TSUKUBA J. MATH.
Vol. 27 No. 2 (2003), 375–387

ALGEBRAIC INDEPENDENCE OF FIBONACCI RECIPROCAL SUMS ASSOCIATED WITH NEWTON'S METHOD

By

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0) \tag{1}$$

and $\{L_n\}_{n\geq 0}$ the sequence of Lucas numbers defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \tag{2}$$

There are many investigations on the arithmetic properties of reciprocal sums of products of Fibonacci or Lucas numbers. André-Jeannin [1] proved that the sums

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}}$$

are expressed as explicit formulas, more precisely as linear combinations over $\mathbf{Q}(\sqrt{5})$ of the values of the Lambert series $\sum_{n=1}^{\infty} z^n/(1-z^n)$ at numbers of $\mathbf{Q}(\sqrt{5})$. It is well-known that

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}.$$

(For the proof see (9) in the next section.) Brousseau [2] proved that

$$S_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} = 2 - \sqrt{5}.$$

It is easily seen that

Mathematics Subject Classification (2000): 11J81.
Key words: Algebraic independence, Fibonacci numbers, Mahler's method, Newton's method.
Received March 19, 2002.
Revised December 4, 2002.

$$S_3 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1.$$

In this paper we consider a new type of reciprocal sums such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+2}}, \quad \sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2}}, \tag{3}$$

where d is an integer greater than 1 and $[x]$ denotes the largest integer not exceeding the real number x. In the following sections it will be apparent for the readers that the sums (3) are transcendental numbers in contrast with the algebraic numbers S_1, S_2, and S_3 mentioned above, due to the factor $[\log_d n]$ in the numerators. In the next section we express such sums, using Newton's method, as the values of Lambert series of the form

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{d^k}}{1 - z^{d^k}}. \tag{4}$$

In the last section we prove the algebraic independence of reciprocal sums (3) of a more general binary linear recurrence $\{R_n\}_{n \geq 0}$ in place of $\{F_n\}_{n \geq 0}$ for distinct values of d by using Mahler's method, in which the functional equation $f(z) = f(z^d) + z^d/(1 - z^d)$ plays an essential role.

REMARK 1. The algebraic independence of the values of Lambert series similar to (4) implies the algebraic independence of reciprocal sums of Fibonacci numbers with their subscripts appearing in a geometric progression. Let $\{b_k\}_{k \geq 0}$ be a periodic sequence of algebraic numbers not identically zero and c a fixed positive integer. Nishioka, Tanaka, and Toshimitsu [10] proved that if $\{b_k\}_{k \geq 0}$ is not a constant sequence, the numbers

$$\sum_{k=0}^{\infty} \frac{b_k}{(F_{cd^k + l})^m} \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0, m \in \mathbf{N}) \tag{5}$$

are algebraically independent, and if $\{b_k\}_{k \geq 0}$ is a constant sequence, the numbers (5) except the algebraic number $\sum_{k=0}^{\infty} b_k / F_{c2^k}$ are algebraically independent; and also the numbers

$$\sum_{k=0}^{\infty} \frac{b_k}{(L_{cd^k + l})^m} \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0, m \in \mathbf{N})$$

are algebraically independent for any $\{b_k\}_{k \geq 0}$.

Recently, Duverney, Kanoko, and Tanaka [3] proved that the numbers

$$\sum_{k\geq 0}{}' \frac{a^k}{F_{cd^k}+h} \quad \text{and} \quad \sum_{k\geq 0}{}' \frac{a^k}{L_{cd^k}+h},$$

where the sum $\sum'_{k\geq 0}$ is taken over those k with $F_{cd^k}+h\neq 0$, $L_{cd^k}+h\neq 0$ respectively, a is a nonzero algebraic number, and c, d, and h are integers with $c\geq 1$ and $d\geq 2$, are transcendental except three algebraic numbers $\sum_{k=0}^{\infty} 1/F_{c2^k}$, $\sum_{k=0}^{\infty} 4^k/(L_{c2^k}+2)$, and $\sum_{k=0}^{\infty}(-2)^k/(L_{c2^k}-1)$.

2. Newton's Method and Algebraic Independence

We state a particular case, Theorem 1 below, related to Newton's method for approximating the roots of polynomials before stating the general theorem including Theorem 1 (see Theorem 3 in Section 3), since a lemma used in the proof of Theorem 1 induces the key formula (11) of the proof of Theorem 3. Let $\{U_n\}_{n\geq 0}$ be the binary linear recurrence defined by

$$U_0=0, \quad U_1=1, \quad U_{n+2}=A_1U_{n+1}+A_2U_n \quad (n\geq 0),$$

where A_1, A_2 are integers with $A_1>0$, $A_2\neq 0$, and $\Delta=A_1^2+4A_2>0$. Then $\{U_n\}_{n\geq 0}$ is expressed as follows:

$$U_n=\frac{\alpha^n-\beta^n}{\sqrt{\Delta}} \quad (n\geq 0),$$

where $\alpha=(A_1+\sqrt{\Delta})/2$ and $\beta=(A_1-\sqrt{\Delta})/2$ are the roots of $\Phi(X)=X^2-A_1X-A_2$, and it is easily seen that $|\alpha|>|\beta|>0$.

THEOREM 1. *The numbers*

$$\sum_{n=2}^{\infty}\frac{(-A_2)^n[\log_2 n]}{U_{n+l}U_{n+l+1}} \quad (l\geq 0)$$

are algebraically independent.

REMARK 2. We note that

$$\sum_{n=2}^{\infty}\frac{(-A_2)^n}{U_{n+l}U_{n+l+1}}\in \mathbf{Q}(\sqrt{\Delta}) \quad (l\geq 0)$$

(see (9) in the proof of Lemma 4).

EXAMPLE 1. Let $\{F_n\}_{n\geq 0}$ be the sequence of the Fibonacci numbers defined by (1). Then the numbers

$$\sum_{n=2}^{\infty}\frac{(-1)^n[\log_2 n]}{F_{n+l}F_{n+l+1}} \quad (l\geq 0)$$

are algebraically independent.

EXAMPLE 2. The numbers

$$\sum_{n=2}^{\infty}\frac{2^n[\log_2 n]}{(2^{n+l}-1)(2^{n+l+1}-1)} \quad (l\geq 0)$$

are algebraically independent. This is the case of $A_1=3$ and $A_2=-2$ in Theorem 1.

In what follows, let

$$\theta_l=\sum_{n=2}^{\infty}\frac{(-A_2)^n[\log_2 n]}{U_{n+l}U_{n+l+1}} \quad (l\geq 0)$$

and let

$$f_l(z)=\sum_{k=1}^{\infty}\frac{z^{2^k}}{1-(\alpha^{-1}\beta)^l z^{2^k}} \quad (l\geq 0).$$

Theorem 1 is proved by using the following lemma.

LEMMA 1.

$$\theta_l=\sqrt{\Delta}\alpha^{-2l}f_l(\alpha^{-1}\beta) \quad (l\geq 0).$$

In order to prove Lemma 1 we prepare three lemmas below. We introduce here the Newton's method for approximating the root α of $\Phi(X)$. Let $\{x_k\}_{k\geq 0}$ be a sequence defined by

$$x_{k+1}=x_k-\frac{\Phi(x_k)}{\Phi'(x_k)} \quad (k\geq 0)$$

or

$$x_{k+1}=\frac{x_k^2+A_2}{2x_k-A_1} \quad (k\geq 0). \tag{6}$$

The sequence $\{x_k\}_{k\geq 0}$ converges to α for suitable choice of x_0.

LEMMA 2. *If* $x_0 = A_1$, *then* $\sum_{k=1}^{\infty}(x_k - \alpha) = \sqrt{\Delta} f_0(\alpha^{-1}\beta)$.

PROOF. If $x_k = \alpha$ for some k, then $x_{k-1} = \alpha$ by (6). Since $x_0 \neq \alpha$, we see that $x_k \neq \alpha$ for any $k \geq 0$. Substituting $x_k = \sqrt{\Delta} y_k^{-1} + \alpha$ in (6), we get

$$y_{k+1} + 1 = (y_k + 1)^2 \quad (k \geq 0).$$

Therefore $y_k + 1 = (y_0 + 1)^{2^k}$ $(k \geq 0)$ and so

$$x_k - \alpha = \frac{\sqrt{\Delta}}{\left(\dfrac{x_0 - \beta}{x_0 - \alpha}\right)^{2^k} - 1} \quad (k \geq 0). \tag{7}$$

Since $x_0 = A_1 = \alpha + \beta$, we have

$$x_k - \alpha = \frac{\sqrt{\Delta}}{(\alpha\beta^{-1})^{2^k} - 1} \quad (k \geq 0),$$

which implies the lemma.

LEMMA 3. *If* $x_0 = A_1$, *then* $x_k = \dfrac{U_{2^k+1}}{U_{2^k}}$ *for all* $k \geq 0$.

PROOF. The lemma is proved by induction on k. The case of $k = 0$ is trivial. Assume that $x_k = U_{2^k+1}/U_{2^k}$ for some k. Then

$$\begin{aligned}
x_{k+1} &= \frac{x_k^2 + A_2}{2x_k - A_1} \\
&= \frac{U_{2^k+1}^2 + A_2 U_{2^k}^2}{2U_{2^k+1}U_{2^k} - A_1 U_{2^k}^2} \\
&= \frac{(\alpha^{2^k+1} - \beta^{2^k+1})^2 - \alpha\beta(\alpha^{2^k} - \beta^{2^k})^2}{2(\alpha^{2^k+1} - \beta^{2^k+1})(\alpha^{2^k} - \beta^{2^k}) - (\alpha + \beta)(\alpha^{2^k} - \beta^{2^k})^2} \\
&= \frac{(\alpha - \beta)(\alpha^{2^{k+1}+1} - \beta^{2^{k+1}+1})}{(\alpha - \beta)(\alpha^{2^{k+1}} - \beta^{2^{k+1}})} \\
&= \frac{U_{2^{k+1}+1}}{U_{2^{k+1}}},
\end{aligned}$$

which implies the lemma.

LEMMA 4.

$$\frac{U_{m+1}}{U_m} - \alpha = \sum_{n=m}^{\infty} \frac{(-A_2)^n}{U_n U_{n+1}} \quad (m \geq 2).$$

PROOF. Since

$$\frac{U_{n+1}}{U_n} - \frac{U_{n+2}}{U_{n+1}} = \frac{(-A_2)^n}{U_n U_{n+1}} \quad (n \geq 1),$$

we have

$$\sum_{n=1}^{m-1} \frac{(-A_2)^n}{U_n U_{n+1}} = \frac{U_2}{U_1} - \frac{U_{m+1}}{U_m}. \tag{8}$$

As $m \to \infty$, this gives

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n}{U_n U_{n+1}} = \frac{U_2}{U_1} - \alpha. \tag{9}$$

Subtracting (8) from (9), we get the lemma.

PROOF OF LEMMA 1. The lemma is proved by induction on l. Let $\{x_k\}_{k \geq 0}$ be defined by (6) with $x_0 = A_1$. Then we have by Lemmas 3 and 4

$$\sum_{k=1}^{\infty}(x_k - \alpha) = \sum_{k=1}^{\infty}\left(\frac{U_{2^k+1}}{U_{2^k}} - \alpha\right) = \sum_{k=1}^{\infty}\sum_{n=2^k}^{\infty} \frac{(-A_2)^n}{U_n U_{n+1}} = \sum_{n=2}^{\infty}\sum_{k=1}^{[\log_2 n]} \frac{(-A_2)^n}{U_n U_{n+1}} = \theta_0.$$

Therefore $\theta_0 = \sqrt{\Delta} f_0(\alpha^{-1}\beta)$ by Lemma 2.

Next assume that $\theta_l = \sqrt{\Delta}\alpha^{-2l} f_l(\alpha^{-1}\beta)$ for some l. We have

$$\theta_l + A_2\theta_{l+1} = \frac{(-A_2)^2}{U_{l+2}U_{l+3}} + \sum_{n=3}^{\infty} \frac{(-A_2)^n([\log_2 n] - [\log_2(n-1)])}{U_{n+l}U_{n+l+1}}.$$

Since

$$[\log_2 n] - [\log_2(n-1)] = \begin{cases} 1 & (n = 2^k, k \in \mathbf{N}) \\ 0 & (\text{otherwise}), \end{cases}$$

we get

$$\theta_l + A_2\theta_{l+1} = \sum_{k=1}^{\infty} \frac{(-A_2)^{2^k}}{U_{2^k+l}U_{2^k+l+1}}.$$

Using $\alpha\beta = -A_2$, we see that

$$\sum_{k=1}^{\infty} \frac{(-A_2)^{2^k}}{U_{2^k+l}U_{2^k+l+1}} = \sum_{k=1}^{\infty} \frac{\Delta(-A_2)^{2^k}}{(\alpha^{2^k+l} - \beta^{2^k+l})(\alpha^{2^k+l+1} - \beta^{2^k+l+1})}$$

$$= \sqrt{\Delta}\sum_{k=1}^{\infty}\left(\frac{\alpha^{-l}\beta^{2^k}}{\alpha^{2^k+l} - \beta^{2^k+l}} - \frac{\alpha^{-l}\beta^{2^k+1}}{\alpha^{2^k+l+1} - \beta^{2^k+l+1}}\right)$$

$$= \sqrt{\Delta}\alpha^{-2l}f_l(\alpha^{-1}\beta) + A_2\sqrt{\Delta}\alpha^{-2(l+1)}f_{l+1}(\alpha^{-1}\beta).$$

Therefore $\theta_{l+1} = \sqrt{\Delta}\alpha^{-2(l+1)}f_{l+1}(\alpha^{-1}\beta)$, and the lemma is proved.

Proof of Theorem 1. It suffices to prove the algebraic independency of θ_l $(0 \le l \le L)$ for any nonnegative integer L. By Lemma 1 it is enough to prove the algebraic independency of $f_l(\alpha^{-1}\beta)$ $(0 \le l \le L)$. We see that $f_l(z)$ satisfies

$$f_l(z) = f_l(z^2) + \frac{z^2}{1 - (\alpha^{-1}\beta)^l z^2}.$$

By Nishioka's lemmas [9, Lemma 2 and Lemma 6] the functions $f_l(z)$ $(0 \le l \le L)$ are linearly independent over $\mathbf{C}$ modulo the rational function field $\mathbf{C}(z)$, namely $\sum_{l=0}^{L} c_l f_l(z) \in \mathbf{C}(z)$ $(c_l \in \mathbf{C})$ holds only if $c_l = 0$ for all l $(0 \le l \le L)$. By Loxton and van der Poorten's theorem [5, Theorem 2] or by Kubota's result [4, Corollary 9] the functions $f_l(z)$ $(0 \le l \le L)$ are algebraically independent over $\mathbf{C}(z)$. Then by Mahler's theorem [6] (see also [7, Theorem 2]), $f_l(\alpha^{-1}\beta)$ $(0 \le l \le L)$ are algebraically independent, and the proof of the theorem is completed.

By (7) in the proof of Lemma 2 we see that, if $0 < |(x_0 - \alpha)/(x_0 - \beta)| < 1$ or equivalently

$$x_0 > \frac{A_1}{2}, \quad x_0 \neq \alpha, \tag{10}$$

then

$$\sum_{k=1}^{\infty}(x_k - \alpha) = \sqrt{\Delta}f_0\left(\frac{x_0 - \alpha}{x_0 - \beta}\right),$$

whose transcendency is seen by the same way as in the above proof of Theorem 1 with $L = 0$. Therefore we have the following:

Theorem 2. *Let A_1, A_2 be real algebraic numbers with $A_1^2 + 4A_2 > 0$. Let*

$\{x_k\}_{k\geq 0}$ *be defined by* (6) *with* x_0 *an algebraic number satisfying* (10). *Then the sum of errors* $\sum_{k=1}^{\infty}(x_k - \alpha)$ *is transcendental.*

3. General Case

Letting $z = \alpha^{-1}\beta$ in Lemma 1, we have

$$\sum_{n=2}^{\infty}[\log_2 n]\left(\frac{z^{n+l}}{1-z^{n+l}} - \frac{z^{n+l+1}}{1-z^{n+l+1}}\right) = \sum_{k=1}^{\infty}\frac{z^{2^k+l}}{1-z^{2^k+l}} \quad (l \geq 0),$$

which is valid inside the unit circle $|z| = 1$. Let d be an integer greater than 1 and γ a complex number with $|\gamma| \leq 1$. We have a more general equation

$$\sum_{n=d}^{\infty}[\log_d n]\left(\frac{z^{n+l}}{1+\gamma z^{n+l}} - \frac{z^{n+l+1}}{1+\gamma z^{n+l+1}}\right) = \sum_{k=1}^{\infty}\frac{z^{d^k+l}}{1+\gamma z^{d^k+l}} \quad (|z| < 1, l \geq 0), \tag{11}$$

since

$$[\log_d n] - [\log_d(n-1)] = \begin{cases} 1 & (n = d^k, k \in \mathbf{N}) \\ 0 & (\text{otherwise}) \end{cases} \tag{12}$$

and so

$$\sum_{n=d}^{m}[\log_d n]\left(\frac{z^{n+l}}{1+\gamma z^{n+l}} - \frac{z^{n+l+1}}{1+\gamma z^{n+l+1}}\right) = \sum_{k=1}^{[\log_d m]}\frac{z^{d^k+l}}{1+\gamma z^{d^k+l}} - \frac{[\log_d m]z^{m+l+1}}{1+\gamma z^{m+l+1}}.$$

Using (11), we prove the following theorem, which is more general than Theorem 1.

THEOREM 3. *Let* $\{R_n\}_{n\geq 0}$ *be the binary linear recurrence defined by*

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0),$$

where A_1, A_2 *are nonzero integers with* $\Delta = A_1^2 + 4A_2 > 0$ *and* R_0, R_1 *are integers with* $R_0 R_2 \neq R_1^2$ *and* $A_1 R_0(A_1 R_0 - 2R_1) \leq 0$. *Then the numbers*

$$\sum_{n=d}^{\infty}\frac{(-A_2)^n[\log_d n]}{R_{n+l}R_{n+l+1}} \quad (d \in \mathbf{N}\backslash\{1\}, l \geq 0)$$

are algebraically independent.

REMARK 3. The condition $A_1 R_0(A_1 R_0 - 2R_1) \leq 0$ assures $R_{n+l}R_{n+l+1} \neq 0$. We can prove the theorem also in the case $A_1 R_0(A_1 R_0 - 2R_1) > 0$ if we exclude the subscripts n with $R_{n+l}R_{n+l+1} = 0$ from the sum; however we have omitted such a case for the sake of simplicity.

COROLLARY 1. *Let $\{R_n\}_{n\geq 0}$ be as in Theorem 3. Then the numbers*

$$\sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+l}R_{n+l+2}} \quad (d \in \mathbf{N}\setminus\{1\}, l \geq 0)$$

are algebraically independent and the numbers

$$\sum_{n=d}^{\infty} \frac{A_2^n [\log_d n]}{R_{n+l}R_{n+l+2}} \quad (d \in \mathbf{N}\setminus\{1\}, l \geq 0)$$

are also algebraically independent.

PROOF. Let

$$\theta_{d,l} = \sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+l}R_{n+l+1}} \quad (d \in \mathbf{N}\setminus\{1\}, l \geq 0).$$

Using $R_{n+2} - A_2 R_n = A_1 R_{n+1}$ $(n \geq 0)$, we have

$$\begin{aligned}\sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+l}R_{n+l+2}} &= A_1^{-1} \sum_{n=d}^{\infty} \left(\frac{(-A_2)^n [\log_d n]}{R_{n+l}R_{n+l+1}} + \frac{(-A_2)^{n+1} [\log_d n]}{R_{n+l+1}R_{n+l+2}} \right) \\ &= A_1^{-1}(\theta_{d,l} - A_2 \theta_{d,l+1})\end{aligned}$$

and

$$\sum_{n=d}^{\infty} \frac{A_2^n [\log_d n]}{R_{n+l}R_{n+l+2}} = A_1^{-1} \sum_{n=d}^{\infty} \left(\frac{A_2^n [\log_d n]}{R_{n+l}R_{n+l+1}} - \frac{A_2^{n+1} [\log_d n]}{R_{n+l+1}R_{n+l+2}} \right). \tag{13}$$

If d is even, $[\log_d(2m)] = [\log_d(2m+1)]$ for any $m \in \mathbf{N}$ by (12) and so the right-hand side of (13) is equal to

$$\begin{aligned}&A_1^{-1} \sum_{m=d/2}^{\infty} \left(\frac{A_2^{2m} [\log_d(2m)]}{R_{2m+l}R_{2m+l+1}} - \frac{A_2^{2m+1} [\log_d(2m+1)]}{R_{2m+l+1}R_{2m+l+2}} \right) \\ &\quad + A_1^{-1} \sum_{m=d/2}^{\infty} \left(\frac{A_2^{2m+1} [\log_d(2m)]}{R_{2m+l+1}R_{2m+l+2}} - \frac{A_2^{2m+2} [\log_d(2m+1)]}{R_{2m+l+2}R_{2m+l+3}} \right) \\ &= A_1^{-1}(\theta_{d,l} + A_2 \theta_{d,l+1}).\end{aligned}$$

If d is odd, $[\log_d(2m-1)] = [\log_d(2m)]$ for any $m \in \mathbf{N}$ by (12) and so the right-hand side of (13) is equal to

$$A_1^{-1} \sum_{m=(d+1)/2}^{\infty} \left(\frac{A_2^{2m-1}[\log_d(2m-1)]}{R_{2m+l-1}R_{2m+l}} - \frac{A_2^{2m}[\log_d(2m)]}{R_{2m+l}R_{2m+l+1}} \right)$$
$$+ A_1^{-1} \sum_{m=(d+1)/2}^{\infty} \left(\frac{A_2^{2m}[\log_d(2m-1)]}{R_{2m+l}R_{2m+l+1}} - \frac{A_2^{2m+1}[\log_d(2m)]}{R_{2m+l+1}R_{2m+l+2}} \right)$$
$$= -A_1^{-1}(\theta_{d,l} + A_2\theta_{d,l+1}).$$

Therefore we have

$$\sum_{n=d}^{\infty} \frac{A_2^n[\log_d n]}{R_{n+l}R_{n+l+2}} = (-1)^d A_1^{-1}(\theta_{d,l} + A_2\theta_{d,l+1}).$$

By Theorem 3 the numbers $A_1^{-1}(\theta_{d,l} - A_2\theta_{d,l+1})$ $(d \in \mathbf{N}\backslash\{1\}, l \geq 0)$ are algebraically independent and the numbers $(-1)^d A_1^{-1}(\theta_{d,l} + A_2\theta_{d,l+1})$ $(d \in \mathbf{N}\backslash\{1\}, l \geq 0)$ are also algebraically independent, which implies the corollary.

EXAMPLE 3. Let $\{F_n\}_{n\geq 0}$ be the sequence of the Fibonacci numbers defined by (1). Then the numbers

$$\sum_{n=d}^{\infty} \frac{(-1)^n[\log_d n]}{F_{n+l}F_{n+l+1}} \quad (d \in \mathbf{N}\backslash\{1\}, l \geq 0)$$

are algebraically independent; moreover, so are the numbers

$$\sum_{n=d}^{\infty} \frac{(-1)^n[\log_d n]}{F_{n+l}F_{n+l+2}} \quad (d \in \mathbf{N}\backslash\{1\}, l \geq 0);$$

furthermore, so are the numbers

$$\sum_{n=d}^{\infty} \frac{[\log_d n]}{F_{n+l}F_{n+l+2}} \quad (d \in \mathbf{N}\backslash\{1\}, l \geq 0).$$

EXAMPLE 4. Let $\{L_n\}_{n\geq 0}$ be the sequence of the Lucas numbers defined by (2). Then the numbers

$$\sum_{n=d}^{\infty} \frac{(-1)^n[\log_d n]}{L_{n+l}L_{n+l+1}} \quad (d \in \mathbf{N}\backslash\{1\}, l \geq 0)$$

are algebraically independent; moreover, so are the numbers

$$\sum_{n=d}^{\infty} \frac{(-1)^n [\log_d n]}{L_{n+l} L_{n+l+2}} \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0);$$

furthermore, so are the numbers

$$\sum_{n=d}^{\infty} \frac{[\log_d n]}{L_{n+l} L_{n+l+2}} \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0).$$

Proof of Theorem 3. We can express $\{R_n\}_{n \geq 0}$ as follows:

$$R_n = a\alpha^n + b\beta^n \quad (n \geq 0),$$

where α, β ($|\alpha| \geq |\beta|$) are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ and $a, b \in \mathbf{Q}(\sqrt{\Delta})$. It is easily seen that $|\alpha| > |\beta| > 0$. Since $R_0 R_2 - R_1^2 = ab\Delta$ and $A_1 R_0 (A_1 R_0 - 2R_1) = (\alpha^2 - \beta^2)(b^2 - a^2)$, we see that $|a| \geq |b| > 0$. Letting

$$g_{dl}(z) = \sum_{k=1}^{\infty} \frac{z^{d^k}}{1 + a^{-1} b (\alpha^{-1}\beta)^l z^{d^k}} \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0)$$

and substituting $\gamma = a^{-1} b$ and $z = \alpha^{-1}\beta$ in (11), we have

$$\sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+l} R_{n+l+1}} = a^{-2} \alpha^{-2l} (\alpha - \beta)^{-1} g_{dl}(\alpha^{-1}\beta) \quad (d \in \mathbf{N} \backslash \{1\}, l \geq 0). \tag{14}$$

Noting that $g_{dl}(z)$ satisfies

$$g_{dl}(z) = g_{dl}(z^d) + \frac{z^d}{1 + a^{-1} b (\alpha^{-1}\beta)^l z^d}, \tag{15}$$

we apply Nishioka's theorem [8, Theorem 1]. Define

$$D = \{d \in \mathbf{N} \mid d \neq a^n \ (a, n \in \mathbf{N}, n \geq 2)\}.$$

Then we have

$$\mathbf{N} \backslash \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\}.$$

We note that if $d, d' \in D$ are distinct, then $\log d / \log d' \notin \mathbf{Q}$. It is enough by (14) to prove the algebraic independency of the values $g_{d^j l}(\alpha^{-1}\beta)$ ($d \in D, 1 \leq j \leq n$, $0 \leq l \leq L$) for any positive integer n and for any nonnegative integer L. Assume on the contrary that the values $g_{d^j l}(\alpha^{-1}\beta)$ ($d \in D, 1 \leq j \leq n, 0 \leq l \leq L$) are algebraically dependent for some positive integer n and nonnegative integer L. Letting $N = n!$ and iterating (15), we have the functional equation

$$g_{d^j l}(z) = g_{d^j l}(z^{d^N}) + \sum_{k=1}^{Nj^{-1}} \frac{z^{d^{jk}}}{1 + a^{-1}b(\alpha^{-1}\beta)^l z^{d^{jk}}} \quad (1 \le j \le n, 0 \le l \le L).$$

By Nishioka's theorem [8, Theorem 1] the functions $g_{d^j l}(z)$ $(1 \le j \le n, 0 \le l \le L)$ are algebraically dependent over $\mathbf{C}(z)$ for some $d \in D$. Then by Loxton and van der Poorten's theorem [5, Theorem 2] or by Kubota's result [4, Corollary 9] the functions $g_{d^j l}(z)$ $(1 \le j \le n, 0 \le l \le L)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$. Thus there are complex numbers c_{jl} $(1 \le j \le n, 0 \le l \le L)$, not all zero, such that

$$\sum_{j=1}^{n} \sum_{l=0}^{L} c_{jl} g_{d^j l}(z) \in \mathbf{C}(z).$$

Letting ζ be a primitive N-th root of unity and letting

$$h_{li}(z) = \sum_{k=1}^{\infty} \frac{\zeta^{ik} z^{d^k}}{1 + a^{-1}b(\alpha^{-1}\beta)^l z^{d^k}} \quad (0 \le l \le L, 0 \le i \le N-1),$$

we see that

$$\sum_{j=1}^{n} c_{jl} g_{d^j l}(z) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{c_{jl} z^{d^{jk}}}{1 + a^{-1}b(\alpha^{-1}\beta)^l z^{d^{jk}}} = \sum_{i=0}^{N-1} c_{li}^* h_{li}(z) \quad (0 \le l \le L),$$

where c_{li}^* $(0 \le l \le L, 0 \le i \le N-1)$ are complex numbers not all zero (cf. Proof of Theorem 1.1 in [10]). Therefore

$$\sum_{l=0}^{L} \sum_{i=0}^{N-1} c_{li}^* h_{li}(z) \in \mathbf{C}(z).$$

Since $h_{li}(z)$ satisfies

$$\zeta^i h_{li}(z^d) = h_{li}(z) - \frac{\zeta^i z^d}{1 + a^{-1}b(\alpha^{-1}\beta)^l z^d}$$

and $1, \zeta, \ldots, \zeta^{N-1}$ are distinct, again by the Loxton and van der Poorten's theorem or by the Kubota's result, the functions $h_{li}(z)$ $(0 \le l \le L)$ are linearly dependent over $\mathbf{C}$ modulo $\mathbf{C}(z)$ for some i, which contradicts Nishioka's lemmas [9, Lemmas 2, 3, and 6]. This completes the proof of the theorem.

Acknowledgment

The author would like to express his gratitude to Professor Daniel Duverney for valuable discussions.

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