 2a\omega^i \wedge \omega^j, \tag{40}$$

and making use of (24), one finds that the curvature forms $\Theta$ of $M$ can be expressed by

$$\Theta^i_j = \alpha \wedge \theta^i_j + \left(\frac{\rho}{2} + a\right)\omega^i \wedge \omega^j. \tag{41}$$

In consequence of (41), one finds that the components $\mathcal{R}_{ij}$ of the Ricci tensor $\mathcal{R}$ are

$$\begin{cases} \mathcal{R}_{ii} = -(n-2)(t_i)^2 - n\left(\frac{\rho}{2} - a\right), \\ \mathcal{R}_{ij} = -(n-2)t_i t_j. \end{cases} \tag{42}$$

Now it can be observed that for (42) to be consistent with (39), the constant $a$ must vanish. Accordingly, (40) yields

$$d\theta_j^i = 2\alpha \wedge \theta_j^i, \tag{43}$$

and also

$$\Theta_j^i = \alpha \wedge \theta_j^i + \frac{\rho}{2}\omega^i \wedge \omega^j. \tag{44}$$

Next, taking the exterior differential of (44), one finds by (34) that

$$d\Theta_j^i = 4\rho\alpha \wedge \Theta_j^i. \tag{45}$$

In terms of cohomology, the above formulas can be interpreted as follows: on the considered manifold, the dual forms, the connection forms, and the curvature forms are $d^{-\alpha}$-exact, $d^{-2\alpha}$-exact, and $d^{-4\rho\alpha}$-exact, respectively.

If we write now $S$ for the scalar curvature of $(M, g)$, then, in consequence of (45) and $a = 0$, one gets that

$$S = -\frac{n^2 + n - 2}{2}\rho, \tag{46}$$

which since $\rho = 2g(\mathcal{T}, \mathcal{T})$ shows that $S$ is always negative. Next, we define

$$E_{ij} = t_i e_j - t_j e_i, \tag{47}$$

for the dual vectors of $\theta_i^j$. Taking the covariant differential of $E_{ij}$, one finds by (29)

$$dE_{ij} = \alpha \otimes E_{ij} - \theta_i^j \otimes \mathcal{T}, \tag{48}$$

and on behalf of (31), one may write

$$[\mathcal{T}, E_{ij}] = 0. \tag{49}$$

Hence, the conformal section $\mathcal{T}$ commutes with all the dual vectors of the connection forms on $M$. Now, by reference to Orsted's lemma [1], it follows in virtue of (49) that

$$\mathcal{L}_{\mathcal{T}}\theta_i^j = \rho\theta_i^j, \tag{50}$$

and by (44) also that

$$\mathcal{L}_{\mathcal{T}}\Theta_i^j = 2\rho\Theta_i^j. \tag{51}$$

The above equations now express that the vector field $\mathcal{T}$ defines an infinitesimal conformal transformation [7], not only of the dual forms of $\mathcal{O}^*$, but also of the connection and the curvature forms.

Further, since $\delta\alpha = -\operatorname{div}\mathcal{T}$ (where $\delta$ denotes the codifferential operator), then, by (37) and (34), one calculates that

$$\Delta\alpha = -np\alpha. \tag{52}$$

This shows that $\alpha$ is an eigenform of the Laplacian with $-np$ as associated eigenvalue. As $p$ is always positive, it follows from the nature of the spectrum of the Laplacian operator that a manifold structured by a local conformal section cannot be compact. With the general formula $\Delta v = -\operatorname{div}\operatorname{grad} v$ and using (34), one gets

$$\Delta t_i = -\frac{n+1}{2}pt_i, \tag{53}$$

which by (30) turns into

$$\Delta \operatorname{div} e_i = -\frac{n+1}{2}p \operatorname{div} e_i. \tag{54}$$

The above equation expresses that the divergencies of the vector basis $\mathcal{O}$ on $M$ form an $n$-dimensional space $E^n(M)$, which is an eigenspace of $\Delta$ corresponding to the eigenvalue $-((n+1)/2)p$. Similarly, one finds by $dt = (2t+a)\alpha$ (see (32)) and (31) that

$$\operatorname{tr}\nabla^2\mathcal{T} = -\frac{n-2}{4}p\mathcal{T}, \tag{55}$$

and

$$\|\nabla\mathcal{T}\|^2 = \frac{np^2}{2}. \tag{56}$$

It can be checked that the above equations, in combination with (52), are indeed consistent with Bochner's theorem [10]

$$2\langle\operatorname{tr}\nabla^2 Z, Z\rangle + 2\|\nabla Z\|^2 + \Delta\|Z\|^2 = 0.$$

Summarizing, we can formulate the following

THEOREM 3.1. *Let $(M,g)$ be an $n$-dimensional Riemannian manifold structured by a local conformal section $\mathcal{T}$ and let $\alpha = \mathcal{T}^\flat$ be the dual form of $\mathcal{T}$. If $\mathcal{O}$ is a local field of orthonormal frames over $M$, then the dual forms, the connection* *forms, and the curvature forms are $d^{-\alpha}$-exact, $d^{-2\alpha}$-exact, and $d^{-4\rho\alpha}$-exact, respectively. Furthermore:*

**(i)** *$\mathscr{T}$ commutes with the dual vectors of the connection forms $\theta$ on $M$;*
**(ii)** *the divergences of the vector basis on $M$ constitute an eigenspace of $\Delta$ which corresponds to the eigenvalue $-((n+1)/2)\rho$;*
**(iii)** *the scalar curvature $S$ of $M$ is negative and is given by $S = -((n^2+n-2)/2)\rho$;*
**(iv)** *$\alpha$ is an eigenform of $\Delta$ and the manifold $M$ under consideration can not be compact.*

## 4 The Lie Algebra of Infinitesimal Transformations

In this section, we discuss some properties of the Lie algebra of infinitesimal transformations generated by the conformal field $\mathscr{T}$. First, by (36), one may write

$$\operatorname{grad}\rho = 2\rho\mathscr{T} \Rightarrow \|\operatorname{grad}\rho\|^2 = 2\rho^3. \tag{57}$$

Therefore,

$$\operatorname{div}(\operatorname{grad}\rho) = (n+2)\rho^2. \tag{58}$$

The above equations show that $\|\operatorname{grad}\rho\|^2$ and $\operatorname{div}(\operatorname{grad}\rho)$ can be expressed as functions of $\rho$. Thus, on behalf of a well known definition [14], it follows that the conformal scalar $\rho$ is an isoparametric function.

Now, by reference to Yano's formula (9) one gets

$$\mathscr{L}_{\mathscr{T}}S = -\frac{n^2+n-2}{2}\rho^2 - \rho S, \tag{59}$$

i.e. $S$ defines an infinitesimal conformal transformation of the scalar curvature $S$.

Next, since

$$\nabla_{\mathscr{T}} \operatorname{grad}\rho = 3\rho^2\mathscr{T},$$

one finds by (11), (12), and (42) that

$$\mathscr{L}_{\mathscr{T}}g(\mathscr{T}Z) = \frac{2n-3}{n-1}\rho g(\mathscr{T}Z), \quad Z \in \Xi(M).$$

Therefore, and on behalf of (57), it follows that

$$\nabla \operatorname{grad}\rho = \rho^2\,dp + 2\alpha \otimes \operatorname{grad}\rho. \tag{60}$$

This shows that $\operatorname{grad}\rho$ is a torse forming vector field [15] [13] [9] with $2\alpha$ as generating form.

We assume from now on that $M$ is of even dimension, say $n = 2m$, and we suppose that the following 2-form of rank $2m$ is globally defined on $M$

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}, \quad i^* = i + m. \tag{61}$$

Exterior differentiation of (61) gives in combination with (29) that

$$d^{-2\alpha}\Omega = 0, \tag{62}$$

which shows that $\Omega$ defines a local conformal symplectic structure with $\alpha$ (resp. $\mathcal{T}$) as covector of Lee (resp. vector of Lee).

Next, by (37) it follows that

$$\mathcal{L}_{\mathcal{T}}\Omega = \rho\Omega, \tag{63}$$

which means that $\mathcal{T}$ defines an infinitesimal conformal transformation of $\Omega$.

Let now $\mathcal{E}_\alpha$ be the vector space such that for every $X_\alpha \in \mathcal{E}_\alpha$

$$\alpha(X_\alpha) = \mathrm{Cst}.$$

Denote by

$$\mu : TM \to T^*M : Z \to i_Z\Omega$$

the bundle isomorphism defined by $\Omega$. Setting then $\beta = \mu(X_\alpha)$, one gets by (62)

$$\mathcal{L}_{X_\alpha}\Omega = d^{-2\alpha}\beta + 2c\Omega,$$

and on behalf of (2) one derives

$$d^{-2\alpha}(\mathcal{L}_{X_\alpha}\Omega).$$

Therefore, we conclude that the Lie derivatives $\mathcal{L}_{X_\alpha}\Omega$ are also $d^{-2\alpha}$-exact. Set now

$$X_\beta = \beta^{\#} = \sum_{i=1}^{m} (t_i e_{i^*} - t_{i^*} e_i),$$

and operate on $X_\beta$ by $\nabla$. By (31) and (33), with $a = 0$, one calculates that

$$\nabla X_\beta = X_\beta \wedge \mathcal{T},$$

which shows that $X_\beta$ is a Killing vector field. Moreover, one can also verify that $[\mathcal{T}, X_\beta]$, i.e. $X_\beta$ commutes with $\mathcal{T}$.

Summarizing, we can formulate the following

THEOREM 4.1. *Let $(M,g)$ be the manifold defined in Section 3. Then, the conformal scalar $\rho$ associated with $\mathcal{T}$ is an isoparametric function and $\mathcal{T}$ defines an infinitesimal conformal transformation of the scalar curvature $S$ on $M$ and of the functions $g(\mathcal{T},Z)$ $(Z \in \Xi(M))$; that is*

$$\mathcal{L}_{\mathcal{T}} S = \rho S,$$

$$\mathcal{L}_{\mathcal{T}} g(\mathcal{T},Z) = \frac{2n-3}{n-1}\rho g(\mathcal{T},Z).$$

*Besides, if $M$ is of even dimension, it admits a conformal symplectic structure $(\Omega,\alpha)$, having $\alpha = \mathcal{T}^{\flat}$ as covector of Lee, i.e. $d^{-2\alpha}\Omega = 0$, and $\mathcal{T}$ defines an infinitesimal conformal transformation of $\Omega$, i.e. $\mathcal{L}_{\mathcal{T}}\Omega = \rho\Omega$. If $\mathcal{E}_\alpha$ is the vector space such that for $X_\alpha \in \mathcal{E}_\alpha$, one has $\alpha(X_\alpha) = Cst.$, then the Lie derivative $\mathcal{L}_{X_\alpha}\Omega$ is $d^{-2\alpha}$-exact and $X_\beta$ is a Killing vector field which commutes with $\mathcal{T}$.*

## 5 Geometry of the Tangent Bundle

Let now $TM$ be the tangent bundle having as basis the manifold $M$ introduced in Section 3, which is now in addition assumed to be of dimension $2m$. In the present section we will study the properties of the lifts to the tangent bundle $TM$ of the tensor fields discussed in the previous sections. Denote by $V(V^i)$ the canonical vector field (or Liouville vector field) [5] and consider $\mathcal{B}^* = \{\omega^i, dV^i \mid i = 1,\dots 2m\}$ as a covectorial basis of $TM$. Recalling that the complete lift [16] of the 2-form $\omega^i \wedge \omega^j$ is defined by

$$(\omega^i \wedge \omega^j)^C = dV^i \wedge \omega^j + \omega^i \wedge dV^j, \tag{64}$$

one derives by reference to (61) that

$$\Omega^C = \sum_{i=1}^{m}(dV^i \wedge \omega^{i^*} + \omega^i \wedge dV^{i^*}), \quad i^* = i+m; \tag{65}$$

we remind that one knows from [16] that $\Omega^C$ defines an almost symplectic structure on $TM$. Taking the exterior differential of $\Omega^C$, one finds by (29)

$$d\Omega^C = \alpha \wedge \Omega^C. \tag{66}$$

Hence, we observe that in the case under consideration the conformal character of $\Omega$ is conserved bij complete lifting; we emphasize the remarkable aspect of this fact, since in general this property is not conserved. Next, since with respect to the vectorial basis $\mathcal{B} = \{e_i, \partial/\partial V^i \mid i = 1,\dots 2m\}$ the Liouville vector field $V$ is expressed by

$$V = \sum_{i=1}^{2m} V^i \frac{\partial}{\partial V^i}, \tag{67}$$

one may compute from this that

$$\mathscr{L}_V \Omega^C = \Omega^C; \tag{68}$$

with reference to [5] this shows that $\Omega^C$ is homogeneous of degree 1. Setting now $\rho = c/f^2$ ($c = \text{const.}$), and on behalf of (34), we can write that

$$\alpha = -\frac{df}{f}. \tag{69}$$

In addition, we put

$$\mathsf{v} = \frac{1}{2}\sum_{i=1}^{2m}(V^i)^2, \tag{70}$$

and consider the function

$$I = f\mathsf{v}. \tag{71}$$

If we operate on $I$ by the vertical differential operator $d_V$, then we find

$$d_V(I) = f\sum_{i=1}^{2m} V^i \omega^i. \tag{72}$$

The basic 1-form

$$\mu = \sum_{i=1}^{2m} V^i \omega^i, \tag{73}$$

is also known [16] as the Liouville form on $TM$ (Alternatively, one can also write that $\mu = V^\flat$). By (69) one can now derive that

$$d(d_V I) = f\sum_{i=1}^{2m} dV^i \wedge \omega^i =: \psi. \tag{74}$$

Since the 2-form $\psi$ is clearly of maximal rank on $TM$, the above equation shows that $\psi$ is an exact (or potential) symplectic form. Since $i_V \psi = f\mu$, then by reference to [16], we call $\psi$ the canonical symplectic form on $TM$. Invoking (69), we can check that the Lie derivative of $\psi$ with respect to $V$ is given by

$$\mathscr{L}_V \psi = \psi. \tag{75}$$

Consequently, $\psi$ is (like $\Omega^C$) also homogeneous of degree 1. Besides, operating on $\psi$ by the vertical derivative operator $i_V$ and invoking (15), leads to

$$i_V(\psi) = 0. \tag{76}$$

On basis of (75) and (76) we conclude that $\psi$ is a Finslerian form [4].

Denote by $\partial_i$ the Pfaffian derivative with respect to $\omega^i$ and set according to [16]

$$\partial = \sum_{i=1}^{2m} V^i \partial_i. \tag{77}$$

Therefore, by reference to [16], the complete lift $\alpha^C$ of $\alpha$ is defined by

$$\alpha^C = (\partial t_i, t_i). \tag{78}$$

Next, setting

$$\beta = \sum_{i=1}^{2m} t_i \, dV^i, \tag{79}$$

one finds

$$\alpha^C = v\alpha + \beta = dv, \tag{80}$$

in which we have used the notation $v := \boldsymbol{L}\alpha$ for the image of the 1-form $\alpha$ under the operator $\boldsymbol{L}$ of Yano and Ishihara (see [16]). Equation (80) shows that the complete lift $\alpha^C$ is, like $\alpha$, also an exact form. Consider now on $TM$ the following 2-form of rank $4m$

$$\phi = v(\alpha \wedge \mu + \psi). \tag{81}$$

By exterior differentiation and taking into account (73) and (74), one obtains

$$d\phi = \left(\frac{\alpha^C}{v} - \frac{\alpha}{f}\right) \wedge \phi. \tag{82}$$

From the above it follows that $\phi$ defines on $TM$ a second conformal symplectic structure having the exact form $\alpha^C/v - \alpha/f$ as covector of Lee. One also finds that

$$\mathscr{L}_V \phi = 2\phi,$$

which shows that $\phi$ is homogeneous of degree 2. Further, let

$$\mathscr{T}^V = \begin{pmatrix} 0 \\ t^i \end{pmatrix}, \quad \text{and} \quad \mathscr{T}^C = \begin{pmatrix} t^i \\ \partial t^i \end{pmatrix}, \tag{83}$$

be the vertical and the complete lift respectively of the conformal section $\mathscr{T}$. By (83) one may write

$$\mathscr{T}^V = \sum_{i=1}^{2m} t^i \frac{\partial}{\partial V^i}, \tag{84}$$

and

$$\mathscr{T}^C = \mathscr{T} + v\mathscr{T}^V. \tag{85}$$

By (74) and (83) one finds that

$$\mathscr{L}_I \psi = 0, \quad \mathscr{L}_{I^V} \psi = 0, \quad \mathscr{L}_{I^C} \psi = 0,$$

which shows that the canonical symplectic form $\psi$ is invariant by $\mathscr{T}, \mathscr{T}^V$, and $\mathscr{T}^C$.

Summarizing, we can formulate the following

THEOREM 5.1. *Let $TM$ be the tangent bundle manifold having as basis the manifold in Section 3 which is now in addition assumed to be of even dimension. Let $V, \mu, \psi$, and $\mathbf{L}$, be the canonical vector field, the Liouville form, the canonical form on $TM$, and the operator which assigns to 1-forms on $M$ functions on $TM$, respectively. Then*:

**(i)** *$\psi$ is a Finslerian form which is invariant under the conformal section $\mathscr{T}$ and its vertical and complete lifts $\mathscr{T}^V$ and $\mathscr{T}^C$, respectively*;

**(ii)** *the complete lift $\Omega^C$ of the conformal symplectic form $\Omega$ on $M$ is a conformal symplectic form on $TM$, which is $d^{-\alpha}$-exact and homogeneous of degree 1*;

**(iii)** *the complete lift $\alpha^C$ of $\alpha$ is also an exact form and the 2-form*

$$\phi = v(\alpha \wedge \mu + \psi)$$

*defines a second conformal symplectic form on $TM$, having the exact form*

$$\frac{\alpha^C}{v} - \frac{\alpha}{f}.$$

*as covector of Lee.*

## References

[1] T. P. Branson, Conformally covariant equations and differential forms, Comm. in partial differential equations **7(4)** (1982), 393–431.

[2] B. Y. Chen, Geometry of Submanifolds, M. Dekker, New York (1973).

[3] J. Dieudonné, Treatise on Analysis Vol. 4, Academic Press, New York (1974).
[4] C. Godbillon, Géometrie différentielle et mécanique analytique, Herman, Paris (1969).
[5] F. Guedira, A. Lichnerowicz, Geometrie des algebres de Lie locales de Kirilov, J. Math. Pures Appl. **63** (1984), 407–494.
[6] J. Klein, Espaces variationnels et mécanique, Ann. Inst. Fourier **4** (1962), 1–124.
[7] P. Libermann, C. M. Marle, Géométrie Symplectique, Bases Théorétiques de la Mécanique, U.E.R. Math. du C.N.R.S. **7** (1986).
[8] A. Lichnerowicz, Les relations intégrales d'invariance et leurs applications a la dynamique, Bull. Sci. Math. **70** (1946), 82–95.
[9] K. Matsumoto, I. Mihai, R. Rosca, On gradient almost torse forming vector fields, Math. J. Toyama Univ. **19** (1996), 149–157.
[10] W. A. Poor, Differential Geometric Structures, Mc Graw Hill, New York (1981).
[11] R. Rosca, On parallel conformal connections, Kodai Math. J. **2** (1979), 1–9.
[12] R. Rosca, Exterior concurrent vectorfields on a conformal cosymplectic manifold admitting a Sasakian structure, Libertas Math. (Univ. Arlington, Texas) **6** (1986), 167–174.
[13] C. Udriste, Properties of torse forming vector fields, Tensor NS **42** (1982), 134–144.
[14] K. Yano, On torse forming directions in Riemannian spaces, Proc. Imp. Acad. Tokyo **20** (1944), 340–345.
[15] K. Yano, Integral formulas in Riemannian Geometry, M. Dekker, New-York (1970).
[16] K. Yano, S. Ishihara, Differential Geometry of Tangent and Cotangent Bundles, M. Dekker, New York (1973).

Filip Defever, Department Industriële Wetenschappen en Technologie
Katholieke Hogeschool Brugge-Oostende
Zeedijk 101, 8400 Oostende, BELGIUM

Radu Rosca, 59 Avenue Emile Zola
75015 Paris, FRANCE