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REDUCTION OF LOCALLY CONFORMAL SYMPLECTIC MANIFOLDS WITH EXAMPLES OF NON-KÄHLER MANIFOLDS

By

Tomonori NODA

Abstract. Let (M, Ω) be a locally conformal symplectic manifold. Ω is a non-degenerate 2-form on M such that there is a closed 1-form ω, called the Lee form, satisfing $d\Omega = \omega \wedge \Omega$. In this paper we consider Marsden-Weinstein reduction theorem which induces Jacobi-Liouville theorem as a special case. For locally conformal Kähler manifolds, this reduction theorem gives a construction of non-Kähler manifolds in general dimension.

1. Introduction

For a nondegenerate 2-form Ω on a connected smooth manifold M of real dimension $2n$ $(n > 1)$, we say that (M, Ω) is *locally conformal symplectic* if there exists a closed 1-form ω, called the *Lee form*, such that $d\Omega = \omega \wedge \Omega$. Furthermore, if ω is exact, (M, Ω) is said to be *globally conformal symplectic*, in which case, M has a natural symplectic structure. For any real-valued smooth function $f \in C^\infty(M)$ on M, let X_f be the associated *Hamiltonian vector field* defined by $i(X_f)\Omega = df - f\omega$. Set $C^\infty(M)^A := \{f \in C^\infty(M); i(X_f)\omega = 0\}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$ which acts differentiably on M preserving Ω. To each $\xi \in \mathfrak{g}$, we associate a vector field ξ_M on M obtained by the infinitesimal action of ξ. Assume, for every $\xi \in \mathfrak{g}$, a smooth function μ_ξ exists in such a way that the Hamiltonian vector field X_{μ_ξ} coincides with ξ_M. Then we can uniquely define a *moment map* $\mu : M \to \mathfrak{g}^*$ by

$$\langle \xi, \mu(x) \rangle = \mu_\xi(x), \quad x \in M.$$

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This map is always G-equivariant. Let $\mathfrak{g}^*_{\mathrm{reg}}$ be the set of all regular values of μ, and for each $\eta \in \mathfrak{g}^*$, let G_η denote the isotropy subgroup of G at η. Put $M_\eta := \mu^{-1}(\eta)/G_\eta$, and let $\pi_\eta : \mu^{-1}(\eta) \to M_\eta$ and $\iota_\eta : \mu^{-1}(\eta) \hookrightarrow M$ be the projection and the inclusion, respectively. We first prove the following reduction theorem:

THEOREM A. (1) *Let $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$ be such that G_η acts on $\mu^{-1}(\eta)$ properly and freely. Assume that $\iota_\eta^*\omega = 0$. Then M_η admits a unique symplectic form Ω_η such that $\pi_\eta^*\Omega_\eta = \iota_\eta^*\Omega$.*

(2) *Assume that $0 \in \mathfrak{g}^*_{\mathrm{reg}}$ and that the isotropy subgroup G_0 of G at 0 acts on $\mu^{-1}(0)$ properly and freely. Then M_0 admits a unique locally conformal symplectic form Ω_0 with Lee form ω_0 satisfing $\pi_0^*\Omega_0 = \iota_0^*\Omega$ and $\pi_0^*\omega_0 = \iota_0^*\omega$.*

(3) *Let $f : M \to \boldsymbol{R}$ be a G-invariant function and F_t the flow on M of the Hamiltonian vector field X_f. Suppose that either $\eta = 0$ or $f \in C^\infty(M)^A$. Then the flow F_t canonically induces a flow $\bar{F}_t$ on M_η satisfing $\pi_\eta \circ F_t = \bar{F}_t \circ \pi_\eta$ and $f_\eta \circ \pi_\eta = f \circ \iota_\eta$ for some $f_\eta \in C^\infty(M_\eta)$. Moreover f_η is constant along the flow $\bar{F}_t$ if $f \in C^\infty(M)^A$.*

We next consider a reduction theorem for locally conformal Kähler structures. Namely, assuming (M, Ω) to be a locally conformal Kähler manifold in Theorem A, we obtain:

THEOREM B. *In theorem A, assume further that (M, Ω) is a locally conformal Kähler manifold.*

(1) *Suppose that $0 \in \mathfrak{g}^*_{\mathrm{reg}}$ and that the isotropy subgroup G_0 of G at 0 acts on $\mu^{-1}(0)$ properly and freely. If M_0 is compact and ω_0 is not d-exact, then M_0 admits no Kähler metrics.*

(2) *For each $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$, $M_\eta = \mu^{-1}(\eta)/G_\eta$ admits a natural complex structure.*

Now we construct non-Kähler manifolds as an application of this theorem. Let us fix $n+1$ complex numbers $\alpha_1, \ldots, \alpha_{n+1}$ such that $|\alpha_1| = \cdots = |\alpha_{n+1}| > 1$. Denote by $\langle\alpha\rangle$ the cyclic group generated by the transformations $\alpha : (z_1, \ldots, z_{n+1}) \mapsto (\alpha_1 z_1, \ldots, \alpha_{n+1} z_{n+1})$ of $\boldsymbol{C}^{n+1} - \{0\}$. This group acts freely and holomorphically on $\boldsymbol{C}^{n+1} - \{0\}$ as a properly discontinuous group. Thus the quotient

Haller and Rybicki [7] also constructed locally conformal symplectic manifolds by analogy with the reduction theorem for Poisson manifolds. The crucial point of our work lies in the key equality $\xi_Y\omega(X) = 0$ in Lemma 3.2, which allows us to obtain a very simple formulation as above.

space $CH^{n+1} := (C^{n+1} - \{0\})/\langle\alpha\rangle$ is a complex manifold, and called a *Hopf manifold.* Consider the real 1-parameter family of nondegenerate 2-forms

$$\Omega^{(t)} = \sqrt{-1}\,\frac{\|z\|^{2t}\sum dz_j \wedge d\bar{z}_j + t\|z\|^{2(t-1)}(\sum \bar{z}_j\, dz_j) \wedge (\sum z_k\, d\bar{z}_k)}{\|z\|^{2(t+1)}}, \quad t > -1$$

on $C^{n+1} - \{0\}$. Each Ω_t define a locally conformal Kähler structure on CH^{n+1} with Lee form

$$\omega^{(t)} = -(1+t)\frac{\sum(z_j\, d\bar{z}_j + \bar{z}_j\, dz_j)}{\|z\|^2}, \quad t > -1.$$

Fix pairwise relatively prime integers $a_1, \dots, a_{n+1}$ with $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$. Define an action of $G = S^1 = \{e^{2\pi\sqrt{-1}\theta}; \theta \in R\}$ on $C^{n+1} := \{z = (z_1, \dots, z_{n+1})\}$ by

$$S^1 \times C^{n+1} \to C^{n+1} : e^{2\pi\sqrt{-1}\theta}, (z_1, \dots, z_{n+1})$$
$$\mapsto (e^{a_1 2\pi\sqrt{-1}\theta} z_1, \dots, e^{a_{n+1} 2\pi\sqrt{-1}\theta} z_{n+1}). \tag{1.1}$$

This leads to an action on CH^{n+1}. Then the corresponding moment map μ is given by

$$\mu(z_1, \dots, z_{n+1}) = -\frac{a_1|z_1|^2 + \cdots + a_{n+1}|z_{n+1}|^2}{\|z\|^2}. \tag{1.2}$$

Let ℓ and k be, respectively, the numbers of positive a_i's and negative a_i's. Assume that $\ell > 0$, $k > 0$ and that $\ell + k = n + 1$. Then by (2) of Theorem A, we obtain the reduction space M_0 over $0 \in \mathfrak{g}^*$. Furthermore, without loss of generality, we may assume $\ell \leq k$. Then

THEOREM C. *In the situation just above, M_0 with natural complex structure admits no Kähler metrics. Moreover, its cohomology ring is*

$$H^*(M_0; Z) \cong ((Z[x_2] \otimes \Lambda[e_{2k-1}])/R) \otimes H^*(S^1; Z),$$

where R is the ideal of $Z[x_2] \otimes \Lambda[e_{2k-1}]$ generated by three elements

$$\sigma^{\ell}_{\ell-1}(a_1, \dots, a_\ell)x_2^{\ell-1}, \quad \sigma^{k}_{k-1}(a_{\ell+1}, \dots, a_{n+1})x_2^{k-1}, \quad x_2^{\ell} e_{2k-1}.$$

Here $\sigma^m_0 := 0$ and each σ^m_i, $1 \leq i \leq m$, denotes the i-th elementary symmetric function of m variables.

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2. Hamiltonian Dynamics for Locally Conformal Symplectic Manifolds

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M such that $\omega = d\sigma_\alpha$ for some $\sigma_\alpha \in C^\infty(U_\alpha)$ on U_α. Then $\Omega_\alpha := e^{-\sigma_\alpha}\Omega$ is a symplectic form on U_α. For $f \in C^\infty(M)$, define the Hamiltonian vector field X_α of $f_\alpha := e^{-\sigma_\alpha} f$ by $i(X_\alpha)\Omega_\alpha = df_\alpha$. Hence, on U_α,

$$i(X_\alpha)\Omega = df - f\, d\sigma_\alpha = df - f\omega.$$

Since the right-hand side is independent of local expressions, the vector fields X_α, $\alpha \in A$, glue together to define the *Hamiltonian vector field* X_f of f such that $X_f|_{U_\alpha} = X_\alpha$. Let $\mathfrak{X}(M)$ be the space of all smooth vector fields on M.

Proposition 2.1. *Let (M, Ω) be a locally conformal symplectic manifold. Then the map of $C^\infty(M)$ to $\mathfrak{X}(M)$ which sends each $f \in C^\infty(M)$ to $X_f \in \mathfrak{X}(M)$ is injective.*

Proof. Assume $X_f = 0$. Then we have $df - f\omega = 0$. Hence f vanishes at some point $x_0 \in M$, because otherwise, ω would be exact in contradiction. Since ω is d-closed,

$$\omega = -d\tau/\tau,$$

on some open neighborhood U of x_0, where τ is nowhere vanishing on U. By this together with $df = f\omega$, we obtain $d(f\tau) = \tau\, df + f\, d\tau = 0$, i.e., $f\tau$ is constant on U. Hence by $f(x_0) = 0$, the function f vanishes everywhere on U. By the connectedness of M, it is now easy to see that f vanishes everywhere on M. □

For (M, Ω) above, the *canonical vector field* A is the Ω-dual of the Lee form ω, i.e., A is the vector field on M defined by $i(A)\Omega = \omega$. Then a smooth function f on M sits in $C^\infty(M)^A$ if and only if $df(A)$ vanishes identically on M in view of the equalities $\omega(X_f) = i(X_f)i(A)\Omega = -i(A)(df - f\omega) = -df(A)$. To each pair (f, g) of functions in $C^\infty(M)$, we associate the Poisson bracket $\{f, g\} := \Omega(X_g, X_f) = X_f g - g\omega(X_f)$. This obviously satisfies the Jacobi identity, though the Leibniz rule does not necessarily hold. Moreover, $(C^\infty(M)^A, \{\,,\})$ is a Poisson algebra such that

$$X_{\{f,g\}} = [X_f, X_g], \quad \text{for all } f, g \in C^\infty(M)^A. \tag{2.1}$$

Lemma 2.2. (1) *If $f \in C^\infty(M)^A$, then f is constant along the flow of X_f.*

(2) *Let* F_t *be the flow of* X_f. *Then* $F_t^*\Omega = \Omega$ *for all* $t \in \boldsymbol{R}$ *if and only if* $f \in C^\infty(M)^A$.

PROOF. (1) By the defition of X_f, we have $df(X_f) - f\omega(X_f) = \Omega(X_f, X_f) = 0$. Hence $df(X_f) = f\omega(X_f) = 0$ if $f \in C^\infty(M)^A$.

(2) Since $(d/dt)F_t^*\Omega = F_t^* L_{X_f}\Omega = F_t^*(i(X_f)(\omega \wedge \Omega) + d(df - f\omega)) = F_t^*\omega(X_f)\Omega$, it follows that $(d/dt)F_t^*\Omega = 0$ if and only if $f \in C^\infty(M)^A$. □

REMARK 2.3. Let Ω_t, $t \in [0,1]$, be a one-parameter family of locally conformal symplectic forms on M. For each (M, Ω_t), let A_t and ω_t be the associated canonical vector field and the Lee form, respectively. If there exists a 1-form σ_t on M satisfing

$$(d/dt)\Omega_t = d\sigma_t - \sigma_t(A_t)\Omega_t - \omega_t \wedge \sigma_t,$$

then there is a one-parameter family of diffeomorphisms φ_t on M such that $\varphi_t^*\Omega_t = \Omega_0$ for all $t \in \boldsymbol{R}$. Indeed, φ_t is the flow of the vector field X_t defined by $i(X_t)\Omega_t = -\sigma_t$.

3. Moment Maps for Locally Conformal Symplectic Manifolds

Let $G \times M \to M$ be a smooth action of a Lie group G on a locally conformal symplectic manifold (M, Ω) such that the action preserves Ω. We here assume that, for every $\xi \in \mathfrak{g}$, the associate vector field ξ_M is Hamiltonian, i.e., ξ_M is expressible as X_{μ_ξ} for some smooth function μ_ξ on M. We first observe the following:

LEMMA 3.1. $\omega(\xi_M) = 0$ *for all* $\xi \in \mathfrak{g}$.

PROOF. By $i(\xi_M)\Omega = d\mu_\xi - \mu_\xi\omega$ and $i(\xi_M) \circ d = L_{\xi_M} - d \circ i(\xi_M)$, we have $i(\xi_M)\, d\Omega = d\mu_\xi \wedge \omega$ in view of $L_{\xi_M}\Omega = 0$. On the other hand,

$$i(\xi_M)\, d\Omega = i(\xi_M)(\omega \wedge \Omega) = (i(\xi_M)\omega)\Omega - \omega \wedge i(\xi_M)\Omega = \omega(\xi_M)\Omega - \omega \wedge d\mu_\xi.$$

Hence $\omega(\xi_M)\Omega = d\mu_\xi \wedge \omega + \omega \wedge d\mu_\xi = 0$. By the nondegeneracy of Ω, we now conclude that $\omega(\xi_M) = 0$. □

Put $\xi_M^g := (g^{-1})_*\xi_M$ for each $g \in G$, where g is regarded as a diffeomorphism of M. Then by $i(\xi_M^g)\Omega = g^*(d\mu_\xi - \mu_\xi\omega) = i(X_{g^*\mu_\xi})\Omega$, we have $g^*\mu_\xi = \mu_{\mathrm{Ad}(g^{-1})\xi}$. This means the equivariance of the moment map $\mu : M \to \mathfrak{g}^*$. By (2.1),

$$X_{\{\mu_\xi,\mu_\eta\}} = [X_{\mu_\xi}, X_{\mu_\eta}] = [\xi_M, \eta_M] = -[\xi,\eta]_M = -X_{\mu_{[\xi,\eta]}}.$$

Hence by Proposition 2.1, we have $\{\mu_\xi, \mu_\eta\} = -\mu_{[\xi,\eta]}$ for all $\xi, \eta \in \mathfrak{g}$.

LEMMA 3.2. *Let $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$ and $p \in \mu^{-1}(\eta)$. Assume that the action of G_η on $\mu^{-1}(\eta)$ is free and proper. Then, on the tangent space $T_p(M)$ of M at p, the following holds*:

(1) $T_p(G_\eta \cdot p) = T_p(G \cdot p) \cap T_p(\mu^{-1}(\eta))$,

(2) *For every $X \in T_p(\mu^{-1}(\eta))$ and $Y \in T_p(G \cdot p)$, there exists an element ξ^Y in $\mathfrak{g}$ such that $\Omega(X, Y) = \mu_{\xi^Y}\omega(X)$. In particular $T_p(\mu^{-1}(\eta))$ is the Ω-orthogonal complement of $T_p(G \cdot p)$ in $T_p(M)$ if and only if $\mu_{\xi^Y}\omega(X) = 0$ for all $X \in T_p(\mu^{-1}(\eta))$ and $Y \in T_p(G \cdot p)$.*

PROOF. (1) Let $\xi \in \mathfrak{g}$ and $\mathfrak{g}_\eta$ be the Lie algebra of the isotropy subgroup G_η. By the equivariance of μ, we have $d\mu(\xi_M)(p) = \mathrm{ad}(\xi)^*(\eta)$, and hence $\xi_M(p) \in T_p(\mu^{-1}(\eta))$ if and only if $\mathrm{ad}(\xi)^*(\eta) = 0$, i.e., $\xi \in \mathfrak{g}_\eta$.

(2) For Y as above, there exists an element ξ^Y in $\mathfrak{g}$ such that the associated vector field ξ^Y_M on M coincides with Y. Then

$$\Omega(X, Y) = -i(X)i(Y)\Omega = -i(X)i(\xi^Y_M)\Omega = -i(X)(d\mu_{\xi^Y} - \mu_{\xi^Y}\omega) = \mu_{\xi^Y}\omega(X)$$

for all X and Y as above, as required. □

PROPOSITION 3.3. *For a G-invariant smooth function $f : M \to \boldsymbol{R}$, let F_t be the flow of X_f. For a point $p \in M$, if either $p \in \mu^{-1}(0)$ or $f \in C^\infty(M)^A$, then $\mu(F_t(p)) = \mu(p)$.*

PROOF. Since f is invariant, we have $i(\xi_M)\, df = 0$ for every $\xi \in \mathfrak{g}$. Then by $i(\xi_M)\, df = -i(X_f)i(\xi_M)\Omega = -i(X_f)\, d\mu_\xi + \mu_\xi i(X_f)\omega$, we obtain $i(X_f)\, d\mu_\xi = \mu_\xi i(X_f)\omega$. The claim is now immediate. □

PROOF OF THEOREM A. For every $X \in T_p(\mu^{-1}(\eta))$, where $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$, let $[X]$ denotes its canonical image in $T_p(\mu^{-1}(\eta))/T_p(G_\eta \cdot p)$. By (2) of Lemma 3.2, if either η is zero or ω vanishes on $T_p(\mu^{-1}(\eta))$, then we can define forms Ω_η and ω_η on M_η by

$$\Omega_\eta([X], [Y]) := \Omega(X, Y) \quad \text{and} \quad \omega_\eta([X]) := \omega(X),$$

where $X, Y \in T_p(\mu^{-1}(\eta))$. This obviously satisfies $\pi^*_\eta \Omega_\eta = \iota^*_\eta \Omega$ and $\pi^*_\eta \omega_\eta = \iota^*_\eta \omega$. Hence $\pi^*_\eta\, d\Omega_\eta = \iota^*_\eta(\omega \wedge \Omega) = \pi^*_\eta(\omega_\eta \wedge \Omega_\eta)$. Then the surjectivity of π_η and $d\pi_\eta$ implies $d\Omega_\eta = \omega_\eta \wedge \Omega_\eta$. From this identity, we obtain (2) by setting $\eta = 0$. The same identity also gives (1), because $d\Omega_\eta = 0$ follows from $\omega_\eta = 0$. We shall finally prove (3) as follows. By Proposition 3.3, $\mu^{-1}(\eta)$ is invariant under the flow F_t of X_f, and hence F_t induces a well-defined flow $\bar{F}_t$ on M_η. Since f is G-invariant, there exists a unique function f_η on M_η such that $f_\eta \circ \pi_\eta = f \circ \iota_\eta$. Now we assume $f \in C^\infty(M)^A$. Then $L_{X_f}\Omega = \omega(X_f)\Omega = 0$. Since $\pi_\eta^* \bar{F}_t^* \Omega_\eta = F_t^* \pi_\eta^* \Omega_\eta = F_t^* \iota_\eta^* \Omega = \iota_\eta^* \Omega = \pi_\eta^* \Omega_\eta$, the surjectivity of π_η implies $\bar{F}_t^* \Omega_\eta = \Omega_\eta$, as required. □

Proof of Theorem B. Let (M, g, J) be a Hermitian manifold whose fundamental 2-form Ω is locally conformal Kähler.

(1) Note that $d\mu_\xi(J\xi'_M)(p) = \Omega(\xi_M, J\xi'_M)(p) = \langle \xi_M, \xi'_M \rangle_\Omega(p)$ for all $\xi, \xi' \in \mathfrak{g}$ and $p \in \mu^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$, where J is the complex structure of M, and $\langle\,,\rangle_\Omega$ is the metric on M associated to Ω. Hence $\mathfrak{g}^*$ is identified with $J\mathfrak{g}$. In particular, M_0 is naturally a complex manifold and admits no Kähler structures by the following general fact by Vaisman ([11]): For a compact locally conformal Kähler manifold (M, Ω), there exists some global Kähler metric on M if and only if (M, Ω) is a globally conformal Kähler manifold.

(2) Fix $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$. On each $p \in \mu^{-1}(\eta)$, we consider subspaces $E_p := \{X(p) \in T_p\mu^{-1}(\eta); d\mu(X) = d\mu(JX) = 0\}$ and $\{\xi_M + \mu_\xi A\}_p := \{(\xi_M + \mu_\xi A)(p); \xi \in \mathfrak{g}\}$ in $T_p\mu^{-1}(\eta)$. Then we obtain an orthogonal decomposition

$$T_pM = E_p \oplus \{\xi_M + \mu_\xi A\}_p \oplus J\{\xi_M + \mu_\xi A\}_p.$$

Set $E = \bigcup_{p \in \mu^{-1}(\eta)} E_p$. It is easily seen that $E^{1,0} = T^{1,0}M|_{\mu^{-1}(\eta)} \cap (T_p\mu^{-1}(\eta) \otimes \boldsymbol{C})$, where $E^{1,0}$ is the $\sqrt{-1}$-eigenspace in $E \otimes \boldsymbol{C}$. Assuming the following Lemma 3.4, $d\pi_\eta|_{E_p} \to T_{\pi_\eta(p)}M_\eta$ is surjective, and then $d\pi_\eta|_E \circ J = J_\eta \circ d\pi_\eta|_E$ define a natural complex structure J_η on M_η, as required.

Lemma 3.4. *If there exist $\xi \in \mathfrak{g}_\eta$ satisfing $\xi_M + \mu_\xi A \in \mathfrak{g}_\eta^\perp \cap T_p\mu^{-1}(\eta)$, then $\xi = 0$.*

Proof. We may prove for all $\xi' \in \mathfrak{g}_\eta$,

$$g(\xi_M + \mu_\xi A, \xi'_M) = 0 \tag{3.1}$$

leads to $\xi = 0$. By the definition of Hamiltonian vector fields, $g(\xi_M + \mu_\xi A, \xi'_M) = -d\mu_\xi(J\xi'_M)$. On the other hand, since g is J invariant, $g(\xi_M + \mu_\xi A, \xi'_M) = d\mu_{\xi'}(J\xi_M) - \mu_{\xi'}\omega(J\xi_M) + \mu_\xi(d\mu_{\xi'}(JA) - \mu_{\xi'}\omega(JA))$. We have then for all $\xi' \in \mathfrak{g}_\eta$

$$\begin{cases} d\mu_\xi(J\xi'_M) = 0, \\ d\mu_{\xi'}(J\xi_M) - \mu_{\xi'}\omega(J\xi_M) + \mu_\xi(d\mu_{\xi'}(JA) - \mu_{\xi'}\omega(JA)) = 0. \end{cases}$$

By the upper equality, we obtain $\mu_{\xi'}\omega(J\xi_M) = \mu_\xi\omega(J\xi'_M) + d\mu_{\xi'}(J\xi_M)$, and substituting this for the lower equality, we have

$$\mu_\xi g(A, \mu_\xi \xi'_M - \xi_M) = 0.$$

If $\mu_\xi \neq 0$, the this shows $A \in \mathfrak{g}^\perp$. The claim is now obtained in consideration of (3.1). □

4. Proof of Theorem C

In this section, we study properties of the reduction space M_0 in Theorem C. For each $t \in (-1, \infty)$, the Lee form $\omega^{(t)}$ in the introduction is not d-exact, where the Lee form ω_0 on M_0 satisfies $\pi_0^*\omega_0 = \iota_0^*\omega^{(t)}$. Hence ω_0 cannot be d-exact. Then by Theorem B, M_0 admits no Kähler metrics.

Let F be the quotient of $S^{2\ell-1} \times S^{2k-1}$ ($\subset \boldsymbol{C}^\ell \times \boldsymbol{C}^k$) by the S^1-action in (1.1) in the introduction. As a differentiable manifold, M_0 is the direct product of a G-invariant circle S^1 and the S^1-bundle F over $(S^{2\ell-1}/S^1) \times (S^{2k-1}/S^1)$. To obtain the cohomology ring of F, we consider the following commutative diagram of fiblations (see Eschenburg [4], [5]):

$$\begin{array}{ccc} F = U(1)\backslash U(\ell) \times U(k)/U(\ell-1) \times U(k-1) & \xrightarrow{\hat{\rho}} & B_{U(\ell)\times U(k)} \\ \downarrow p & & \downarrow p' \\ B_{U(1)\times U(\ell-1)\times U(k-1)} & \xrightarrow{\rho} & B_{(U(\ell)\times U(k))^2}, \end{array}$$

where $U(1)$ acts on $U(\ell) \times U(k)$ from the left with weight $a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_{n+1}$. Recall that $H^*B_{U(n)} \cong \boldsymbol{Z}[c_1, c_2, \dots, c_n]$ for each positive integer n, where $c_i \in H^{2i}B_{U(n)}$. By setting $c'_i := c_i \otimes 1$ and $c''_j = 1 \otimes c_j$, we have $H^*B_{U(\ell)\times U(k)} \cong \boldsymbol{Z}[c'_1, \dots, c'_\ell, c''_1, \dots, c''_k]$. Then

$$H^*B_{(U(\ell)\times U(k))^2} \cong \boldsymbol{Z}[x'_1, \dots, x'_\ell, y'_1, \dots, y'_\ell, x''_1, \dots, x''_k, y''_1, \dots, y''_k],$$

where $x'_i := c'_i \otimes 1$, $y'_i := 1 \otimes c'_i$ and $x''_j := c''_j \otimes 1$, $y''_j := 1 \otimes c''_j$. The Serre spectral sequence associated to the fibration $p' : B_{U(\ell)\times U(k)} \to B_{(U(\ell)\times U(k))^2}$ is isomorphic to $E_2^{s,t}(p') \cong H^s B_{(U(\ell)\times U(k))^2} \otimes H^t(U(\ell) \times U(k))$. If we denote by $k_r : H^*B_{(U(\ell)\times U(k))^2} \to E_r^{*,0}(p')$ the natural projection of $E_2^{*,0}(p')$-term, then $p'^* = k_\infty : H^*B_{(U(\ell)\times U(k))^2} \to E_\infty^{*,0}(p') \subset H^*B_{U(\ell)\times U(k)}$ by Borel [2].

LEMMA 4.1. *The differentials $d_r : E_r^{s,t}(p') \to E_r^{s+r,t-r+1}(p')$ in cohomology spectral sequence $E_r^{*,*}(p')$ converging to $H^*B_{U(\ell)\times U(k)}$ are*

(1) $d_r(e'_{2i-1}) = 0$ *and* $d_{2i}(e'_{2i-1}) = \pm k_{2i}(x'_i - y'_i)$, *if* $r \leq 2i-1$ *and* $1 \leq i \leq \ell$

(2) $d_r(e''_{2j-1}) = 0$ *and* $d_{2j}(e''_{2j-1}) = \pm k_{2j}(x''_j - y''_j)$, *if* $r \leq 2j-1$ *and* $1 \leq j \leq k$, *where* $e'_{2i-1} := e^{\ell}_{2i-1} \otimes 1$ *and* $e''_{2j-1} := 1 \otimes e^{k}_{2j-1}$ *for generators* e^{ℓ}_{2i-1} *and* e^{k}_{2j-1} *of* $H^*U(\ell)$ *and* $H^*U(k)$, *respectively.*

Let u be a 2-dimensional generator of $H^2(B_{U(1)}; \boldsymbol{Z})$, and let v'_i and v''_j be the i-th and j-th generators in $H^*U(\ell)$ and $H^*U(k)$. The inclusion $U(1) \times U(\ell-1) \times U(k-1) \to (U(\ell) \times U(k))^2$ is the product of

$$\imath(p) : U(1) \to U(\ell) \times U(k)$$

$$e^{2\pi\sqrt{-1}\theta} \mapsto (e^{2\pi\sqrt{-1}a_1\theta}, \ldots, e^{2\pi\sqrt{-1}a_\ell\theta}, e^{2\pi\sqrt{-1}a_{\ell+1}\theta}, \ldots, e^{2\pi\sqrt{-1}a_{n+1}\theta})$$

and the natural inclusion

$$\tau : U(\ell-1) \times U(k-1) \to U(\ell) \times U(k).$$

We have then $\rho^*(x'_i) = \sigma^{\ell}_i(a_1, \ldots, a_\ell)u^{2i} \otimes 1$, $\rho^*(y'_j) = 1 \otimes v'_i$, $\rho^*(x''_j) = \sigma^k_j(a_{\ell+1}, \ldots, a_{n+1})u^{2j} \otimes 1$, and $\rho^*(y''_j) = 1 \otimes v''_j$. Theorem C is now immediate consequence of the following lemma:

LEMMA 4.2. *On the cohomology spectral sequence* $E^{*,*}_r(p)$ *converging to* H^*M_0, *the* $E^{*,*}_2$ *term is isomorphic to*

$$\boldsymbol{Z}[u \otimes 1, 1 \otimes v'_1, \ldots, 1 \otimes v'_{\ell-1}, 1 \otimes v''_1, \ldots, 1 \otimes v''_{k-1}] \otimes \Lambda[e'_1, \ldots, e'_{2\ell-1}, e''_1, \ldots, e''_{k-1}],$$

and the differentials $d_r : E^{s,t}_r(p) \to E^{s+r,t-r+1}_r(p)$ *are*

(1) $d_r(e'_{2i-1}) = 0$ *and* $d_{2i}(e'_{2i-1}) = \pm k_{2i}(\sigma^{\ell}_i(a_1, \ldots, a_\ell)u^{2i} \otimes 1 - 1 \otimes v'_i)$, *if* $r \leq 2i-1$ *and* $1 \leq i \leq \ell-1$,

(2) $d_r(e''_{2j-1}) = 0$ *and* $d_{2j}(e''_{2j-1}) = \pm k_{2j}(\sigma^k_j(a_{\ell+1}, \ldots, a_{n+1})u^{2j} \otimes 1 - 1 \otimes v''_j)$, *if* $r \leq 2j-1$ *and* $1 \leq j \leq k-1$,

(3) $d_{2\ell}(e'_{2\ell-1}) = \pm k_{2\ell}(\sigma^{\ell}_\ell(a_1, \ldots, a_\ell)u^{2\ell} \otimes 1)$ *and* $d_{2k}(e''_{2k-1}) = \pm k_{2k}(\sigma^k_k(a_{\ell+1}, \ldots, a_{n+1})u^{2k} \otimes 1)$.

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Department of Mathematics
Graduate school of science
Osaka University
1-1 Machikaneyama, Toyonaka
Osaka 560-0043, Japan
e-mail: noda@gaia.math.wani.osaka-u.ac.jp