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QUANTIFIER ELIMINATION RESULTS FOR PRODUCTS OF ORDERED ABELIAN GROUPS

By

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1 Introduction

Komori [1] introduced the notion of semi-discrete ordered Abelian group with divisible infinitesimals. Roughly speaking, such groups are products of a $\boldsymbol{Z}$-like group and a $\boldsymbol{Q}$-like group. In [1], he showed that such groups are axiomatized by his set SC of axioms. In fact he showed that SC is complete and admits quantifier elimination (QE) in some language expanding $L_{\mathrm{og}} = \{0, +, -, <\}$. In this paper, we shall evolve his study and prove QE for products of ordered Abelian groups H and K, where H admits QE and K is divisible. However, like him, we need to expand the language slightly. First let us explain Komori's axiom. SC is the following set of sentences:

1. the axioms for ordered Abelian groups;
2. the axioms for a semi-discrete ordering

$$0 < 1, \quad \forall x(2x < 1 \vee 1 < 2x);$$

3. the axioms for infinitesimals

$$\forall x(2x < 1 \rightarrow nx < 1) \quad (n = 2, 3, \ldots);$$

4. the axioms for D_n's

$$\forall x(D_n(x) \leftrightarrow \exists y \exists z(-1 < 2z < 1 \wedge x = ny + z) \quad (n = 2, 3, \ldots)$$

$$\forall x(D_n(x) \vee D_n(x+1) \vee \cdots \vee D_n(x+n-1)) \quad (n = 2, 3, \ldots);$$

5. the axioms for divisible infinitesimals

$$\forall x(-1 < 2x < 1 \rightarrow \exists y(x = ny) \quad (n = 2, 3, \ldots);$$

6. the axiom for existence of infinitesimals

$$\exists x(0 < x < 1);$$

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Notice that SC is not formulated in the pure ordered group language. Its language is $L = L_{\mathrm{og}} \cup \{D_n : n = 2, 3, \dots\} \cup \{1\}$. A canonical model of SC is the direct product group $\boldsymbol{Z} \times \boldsymbol{Q}$, where

1. the constants 0 and 1 are interpreted to the elements $(0,0)$ and $(1,0)$, respectively
2. the predicate symbol $<$ is interpreted as the lexicographic order of $\boldsymbol{Z}$ and $\boldsymbol{Q}$,
3. the predicate symbols $D_n(x)$ $(n = 2, 3, \dots)$ means that x is divisible by n.

Notice that $\boldsymbol{Z}$ admits QE in L and that $\boldsymbol{Q}$ admits QE in L_{og}. So, in a sense, Komori's result can be considered a quantifier elimination result for the product group $H \times K$ where both H and K have QE. The above L-structure $\boldsymbol{Z} \times \boldsymbol{Q}$ seems to have two important properties that are essential in Komori's proof. One is that the infinitesimal set $I = \{0\} \times \boldsymbol{Q}$ is definable (by the quantifier free formula $-1 < 2x < 1$). The other is that $\boldsymbol{Q}$ is divisible. In this paper, very roughly, we show that if the two properties are satisfied, then we can show QE for the product group $H \times K$ in some expanded language. (See section 3).

For stating our main result more precisely, we need some definition. Let L_{r} and L_{c} respectively be sets of predicate and constant symbols. Let L be the language $L_{\mathrm{og}} \cup L_{\mathrm{r}} \cup L_{\mathrm{c}}$. Let H be an L-structure such that $H|L_{\mathrm{og}}$ is an ordered Abelian group. Let K be an L_{og}-structure such that K is an ordered Abelian group. We will consider $G := H \times K$ as an $L \cup \{I\}$-structure by the following interpretation:

1. $0^G := (0^H, 0^K)$.
2. $c^G := (c^G, 0^K)$ $(c \in L_{\mathrm{c}})$.
3. $+, -$ are defined coordinatewise.
4. $<$ is the lexicographic order of H and K.
5. Each n-ary predicate symbol R of L_{r} is defined by

$$R^G := \{(\bar{g}) \in G^n : \bar{h} \in R^H\}$$

where $\bar{g} = (g_1, \dots, g_n)$ with $g_i = (h_i, k_i)$ $(i = 1, \dots, n)$ and $\bar{h} = (h_1, \dots, h_n)$.

MAIN RESLUT. *Let L be the language $L_{\mathrm{og}} \cup L_{\mathrm{r}} \cup L_{\mathrm{c}}$ where L_{og} is the language $\{0, +, -, <\}$, L_{r} and L_{c} are sets of predicate symbols and constant symbols respectively. Let H be an L-structure such that $H|L_{\mathrm{og}}$ is an ordered Abelian group. Let K be a divisible ordered Abelian group. (We consider K as an L_{og}-structure.) Let $G := H \times K$ be an L-structure given by the interpretation above. Let $I = \{0\} \times K$ be defined by some quantifier free L-formula in G. If H admits QE in L,* *then G admits QE in L. Moreover in the result above, if H is recursively axiomatizable, then so is G.*

2 Preliminaries

In this paper we require some basic knowledge of model theory. Terminologies we use are rather standard. However, let us explain some of them. L denotes a language and T denotes a consistent set of L-sentences. M denotes an L-structure. Finite tuples of variables are denoted by $\bar{x}, \bar{y}, \ldots$. Finite tuples of elements in M are denoted by $\bar{a}, \bar{b}, \ldots$ Subsets of M are denoted by $A, B, \ldots$ If $\bar{a} = a_1, \ldots, a_n$, we simply write $\bar{a} \in M$ instead of writing $a_1 \in M, \ldots, a_n \in M$. An $L(A)$-formula means an L-formula with parameters from A. Similarly an $L(A)$-term means an L-term with parameters form A.

We say that T is an L-theory if there exists a model M of T. $\mathrm{Th}_L(M)$ denotes the theory of M, i.e. the set of all L-sentences which hold in M. If L is clear from the context, L will be omitted, and we will simply write $\mathrm{Th}(M)$ instead of writing $\mathrm{Th}_L(M)$. We say that a theory T is complete if for any L-sentence ϕ, T proves ϕ or $\neg\phi$.

We say that T admits quantifier elimination in the language L if for any L-formula $\phi(\bar{x})$, there exists a quantifier free L-formula $\psi(\bar{x})$ such that T proves $\forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. We say that M admits quantifier elimination in L if $Th_L(M)$ admits quantifier elimination in L.

Let $A \subset M$. We say that a set $p(\bar{x})$ of $L(A)$-formulas (with free variables $\bar{x}$) is a type if any finite subset of $p(\bar{x})$ has a solution in M. We define the type of $\bar{a} \in M$ over A to be the set of $L(A)$-formulas $\psi(\bar{x})$ such that $\bar{a}$ is a solution of $\psi(\bar{x})$. The type of $\bar{a} \in M$ over A is denoted by $\mathrm{tp}(\bar{a}/A)$. If $A = \emptyset$, we simply write $\mathrm{tp}(\bar{a})$ instead of $\mathrm{tp}(\bar{a}/A)$. We define the quantifier free type of $\bar{a}$ over A to be the set of quantifier free $L(A)$-formula $\psi(\bar{x})$'s such that $\bar{a}$ is a solution of $\psi(\bar{x})$. The quantifier free type of $\bar{a}$ over A is denoted by $\mathrm{qftp}(\bar{a}/A)$. Similarly if $A = \emptyset$, we write $\mathrm{qftp}(\bar{a})$ instead of $\mathrm{qftp}(\bar{a}/A)$.

We say that a model M of T is κ-saturated if whenever A is a subset of M with $|A| < \kappa$ then any type over A has a solution in M.

In this paper we use the following well-known fact:

FACT 1. *Let L be a language. Let T be an L-theory such that T is complete for quantifier free sentences. Then the following are equivalent;*

1. *T is complete and admits quantifier elimination in L.*
2. *Let M and N be $\aleph_0$-saturated models of T. Suppose $\bar{a} \in M$ and $\bar{b} \in N$ have*

the same quantifier free type, i.e. $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$. *Then for any* $a \in M$ *there exists* $b \in N$ *such that* $\mathrm{qftp}(\bar{a}, a) = \mathrm{qftp}(\bar{b}, b)$.

3 Product of Ordered Abelian Groups

In this section we introduce the notion of the product interpretation. Let G be a group. We say that a subset A of G is free if whenever $\sum_{i \in N} m_i a_i = 0$ for some finite subsets $\{a_i\}_{i \in N}$ of A and $\{m_i\}_{i \in N}$ of $\boldsymbol{Z}$, then $m_i = 0$ $(i \in N)$.

DEFINITION 2. Let G be a group. For any $A \subset G$,

$$H(A) := \{h \in G : mh \in \langle A \rangle \text{ for some } m \in \boldsymbol{Z} \backslash \{0\}\},$$

where $\langle A \rangle$ is the subgroup of G generated by A.

LEMMA 3. *Let* $G (\neq \{0\})$ *be a torsion free Abelian group. Then for any free subset* S *of* G, *there exists some free subset* A *of* G *with the following conditions*;

1. $S \subset A$,
2. $G = H(A)$,
3. *If* $mg = \sum m_i a_i$ *and* $ng = \sum n_i a_i$ *for some element* g *of* G, *some finite subset* $\{a_i\}_{i \in N}$ *of* A, *some* m, n *of* $\boldsymbol{Z} \backslash \{0\}$ *and some* $m_i, n_i \in \boldsymbol{Z}$ $(i \in N)$, *then* $nm_i = mn_i$ $(i \in N)$.

PROOF. Since G is torsion free, by the Zorn's lemma, there exists a maximal free subset A of G containing S. Then A satisfies the condition of the lemma. ■

Let L_{og} be the language $\{0, +, -, <\}$ of ordered groups. Let L_{r} and L_{c} be sets of predicate and constant symbols, respectively. Let L be the language $L_{\mathrm{og}} \cup L_{\mathrm{r}} \cup L_{\mathrm{c}}$. Let H be an L-structure such that $H|L_{\mathrm{og}}$ is an ordered Abelian group. Let K be an L_{og}-structure such that K is an ordered Abelian group. Let I be a new unary predicate symbol. In what follows, we will consider $G := H \times K$ as an $L \cup \{I\}$-structure by the following interpretation:

1. $0^G := (0^H, 0^K)$.
2. $c^G := (c^H, 0^K)$ $(c \in L_{\mathrm{c}})$.
3. $+, -$ are defined coordinatewise.
4. $<$ is the lexicographic order of H and K.
5. Each n-ary predicate symbol R of L_{r} is defined by

$$R^G := \{\bar{g} \in G^n : \bar{h} \in R^H\}$$

where $\bar{g} = (g_1, \ldots, g_n)$ with $g_i = (h_i, k_i)$ $(i = 1, \ldots, n)$ and $\bar{h} = (h_1, \ldots, h_n)$.

6. $I^G := \{0^H\} \times K$.

We call this interpretation the *product interpretation* of H and K.

Let $L = L_{\text{og}} \cup L_{\text{r}} \cup L_{\text{c}}$. Let H be an L-structure such that $H|L_{\text{og}}$ is an ordered Abelian group. Let K be an L_{og}-structure such that K is an ordered Abelian group. Let $G := H \times K$ be an $L \cup \{I\}$-structure given by the product interpretation of H and K.

Let $G^* \models Th(G)$. Let $I^* := \{g \in G^* : g \models I(x)\}$. An equivalent relation $\sim$ on G^* is defined by $a \sim b$ if $a - b \in I^*$. Let $[g]$ be the equivalent class of g. Let $H^* := \{[g] : g \in G^*\}$ and $K^* := I^*$. We will consider H^* as an L-structure by the following interpretation:

1. $0, c$ $(c \in L_{\text{c}})$, $+$ and $-$ are defined naturally.
2. Let g_1 and $g_2 \in G^*$. $[g_1] < [g_2]$ is defined by $g_1 < g_2$ and $g_1 - g_2 \notin I^*$.
3. Each n-ary predicate R of L_{r} is defined by

$$R^{H^*} := \{[\bar{g}] \in (H^*)^n : \bar{g} \in R^{G^*}\}$$

where $\bar{g} = (g_1, \ldots, g_n)$ and $[\bar{g}] = ([g_1], \ldots, [g_n])$

and consider K^* as an L_{og}-substructure of G^*.

Remark 4. $H^* \equiv H$ and $K^* \equiv K$.

This can be shown as follows: It is trivial that $K^* \equiv K$. So we show that $H^* \equiv H$. Let g_1 and $g_2 \in G^*$. Let $\bar{g}$ be an tuple of elements of G^*. By the definition of H^*, the followings are hold.

1. $[g_1] = [g_2]$ holds in $H^* \leftrightarrow g_1 - g_2 \in I^*$ holds in G^*.
2. $[g_1] < [g_2]$ holds in $H^* \leftrightarrow$ both $g_1 < g_2$ and $g_1 - g_2 \notin I^*$ hold in G^*.
3. $R([\bar{g}])$ holds in $H^* \leftrightarrow R(\bar{g})$ holds in G^* $(R \in L_{\text{r}})$.

So for any L-sentence ϕ there exists an $L \cup \{I\}$-sentence ψ such that ϕ holds in H^* iff ψ holds in G^*. Since $G^* \equiv G$, we have $H^* \equiv H$.

Let $H^* \times K^*$ be the $L \cup \{I\}$-structure given by the product interpretation of H^* and K^*.

Lemma 5. *Let K be divisible. Then there exists some $L \cup \{I\}$-isomorphism σ from G^* to $H^* \times K^*$.*

PROOF. Suppose that $H = \{0\}$. Then $H^* = \{0\}$ and $G^* = K^*$. In this case, it is trivial. So we can assume that $H \neq \{0\}$. Then H^* is nontrivial torsion free group. Let S be a maximal free subset of $\{c^* : c \in L_c\}$ where c^* is the interpretation of c in G^*. We claim that $[S] := \{[c^*] : c^* \in S\}$ is free. Suppose that $\sum m_i[c_i^*] = 0$ for some finite subsets $\{c_i^*\}_{i \in N}$ of S and $\{m_i\}_{i \in N}$ of $\boldsymbol{Z}$. Then $\sum m_i c_i^* \in I^*$. By the definition of the product interpretation and $G^* \equiv G$, $\sum m_i c_i^* = 0$. Since S is free, $m_i = 0$ $(i \in N)$.

So by lemma 3, there exists some subset H_0 of H^* with the following conditions;

1. $[S] \subset H_0$.
2. $H^* = H(H_0)$.
3. If $m[g] = \sum m_i[g_i]$ and $n[g] = \sum n_i[g_i]$ for some element $[g]$ of H^*, some finite subset $\{[g_i]\}_{i \in N}$ of H_0, some m, n of $\boldsymbol{Z} \backslash \{0\}$ and some $m_i, n_i \in \boldsymbol{Z}$ $(i \in N)$, then $nm_i = mn_i$ $(i \in N)$.

We fix a subset G_0 of G^* with the following conditions;

1. $S \subset G_0$.
2. $H_0 = \{[g] : g \in G_0\}$.
3. If $g_1 \neq g_2 \in G_0$, then $[g_1] \neq [g_2]$.

Let σ be the map from G^* to K^* defined by

$$\sigma(g) := 1/m\left(mg - \sum m_i g_i\right)$$

where $m[g] = \sum m_i[g_i]$ for some subset $\{g_i\}_{i \in N}$ of G_0, $m \in \boldsymbol{Z} \backslash \{0\}$ and $m_i \in \boldsymbol{Z}$ $(i \in N)$. Note that σ is well-defined by the divisibility of K and the conditions of H_0 and G_0. Let $\sigma^* : G^* \to H^* \times K^*$ be the map defined by

$$\sigma^*(g) = ([g], \sigma(g)).$$

CLAIM. *σ^* is an $L \cup \{I\}$-isomorphism.*

First we claim that σ^* is $\{+, -, 0\} \cup L_c$-isomorphic. In the case of $+$, we show that $\sigma(g_1) + \sigma(g_2) = \sigma(g_1 + g_2)$ for any $g_1, g_2 \in G^*$. Note that $mg_1 = \sum m_i g_i + m\sigma(g_1)$ and $ng_2 = \sum n_i g_i + n\sigma(g_2)$ for some finite subset $\{g_i\}_{i \in N}$ of G_0, some m and $n \in \boldsymbol{Z} \backslash \{0\}$ and some m_i and $n_i \in \boldsymbol{Z}$ $(i \in N)$. So $mn(g_1 + g_2) = \sum (nm_i + mn_i) g_i + mn(\sigma(g_1) + \sigma(g_2))$. Then $\sigma(g_1 + g_2) = \sigma(g_1) + \sigma(g_2)$. In the case of L_c, we show that $\sigma(c^*) = 0$. Since S is a maximal free subset of $\{c^* : c \in L_c\}$, for any $c \in L_c$ there exist some $m \in \boldsymbol{Z} \backslash \{0\}$, finite subsets $\{c_i^*\}_{i \in N}$

of S and $\{m_i\}_{i\in N}$ of $\boldsymbol{Z}$ such that $mc^* = \sum m_i c_i^*$. So $m[c^*] = \sum m_i[c_i^*]$ and $\{c_i^*\}_{i\in N} \subset G_0$. By the definition of σ, $\sigma(c^*) = 1/m(mc^* - \sum m_i c_i^*) = 0$. In the case of 0 and $-$, it is similar.

Second we claim that σ^* is injective and surjective. (injective) Suppose that $\sigma^*(g) = (0,0)$. Then $[g] = 0$ and $1/m(mg - \sum m_i g_i) = 0$ for some subset $\{g_i\}_{i\in N}$ of G_0, $m \in \boldsymbol{Z}\backslash\{0\}$ and $m_i \in \boldsymbol{Z}$ $(i \in N)$. Then $0 = m[g] = \sum m_i[g_i]$. Since H_0 is free, $m_i = 0$ $(i \in N)$. So we have $g = 0$. (surjective) For any $([g],k) \in H^* \times K^*$, we pick a finite subset $\{g_i\}_{i\in N}$ of G_0, $m \in \boldsymbol{Z}\backslash\{0\}$ and $m_i \in \boldsymbol{Z}$ $(i \in N)$ such that $m[g] = \sum m_i[g_i]$. We put $g_0 := g - 1/m(mg - \sum m_i g_i) + k$. Then we have $[g_0] = [g]$ and $\sigma(g_0) = 1/m(mg_0 - \sum m_i g_i) = k$.

Next we claim that σ^* is $\{<\}$-isomorphic. Suppose that $g_1 < g_2$. If $g_1 - g_2 \notin I$, by the definition of $<$, it is trivial. If $g_1 - g_2 \in I$, $m[g_1] = m[g_2] = \sum m_i[g_i]$ for some finite subset $\{g_i\}_{i\in N}$ of G_0, $m \in \boldsymbol{Z}\backslash\{0\}$ and $m_i \in \boldsymbol{Z}$ $(i \in N)$. So $\sigma(g_1) = 1/m(mg_1 - \sum m_i g_i) < 1/m(mg_2 - \sum m_i g_i) = \sigma(g_2)$.

Last by the definition, we have that σ^* is $L_{\mathrm{r}} \cup \{I\}$-isomorphic. ■

4 Main Theorem

In this section, $L = L_{\mathrm{og}} \cup L_{\mathrm{r}} \cup L_{\mathrm{c}}$, where L_{og} is the language $\{0, +, -, <\}$, and L_{r} and L_{c} respectively are a set of predicate symbols and a set of constant symbols. I is a fixed unary predicate symbol not contained in L.

THEOREM 6. *Let H be an L-structure such that $H|L_{\mathrm{og}}$ is an ordered Abelian group. Let K be a divisible ordered Abelian group. We consider K as an L_{og}-structure. Let $G := H \times K$ be an $L \cup \{I\}$-structure given by the product interpretation of H and K. Then if H admits QE in L, G admits QE in $L \cup \{I\}$. Moreover H is recursively axiomatizable, so is G.*

PROOF. It is clear that $Th_{L\cup\{I\}}(G)$ is complete for quantifier free sentences. By fact 1, it is sufficient to show that:

CLAIM. *Let G_1, G_2 be $\aleph_0$-saturated models of $Th_{L\cup\{I\}}(G)$. Suppose $\bar{g}^1 \in G_1$ and $\bar{g}^2 \in G_2$ such that* $\mathrm{qftp}(\bar{g}^1) = \mathrm{qftp}(\bar{g}^2)$. *Then for any $g^1 \in G_1$ there exists $g^2 \in G_2$ such that* $\mathrm{qftp}(\bar{g}^1, g^1) = \mathrm{qftp}(\bar{g}^2, g^2)$.

Before proving the claim above, we need some preparation. By lemma 5, for $j = 1, 2$ we can assume that $G_j = H_j \times K_j$ where H_j is an L-structure, K_j is an L_{og}-structure and G_j is the $L \cup \{I\}$-structure given by the product interpretation of H_j and K_j. Let $\bar{g}^j$ be an tuple $(g_1^j, \dots, g_n^j)$ of G_j with $g_i^j = (h_i^j, k_i^j)$. Let $\bar{h}^j$ be the tuple $(h_1^j, \dots, h_n^j)$ of H_j. Let $\bar{k}^j$ be the tuple $(k_1^j, \dots, k_n^j)$ of K_i.

REMARK 7. Since the language of G_j contains I, if $\bar{g}^1$ and $\bar{g}^2$ have the same quantifier free type, then $\bar{h}^1$ and $\bar{h}^2$ have the same quantifier free type. ($\bar{k}^1$ and $\bar{k}^2$ may not have the same quantifier free type.) Moreover since H admits QE, $\bar{h}^1$ and $\bar{h}^2$ have the same type.

Similarly as in remark 4, for any quantifier free L-formula $\phi(\bar{y})$, there exists a quantifier free $L \cup \{I\}$-formula $\psi(\bar{x})$ such that for $j = 1, 2$, $\bar{g}^j$ is a solution of $\psi(\bar{x})$ if and only if $\bar{h}^j$ is a solution of $\phi(\bar{y})$. Thus $\bar{h}^1$ and $\bar{h}^2$ have the same quantifier free type.

We begin our proof of the claim. We fix $g^1 \in G_1$ and choose $\varphi_1(x, \bar{g}^1), \dots, \varphi_n(x, \bar{g}^1) \in \mathrm{qftp}(g^1/\bar{g}^1)$. Let $\Phi(x, \bar{g}^1)$ be the set $\{\varphi_1(x, \bar{g}^1), \dots, \varphi_n(x, \bar{g}^1)\}$. We need to show that $\Phi(x, \bar{g}^2)$ (the set obtained from $\Phi(x, \bar{g}^1)$ replacing $\bar{g}^1$ by $\bar{g}^2$.) is satisfied in G_2. Let $\Phi(x, \bar{x})$ be the set of formulas obtained from $\Phi(x, \bar{g}^1)$ replacing $\bar{g}^1$ by the tuples $\bar{x}$ of variables without x. Note that the formula in the form $t \neq s$ or $\neg(t < s)$ is equivalent a disjunction of formulas in the form $t = s$ or $t < s$. So we can assume that the set $\Phi(x, \bar{x})$ has the following form:

$$\{t_i(\bar{x}) < n_i x\}_{i \in I_1} \cup \{n_i x = t_i(\bar{x})\}_{i \in I_2} \cup \{n_i x < t_i(\bar{x})\}_{i \in I_3} \cup \Phi_0(x, \bar{x})$$

where $t_i(\bar{x})$ are terms without x and $n_i \in \boldsymbol{N}$ and $\Phi_0(x, \bar{x})$ is a finite set of $L \cup \{I\}$-formulas in the form $I(t(x, \bar{x})), R(s(x, \bar{x}))$ or these negations with terms $t(x, \bar{x})$ and $s(x, \bar{x})$. For any $m \in \boldsymbol{N} \backslash \{0\}$, formulas $t < s$ and $t = s$ are equivalent to $mt < ms$ and $mt = ms$, respectively. Then we can assume that $\Phi(x, \bar{x})$ is the following set:

$$\{s_i(\bar{x}) < Nx\}_{i \in I_1} \cup \{Nx = s_i(\bar{x})\}_{i \in I_2} \cup \{Nx < s_i(\bar{x})\}_{i \in I_3} \cup \Phi_0(x, \bar{x})$$

where $s_i(\bar{x})$ are new terms without x and $N \in \boldsymbol{N}$.

There are two cases to be considered in the following:

Case 1. First we assume that $I_2 \neq \emptyset$. We fix a term $s(\bar{x})$ of $\{s_i(\bar{x})\}_{i \in I_2}$. We remark that for $j = 1$ and 2, finding $x \in G_j$ satisfying that

$$\{s_i(\bar{g}^j) < Nx\}_{i \in I_1} \cup \{Nx = s_i(\bar{g}^j)\}_{i \in I_2} \cup \{Nx < s_i(\bar{g}^j)\}_{i \in I_3}$$

is equivalent to finding $x \in G_j$ satisfying that

$$\{Nx = s(\bar{g}^j)\}.$$

Then the condition above is equivalent to finding $h^j \in H_j$ satisfying that $Ny = s(\bar{h}^j)$ and finding $k^j \in K_j$ satisfying that $Nz = s(\bar{k}^j)$. By the definition of

R ($R \in L_r$) and I, for $j = 1, 2$, finding $g^j \in G_j$ satisfying that $\Phi_0(x, \bar{g}^1)$ is equivalent to finding $h^j \in H_j$ satisfying that $\Psi(y, \bar{h}^j)$ where $\Psi(y, \bar{h}^j)$ is the set of L-formulas obtained from $\Phi_0(x, \bar{g}^j)$ replacing $I(t(x, \bar{g}^j))$ and $R(s(x, \bar{g}^j))$ by $t(y, \bar{h}^j) = 0$ and $R(s(y, \bar{h}^j))$, respectively. So for $j = 1, 2$ finding $x \in G_j$ satisfying that $\Phi(x, \bar{g}^j)$ is equivalent to finding $y \in H_j$ satisfying that

$$\{Ny = s(\bar{h}^j)\} \cup \Psi(y, \bar{h}^j)$$

and $z \in K_j$ satisfying that

$$\{Nz = s(\bar{k}^j)\}.$$

By remark 7, $\bar{h}^1$ and $\bar{h}^2$ have the same type. By the assumption, there exists some solution $h^1 \in H_1$ of $\{Ny = s(\bar{h}^1)\} \cup \Psi(y, \bar{h}^1)$. So there exists some solution $h^2 \in H_2$ of $\{Ny = s(\bar{h}^2)\} \cup \Psi(y, \bar{h}^2)$. By the divisibility of K_2, there exists $k^2 \in K_2$ such that $Nk^2 = s(\bar{k}^2)$. Then $(h^2, k^2) \in G_2$ is a solution of $\{Nx = u(\bar{g}^2)\} \cup \Phi_0(x, \bar{g}^2)$. Thus (h^2, k^2) is a solution of $\Phi(x, \bar{g}^2)$.

Case 2. Second we assume that $I_2 = \varnothing$. We can assume that I_1 and $I_3 \neq \varnothing$ since other cases can be treated similarly. Since $\bar{g}^1$ and $\bar{g}^2$ have the same quantifier free type, there exists $1 \in I_1$ such that $s_l(\bar{g}^1)$ and $s_l(\bar{g}^2)$ are the maximums of $\{s_i(\bar{g}^1)\}_{i \in I_1}$ and $\{s_i(\bar{g}^2)\}_{i \in I_1}$ respectively, and there exists $u \in I_3$ such that $s_u(\bar{g}^1)$ and $s_u(\bar{g}^2)$ are the minimums of $\{s_i(\bar{g}^1)\}_{i \in I_3}$ and $\{s_i(\bar{g}^2)\}_{i \in I_3}$ respectively. Similarly as in the case 1, for $j = 1$ and 2, finding $x \in G^j$ satisfying that

$$\{s_i(\bar{g}^j) < Nx\}_{i \in I_1} \cup \{Nx < s_i(\bar{g}^j)\}_{i \in I_3}$$

is equivalent to finding $x \in G^j$ satisfying that

$$\{s_l(\bar{g}^j) < Nx < s_u(\bar{g}^j)\}.$$

By the definition of $<$, for $j = 1, 2$, finding $x \in G_j$ satisfying $\Phi(x, \bar{g}^j)$ is equivalent to either (a), (b), (c) or (d) in the following:

(a) finding $y \in H_j$ satisfying $\{s_l(\bar{h}^j) < Ny < s_u(\bar{h}^j)\} \cup \Psi(y, \bar{h}^j)$
(b) finding $y \in H_j$ satisfying $\{s_l(\bar{h}^j) = Ny < s_u(\bar{h}^j)\} \cup \Psi(y, \bar{h}^j)$ and $z \in K_j$ satisfying $\{s_l(\bar{k}^j) < Nz\}$
(c) finding $y \in H_j$ satisfying $\{s_l(\bar{h}^j) < Ny = s_u(\bar{h}^j)\} \cup \Psi(y, \bar{h}^j)$ and $z \in K_j$ satisfying $\{Nz < s_u(\bar{k}^j)\}$
(d) finding $y \in H_j$ satisfying $\{s_l(\bar{h}^j) = Ny = s_u(\bar{h}^j)\} \cup \Psi(y, \bar{h}^j)$ and $z \in K_j$ satisfying $\{s_l(\bar{k}^j) < Nz < s_u(\bar{k}^j)\}$.

In the case (a). Since $\bar{h}^1$ and $\bar{h}^2$ have the same type, there exists some solution $h^2 \in H_2$ of $\{s_l(\bar{h}^2) < Ny < s_u(\bar{h}^2)\} \cup \Psi(y, \bar{h}^2)$. Thus for any $k^2 \in K_2$, $(h^2, k^2) \in G_2$ is a solution of $\Phi(x, \bar{g}^2)$.

In the case (b). For a similar reason as in the case (a), there exists some solution $h^2 \in H_2$ of $\{s_l(\bar{h}^2) = Ny < s_u(\bar{h}^2)\} \cup \Psi(y, \bar{h}^2)$. Since there exists $k^1 \in K_1$ such that $s_l(\bar{k}^1) < Nk^1$, $K_1 \neq \{0\}$. Since $K_1 \equiv K_2$, $K_2 \neq \{0\}$. So there exists $k^2 \in K_2$ such that $s_l(\bar{k}^2) < Nk^2$. Then $(h^2, k^2) \in G_2$ is a solution of $\Phi(x, \bar{g}^2)$.

In the case (c). Similarly above, $\Phi(x, \bar{g}^2)$ has a solution of G_2.

In the case (d). Similarly there exists some solution $h^2 \in H_2$ of $\{s_l(\bar{h}^2) = Ny = s_u(\bar{h}^2)\} \cup \Psi(y, \bar{h}^2)$. By the definition of the product interpretation, for $j = 1$ and 2, both $s_l(\bar{h}^j) = s_u(\bar{h}^j)$ and $s_l(\bar{k}^j) < s_u(\bar{k}^j)$ hold in H_j and K_j respectively if and only if both $s_l(\bar{g}^j) < s_u(\bar{g}^j)$ and $s_l(\bar{g}^j) - s_u(\bar{g}^j) \in I$ hold in G_j. Since $\bar{g}^1$ and $\bar{g}^2$ have the same quantifier free type, $s_l(\bar{k}^2) < s_u(\bar{k}^2)$ holds in K_2. By the divisibility of K_2, there exists $k^2 \in K_2$ such that $s_l(\bar{k}^2) < Nk^2 < s_u(\bar{k}^2)$. Then $(h^2, k^2) \in G_2$ is a solution of $\Phi(x, \bar{g}^2)$.

Let $q(x) := \{\varphi(x, \bar{g}^2) : \varphi(x, \bar{g}^1) \in \mathrm{qftp}(g^1/\bar{g}^1)\}$. We have shown that each finite subset of $q(x)$ has a solution in G_2. By the $\aleph_0$-saturation of G_2, there exists a solution g^2 of $q(x)$. Thus we have $\mathrm{qftp}(\bar{g}^1, g^1) = \mathrm{qftp}(\bar{g}^2, g^2)$.

Last we show that in the theorem, if H is recursively axiomatizable, then so is G. In proof of the theorem, we only use the four sets $T_1, \dots, T_4$ of axioms as follows;

1. T_1 says that I is a divisible ordered abelian group.
2. T_2 says that for any model G^* of T_2, H^* is well defined as an L-structure.
3. T_3 says that for any model G^* of T_3, H^* is equivalent to H.
4. T_4 says that any model G^* of T_4 is equivalent to G for quantifier free sentences.

The sets T_1, T_2 and T_3 need to satisfy that $H^* \equiv H$, $K^* \equiv K$, $G^* \cong H^* \times K^*$. The set T_4 needs to satisfy the assumption of fact 1 used in the proof of the theorem. It is easy that T_1 and T_2 are recursively axiomatizable. So we will show in the case of T_3 and T_4.

In the case of T_3, as in remark 4, for any L-sentence ϕ, there exists some $L \cup \{I\}$-sentence ψ_ϕ such that $H \models \phi \leftrightarrow G \models \psi_\phi$. Then $T_3 = \{\psi_\phi \mid H \models \phi\}$. Since H is recursively axiomatizable, so is T_3.

In the case of T_4, $T_4 = \{\phi \mid \phi$ is quantifier free $L \cup \{I\}$-sentence such that $G \models \phi\}$. By the interpretation of constant symbols, for any closed term t, any formula $I(t)$ is equivalent to the formula $t = 0$ in G. Then any quantifier free $L \cup \{I\}$-sentence is defined by some quantifier free L-sentence in G. By the definition of the product interpretation, G is equivalent to H for quantifier free L-sentences. Since H is recursively axiomatizable, so is T_4. ■

By the previous theorem, the following is trivial.

COROLLARY 8. *In previous theorem, we suppose that $I = \{0\} \times K$ is defined by some quantifier free L-formula in G. If H admits QE in L, then G admits QE in L. Moreover H is recursively axiomatizable, so is G.*

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