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CAUCHY-RIEMANN ORBIFOLDS

By

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Abstract. For any *CR* orbifold[1] B, of *CR* dimension n, we build a vector bundle (in the sense of J. Girbau & M. Nicolau, [13]) $T_{1,0}(B)$ over B, so that $T_{1,0}(B)_p \approx \boldsymbol{C}^n/G_x$ at any singular point $p = \varphi(x) \in B$ (and the portion of $T_{1,0}(B)$ over the regular part of B is an ordinary *CR* structure), hence study the tangential Cauchy-Riemann equations on orbifolds. As an application, we build a two-sided parametrix for the Kohn-Rossi laplacian $\square_\Omega$ (on the domain Ω of a local uniformizing system $\{\Omega, G, \varphi\}$ of B) inverting $\square_\Omega$ over the G-invariant $(0,q)$-forms $(1 \leq q \leq n-1)$ up to (smoothing) operators of type 1 (in the sense[2] of G. B. Folland & E. M. Stein, [12]).

1. Introduction

An N-dimensional orbifold (or V-manifold, cf. I. Satake, [20], to whom the notion is due) is a Hausdorff space B looking locally like a quotient of (an open set in) the Euclidean space, by the action of some finite group of C^∞ diffeomorphisms (cf. [1]–[3], [7], [19]–[22]). That is, each point $p \in B$ admits a neighborhood U which is uniformized by a domain $\Omega \subset \boldsymbol{R}^N$ and a continuous map $\varphi : \Omega \to U$, in the sense that there is a finite subgroup $G \subset \mathit{Diff}^\infty(\Omega)$ so that φ is G-invariant and factors to a homeomorphism $\Omega/G \approx U$. Such (local) uniformizing systems $\{\Omega, G, \varphi\}$ (shortly l.u.s.'s) play the role of local coordinate charts in manifold theory, and as well as for ordinary manifolds, are required to agree smoothly on overlaps: if $p \in U' \cap V$ and $\{\Omega', G', \varphi'\}$, $\{D, H, \psi\}$ uniformize U', V respectively, then there is a neighborhood $U \subset U' \cap V$ of p uniformized by some $\{\Omega, G, \varphi\}$, and an *injection* $\lambda : \Omega \to \Omega'$, i.e. a smooth map which is a C^∞ diffeomorphism on some open subset of Ω' and satisfies $\varphi' \circ \lambda = \varphi$. This being the

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case, various G-structures of current use in differential geometry, such as Riemannian metrics, complex structures, etc., may be prescribed on orbifolds, by merely assigning an ordinary G-structure to Ω, for each l.u.s. $\{\Omega, G, \varphi\}$, and requiring that injections preserve these (local) G-structures (cf. [5], [8], [16], [23]). For instance, if B is a $(2n+k)$-dimensional orbifold, whose V-manifold structure is described by some fixed family of l.u.s.'s $\mathscr{A}$, then a *CR structure* on B is a set

$$\{T_{1,0}(\Omega) : \{\Omega, G, \varphi\} \in \mathscr{A}\} \tag{1}$$

where $T_{1,0}(\Omega)$ is a CR structure (of type (n,k)) on Ω and each injection $\lambda : \Omega \to \Omega'$ is a CR map (i.e. $(d_x\lambda)T_{1,0}(\Omega)_x \subseteq T_{1,0}(\Omega')_{\lambda(x)}$, $x \in \Omega$). A CR structure (1) on B is easily seen to be a vector bundle over B, in the sense of W. L. Baily, [3], p. 863, i.e. there is a group monomorphism

$$h_\Omega : G \to Hom(T_{1,0}(\Omega), T_{1,0}(\Omega))$$

for each l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{A}$, and a bundle map

$$\lambda^* : T_{1,0}(\Omega')|_{\lambda(\Omega)} \to T_{1,0}(\Omega)$$

for each injection $\lambda : \Omega \to \Omega'$, so that 1) $h_\Omega(\sigma)T_{1,0}(\Omega)_x \subseteq T_{1,0}(\Omega)_{\sigma^{-1}(x)}$, $x \in \Omega$, 2) $h_\Omega(\sigma) \circ \lambda^* = \lambda^* \circ h_{\Omega'}(\eta(\sigma))$, $\sigma \in G$, and 3) $(\mu \circ \lambda)^* = \lambda^* \circ \mu^*$, for any pair of injections $\lambda : \Omega \to \Omega'$ and $\mu : \Omega' \to \Omega''$, where $\eta : G \to G'$ is a natural group monomorphism associated with λ (cf. our section 3). Indeed, $h_\Omega(\sigma)_x := d_x\sigma^{-1}$, $\sigma \in G$, $x \in \Omega$, respectively $\lambda^*(v') = (d_{\lambda(x)}\mu)v'$, $v' \in T_{1,0}(\Omega')_{\lambda(x)}$, $x \in \Omega$, where $\mu := (\lambda : \Omega \to \lambda(\Omega))^{-1}$, satisfy the requirements (1) to (3) (each $\sigma \in G$ is in particular an injection, hence $G \subset Aut_{CR}(\Omega)$). One may proceed to define *CR functions* as continuous functions $f : B \to C$ for which each $f_\Omega := f \circ \varphi : \Omega \to C$ is smooth and

$$\bar{\partial}_\Omega f_\Omega = 0 \tag{2}$$

in Ω, where $\bar{\partial}_\Omega$ is the tangential Cauchy-Riemann operator on $(\Omega, T_{1,0}(\Omega))$. The equations (2) may then be referred to as the tangential Cauchy-Riemann equations on (the CR orbifold) B and it appears that a satisfactory scheme for recovering CR geometry and analysis, on V-manifolds, has been devised.

The weakness of this approach consists in the lack of relationship between the G-structure (here CR structure) so assigned to B and its singular locus. A point $p \in B$ is *singular* if it admits a neighborhood U, uniformized by some l.u.s. $\{\Omega, G, \varphi\}$ for which a point $x \in \Omega$ with nontrivial isotropy group (i.e. $G_x := \{\sigma \in G : \sigma(x) = x\} \neq \{1_\Omega\}$) and lying over p (i.e. $\varphi(x) = p$) may be found. If Σ is the set of all singular points of B (its *singular locus*) then $B_{reg} := B \backslash \Sigma$ is an ordinary CR manifold. Although Σ has a quite simple local structure (locally, it is a finite union of real algebraic CR submanifolds) there is no obvious relationship between $T_{1,0}(\Omega)$ and $S := \{x \in \Omega : G_x \neq \{1_\Omega\}\}$, and generally speaking, expressions such as the behaviour of the CR structure $T_{1,0}(B_{reg})$ (a bundle over $B \backslash \Sigma$), or of a CR function $f \in CR^\infty(B_{reg})$, near Σ, lack a precise meaning. To ask a more concrete question, given a CR orbifold B, can one construct a 'bundle' $T_{1,0}(B)$ over the *whole* of B so that $T_{1,0}(B)|_{B_{reg}} = T_{1,0}(B_{reg})$ and the fibres $T_{1,0}(B)_p$ reflect the nature of p (i.e. whether p is singular or regular)? In other words, can one write a set of equations on B reducing to the ordinary Cauchy-Riemann equations $\bar{\partial}_{B_{reg}} f = 0$ on the regular part of B, and exhibiting at Σ a feature related to the nature of Σ?

The scope of the present paper is to answer some fundamental questions of this sort, i.e. regarding (the Cauchy-Riemann equations on) CR orbifolds. Precisely, for each CR orbifold B, we build a bundle $T_{1,0}(B) \to B$ in the sense of J. Girbau & M. Nicolau, [13], p. 257–259, so that

$$T_{1,0}(B) \approx \boldsymbol{C}^n / G_x, \quad p = \varphi(x) \in B, \tag{3}$$

a bijection (hence when $p \in \Sigma$, $T_{1,0}(B)_p$ is not even a vector space) and $T_{1,0}(B)_p = T_{1,0}(B_{reg})_p$ for any $p \in B \backslash \Sigma$. Moreover, by adapting (from real to complex geometry) an ideea of I. Satake, [22], p. 473, who observed that G_x-invariant tangent vectors at $x \in \Omega$ give rise, in our context, to a subset of $T_{1,0}(B)_p$ depending only on $p = \varphi(x)$ and *possessing* a $\boldsymbol{C}$-linear space structure, we are led to the equations

$$\sum_{\alpha=1}^{n} \bar{\zeta}^\alpha L_{\bar{\alpha}}(f)_x = 0, \tag{4}$$

$f \in C^\infty(\Omega)$, $x \in \Omega$, $\zeta = (\zeta^1, \dots, \zeta^n) \in \bigcap_{\sigma \in G_x} Ker[g_\sigma(x) - I_n]$, where $\{L_\alpha\}$ is a frame of $T_{1,0}(\Omega)$, which may be thought of w.l.o.g. as being defined on the whole of Ω, and $g_\sigma(x) \in GL(n, \boldsymbol{C})$ is given by

$$(d_x \sigma) L_{\alpha, x} = g_\sigma(x)^\beta_\alpha L_{\beta, \sigma(x)}, \quad x \in \Omega.$$

Clearly (4) reduces to (2) in $\Omega \backslash S$; we show that for each singular point $x \in S$ there is a neighborhood D of x in Ω and an algebraic CR submanifold $F_x \subset S \cap D$ so that each smooth solution f of (4) is a CR function on F_x.

Any (smooth) function $f : B \to \boldsymbol{C}$ gives rise to a G-invariant function $f_\Omega := f \circ \varphi$ on Ω. In general, a (geometric) object prescribed on (each) Ω must be preserved by injections, hence by each $\sigma \in G$, hence it is G-invariant. Therefore, another fundamental feature of any attempt to recover known facts from CR geometry (on CR orbifolds) is, locally, to prove G-invariant analogues of the facts of interest. In view of [3] (which uses a G-average of a fundamental solution of an elliptic operator to prove a Kodaira-Hodge-de Rham decomposition theorem on V-manifolds) this part of the task is rather well understood. To illustrate this line of thought, given a domain Ω in $\boldsymbol{R}^{2n+1}$ carrying a G-invariant strictly pseudoconvex CR structure $T_{1,0}(\Omega)$ and a pseudohermitian structure θ so that G consists of pseudohermitian transformations of (Ω, θ), we build a two-sided parametrix inverting the Kohn-Rossi operator $\Box_\Omega$ on the G-invariant forms of degree $0 < q < n-1$, up to operators of type 1, cf. [12]; these are smoothing, in the sense that they are bounded operators $S_k^p(\Omega) \to S_{k+1}^p(\Omega)$ of Folland-Stein spaces. Our methods in section 6 resemble closely those in [3], p. 870–874, and [13], p. 71–74.

The paper is organized as follows. In section 2 we recall the material we need as to CR manifolds and pseudohermitian geometry. In section 3 we discuss the case of complex orbifolds (CR codimension $k = 0$), the local structure of their singular locus, and V-holomorphic functions. Sections 4 and 5 are devoted to CR orbifolds of CR codimension 1 (certain local aspects are examined in section 4). In section 6 we prove our main result (inverting the Kohn-Rossi operator over the G-invariant forms).

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2. *CR* Geometry

In this section we discuss basic notions such as pseudohermitian structures, the Levi form (of a CR manifold of hypersurface type), and pseudohermitian transformations. The main tool is the Tanaka-Webster connection (of a nondegenerate CR manifold endowed with a contact form) and the corresponding parabolic exponential map (leading to a choice of pseudohermitian normal coordinates at each point of the given CR manifold). The notion is due to D. Jerison & J. M. Lee, [15]; Lemma 1 is however new.

Let $(M, T_{1,0}(M))$ be a CR manifold, of type $(n, 1)$, i.e. of CR dimension n and CR codimension 1 (cf. e.g. [4], p. 120). The *maximally complex* (or *Levi*) distribution of M

$$H(M) = Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$$

carries the complex structure

$$J(Z + \bar{Z}) = i(Z - \bar{Z}), \quad Z \in T_{1,0}(M),$$

where $i = \sqrt{-1}$. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and an overbar denotes complex conjugation. If M is oriented then the conormal bundle $H(M)^{\perp} := \{\omega \in T^*(M) : Ker(\omega) \supset H(M)\}$ (a line bundle over M) is trivial, and each global nowhere zero section $\theta \in \Gamma^{\infty}(H(M)^{\perp})$ is a *pseudohermitian structure* on M. Given two pseudohermitian structures θ and $\hat{\theta}$ there is a unique C^{∞} function $u : M \to \boldsymbol{R}\backslash\{0\}$ so that $\hat{\theta} = u\theta$. The *Levi form* is

$$L_{\theta}(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

A CR manifold is *nondegenerate* (respectively *strictly pseudoconvex*) if L_{θ} is nondegenerate (respectively positive-definite) for some θ.

A C^{∞} map $f : M \to N$ of CR manifolds is a *CR map* if $(d_x f)T_{1,0}(M)_x \subseteq T_{1,0}(N)_{f(x)}$, for any $x \in M$. A *CR isomorphism* is a C^{∞} diffeomorphism and a CR map, and $Aut_{CR}(M)$ is the group of all CR isomorphisms of M in itself. A *pseudohermitian transformation* is a CR isomorphism between two CR manifolds M, N on which pseudohermitian structures θ, θ_N have been fixed, so that $f^*\theta_N = a(f)\theta$, for some $a(f) \in \boldsymbol{R}\backslash\{0\}$. If $a(f) \equiv 1$ then f is *isopseudohermitian*.

Let M be a nondegenerate CR manifold. Then any pseudohermitian structure θ is a contact form on M, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. Once a contact form θ has been fixed, there is a globally defined nowhere zero vector field T on M, transverse to $H(M)$, determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$ (the *characteristic direction* of (M, θ)). Let $\pi_H : T(M) \to H(M)$ be the projection associated with the direct sum decomposition $T(M) = H(M) \oplus \boldsymbol{R}T$, i.e. $\pi_H(X) := X - \theta(X)T$. The *Webster metric* is the semi-Riemannian (i.e. nondegenerate, of constant index) metric

$$g_{\theta}(X, Y) = (d\theta)(\pi_H X, J\pi_H Y) + \theta(X)\theta(Y), \quad X, Y \in T(M).$$

If (r, s) is the signature of the Levi form $(r + s = n)$ then g_{θ} has signature $(2r + 1, 2s)$.

By a result of N. Tanaka, [24], and S. Webster, [25], for any nondegenerate CR manifold, on which a contact form θ has been fixed, there is a unique linear connection ∇ (the *Tanaka-Webster connection* of (M, θ)) so that 1) $H(M)$ is parallel with respect to ∇, 2) $\nabla J = 0$ and $\nabla g_{\theta} = 0$, 3) $T_{\nabla}(Z, W) = 0$ and $T_{\nabla}(Z, \overline{W}) = 2iL_{\theta}(Z, \overline{W})T$, for any $Z, W \in T_{1,0}(M)$, and 4) $\tau \circ J + J \circ \tau = 0$. Here T_{∇} is the torsion tensor field of ∇ and $\tau(X) := T_{\nabla}(T, X)$, $X \in T(M)$ (the *pseudohermitian torsion* of ∇).

If $\Omega \subset \boldsymbol{C}^{n+1}$ is a domain with smooth boundary, i.e. there is a $\boldsymbol{R}$-valued function $\rho \in C^{\infty}(U)$, for some open set $U \subseteq \boldsymbol{C}^{n+1}$ with $U \supset \bar{\Omega}$, so that $\Omega = \{z \in U : \rho(z) > 0\}$, $\partial\Omega = \{z \in U : \rho(z) = 0\}$, and $\nabla\rho(z) \neq 0$ for any $z \in \partial\Omega$, then

$\partial\Omega$ admits a natural CR structure, recalled in some detail in section 4. The pullback θ of $\frac{i}{2}(\bar{\partial}-\partial)\rho$, via $j:\partial\Omega\subset \boldsymbol{C}^{n+1}$, is a pseudohermitian structure on $\partial\Omega$. The bundle-theoretic recast of (13)–(14) in section 4 consists in observing that

$$T_{1,0}(M)=T_{1,0}(\boldsymbol{C}^{n+1})\cap[T(M)\otimes\boldsymbol{C}],\quad M=\partial\Omega,$$

and any CR manifold obtained this way is said to be *embedded*. Here $T_{1,0}(\boldsymbol{C}^{n+1})$ is the holomorphic tangent bundle over $\boldsymbol{C}^{n+1}$. A CR manifold is (locally) *embeddable* if there is a CR isomorphism of M (respectively of a neighborhood of each point of M) onto some embedded CR manifold.

Let $(M,T_{1,0}(M))$ be a nondegenerate CR manifold and θ a contact form on M. A *$(0,q)$-form* on M is a complex q-form η so that $T_{1,0}(M)\rfloor\eta=0$ and $T\rfloor\eta=0$. Let $\Lambda^{0,q}(M)\to M$ be the bundle of all $(0,q)$-forms on M. The *tangential Cauchy-Riemann operator* is the first order differential operator

$$\bar{\partial}_M:\Gamma^\infty(\Lambda^{0,q}(M))\to\Gamma^\infty(\Lambda^{0,q+1}(M)),\quad q\geq 0,$$

defined as follows. If η is a $(0,q)$-form then $\bar{\partial}_M\eta$ is the unique $(0,q+1)$-form on M coinciding with $d\eta$ on $T_{0,1}(M)\otimes\cdots\otimes T_{0,1}(M)$ ($q+1$ terms). Let $\bar{\partial}_M^*$ be the (formal) adjoint of $\bar{\partial}_M$ with respect to the L^2 inner product

$$(\varphi,\psi)=\int_M L_\theta^*(\varphi,\bar{\psi})\theta\wedge(d\theta)^n,$$

for any $\varphi,\psi\in\Omega^{0,q}(M)$ (at least one of compact support). The *Kohn-Rossi laplacian* is

$$\Box_M=\bar{\partial}_M\bar{\partial}_M^*+\bar{\partial}_M^*\bar{\partial}_M.$$

If $f:M\to N$ is an isopseudohermitian transformation then

$$\Box_M^f v=\Box_N v,\quad v\in C^\infty(N),\tag{5}$$

where $\Box_M^f v:=(\Box_M v^{f^{-1}})^f$ and $u^f:=u\circ f^{-1}$, $u\in C^\infty(M)$.

Let M be a strictly pseudoconvex CR manifold and θ a contact form with L_θ positive definite. A smooth curve $\gamma(t)$ in M satisfying the ODE

$$\left(\nabla_{d\gamma/dt}\frac{d\gamma}{dt}\right)_{\gamma(t)}=2cT_{\gamma(t)},\tag{6}$$

for some $c\in\boldsymbol{R}$ and any value of the parameter t is a *parabolic geodesic* on M. Let $x\in M$ and $W\in H(M)_x$. By standard theorems on ODEs, there is $\delta>0$ so that whenever $g_{\theta,x}(W,W)^{1/2}<\delta$ the unique solution $\gamma_{W,c}(t)$ to (6) of initial data (x, W) may be uniquely continued to an interval containing $t = 1$ and the map $\Psi_x : B(0,\delta) \subset T_x(M) \to M$ given by $\Psi_x(W + cT_x) := \gamma_{W,c}(1)$ (the *parabolic exponential map*) is a diffeomorphism of a sufficiently small neighborhood of $0 \in T_x(M)$ onto a neighborhood of $x \in M$. The terminology is justified by the fact that Ψ_x maps any parabola $t \mapsto tW + t^2 cT_x$ in the tangent space onto $\gamma_{W,c}$.

Let now $\{T_\alpha\}$ be a local orthonormal frame of $T_{1,0}(M)$, defined on a neighborhood U of x in M. It determines an isomorphism $\lambda_x : T_x(M) \to \boldsymbol{H}_n$ given by

$$\lambda_x(v) = (\theta_x^\alpha(v)e_\alpha, \theta_x(v)),$$

for any $v \in T_x(M)$. Here $\boldsymbol{H}_n = \boldsymbol{C}^n \times \boldsymbol{R}$ is the *Heisenberg group* (cf. e.g. [12], p. 434–435) and $\{\theta^\alpha\}$ is the frame of $T_{1,0}(M)^*$ determined by

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T_{\bar\beta}) = \theta^\alpha(T) = 0.$$

The resulting local coordinates $(z, t) := \lambda_x \circ \Psi_x^{-1}$, defined in some neighborhood of x, are the *pseudohermitian normal coordinates* at x, determined by $\{T_\alpha\}$. By Prop. 2.5 in [15], p. 313, these coordinates are also normal coordinates at x in the sense of G. B. Folland & E. M. Stein (cf. [12], p. 471–472). We shall need the following

Lemma 1. *Let M be a nondegenerate CR manifold and θ a contact form on M. Let $\sigma : M \to M$ be a CR automorphism so that $\sigma^*\theta = a(\sigma)\theta$ for some $a(\sigma) \in \boldsymbol{R}\setminus\{0\}$. Let $\gamma_{W,c}(s)$ be the solution to $\nabla_{d\gamma/dt}(d\gamma/dt) = 2cT \circ \gamma$ of initial data (η, W), $\eta \in M$, $W \in H(M)_\eta$. Then $\sigma \circ \gamma_{W,c} = \gamma_{W_{\sigma,a(\sigma)c}}$, where $W_\sigma := (d_\eta\sigma)W \in H(M)_{\sigma(\eta)}$, i.e. $\sigma \circ \gamma_{W,c}$ is the solution to $\nabla_{d\gamma/dt}(d\gamma/dt) = 2ca(\sigma)T \circ \gamma$ of initial data $(\sigma(\eta), W_\sigma)$.*

Proof. For each $y \in M$ and $X \in \mathscr{X}(M)$ consider

$$(\sigma_* X)_y := (d_{\sigma^{-1}(y)}\sigma)X_{\sigma^{-1}(y)}$$

(hence $\sigma_* : \mathscr{X}(M) \approx \mathscr{X}(M)$, an isomorphism) and set

$$\nabla^\sigma_X Y := (\sigma_*)^{-1}\nabla_{\sigma_* X}\sigma_* Y.$$

Then $\nabla^\sigma\theta = 0$. Using $\sigma^* g_\theta = a(\sigma)g_\theta + [a(\sigma)^2 - a(\sigma)]\theta \otimes \theta$ one may show that $\nabla^\sigma g_\theta = 0$. Also, it is easy to check that $\nabla^\sigma J = 0$. Next $\sigma_* T = a(\sigma)T$ so that $T_{\nabla^\sigma}(Z, W) = 0$, $T_{\nabla^\sigma}(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})T$ and $T_{\nabla^\sigma}(T, JX) + JT_{\nabla^\sigma}(T, X) = 0$, for any $Z, W \in T_{1,0}(M)$ and $X \in T(M)$. We may conclude that $\nabla^\sigma = \nabla$, the

Tanaka-Webster connection of (M,θ). Set $\gamma := \gamma_{W,c}$ and $\gamma_\sigma := \sigma \circ \gamma$. Then $\gamma_\sigma(0) = \sigma(\eta)$ and $(d\gamma_\sigma/ds)(0) = W_\sigma$. Finally

$$\nabla_{d\gamma_\sigma/ds}\frac{d\gamma_\sigma}{ds} = \sigma_*\nabla^\sigma_{d\gamma/ds}\frac{d\gamma}{ds} = \sigma_*\nabla_{d\gamma/ds}\frac{d\gamma}{ds} = \sigma_*(2cT\circ\gamma) = 2ca(\sigma)T\circ\gamma_\sigma,$$

hence $\gamma_\sigma = \gamma_{W_\sigma,a(\sigma)c}$, that is a pseudohermitian transformation σ maps the parabolic geodesic $\gamma_{W,c}$ into the parabolic geodesic $\gamma_{W_\sigma,a(\sigma)c}$. Q.e.d..

We have specified the behaviour (5) of the Kohn-Rossi laplacian on functions, with respect to isopseudohermitian transformations. In general, if φ is a $(0,q)$-form and $\sigma: M \to M$ a pseudohermitian transformation of a nondegenerate CR manifold then

$$\square_M(\sigma^*\varphi) = a(\sigma)\sigma^*\square_M\varphi. \tag{7}$$

Indeed, on one hand $\sigma^*\bar\partial_M\varphi = \bar\partial_M\sigma^*\varphi$, as it easily follows from the axioms defining $\bar\partial_M$. On the other hand,

$$\bar\partial^*_M\psi = (-1)^{q+1}(q+1)h^{\lambda\bar\mu}(\nabla_\lambda\psi_{\bar\alpha_1\cdots\bar\alpha_q\bar\mu})\theta^{\bar\alpha_1}\wedge\cdots\wedge\theta^{\bar\alpha_q}$$

for any $(0,q+1)$-form ψ on M, where covariant derivatives are meant with respect to the Tanaka-Webster connection of (M,θ). For instance, if φ is a $(0,1)$-form

$$\bar\partial^*_M\varphi = -h^{\lambda\bar\mu}\nabla_\lambda\varphi_{\bar\mu}$$

hence

$$\bar\partial^*_M(\sigma^*\varphi) = -h^{\lambda\bar\mu}\{T_\lambda((g_\sigma)^{\bar\nu}_{\bar\mu})(\varphi_{\bar\nu}\circ\sigma) + (g_\sigma)^{\bar\nu}_{\bar\mu}(g_\sigma)^\rho_\lambda[T_\rho(\varphi_{\bar\nu})\circ\sigma] - \Gamma^{\bar\nu}_{\lambda\bar\mu}(g_\sigma)^{\bar\rho}_{\bar\nu}(\varphi_{\bar\rho}\circ\sigma)\}$$

and the identity

$$\Gamma^{\bar\mu}_{\alpha\bar\beta}(g_\sigma)^{\bar\nu}_{\bar\mu} = T_\alpha((g_\sigma)^{\bar\nu}_{\bar\beta}) + (g_\sigma)^\mu_\alpha(g_\sigma)^{\bar\rho}_{\bar\beta}(\Gamma^{\bar\nu}_{\mu\bar\rho}\circ\sigma)$$

(a consequence of $\nabla = \nabla^\sigma$) lead to

$$\bar\partial^*_M(\sigma^*\varphi) = a(\sigma)(\bar\partial^*_M\varphi)\circ\sigma.$$

Q.e.d.. Here Γ^A_{BC} denote the Christoffel symbols (of ∇ with respect to $\{T_\alpha\}$) and $\sigma_*T_\alpha = (g_\sigma)^\beta_\alpha T_\beta$.

3. Complex Orbifolds

In this section we review the notion of complex orbifold (complex analytic V-manifold) and, given a complex orbifold X, we build an analogue of the holomorphic tangent bundle (of a complex manifold) which turns out to be a complex vector bundle $T_{1,0}(X)$ in the sense of J. Girbau & M. Nicolau, [13]. In particular (cf. Step 2 below) each fibre $\pi^{-1}(p)$ of the projection $\pi : T_{1,0}(X) \to X$ is shown to contain a natural vector space $T_{1,0}(X)_p$ [coinciding with $\pi^{-1}(p)$ when p is a regular point]. We show that the smooth functions $f : X \to \boldsymbol{C}$ satisfying $Z(\bar{f}) = 0$ for any section Z in $T_{1,0}(X)$ are precisely those whose local expressions $f \circ \varphi$ are holomorphic in Ω, for each l.u.s. $\{\Omega, G, \varphi\}$ of X (cf. 3) in Theorem 1). The weaker requirement that $Z(\bar{f}) = 0$ only for those sections Z with $Z_p \in T_{1,0}(X)_p$, $p \in X$, leads to the notion of a V-holomorphic function. Locally, i.e. on a fixed l.u.s. $\{\Omega, G, \varphi\}$, one deals with G-invariant C^1 functions satisfying (11). V-holomorphic functions are holomorphic except along the singular locus and exhibit a particular behaviour at singular points $x \in S$ (such that the isotropy group G_x acts on $\boldsymbol{C}^n$ with fixed points): each V-holomorphic function in Ω is holomorphic on a certain complex submanifold F_x passing through x (and there are complex local coordinates at x with respect to which F_x is an affine set in $\boldsymbol{C}^n$), cf. b) in Theorem 2.

Let X be a Hausdorff space and $U \subseteq X$ an open subset. A *local uniformizing system* (l.u.s.) *of dimension n of X over U* is a synthetic object $\{\Omega, G, \varphi\}$ consisting of a domain $\Omega \subseteq \boldsymbol{C}^n$, a finite subgroup $G \subset Aut(\Omega)$ of biholomorphisms of Ω in itself, and a continuous map $\varphi : \Omega \to U$ so that the induced map $\varphi_G : \Omega/G \to U$ is a homeomorphism. An *injection* of $\{\Omega, G, \varphi\}$ into $\{\Omega', G', \varphi'\}$ is a C^∞ map $\lambda : \Omega \to \Omega'$ so that λ is a biholomorphism of Ω onto some open subset of Ω' and $\varphi' \circ \lambda = \varphi$. The set $U = \varphi(\Omega)$ is the *support* of the l.u.s. $\{\Omega, G, \varphi\}$.

Given a family $\mathscr{F}$ of l.u.s.'s of dimension n of X, let $\mathscr{H}$ be the family of all supports of all l.u.s.'s in $\mathscr{F}$. Then $\mathscr{F}$ is a *defining family* for X if 1) for any $\{\Omega, G, \varphi\}$, $\{\Omega', G', \varphi'\} \in \mathscr{F}$ of supports U, U', if $U \subseteq U'$ then there is an injection λ of $\{\Omega, G, \varphi\}$ into $\{\Omega', G', \varphi'\}$, and 2) $\mathscr{H}$ is a basis of open sets for the topology of X. Two defining families $\mathscr{F}, \mathscr{F}'$ are *directly equivalent* if there is a third defining family $\mathscr{F}''$ so that $\mathscr{F} \cup \mathscr{F}' \subseteq \mathscr{F}''$. Also, $\mathscr{F}, \mathscr{F}'$ are *equivalent* if there is a set $\{\mathscr{F}_1, \dots, \mathscr{F}_r\}$ of defining families so that $\mathscr{F}_1 = \mathscr{F}, \mathscr{F}_r = \mathscr{F}''$, and $\mathscr{F}_i, \mathscr{F}_{i+1}$ are directly equivalent for each $1 \le i \le r-1$. A n-dimensional *complex orbifold* is a connected paracompact Hausdorff space X together with an equivalence class of defining families; as in ordinary complex manifold theory, it is customary to choose a defining family $\mathscr{F}$ in the class and refer to $(X, \mathscr{F})$ as a complex orbifold. Cf. I. Satake, [21], p. 261–262 (where complex orbifolds are referred to as complex analytic V-manifolds). Clearly, any complex orbifold, of complex dimension n as above, is a real $2n$-dimensional V-manifold (in the sense of [20], p. 359–360, or [3], p. 862–863).

Let $(X, \mathscr{F})$ be a V-manifold. By a result in [13], given l.u.s.'s $\{\Omega, G, \varphi\}$ and $\{\Omega', G', \varphi'\}$, of supports U, U' respectively, and given injections $\lambda, \mu : \Omega \to \Omega'$, if $U \subseteq U'$ then there is a unique element $\sigma_1' \in G'$ so that $\mu = \sigma_1' \circ \lambda$. As a corollary, with any injection $\lambda : \Omega \to \Omega'$ one may associate a group monomorphism $\eta : G \to G'$ so that $\lambda \circ \sigma = \eta(\sigma) \circ \lambda$, for any $\sigma \in G$. It is noteworthy that the existence of the monomorphism η is postulated in both [3] and the more recent [6] (and it is a merit of J. Girbau & M. Nicolau, [13], to have provided a remedy to this inadequacy). A point $p \in X$ is *singular* if there is $U \in \mathscr{H}$ with $p \in U$ and there is a l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{F}$ over U, and an element $x \in \Omega$ so that $\varphi(x) = p$ and $G_x \neq \{e\}$. Here $G_x := \{\sigma \in G : \sigma(x) = x\}$ is the isotropy group at x and $e = 1_\Omega$. By Prop. 1.5 in [13], p. 257, if $p \in U'$, where $U' \in \mathscr{H}$, and $\{\Omega', G', \varphi'\}$ is a l.u.s. of support U' then $G_x \approx G_y'$ (a group isomorphism) for any $y \in \Omega'$ with $\varphi'(y) = p$, hence the notion of singular point of X is unambigously defined. Set $S = \{x \in \Omega : G_x \neq \{e\}\}$ (a closed subset of Ω). Then $\Sigma := \bigcup_{\{\Omega, G, \varphi\} \in \mathscr{F}} \varphi(S)$ is the *singular locus* of X and $X_{reg} := X \backslash \Sigma$ its *regular part*. X_{reg} is an ordinary C^∞ manifold.

Let E be a connected paracompact Hausdorff space and $\pi : E \to X$ a continuous surjective map. Then (E, π, X) is a *vector bundle*, of standard fibre K^m, $K \in \{\boldsymbol{R}, \boldsymbol{C}\}$, if the following requirements are fulfilled

1) for any l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{F}$ there is a continuous map $\varphi_* : \Omega \times K^m \to E$ such that $\pi \circ \varphi_* = \varphi \circ \pi_\Omega$, where $\pi_\Omega(x, \zeta) = x$ for any $(x, \zeta) \in \Omega \times K^m$. Moreover

2) for any injection λ of $\{\Omega, G, \varphi\}$ into $\{\Omega', G', \varphi'\}$ there is a C^∞ map $g_\lambda :$ $\Omega \to GL(m, K)$ such that $g_e(x) = I_m$, the unit $m \times m$ matrix, for any $x \in \Omega$ and

i) $\{\Omega \times K^m, G_*, \varphi_*\}$ is a l.u.s. of dimension $d(K)m + N$ of E over $\pi^{-1}(U)$ (an open subset of E), where $G_* = \{\sigma_* : \sigma \in G\}$, with $\sigma_*(x, \zeta) := (\sigma(x), g_\sigma(x)\zeta)$ for any $(x, \zeta) \in \Omega \times K^m$, and $d(K) = \dim_{\boldsymbol{R}} K$, $N = \dim(X)$,

ii) the family of l.u.s.'s $\{\Omega \times K^m, G_*, \varphi_*\}$, obtained as $\{\Omega, G, \varphi\}$ ranges over $\mathscr{F}$, is a defining family for E, thus organizing E as a $(d(K)m + N)$-dimensional V-manifold of class C^∞,

iii) the map $\lambda_* : \Omega \times K^m \to \Omega' \times K^m$ given by $\lambda_*(x, \zeta) = (\lambda(x), g_\lambda(x)\zeta)$, is an injection of $\{\Omega \times K^m, G_*, \varphi_*\}$ into $\{\Omega' \times K^m, G_*', \varphi_*'\}$. Finally

3) for any pair of injections $\Omega \xrightarrow{\lambda} \Omega' \xrightarrow{\mu} \Omega''$ one requests that

$$g_\mu(\lambda(x))g_\lambda(x) = g_{\mu \circ \lambda}(x),$$

for any $x \in \Omega$. Cf. [13], p. 258. We underline the slight discrepancy in terminology: for a vector bundle of standard fibre K^m the fibre $\pi^{-1}(p)$ over a point $p \in X$ is (isomorphic to) K^m if and only if $p \in X_{reg}$ (and if $p \in \Sigma$ then $\pi^{-1}(p)$ has no natural vector space structure), cf. [13], p. 259.

A function $f : X \to \boldsymbol{C}$ on a V-manifold $(X, \mathscr{F})$ is *smooth* (of class C^∞) if $f_\Omega := f \circ \varphi$ is C^∞ for any $\{\Omega, G, \varphi\} \in \mathscr{F}$, and $\mathscr{E}(X)$ is the ring of all complex valued smooth functions on X. We shall prove the following

THEOREM 1. *For any complex orbifold $(X, \mathscr{F})$, of complex dimension n, there is a vector bundle $(T_{1,0}(X), \pi, X)$ so that*

1) *for any $p \in X$, if $p \in U \in \mathscr{H}$ and $\{\Omega, G, \varphi\} \in \mathscr{F}$ is a l.u.s. over U then $\pi^{-1}(x) \approx \boldsymbol{C}^n/G_x$ (a bijection) for any $x \in \Omega$ with $\varphi(x) = p$.*

2) *X_{reg} is a complex manifold and $T_{1,0}(X)|_{X_{reg}}$ its holomorphic tangent bundle. The singular locus of $T_{1,0}(X)$ (as a 4n-dimensional V-manifold) is contained in $\pi^{-1}(\Sigma)$.*

3) *For any section Z in $T_{1,0}(X)$ (i.e. any continuous map $Z : X \to T_{1,0}(X)$ so that $Z(p) \in \pi^{-1}(p)$ for any $p \in X$) and any $f \in \mathscr{E}(X)$ there is a (naturally defined) function $Z(f) : X \to \boldsymbol{C}$; if $Z(\bar{f}) = 0$ for all sections Z then f_Ω is holomorphic in Ω for any l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{F}$, and conversely.*

We organize the proof in several steps, as follows.

STEP 1. *The construction of $T_{1,0}(X)$.*

Define $g_\lambda : \Omega \to GL(n, \boldsymbol{C})$ by setting

$$g_\lambda(x)\zeta = \zeta^k \frac{\partial(z^j \circ \lambda)}{\partial z^k}(x) e_j,$$

where (z^j) are the natural complex coordinates on $\boldsymbol{C}^n$, and $\{e_j\}$ its canonical linear basis. Then $G_* = \{\sigma_* : \sigma \in G\}$ acts on $\Omega \times \boldsymbol{C}^n$ as a (finite) group of biholomorphisms. Set

$$\hat{T}_{1,0}(X) := \bigcup_{\{\Omega, G, \varphi\} \in \mathscr{F}} (\Omega \times \boldsymbol{C}^n)/G_*$$

(disjoint union). Then $\hat{T}_{1,0}(X)$ is a Hausdorff space, in a natural manner. We define an equivalence relation $\sim$ on $\hat{T}_{1,0}(X)$ as follows. Let $\hat{x}, \hat{y} \in \hat{T}_{1,0}(X)$. If $\hat{x}$ is the G_*-orbit $orb_{G_*}(x, \zeta)$ of some $(x, \zeta) \in \Omega \times \boldsymbol{C}^n$, for some l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{F}$, then we say that $\hat{x} \sim \hat{y}$ if there is an injection $\lambda : \Omega \to \Omega'$ to that

$$\hat{y} = orb_{G'_*}(\lambda(x), g_\lambda(x)\zeta).$$

If $(\sigma(x), g_\sigma(x)\zeta) \in \hat{x}$ is another representative of $\hat{x}$ then

$$\begin{aligned} orb_{G'_*}(\lambda(\sigma(x)), g_\lambda(\sigma(x))g_\sigma(x)\zeta) &= orb_{G'_*}(\eta(\sigma)\lambda(x), g_{\lambda\circ\sigma}(x)\zeta) \\ &= orb_{G'_*}[\eta(\sigma)_*(\lambda(x), g_\lambda(x)\zeta)] = orb_{G'_*}(\lambda(x), g_\lambda(x)\zeta), \end{aligned}$$

(where $\eta : G \to G'$ is the group monomorphism associated with λ) hence $\hat{x} \sim \hat{y}$ is well defined. Clearly $\sim$ is refexive and transitive. The only issue which needs a bit of care is the symmetry property. Note that, for any injection $\lambda : \Omega \to \Omega'$ the synthetic object $\{\lambda(\Omega), \eta(G), \psi\}$, where $\psi = \varphi'|_{\lambda(\Omega)}$, is a l.u.s. of support $U = \varphi(\Omega)$. Indeed $\eta(G)$ acts on $\lambda(\Omega)$ as a group of complex analytic transformations and ψ is $\eta(G)$-invariant. Moreover λ is equivariant hence it induces a homeomorphism $\lambda_G : \Omega/G \approx \lambda(\Omega)/\eta(G)$. The map $\psi_G : \lambda(\Omega)/\eta(G) \to U'$ (induced by ψ) correstricts to U and $\psi_G \circ \lambda_G = \varphi_G$ hence $\psi_G : \lambda(\Omega)/\eta(G) \approx U$ (a homeomorphism). Then $\hat{x} \sim \hat{y}$ yields $\hat{y} \sim \hat{x}$, as we may think of $(\lambda(x), g_\lambda(x)\zeta)$ as a representative of $\hat{y}$ with respect to the l.u.s. $\{\lambda(\Omega), \eta(G), \psi\}$ and rewrite $\hat{x}$ as

$$\hat{x} = orb_{G_*}(\mu(\lambda(x)), g_\mu(\lambda(x))g_\lambda(x)\zeta),$$

where μ is the injection $(\lambda : \Omega \to \lambda(\Omega))^{-1}$.

Next $T_{1,0}(X) := \hat{T}_{1,0}(X)/\sim$ carries the quotient topology and

$$\pi : T_{1,0}(X) \to X, \quad \pi([orb_{G_*}(x,\zeta)]) := \varphi(x),$$

is continuous (square brackets indicate classes mod $\sim$, i.e. $T_{1,0}(X) = \{[\hat{x}] : \hat{x} \in \hat{T}_{1,0}(X)\}$). The definition doesn't depend upon the choice of representatives; indeed, if $\hat{x} = orb_{G_*}(x,\zeta)$ and $\hat{y} \in [\hat{x}]$ then $\hat{y} = orb_{G'_*}(\lambda(x), g_\lambda(x)\zeta)$ for some injection $\lambda : \Omega \to \Omega'$, and $\varphi'(\lambda(x)) = \varphi(x)$.

We wish to show that $(T_{1,0}(X), \pi, X)$ is a vector bundle of standard fibre $\boldsymbol{C}^n$. To this end, let $\varphi_* : \Omega \times \boldsymbol{C}^n \to T_{1,0}(X)$ be the (continuous) map given by $\varphi_*(x,\zeta) = [orb_{G_*}(x,\zeta)]$. Then $\pi \circ \varphi_* = \varphi \circ \pi_\Omega$. Also φ_* is G_*-invariant and the induced map $(\varphi_*)_{G_*} : (\Omega \times \boldsymbol{C}^n)/G_* \to T_{1,0}(X)$ is injective. Finally, it is straightforward that $\lambda_*(x,\zeta) = (\lambda(x), g_\lambda(x)\zeta)$ is an injection of $\{\Omega \times \boldsymbol{C}^n, G_*, \varphi_*\}$ into $\{\Omega' \times \boldsymbol{C}^n, G'_*, \varphi'_*\}$.

Let $p \in X$ be an arbitrary point (eventually singular) and $U \in \mathscr{H}$ so that $p \in U$. Let $\{\Omega, G, \varphi\} \in \mathscr{F}$ be a l.u.s. of support U and $x \in \Omega$ so that $\varphi(x) = p$. Let $\{\Omega_*, G_*, \varphi_*\}$ be a l.u.s. of $T_{1,0}(X)$ corresponding to $\{\Omega, G, \varphi\}$ as above, where $\Omega_* = \Omega \times \boldsymbol{C}^n$. Then $\pi(\varphi_*(x,\zeta)) = \varphi(x) = p$ hence $\varphi_*(x,\zeta) \in \pi^{-1}(p)$ for any $\zeta \in \boldsymbol{C}^n$. There is a natural action of G_x on $\boldsymbol{C}^n$ given by $(\sigma,\zeta) \mapsto g_\sigma(x)\zeta$. We may consider the map

$$\boldsymbol{C}^n/G_x \to \pi^{-1}(p), \quad [\zeta] \mapsto \varphi_*(x,\zeta), \tag{8}$$

where $[\zeta]$ is the G_x-orbit of ζ. If $[\zeta] = [\xi]$ then $\xi = g_\sigma(x)\zeta$ for some $\sigma \in G$ and

$$\varphi_*(x,\xi) = \varphi_*(\sigma(x), g_\sigma(x)\zeta) = \varphi_*(\sigma_*(x,\zeta)) = \varphi_*(x,\zeta),$$

i.e. (8) is well defined. To see that (8) is injective, let $\varphi_*(x,\xi) = \varphi_*(x,\zeta)$. As $\{\Omega_*, G_*, \varphi_*\}$ is a l.u.s., there is $\sigma \in G$ so that $(x,\zeta) = \sigma_*(x,\xi)$ hence $\sigma \in G_x$ and $g_\sigma(x)\xi = \zeta$, i.e. ξ, ζ are G_x-equivalent. To see that (8) is surjective, let $f \in \pi^{-1}(p)$. As φ_* induces a bijection $\Omega_*/G_* \approx \pi^{-1}(U)$ there is $\tilde{f} = (y,\xi) \in \Omega_*$ so that $\varphi_*(\tilde{f}) = f$. Then

$$\varphi(x) = p = \pi(f) = \pi(\varphi_*(\tilde{f})) = \varphi(\pi_\Omega(\tilde{f})) = \varphi(y),$$

hence there is $\sigma \in G$ so that $y = \sigma(x)$. At this point, set $\tilde{f}_* := (\sigma^{-1})_*\tilde{f} \in \Omega_*$. Then $\varphi_*(\tilde{f}_*) = f$ and $\tilde{f}_*$ is an element of the form (x,ζ) with $\zeta = g_{\sigma^{-1}}(\sigma(x))\xi \in [\xi]$, so we are done.

Step 2. *The image $T_{1,0}(X)_p$ of $T_{1,0}(\Omega)_{G_x} := \{v \in T_{1,0}(\Omega)_x : (d_x\sigma)v = v,$ $\forall \sigma \in G_x\}$ via the map $T_{1,0}(\Omega) \approx \Omega \times \boldsymbol{C}^n \xrightarrow{\varphi_*} T_{1,0}(X)$ depends only on p (i.e. doesn't depend upon the choice of $\{\Omega, G, \varphi\} \in \mathscr{F}$ and $x \in \Omega$ with $\varphi(x) = p$) and $T_{1,0}(X)_p$ has a natural $\boldsymbol{C}$-vector space structure so that*

$$\dim_{\boldsymbol{C}} T_{1,0}(X)_p = \dim_{\boldsymbol{C}} \bigcap_{\sigma \in G_x} Ker[g_\sigma(x) - I_n] \tag{9}$$

Let $p \in U' \in \mathscr{H}$ and $\{\Omega', G', \varphi'\} \in \mathscr{F}$ over U', and consider $x' \in \Omega'$ so that $\varphi'(x') = p$. As $\mathscr{H}$ is a basis of open sets for the topology of X, let $V \subseteq U \cap U'$ with $p \in V \in \mathscr{H}$ and let $\{D, H, \psi\} \in \mathscr{F}$ be a l.u.s. over V. Then there exist injections $\lambda : D \to \Omega$ and $\lambda' : D \to \Omega'$. Let $y \in D$ so that $\psi(y) = p$. We wish to show that $\{\varphi_*(x,\zeta) : \zeta \in (\boldsymbol{C}^n)_{G_x}\}$ depends only on p, where

$$(\boldsymbol{C}^n)_{G_x} := \{\zeta \in \boldsymbol{C}^n : g_\sigma(x)\zeta = \zeta, \ \forall \sigma \in G_x\}.$$

As $\varphi(\lambda(y)) = \varphi(x)$, there is $\sigma \in G$ with $\lambda(y) = \sigma(x)$ hence

$$(\sigma(x), g_\lambda(y)\xi) = \sigma_*(x, g_{\sigma^{-1}\circ\lambda}(y)\xi)$$

and we have

$$\{\psi_*(y,\xi) : \xi \in (\boldsymbol{C}^n)_{H_y}\} = \{\varphi_*(\lambda(y), g_\lambda(y)\xi) : \xi \in (\boldsymbol{C}^n)_{H_y}\}$$
$$= \{\varphi_*(x, g_{\sigma^{-1}\circ\lambda}(y)\xi) : \xi \in (\boldsymbol{C}^n)_{H_y}\}$$

At this point, it suffices to show that the map

$$(\boldsymbol{C}^n)_{H_y} \to (\boldsymbol{C}^n)_{G_x}, \quad \xi \mapsto g_{\sigma^{-1}\circ\lambda}(y)\xi, \tag{10}$$

is a well defined bijection. $\sigma^{-1} \circ \lambda : D \to \Omega$ is an injection. Let $\eta_\sigma : H \to G$ be the corresponding group monomorphism. As $\varphi(x) = p = \psi(y)$, $\eta_\sigma : H_y \to G_x$ is an isomorphism (cf. Prop. 1.5 in [13], p. 257). Given $\tau \in G_x$ let $\rho \in H_y$ so that $\eta_\sigma(\rho) = \tau$. Then

$$g_\tau(x) g_{\sigma^{-1}\circ\lambda}(y)\xi = g_{\tau\circ\sigma^{-1}\circ\lambda}(y)\xi = g_{\eta_\sigma(\rho)\circ\sigma^{-1}\circ\lambda}(y)\xi$$
$$= g_{(\sigma^{-1}\circ\lambda)\circ\rho}(y)\xi = g_{\sigma^{-1}\circ\lambda}(y) g_\rho(y)\xi = g_{\sigma^{-1}\circ\lambda}(y)\xi,$$

hence (10) is well defined. Also, a similar computation shows that

$$g_{\sigma^{-1}\circ\lambda}(y)(\boldsymbol{C}^n)_{H_y} = (\boldsymbol{C}^n)_{G_x}$$

and (10) is clearly injective. The same proof applies to λ', so we are done.

Note that $T_{1,0}(X)_p$ is a $\boldsymbol{C}$-linear space [with $\alpha\varphi_*(x,\zeta) + \beta\varphi_*(x,\xi) := \varphi_*(x, \alpha\zeta + \beta\xi)$ (while the same operation on the image of the whole $\boldsymbol{C}^n/G_x$ is not well defined)]. To see that X_{reg} is a complex manifold we need to review the differentiable structure of X_{reg} in some detail. Let $\{D, H, \psi\} \in \mathscr{F}$ be a l.u.s. of X over $V \in \mathscr{H}$. Set $\Omega = \psi^{-1}(U)$ where $U := V \cap X_{reg}$. Then $\sigma \in H \Rightarrow \sigma(\Omega) = \Omega$. [Indeed, let $x \in \Omega$ and $p := \psi(x)$. Then $p \in U$ and $U \subseteq X \backslash \Sigma$ hence each point of $\psi^{-1}(p)$ has a trivial isotropy group. Yet $\sigma(x) \in \psi^{-1}(p)$ hence $G_{\sigma(x)} = \{e\}$. It follows that $\psi(\sigma(x)) \in X\backslash\Sigma$ and $\psi(\sigma(x)) = \psi(x) = p \in U$, i.e. $\sigma(x) \in \Omega$, q.e.d.]. Set $G := \{\sigma|_\Omega : \sigma \in H\}$ and $\varphi := \psi|_\Omega$. Then $\{\Omega, G, \varphi\}$ is a l.u.s. of X_{reg} over U. As $\{D, H, \psi\}$ runs over $\mathscr{F}$, the l.u.s.'s $\{\Omega, G, \varphi\}$ form a defining family of X_{reg}, hence X_{reg} is a $2n$-dimensional V-manifold. To see that it actually possesses a C^∞ manifold structure note first that G acts freely on Ω, as a mere consequence of definitions. Let $y \in \Omega$. Then $\sigma(y) \neq y$ for any $\sigma \in G\backslash\{e\}$ (as $G_y = \{e\}$) hence there is an open neighborhood Ω_σ of y in Ω so that $\sigma(\Omega_\sigma) \cap \Omega_\sigma = \varnothing$. Set $D_y := \bigcap_{\sigma \in G\backslash\{e\}} \Omega_\sigma$. As G is finite D_y is open, $y \in D_y \subseteq \Omega$, and $\sigma(D_y) \cap D_y = \varnothing$ for any $\sigma \in G\backslash\{e\}$, hence G acts on Ω as a properly discontinuous group of C^∞ diffeomorphisms. Thus Ω/G is a real $2n$-dimensional C^∞ manifold, and each $U \in \mathscr{H}_{reg} := \{V \cap (X\backslash\Sigma) : V \in \mathscr{H}\}$ inherits a manifold structure via φ_G. Once Ω/G is organized as a manifold, the projection $\Omega \to \Omega/G$ is a local diffeomorphism and its local inverses form a C^∞ atlas $\mathscr{F}_\Omega$. Then $\mathscr{F}_U := \{\chi \circ \varphi_G^{-1} : \chi \in \mathscr{F}_\Omega\}$ is an atlas on U and $\mathscr{F}_{reg} := \bigcup_{U \in \mathscr{H}_{reg}} \mathscr{F}_U$ an atlas on X_{reg}. Also $\varphi : \Omega \to U$ is differentiable (and φ_G a diffeomorphism). As Ω and U are locally diffeomorphic there is a unique complex structure on U so that $T_{1,0}(U)_{\varphi(x)} = (d_x\varphi)T_{1,0}(\Omega)_x$, for any $x \in \Omega$. Let $p \in X_{reg}$ and $U, U' \in \mathscr{H}_{reg}$ so that $p \in U \cap U'$. We need to show that $T_{1,0}(U)_p = T_{1,0}(U')_p$, i.e. the complex structures $\{T_{1,0}(U) : U \in \mathscr{H}_{reg}\}$ glue up to a globally defined complex structure on X_{reg}. To this end let $V \in \mathscr{H}_{reg}$ so that $p \in V \subseteq U \cap U'$ and $\{D, H, \psi\}$ a l.u.s. of X_{reg} over V. Let $\lambda : D \to \Omega$ and

$\lambda' : D \to \Omega'$ be injections and let $y \in D$ so that $\psi(y) = p$. Set $x := \lambda(y) \in \Omega$ and $x' := \lambda'(y) \in \Omega'$. Then

$$T_{1,0}(U)_p = (d_y\psi)T_{1,0}(D)_y = T_{1,0}(U')_p,$$

as both λ, λ' are holomorphic maps and $\varphi \circ \lambda = \psi = \varphi' \circ \lambda'$. So X_{reg} is a complex manifold, in a natural way. Next $\pi^{-1}(X_{reg}) = T_{1,0}(X_{reg})$ because of the isomorphism

$$T_{1,0}(X)_p \to T_{1,0}(X_{reg})_p, \quad \varphi_*(x,\zeta) \mapsto (d_x\varphi)\zeta^j \frac{\partial}{\partial z^j}\bigg|_x, \quad p \in U \in \mathscr{H}_{reg}.$$

If v is a singular point of $T_{1,0}(X)$ with $p := \pi(v)$, there is $U \in \mathscr{H}$ with $p \in U$, and there is a l.u.s. $\{\Omega, G, \varphi\}$ over U so that $(G_*)_{(x,\zeta)} \neq \{e_*\}$, for some $(x,\zeta) \in \Omega \times \boldsymbol{C}^n$. That is $\sigma_*(x,\zeta) = (x,\zeta)$ for some $\sigma \in G\backslash\{e\}$, hence $\sigma(x) = x$, i.e. $G_x \neq \{e\}$. It follows that $p \in \Sigma$, i.e. the singular locus of $T_{1,0}(X)$ projects on Σ. Statement 2 in Theorem 1 is proved.

It remains that we prove 3. Let $Z : X \to T_{1,0}(X)$ be a continuous map so that $\pi \circ Z = 1_X$. Let $f \in \mathscr{E}(X)$ and $p \in X$. Let $U \in \mathscr{H}$ so that $p \in U$ and let $\{\Omega, G, \varphi\} \in \mathscr{F}$ over U. Let $x \in \Omega$ so that $\varphi(x) = p$ and set

$$Z(f)_p := \sum_{j=1}^{n} \zeta^j \frac{\partial f_\Omega}{\partial z^j}(x),$$

where $[\zeta] \in \boldsymbol{C}^n/G_x$ corresponds to $Z_p \in \pi^{-1}(p)$ under the bijection $\boldsymbol{C}^n/G_x \approx \pi^{-1}(p)$.

Step 3. *$Z(f)_p$ is well defined.*

If $[\xi] = [\zeta]$ then $\xi = g_\sigma(x)\zeta$ for some $\sigma \in G_x$ and then

$$\xi^j \frac{\partial f_\Omega}{\partial z^j}(x) = g_\sigma(x)^j_k \zeta^k \frac{\partial f_\Omega}{\partial z^j}(x) = \zeta^k \frac{\partial (f_\Omega \circ \sigma)}{\partial z^k}(x).$$

If another open neighborhood $U' \in \mathscr{H}$ of p is used, let $\{\Omega', G', \varphi'\}$ over U' and $x' \in \Omega'$ with $\varphi'(x') = p$. Then, consider $p \in V \subseteq U \cap U'$ and $\{D, H, \psi\}$ over V, and two injections $\lambda : D \to \Omega$, $\lambda' : D \to \Omega'$. Let $y \in D$ with $\psi(y) = p$. Let $[\zeta] \in \boldsymbol{C}^n/G_x$ and $[\zeta'] \in \boldsymbol{C}^n/G'_{x'}$ correspond to Z_p. If $[\xi] \in \boldsymbol{C}^n/H_y$ corresponds to Z_p then

$$\begin{aligned}\varphi_*(x,\zeta) = Z_p = \psi_*(y,\xi) &= [orb_{H_*}(y,\xi)]\\ &= [orb_{G_*}(\lambda(y), g_\lambda(y)\xi)] = \varphi_*(\lambda(y), g_\lambda(y)\xi),\end{aligned}$$

hence there is $\tau \in G$ so that

$$\tau_*(x, \zeta) = (\lambda(y), g_\lambda(y)\xi),$$

i.e. $\tau(x) = \lambda(y)$ and $\zeta = g_{\tau^{-1}}(\tau(x))g_\lambda(y)\xi$. As $f_\Omega \circ \lambda = f_D$

$$\zeta^j \frac{\partial f_\Omega}{\partial z^j}(x) = g_{\tau^{-1}}(\tau(x))^j_k g_\lambda(y)^k_\ell \xi^\ell \frac{\partial f_\Omega}{\partial z^j}(x) = \frac{\partial(f_\Omega \circ \tau^{-1})}{\partial z^k}(\tau(x))g_\lambda(y)^k_\ell \xi^\ell =$$

(as f_Ω is G-invariant and $\tau(x) = \lambda(y)$)

$$= \frac{\partial(f_\Omega \circ \lambda)}{\partial z^\ell}(y)\xi^\ell = \xi^\ell \frac{\partial f_D}{\partial z^\ell}(y).$$

The same argument holds for λ', hence

$$\zeta'^j \frac{\partial f_{\Omega'}}{\partial z^j}(x') = \zeta^j \frac{\partial f_\Omega}{\partial z^j}(x),$$

and Step 3 is proved. Let $Z_p \in \pi^{-1}(p)$ correspond to $[e_j] \in \boldsymbol{C}^n/G_x$, with $\varphi(x) = p$. Then $Z(\bar{f})_p = 0$ yields $(\partial f_\Omega/\partial \bar{z}^j)(x) = 0$, i.e. $f \in \mathcal{O}(\Omega)$. Theorem 1 is completely proved.

Throughout, if Y is a complex manifold, $\mathcal{O}(Y)$ denotes the space of all holomorphic functions on Y. The last statement in Theorem 1 shows that the requirement $Z(\bar{f}) = 0$ for all sections Z in $T_{1,0}(X)$ is too restrictive for our purposes. In the sequel, we restrict ourselves to sections Z such that $Z_p \in T_{1,0}(X)_p = \{\varphi_*(x, \zeta) : \zeta \in (\boldsymbol{C}^n)_{G_x}\}$, as mentioned in the Introduction. Locally, we are led to a new notion, termed *V-holomorphic function*. Let $\Omega \subseteq \boldsymbol{C}^n$ be a domain and $G \subset Aut(\Omega)$ a finite group of biholomorphisms. A C^1 function $f : \Omega \to \boldsymbol{C}$ is called *V-holomorphic* if it is G-invariant and

$$\sum_{j=1}^n \bar{\zeta}^j \frac{\partial f}{\partial \bar{z}^j}(x) = 0 \tag{11}$$

for any $x \in \Omega$ and any $\zeta \in (\boldsymbol{C}^n)_{G_x}$. Let $\mathcal{O}_V(\Omega)$ be the space of all V-holomorphic functions in Ω. Let $\mathcal{O}_G(\Omega)$ consist of all G-invariant functions $f \in \mathcal{O}(\Omega)$. Then $\mathcal{O}_G(\Omega) \subseteq \mathcal{O}_V(\Omega) \subseteq \mathcal{O}_G(\Omega \backslash S)$. Note that the requirement (11) is empty at the points of $C := \{x \in \Omega : (\boldsymbol{C}^n)_{G_x} = (0)\} \subseteq S$. When $n = 1$, $\mathcal{O}_V(\Omega) \subseteq \mathcal{O}_G(\Omega \backslash C)$.

The following result describes the local structure of S and the behaviour of V-holomorphic functions at the points of $S \backslash C$.

THEOREM 2. *For any $x \in S$ there is a neighborhood D of x in Ω so that*
1) *$D \cap S$ is a finite union of complex submanifolds of Ω of dimension $< n$.*
2) *For any $y \in D$, G_y is a subgroup of G_x.* 3) *If $x \in S \backslash C$ there is a complex submanifold $F_x \subset D$ passing through x so that* a) *for each G-invariant function $f : \Omega \to \boldsymbol{C}$, f satisfies* (11) *at x if and only if the trace of f on F_x is holomorphic at x. Moreover* b) *$F_x \subset \Omega \backslash C$ and if $f \in \mathcal{O}_V(\Omega)$ then $f|_{F_x} \in \mathcal{O}(F_x)$.*

PROOF. Let $x \in S$ and set

$$w^j := \frac{1}{|G_x|} \sum_{\sigma \in G_x} g_{\sigma^{-1}}(x)^j_k (z^k \circ \sigma)$$

(for a set A, $|A|$ denotes its cardinality). Then $(\partial w^j / \partial z^k)(x) = \delta^j_k$ hence there is an open neighborhood V of x in Ω so that $\Phi := (w^1, \dots, w^n) : V \to \boldsymbol{C}^n$ is a biholomorphism on its image. Let $\sigma \in G \backslash G_x$. Then $\sigma(x) \neq x$ hence there is an open neighborhood Ω_σ of x in V so that $\sigma(\Omega_\sigma) \cap \Omega_\sigma = \emptyset$. Set $D_0 := \bigcap_{\sigma \in G \backslash G_x} \Omega_\sigma$ and $D := \bigcap_{\sigma \in G_x} \sigma(D_0)$. As G is finite D_0, and then D, are open. What we just built is an open neighborhood D of x in V so that i) $\sigma(D) \subseteq D$ for any $\sigma \in G_x$ and ii) $\sigma(D) \cap D = \emptyset$ for any $\sigma \in G \backslash G_x$. The first statement in Theorem 2 is a complex analogue of Prop. 1.1 in [13], p. 251–252. For each $\tau \in G_x$ set

$$F_\tau = \{ y \in D : \tau(y) = y \}.$$

Note that $w^j \circ \tau = g_\tau(x)^j_k \circ w^k$. Consequently

$$\Phi(F_\tau) = \Phi(D) \cap Ker[g_\tau(x) - I_n],$$

hence F_τ is a complex submanifold of D, of complex dimension $< n$. Next $S \cap D = Y_x$, where

$$Y_x := \bigcup_{\tau \in G_x \backslash \{e\}} F_\tau.$$

To prove the third statement note that $\bar{\zeta}^j (\partial / \partial \bar{z}^j)_x \in T_x(F_\tau) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ if and only if $\zeta \in Ker[g_\tau(x) - I_n]$. Indeed, if $\rho^j_\sigma(z) := g_\sigma(x)^j_k w^k - w^j$, $\sigma \in G_x$, then

$$\left(\zeta^k \frac{\partial}{\partial z^k} \bigg|_x \right) (\rho^j_\sigma) = \zeta^k [g_\sigma(x)^j_\ell - \delta^j_\ell] \frac{\partial w^\ell}{\partial z^k}(x) = \zeta^k g_\sigma(x)^j_k - \zeta^j.$$

Set

$$F_x := \bigcap_{\tau \in G_x \backslash \{e\}} F_\tau.$$

If $x \in S \backslash C$ then F_x is a complex manifold of dimension $\dim_C(C^n)_{G_x}$. Let us prove (b). To this end, let $y \in F_x$ and $D' \subset V'$ as in the first part of the proof (got by replacing x by y). Then $F'_\sigma \supseteq D' \cap F_x \ni y$ for any $\sigma \in G_y \backslash \{e\}$ hence (by a dimension argument)

$$T_{1,0}(F'_y)_y = T_{1,0}(F_x)_y \approx (C^n)_{G_x} \neq (0). \tag{12}$$

Thus $(C^n)_{G_y} \approx T_{1,0}(F'_y)_y \neq (0)$, a fact which yields $y \in \Omega \backslash C$, i.e. $F_x \subset \Omega \backslash C$. Finally, let $f \in \mathcal{O}_V(\Omega)$. Then $f|_{F'_y}$ is holomorphic in y hence (by (12)) $f|_{F_x}$ is holomorphic in y. Q.e.d..

If $(X, \mathscr{F})$ is a complex orbifold, a function $f \in C^1(X)$ (i.e. a continuous function $f : X \to C$ so that $f_\Omega \in C^1(\Omega)$ for each l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{F}$) is *V-holomorphic* if each f_Ω is V-holomorphic in Ω. In the sequel, we shall study traces of such functions on smooth real hypersurfaces.

4. Real Hypersurfaces

The purpose of this section is to discuss traces of V-holomorphic functions on real hypersurfaces $M \subset \Omega$ preserved by G. This situation is realizable (by a result of B. Coupet & A. Sukhov, [9], as detailed below) when M is the boundary of a C^ω bounded pseudoconvex domain. We are led to a generalization of the notion of CR function, i.e. the solutions to (16). These are CR everywhere except at singular points and exhibit, at a singular point x, the behaviour mentioned in the Introduction (i.e. are CR functions along a CR submanifold passing through x, of smaller CR dimension).

Let $D \subset C^n$ be a bounded pseudoconvex domain with real analytic boundary ∂D and $H \subset Aut(D)$ a finite (hence compact) group of automorphisms of D. By a result of B. Coupet & A. Sukhov, [9], there is a domain Ω so that $\bar{D} \subset \Omega$ and each $\tau \in H$ extends holomorphically on Ω as an automorphism of Ω. Let $G_{\partial D}$ consist of all $\tilde{\tau}|_{\partial D}$ for $\tau \in H$ and some holomorphic extension $\tilde{\tau} \in Aut(\Omega)$ of τ. By the identity principle for holomorphic functions $G_{\partial D}$ is a well defined finite group of CR automorphisms of ∂D. In general, let $\Omega \subseteq C^n$ be a domain, $G \subset Aut(\Omega)$ a finite group of biholomorphims, and $M \subset \Omega$ an embedded real hypersurface such that $\sigma(M) = M$ for each $\sigma \in G$. Set $G_M := \{\sigma|_M : \sigma \in G\}$ and $S_M := \{x \in M : (G_M)_x \neq \{1_M\}\}$. Then $S_M = M \cap S$. For any $x \in M$ there is a neighborhood U of x in C^n and a function $\rho \in C^\infty(U)$ such that $M \cap U = \{z \in U : \rho(z) = 0\}$ and $\nabla\rho(z) \neq 0$ for any $z \in M$. The Cauchy-Riemann equations in C^n

induce on M an overdetermined system of PDEs with smooth complex valued coefficients

$$\bar{L}_\alpha u(z) \equiv \sum_{j=1}^{n} a_\alpha^j(z) \frac{\partial u}{\partial \bar{z}^j} = 0, \quad 1 \le \alpha \le n-1, \tag{13}$$

(the *tangential Cauchy-Riemann equations*) $z \in V$, with $V \subseteq M \cap U$ open. Here

$$\sum_{j=1}^{n} \bar{a}_\alpha^j(z) \frac{\partial \rho}{\partial z^j} = 0, \quad 1 \le \alpha \le n-1, \tag{14}$$

for any $z \in V$, i.e. L_α are purely tangential first order differential operators (tangent vector fields on M). Also

$$[L_\alpha, L_\beta] = C_{\alpha\beta}^\gamma(z) L_\gamma \tag{15}$$

for some complex valued C^∞ functions $C_{\alpha\beta}^\gamma$ on V. At each point $z \in V$ the $L_{\alpha,z}$'s span a complex $(n-1)$-dimensional subspace $T_{1,0}(M)_z$ of the complexified tangent space $T_z(M) \otimes_{\boldsymbol{R}} \boldsymbol{C}$. The bundle $T_{1,0}(M) \to M$ is the *CR* structure of M. A C^1 function $u: M \to \boldsymbol{C}$ is a *CR* function if $\bar{Z}(u) = 0$ for any $Z \in T_{1,0}(M)$. Locally, a *CR* function is a solution of (13). $G \subset Aut(\Omega)$ yields $G_M \subset Aut_{CR}(M)$ hence

$$(d_x\tau) L_{\alpha,x} = \sum_{\beta=1}^{n-1} \tau_\alpha^\beta(x) L_{\beta,\tau(x)}, \quad x \in V,$$

for each $\tau \in G_M$ and some (unique) system of C^∞ functions $\tau_\alpha^\beta : V \to \boldsymbol{C}$. For each $\tau \in G_M$ let $g_{M,\tau} : V \to GL(n-1, \boldsymbol{C})$ be given by $g_{M,\tau}(x)\zeta = \tau_\beta^\alpha(x)\zeta^\beta e_\alpha$ for any $\zeta \in \boldsymbol{C}^{n-1}$. Set

$$(\boldsymbol{C}^{n-1})_{(G_M)_x} = Ker[g_{M,\tau}(x) - I_{n-1}]$$

and $C_M = \{x \in M : (\boldsymbol{C}^{n-1})_{(G_M)_x} = (0)\} \subseteq S_M$. We need the following

Lemma 2. *The trace $u = f|_M$ of any V-holomorphic function $f \in \mathcal{O}_V(\Omega)$ satisfies*

$$\sum_{\alpha=1}^{n-1} \bar{\xi}^\alpha L_{\bar{\alpha},x} u = 0 \tag{16}$$

for any $x \in V$ and any $\xi \in (\boldsymbol{C}^{n-1})_{(G_M)_x}$. In particular u is a CR function on $M \setminus S_M$ (and if $n = 2$ then u is CR on $M \setminus C_M$).

PROOF. Let $\zeta \in (C^{n-1})_{(G_M)_x}$, $x \in V$, and set $\zeta^j = a_\alpha^j(x)\xi^\alpha$. Then

$$a_\alpha^j(x) g_\sigma(x)_j^k = \tau_\alpha^\beta(x) a_\beta^k(x)$$

yields $\zeta \in (C^n)_{G_x}$ hence

$$0 = \bar{\zeta}^j \frac{\partial f}{\partial \bar{z}^j}(x) = \bar{\xi}^\alpha L_{\bar{\alpha},x} u. \qquad \text{Q.e.d..}$$

In view of the result in [18], it is an open problem whether the real analytic solutions to (16) extend to V-holomorphic functions on a neighborhood of M in Ω (provided $M \in C^\omega$).

THEOREM 3. *For any $x \in S_M$ there is an open neighborhood D of x in Ω such that $S_M \cap D$ is a finite union of CR manifolds of CR dimension $< n-1$. For any $y \in V := M \cap D$, $(G_M)_y$ is a subgroup of $(G_M)_x$. If $x \in S_M \backslash C_M$ there is a CR manifold $F_{M,x}$ such that a C^1 function $u : V \to C$ satisfies* (16) *for any $\xi \in (C^{n-1})_{(G_M)_x}$ if and only if the trace of u on $F_{M,x}$ is CR at x.*

The proof of Theorem 3 is similar to that of Theorem 2, so we only emphasize on the main steps. As $x \in S_M \subseteq S$, let D be a neighborhood of x in Ω as in (the proof of) Theorem 2. By eventually shrinking D let (u^a) be local coordinates on $V = M \cap D$ and set

$$v^a = \frac{1}{|G_x|} \sum_{\tau \in (G_M)_x} h_{\tau^{-1}}(x)_b^a (u^b \circ \tau), \quad 1 \le a \le 2n-1,$$

where $h_\tau(x) = [(\partial(u^a \circ \tau)/\partial u^b)(x)]$. Then $(\partial v^a/\partial u^b)(x) = \delta_b^a$ hence $\phi = (v^1, \dots, v^{2n-1})$ is a C^∞ diffeomorphism of (a perhaps smaller open neighborhood of x in) V onto its image. Given $\tau \in (G_M)_x \backslash \{1_M\}$ set $F_{M,\tau} = \{y \in V : \tau(y) = y\}$. Then $\phi(F_{M,\tau}) = \phi(V) \cap Ker[h_\tau(x) - I_{2n-1}]$ hence $F_{M,\tau}$ is a manifold (of dimension $\dim_R Ker[h_\tau(x) - I_{2n-1}] < 2n-1$ if $\tau \ne 1_M$) and $S_M \cap V = \bigcup_{\tau \in (G_M)_x \backslash \{1_M\}} F_{M,\tau}$. Note that $F_{M,\tau} = M \cap F_\sigma$ for any $\sigma \in G_x$ with $\sigma|_M = \tau$. Hence $F_{M,\tau}$ is a CR submanifold of (the complex manifold) F_σ. If $x \in S_M \backslash C_M \subseteq S \backslash C$ then set $F_{M,x} = \bigcap_{\tau \in (G_M)_x \backslash \{1_M\}} F_{M,\tau}$. Then $F_{M,x} = M \cap F_x$ hence $F_{M,x}$ is a CR submanifold of F_x. Let $T_{1,0}(F_{M,x})$ be the CR structure induced from (the complex structure of) F_x. The inclusion $F_{M,x} \subset M$ is a CR immersion (i.e. an immersion and a CR map) and $\bar{\zeta}^\alpha L_{\bar{\alpha},x} \in T_{1,0}(F_{M,x})_x$ if and only if $\zeta \in (C^{n-1})_{(G_M)_x}$. Q.e.d..

5. *CR* Orbifolds

The scope of this section is to introduce the class of CR orbifolds of arbitrary type (n, k) (containing the class of complex orbifolds, $k = 0$). The CR structure of a CR orbifold B and CR functions on B are discussed in Theorem 4. We consider an analogue $\Box_B$ of the Kohn-Rossi laplacian and state the problem of building a parametrix for $\Box_B$, the local approach to which is dealt with in section 6 (the solution to the global problem is delegated to a further paper).

Let $(B, \mathscr{A})$ be a $(2n+k)$-dimensional V-manifold, of class C^∞. A *CR structure* on B is a family

$$T_{1,0}(B) = \{T_{1,0}(\Omega) : \{\Omega, G, \varphi\} \in \mathscr{A}\}$$

where each $(\Omega, T_{1,0}(\Omega))$ is a CR manifold, of type (n, k), i.e. of CR dimension n and CR codimension k, and each injection $\lambda : \Omega \to \Omega'$ is a CR map. In particular, $G \subset Aut_{CR}(\Omega)$ for any l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{A}$. A pair $(B, T_{1,0}(B))$ is a *CR orbifold* (of type (n, k)). When $k = 0$, B is a complex orbifold (of complex dimension n). We shall deal mainly with CR orbifolds of CR codimension $k = 1$.

Let $(B, \mathscr{A})$ be an N-dimensional V-manifold. A continuous map $\Psi : B \to M$ into a C^∞ manifold M is an *immersion* if, for any $\{\Omega, G, \varphi\} \in \mathscr{A}$, the map $\Psi_\Omega := \Psi \circ \varphi : \Omega \to M$ is a C^∞ immersion (i.e. $rank[d_x\Psi_\Omega] = N \leq \dim(M)$, $x \in \Omega$). To give an example of CR orbifold, assume that $N = 2n + 1$ and let $\Psi : B \to \boldsymbol{C}^{n+1}$ be an immersion. Let $T_{1,0}(\Omega)$ be the CR structure on Ω given by

$$(d_x\Psi_\Omega)T_{1,0}(\Omega)_x = T_{1,0}(\boldsymbol{C}^{n+1})_{\Psi(\varphi(x))} \cap [(d_x\Psi_\Omega)T_x(\Omega) \otimes_{\boldsymbol{R}} \boldsymbol{C}], \quad x \in \Omega. \tag{17}$$

Note that $\Psi_{\Omega'} \circ \lambda = \Psi_\Omega$, for any injection $\lambda : \Omega \to \Omega'$; as a consequence, it is easy to see that λ must be a CR map, hence B together with the family of CR structures (17) is a CR orbifold.

Let $(B, \mathscr{A}, T_{1,0}(B))$ be a CR orbifold, of CR codimension 1. A family $\theta = \{\theta_\Omega : \{\Omega, G, \varphi\} \in \mathscr{A}\}$ is a *pseudohermitian structure* on B if each θ_Ω is a pseudohermitian structure on Ω and $\lambda^*\theta_{\Omega'} = a(\lambda)\theta_\Omega$ for any injection $\lambda : \Omega \to \Omega'$ and some constant $a(\lambda) \in \boldsymbol{R}\backslash\{0\}$, i.e. injections are pseudohermitian maps. We shall need

Lemma 3. *Let $(B, \mathscr{A}, T_{1,0}(B))$ be a CR orbifold and two pseudohermitian structures θ, $\hat{\theta}$ on B. If each injection $\lambda : \Omega \to \Omega'$ is isopseudohermitian, i.e. $a(\lambda) \equiv 1$, there is a unique C^∞ function $u : B \to \boldsymbol{R}\backslash\{0\}$ so that $\hat{\theta}_\Omega = u_\Omega\theta_\Omega$, for any l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{A}$.*

Proof. Let $u_\Omega : \Omega \to \boldsymbol{R}\backslash\{0\}$ be a C^∞ function satisfying $\hat{\theta}_\Omega = u_\Omega\theta_\Omega$. Next, consider an injection $\lambda : \Omega \to \Omega'$. The identities $\lambda^*\theta_{\Omega'} = \theta_\Omega$ and $\lambda^*\hat{\theta}_{\Omega'} = \hat{\theta}_\Omega$ lead to

$$u_{\Omega'} \circ \lambda = u_\Omega \tag{18}$$

In particular u_Ω is G-invariant. Define $u : B \to \mathbf{R}\setminus\{0\}$ as follows. Let $p \in B$ and $U \in \mathscr{H}$ so that $p \in U$. Let $\{\Omega, G, \varphi\} \in \mathscr{A}$ be a l.u.s. of support U. Let $x \in \Omega$ so that $\varphi(x) = p$. Finally, set $u(p) := u_\Omega(x)$. One needs to check that the definition of $u(p)$ doesn't depend upon the various choices involved. Let $U' \in \mathscr{H}$ so that $p \in U'$. Then there is $V \in \mathscr{H}$ so that $p \in V \subseteq U \cap U'$. Let $\{\Omega', G', \varphi'\}$ over U' and $x' \in \Omega'$ so that $\varphi'(x') = p$. Let $\{D, H, \psi\}$ be a l.u.s. of support V and consider two injections $\lambda : D \to \Omega$ and $\lambda' : D \to \Omega'$. Let $y \in D$ so that $\psi(y) = p$. From $\varphi(x) = \psi(y) = \varphi(\lambda(y))$, there is $\sigma \in G$ so that

$$\lambda(y) = \sigma(x). \tag{19}$$

Similarly

$$\lambda'(y) = \sigma'(x'), \tag{20}$$

for some $\sigma' \in G'$. Finally, using (18)–(20), one may conduct the following calculation

$$\begin{aligned} u_{\Omega'}(x') &= u_{\Omega'}((\sigma')^{-1}\lambda'(y)) = u_{\Omega'}(\lambda'(y)) \\ &= u_D(y) = u_\Omega(\lambda(y)) = u_\Omega(\sigma(x)) = u_\Omega(x). \end{aligned}$$

Q.e.d..

A *Riemannian orbifold* is a V-manifold B together with a family $g = \{g_\Omega : \{\Omega, G, \varphi\} \in \mathscr{A}\}$, where g_Ω is a Riemannian metric on Ω, so that each injection $\lambda : \Omega \to \Omega'$ is an isometry ($\lambda^* g_{\Omega'} = g_\Omega$). Let $(B, \mathscr{A}, T_{1,0}(B))$ be a *strictly pseudoconvex CR* orbifold, i.e. each $(\Omega, T_{1,0}(\Omega))$ is a strictly pseudoconvex CR manifold. Let θ be a pseudohermitian structure on B. Then each θ_Ω is a contact 1-form on Ω. Let g_Ω be the Webster metric of (Ω, θ_Ω) and set $g := \{g_\Omega : \{\Omega, G, \varphi\} \in \mathscr{A}\}$. If each injection λ is isopseudohermitian then λ preserves the Webster metrics, hence (B, g) is a Riemannian orbifold. The following result is similar to Theorem 1.

THEOREM 4. *For any CR orbifold $(B, \mathscr{A}, T_{1,0}(B))$, of type $(n, 1)$, there is a vector bundle $(E_{1,0}, \pi, B)$ so that for any $p \in B$, if $p \in U \in \mathscr{H}$* and $\{\Omega, G, \varphi\} \in \mathscr{A}$ *is a l.u.s. over U then $\pi^{-1}(p) \approx \mathbf{C}^n/G_x$ for any $x \in \Omega$ with $\varphi(x) = p$. B_{reg} is a CR manifold (of type $(n, 1)$) and $E_{1,0}|_{B_{reg}}$ is its CR structure. $T_{1,0}(B_{reg})$ is contained in $(E_{1,0})_{reg}$, the regular part of $E_{1,0}$ as a V-manifold. The image $T_{1,0}(B)_p \subseteq \pi^{-1}(p)$ of $T_{1,0}(\Omega)_{G_x}$ via the map $T_{1,0}(\Omega) \approx \Omega \times \mathbf{C}^n \to E_{1,0}$ depends only on $p = \varphi(x)$. $T_{1,0}(B)_p$ is a $\mathbf{C}$-vector space of dimension* $\dim_{\mathbf{C}}(\mathbf{C}^n)_{G_x}$. *If Z is a section in $E_{1,0}$ and $f \in \mathscr{E}(B)$ there is a (naturally defined) function $Z(f) : B \to \mathbf{C}$. If $Z(\bar{f}) = 0$ for any Z then $f_\Omega = f \circ \varphi$ is a CR function on Ω, for any $\{\Omega, G, \varphi\} \in \mathscr{A}$, and conversely.*

The bundle $E_{1,0}$ is recovered from the transition functions $g_\lambda(x) = [\lambda^\alpha_\beta(x)]$, where $(d_x\lambda)L_{\alpha,x} = \lambda^\beta_\alpha(x)L'_{\beta,\lambda(x)}$, $x \in \Omega$ (we assume w.l.o.g. that a frame $\{L_\alpha\}$ of $T_{1,0}(\Omega)$, defined on the whole of Ω, is prescribed on each Ω). We omit the details.

Let B be a V-manifold. A linear map $D : \mathscr{E}(B) \to \mathscr{E}(B)$ is a *differential operator* (*of order* k) if for any l.u.s. $\{\Omega, G, \varphi\} \in \mathscr{A}$ there is a differential operator D_Ω of order k on Ω so that $(Du)_\Omega = D_\Omega u_\Omega$ for any $u \in \mathscr{E}(B)$. We say D is *elliptic* (respectively *subelliptic* (*of order* ε)) if D_Ω is elliptic (respectively subelliptic of order ε, (cf. [11], p. 373)) for each l.u.s. $\{\Omega, G, \varphi\}$.

Let $(B, T_{1,0}(B))$ be a nondegenerate CR orbifold, $\theta = \{\theta_\Omega\}$ a fixed pseudohermitian structure on B, and $\Box_\Omega$ the Kohn-Rossi laplacian of (Ω, θ_Ω), cf. section 2. If each injection is isopseudohermitian, we may build a differential operator $\Box_B : \mathscr{E}(B) \to \mathscr{E}(B)$ by setting

$$(\Box_B u)_\Omega = \Box_\Omega u_\Omega$$

for any $u \in \mathscr{E}(B)$. Then $\Box_B u$ is a well defined element of $\mathscr{E}(B)$ if the functions $f_\Omega = \Box_\Omega u_\Omega$ satisfy $f_{\Omega'} \circ \lambda = f_\Omega$ for any injection $\lambda : \Omega \to \Omega'$. This may be seen as follows. By applying (5) we get $\Box^\lambda_\Omega = \Box_{\lambda(\Omega)}$ or

$$(\Box_\Omega(v \circ \lambda)) \circ \lambda^{-1} = \Box_{\lambda(\Omega)} v,$$

for any $v \in C^\infty(\lambda(\Omega))$. In particular, let us consider the functions

$$v = u_{\Omega'}|_{\lambda(\Omega)} \in C^\infty(\lambda(\Omega)).$$

Then

$$\Box_\Omega(u_\Omega|_{\lambda(\Omega)}) \circ \lambda) \circ \lambda^{-1} = \Box_{\lambda(\Omega)}(u_{\Omega'}|_{\lambda(\Omega)})$$

may be written as

$$\Box_\Omega u_\Omega = (\Box_{\Omega'} u_{\Omega'}) \circ \lambda.$$

Q.e.d.. Let T_Ω be the characteristic direction of (Ω, θ_Ω). We define a differential operator $T : \mathscr{E}(B) \to \mathscr{E}(B)$ by setting $(Tu)_\Omega = T_\Omega u_\Omega$ for any $u \in \mathscr{E}(\Omega)$. Again, the functions $T_\Omega u_\Omega$ give rise to a well defined element Tu of $\mathscr{E}(B)$ provided that each injection λ is isopseudohermitian; indeed, if this is the case then $(d_x\lambda)T_{\Omega,x} = T_{\Omega',\lambda(x)}$ for any $x \in \Omega$, and one may perform the calculation

$$T_{\Omega',\lambda(x)}(u_{\Omega'}) = [(d_x\lambda)T_{\Omega,x}](u_{\Omega'}) = T_{\Omega,x}(u_{\Omega'} \circ \lambda) = T_{\Omega,x}(u_\Omega).$$

Q.e.d.. Finally, let $(B, T_{1,0}(B))$ be a strictly pseudoconvex CR orbifold and $\theta = \{\theta_\Omega\}$ a pseudohermitian structure on B so that each Levi form L_{θ_Ω} is positive definite, and each injection is isopseudohermitian. Consider the second order differential operator $\Delta_B : \mathscr{E}(B) \to \mathscr{E}(B)$ given by $\Delta_B u = \square_B u - inT(u)$ for any $u \in B$. Then Δ_B is a subelliptic operator of order 1/2 on B. J. Girbau & M. Nicolau have developed (cf. [13]) a pseudo-differential calculus on V-manifolds (inverting a given elliptic differential operator up to infinitely smoothing operators). The same problem for subelliptic operators on V-manifolds, e.g. for Δ_B on a CR orbifold, is not solved (presumably, one needs to adapt the methods in [17]). Also, see [12], p. 493–498, for a parametrix and the regularity of $\square_M$ for an ordinary strictly pseudoconvex CR manifold M. The problem of building a parametrix for $\square_B$ on a strictly pseudoconvex CR orbifold B is open. In the next section we solve the local problem.

6. A Parametrix for $\square_\Omega$

Let $\Omega \subset \boldsymbol{R}^{2n+1}$ be a domain and $T_{1,0}(\Omega)$ a G-invariant strictly pseudoconvex CR structure on Ω, for some finite group of CR automorphisms $G \subset Aut_{CR}(\Omega)$. Let θ be a pseudohermitian structure on Ω so that the corresponding Levi form L_θ be positive definite and $\sigma^*\theta = a(\sigma)\theta$, for any $\sigma \in G$ and some $a(\sigma) \in (0, +\infty)$. Let $\{T_\alpha\}$ be an orthonormal ($L_\theta(T_\alpha, T_{\bar\beta}) = \delta_{\alpha\beta}$) frame of $T_{1,0}(\Omega)$, defined everywhere in Ω. Let $(z,t) = \Theta_x : V_x \to \boldsymbol{H}_n$ be the pseudohermitian normal coordinates at $x \in \Omega$, determined by $\{T_\alpha\}$ as in section 2, and set

$$D := \bigcup_{x \in \Omega} \{x\} \times V_x,$$

a neighborhood of the diagonal in $\Omega \times \Omega$. Next, we set $\Theta(x,y) := \Theta_x(y)$ and $\rho(x,y) := |\Theta(x,y)|$, for any $(x,y) \in D$. Here $|(z,t)| = (\|z\|^4 + t^2)^{1/4}$ is the *Heisenberg norm* of $(z,t) \in \boldsymbol{H}_n$.

A function $K(x,y)$ on $\Omega \times \Omega$ is a *kernel of type* λ ($\lambda > 0$) if for any $m \in \boldsymbol{Z}$, $m > 0$, one may write $K(x,y)$ as

$$K(x,y) = \sum_{i=1}^{N} a_i(x) K_i(x,y) b_i(y) + E_m(x,y) \tag{21}$$

where $N \geq 1$ and 1) $E_m \in C_0^m(\Omega \times \Omega)$, 2) $a_i, b_i \in C_0^\infty(\Omega)$, $1 \leq i \leq N$, and 3) K_i is C^∞ away from the diagonal and is supported in $\{(x,y) \in D : \rho(x,y) \leq 1\}$ and $K_i(x,y) = k_i(\Theta(y,x))$ for $\rho(x,y)$ sufficiently small, where k_i is homogeneous of degree $\lambda_i := \lambda - 2n - 2 + \mu_i$, i.e.

$$k_i(\delta_r(z,t)) = r^{\lambda_i} k_i(z,t), \quad r > 0, (z,t) \in \boldsymbol{H}_n,$$

for some $\mu_i \geq 0$. Also $\delta_r(z,t) = (rz, r^2 t)$ is the (parabolic) *dilation* of factor $r > 0$. Next

$$(Af)(x) = \int_\Omega K(x,y) f(y)\, dy$$

is an *operator of type* λ $(\lambda > 0)$ if $K(x,y)$ is a kernel of type λ. Here dy is short for $\omega(y) := (\theta \wedge (d\theta)^n)(y)$.

Set $X_\alpha := T_\alpha + T_{\bar\alpha}$ and $Y_\alpha := i(T_{\bar\alpha} - T_\alpha)$ and $\{X_j : 1 \leq j \leq 2n\} := \{X_\alpha, Y_\alpha\}$, where $X_{\alpha+n} = Y_\alpha$. Also, set

$$\mathscr{B}_k = \{X_{j_1} \cdots X_{j\ell} : 1 \leq j_s \leq 2n, 1 \leq s \leq \ell, 1 \leq \ell \leq k\}$$

and let $\mathscr{A}_k$ be the span over $\boldsymbol{C}$ of $\mathscr{B}_k \cup \{I\}$, where I is the identity. The *Folland-Stein spaces* are $S_k^p(\Omega) = \{f \in L^p(\Omega) : Lf \in L^p(\Omega), \forall L \in \mathscr{A}_k\}$ where Lf is intended in distributional sense. The Folland-Stein spaces are Banach spaces under the norms $\|f\|_{p,k} = \|f\|_p + \sum_{L \in \mathscr{B}_k} \|Lf\|_p$. An important feature of the operators of type $\lambda = m \in \{1, 2, \ldots\}$ is that they are bounded operators from $S_k^p(\Omega)$ to $S_{k+m}^p(\Omega)$ (and in this sense smoothing) for $k \in \{0, 1, 2, \ldots\}$ and $1 < p < \infty$ (cf. Theor. 15.19 in [12], p. 491). We shall prove the following result

THEOREM 5. *Let W_0 be a G-invariant compact subset of Ω. For each $0 < q < n$ there is an operator $A_{q,\Omega} : \Gamma_0^\infty(\Lambda^{0,q}(\Omega)) \to \Gamma_0^\infty(\Lambda^{0,q}(\Omega))$, of type 2, so that 1) $A_{q,\Omega} \circ \Box_\Omega - I$ and $\Box_\Omega \circ A_{q,\Omega} - I$ are operators of type 1 on the G-invariant C^∞ forms of support contained in W_0, and 2) $A_{q,\Omega}$ maps G-invariant forms in G-invariant forms.*

A $(0,q)$-form φ on Ω may be written locally $\varphi = \varphi_{\bar I} \theta^{\bar I}$ where $I = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $\theta^{\bar I} = \theta^{\bar\alpha_1} \wedge \cdots \wedge \theta^{\bar\alpha_n}$. Since

$$(\sigma^* \theta^\alpha)_x = g_\sigma(x)^\alpha_\beta \theta^\beta_x, \quad x \in \Omega,$$

if φ is G-invariant (i.e. $\sigma^* \varphi = \varphi$ for any $\sigma \in G$) then

$$\varphi_{\bar I}(x) = g_\sigma(x)^{\bar J}_{\bar I} \varphi_{\bar J}(\sigma(x)), \quad x \in \Omega, \sigma \in G,$$

$$g_\sigma(x)^{\bar J}_{\bar I} := g_\sigma(x)^{\bar\beta_1}_{\bar\alpha_1} \cdots g_\sigma(x)^{\bar\beta_n}_{\bar\alpha_n}, \quad J = (\beta_1, \ldots, \beta_n).$$

By Prop. 16.5 in [12], p. 496, for any $1 \leq q \leq n-1$ we may build an operator A_q of type 2 so that $I - \Box_\Omega A_q$ and $I - A_q \Box_\Omega$ are operators of type 1 on forms $\varphi \in \Gamma_0^\infty(\Lambda^{0,q}(\Omega))$ of support $\subset W_0$. Assuming this is done, set

$$A_{q,\sigma}\varphi := \sigma^* A_q (\sigma^{-1})^* \varphi, \quad A_{q,\Omega} := \frac{1}{|G|} \sum_{\sigma \in G} A_{q,\sigma}.$$

From now on, for the sake of simplicity, we drop the index q. If φ is G-invariant then

$$\tau^* A_\sigma \varphi = (\sigma\tau)^* A(\sigma^{-1})^* \varphi = (\sigma\tau)^* A((\sigma\tau)^{-1})^* \varphi,$$

i.e.

$$\tau^*(A_\sigma \varphi) = A_{\sigma\tau}\varphi.$$

Therefore

$$\tau^* A_\Omega \varphi = \frac{1}{|G|}\sum_{\sigma\in G} \tau^* A_\sigma \varphi = \frac{1}{|G|}\sum_{\sigma\in G} A_{\sigma\tau}\varphi = A_\Omega \varphi,$$

i.e. A_Ω maps G-invariant forms in G-invariant forms.

For each $\xi \in \Omega$ let $\delta(\xi) > 0$ be fixed so that $\Psi_\xi : B(0,\delta(\xi)) \subset T_\xi(\Omega) \to \Omega$ is well defined and a diffeomorphism on its image $V_\xi = \Psi_\xi(B(0,\delta(\xi))$. Next, fix a number

$$0 < \delta_G(\xi) \le \min\left(\left\{\frac{\delta(\sigma(\xi))}{\sqrt{a(\sigma)^2 + a(\sigma)}} : \sigma \in G\right\} \cup \{\delta(\xi)\}\right)$$

and set

$$V_G(\xi) := \Psi_\xi(B(0,\delta_G(\xi))) \subseteq V_\xi \subset \Omega.$$

LEMMA 4. $\sigma[V_G(\xi)] \subseteq V_{\sigma(\xi)}$.

PROOF. Let $\eta \in V_G(\xi) \subset V_\xi$, i.e. there is $W + cT_\xi \in B(0,\delta_G(\xi))$ so that $W \in H(\Omega)_\xi$ and $\eta = \Psi_\xi(W + cT_\xi) = \gamma_{W,c}(1)$. Thus (by Lemma 1 in section 2) $\sigma(\eta) = (\sigma \circ \gamma_{W,c})(1) = \gamma_{W_\sigma, a(\sigma)c}(1)$. On the other hand

$$\begin{aligned}\|W_\sigma + a(\sigma)cT_{\sigma(\xi)}\|^2 &= \|W_\sigma\|^2 + a(\sigma)^2c^2 \\ &= a(\sigma)\|W\|^2 + a(\sigma)^2c^2 < [a(\sigma) + a(\sigma)^2]\delta_G(\xi)^2 \le \delta(\sigma(\xi))^2,\end{aligned}$$

hence $\gamma_{W_\sigma, a(\sigma)c}(1) \in V_{\sigma(\xi)}$. Q.e.d..

Set

$$D_G := \bigcup_{\xi\in\Omega} \{\xi\} \times V_G(\xi).$$

Let us go back to the construction of A. Consider

$$A\varphi(\xi) = \left(\int_\Omega K(\xi,\eta)\varphi_{\bar J}(\eta)\, d\eta\right)\theta_\xi^{\bar J},$$

where K is the kernel of type 2

$$K(\xi,\eta) = \psi(\xi,\eta)\Phi_{n-2q}(\Theta(\eta,\xi)).$$

Here $\psi(\xi,\eta)$ is a C_0^∞ function on $\Omega \times \Omega$, supported in

$$\{(\xi,\eta) \in D_G : \rho(\xi,\eta) \le r\},$$

where

$$r := \min(\{a(\sigma)^{1/2} : \sigma \in G\} \cup \{1\}),$$

and so that $\psi(\xi,\eta) = \psi(\eta,\xi)$ and $\psi(\xi,\eta) = 1$ in a neighborhood $\mathcal{N}$ of the diagonal Δ of $W_0 \times W_0$ ($\Delta \subset \mathcal{N} \subseteq \{(\xi,\eta) \in D : \rho(\xi,\eta) < r\}$). Also Φ_α is the fundamental solution ($\mathcal{S}_\alpha \Phi_\alpha = \delta$) to

$$\mathcal{S}_\alpha = -\sum_{j=1}^{n} L_j L_{\bar{j}} + i(\alpha - n)\frac{\partial}{\partial t}, \tag{22}$$

(the *Folland-Stein operators*) where

$$L_j := \frac{\partial}{\partial z^j} + i\bar{z}^j \frac{\partial}{\partial t}$$

(the *Lewy operators*) i.e.

$$\Phi_\alpha = b_\alpha(\|z\|^2 - it)^{-(n+\alpha)/2}(\|z\|^2 + it)^{-(n-\alpha)/2}, \tag{23}$$

for any $\alpha \in \boldsymbol{C}\backslash\{\pm n, \pm(n+2), \pm(n+4), \ldots\}$, where

$$b_\alpha = \frac{\Gamma((n+\alpha)/2)\Gamma((n-\alpha)/2)}{2^{2-2n}\pi^{n+1}}.$$

Then

$$A_\sigma \varphi(\xi) = \left(\int K(\sigma(\xi),\eta)((\sigma^{-1})^*\varphi)_{\bar{I}}(\eta)\, d\eta\right)\theta^{\bar{I}}_{\sigma(\xi)} \circ (d_\xi \sigma). \tag{24}$$

By $\sigma^*\omega = a(\sigma)^{2n+1}\omega$ and a change of coordinates $\eta' = \sigma(\eta)$ in (24) we get

$$A_\sigma \varphi(\xi) = a(\sigma)^{2n+1}\left(\int g_\sigma(\xi)^{\bar{I}}_{\bar{J}} K(\sigma(\xi),\sigma(\eta)) g_{\sigma^{-1}}(\sigma(\eta))^{\bar{L}}_{\bar{I}} \varphi_{\bar{L}}(\eta)\, d\eta\right)\theta^{\bar{J}}_\xi.$$

LEMMA 5. *For any* $(\xi,\eta) \in D_G$

$$\Theta(\sigma(\xi),\sigma(\eta)) = (g_\sigma(\xi)^\beta_\alpha z^\alpha(\eta) e_\beta, a(\sigma)t(\eta)),$$

where $(z,t) = \Theta_\xi = \lambda_\xi \circ \Psi_\xi^{-1}$ *are the pseudohermitian normal coordinates centered at* ξ.

PROOF. As $(\xi,\eta)\in D_G$ we have $\eta\in V_G(\xi)$ hence (by Lemma 4) $\sigma(\eta)\in\sigma[V_G(\xi)]\subseteq V_{\sigma(\xi)}$ and then

$$\Theta(\sigma(\xi),\sigma(\eta))=\Theta_{\sigma(\xi)}(\sigma(\eta))=\lambda_{\sigma(\xi)}\circ\Psi_{\sigma(\xi)}^{-1}(\sigma(\eta))$$

makes sense. As $\eta\in V_G(\xi)\subseteq V_\xi$, set $W:=z^\alpha(\eta)T_{\alpha,\eta}+z^{\bar\alpha}(\eta)T_{\bar\alpha,\eta}$ and $c:=t(\eta)$. Then

$$\begin{aligned}\Psi_{\sigma(\xi)}(W_\sigma+ca(\sigma)T_{\sigma(\eta)})&=\gamma_{W_\sigma,ca(\sigma)}(1)\quad\text{(by Lemma 1)}\\&=\sigma(\gamma_{W,c}(1))=\sigma(\Psi_\xi(W+cT_\eta))=\sigma(\eta),\end{aligned}$$

hence

$$\Theta(\sigma(\xi),\sigma(\eta))=\lambda_{\sigma(\xi)}(W_\sigma+ca(\sigma)T_{\sigma(\eta)}).\qquad\text{Q.e.d..}$$

For any $\sigma\in G$, $\sigma^*L_\theta=a(\sigma)L_\theta$ hence

$$\sum_\mu g_\sigma(\eta)^\mu_\alpha g_\sigma(\eta)^{\bar\mu}_{\bar\beta}=a(\sigma)\delta_{\alpha\beta},$$

i.e. $a(\sigma)^{-1/2}g_\sigma(\eta)\in U(n)$. Consequently $\|g_\sigma(\eta)z\|^2=a(\sigma)\|z\|^2$ and (by (23) and Lemma 5)

$$\Phi_{n-2q}(\Theta(\sigma(\eta),\sigma(\xi)))=a(\sigma)^{-n}\Phi_{n-2q}(\Theta(\eta,\xi)),$$

and we obtain

$$\begin{aligned}&a(\sigma)^{-n-1}A_\sigma\varphi(\xi)\\&=\left(\int g_\sigma(\xi)^{\bar I}_{\bar J}\psi_\sigma(\xi,\eta)\Phi_{n-2q}(\Theta(\eta,\xi))g_{\sigma^{-1}}(\sigma(\eta))^{\bar K}_{\bar I}\varphi_{\bar K}(\eta)\,d\eta\right)\theta^{\bar J}_\xi,\end{aligned}$$

where $\psi_\sigma(\xi,\eta):=\psi(\sigma(\xi),\sigma(\eta))$. Note that $\psi_\sigma\in C_0^\infty$ and $\psi_\sigma(\xi,\eta)=\psi_\sigma(\eta,\xi)$. Let $\sigma^2:=\sigma\times\sigma$ (direct product). Set

$$\mathcal{N}_G:=\bigcap_{\sigma\in G}\sigma^2(\mathcal{N})\subset\mathcal{N}.$$

As W_0 is G-invariant $\Delta=\sigma^2(\Delta)\subset\sigma^2(\mathcal{N})$ for any $\sigma\in G$, hence $\mathcal{N}_G$ is an open neighborhood of Δ. Also $\psi(\xi,\eta)=1$ on $\mathcal{N}$ yields $\psi_\sigma(\xi,\eta)=1$ on $\mathcal{N}_G$.

Let $(\xi,\eta)\in D_G$. Then (by Lemma 5)

$$\begin{aligned}|\Theta(\sigma(\xi),\sigma(\eta))|&=|(g_\sigma(\xi)z(\eta),a(\sigma)t(\eta))|\\&=(\|g_\sigma(\xi)z(\eta)\|^4+a(\sigma)^2t(\eta)^2)^{1/4}\\&=a(\sigma)^{1/2}|(z(\eta),t(\eta))|=a(\sigma)^{1/2}|\Theta(\xi,\eta)|,\end{aligned}$$

that is

$$\rho(\sigma(\xi),\sigma(\eta)) = a(\sigma)^{1/2}\rho(\xi,\eta). \tag{25}$$

Let Γ and Γ_σ be respectively the supports of ψ and ψ_σ. Then $\sigma^2(\Gamma_\sigma) \subseteq \Gamma \subset \{(\xi,\eta) \in D_G : \rho(\xi,\eta) \le r\}$. Also (by Lemma 4) $\sigma^{-1}(D_G) \subseteq D$. Thus (by (25)) $\Gamma_\sigma \subset \{(\xi,\eta) \in D : \rho(\xi,\eta) \le 1\}$. Then (as in [12], p. 494) we may conclude that

$$K_\sigma(\xi,\eta) = \psi_\sigma(\xi,\eta)\Phi_{n-2q}(\Theta(\eta,\xi))$$

is a kernel of type 2. In general, if $K(\xi,\eta)$ is a kernel of type λ then

$$K_{\bar J}^{\bar I}(\xi,\eta) := g_\sigma(\xi)_{\bar J}^{\bar L} K(\xi,\eta) g_{\sigma^{-1}}(\sigma(\eta))_{\bar L}^{\bar I}$$

is another kernel of type λ, as it easily follows from (21). We have proved that A_σ, and therefore A_Ω, is an operator of type 2.

Set $a(G) := (1/|G|)\sum_{\sigma\in G} a(\sigma) > 0$. We wish to check that $a(G)^{-1}A_\Omega$ inverts $\Box_\Omega$. Set $B := I - \Box_\Omega A$. If φ is a G-invariant $(0,q)$-form then (by (7))

$$\begin{aligned}\Box_\Omega A_\Omega\varphi(\xi) &= \frac{1}{|G|}\sum_{\sigma\in G}\Box_\Omega\sigma^* A(\sigma^{-1})^*\varphi(\xi)\\ &= \frac{1}{|G|}\sum_{\sigma\in G} a(\sigma)\sigma^*\Box_\Omega A\varphi(\xi) = \frac{1}{|G|}\sum_{\sigma\in G} a(\sigma)\sigma^*(\varphi - B\varphi)(\xi)\end{aligned}$$

that is

$$\Box_\Omega A_\Omega\varphi(\xi) = a(G)\varphi(\xi) - \frac{1}{|G|}\sum_{\sigma\in G} a(\sigma)B_\sigma\varphi(\xi),$$

where $B_\sigma := \sigma^* B(\sigma^{-1})^*$. We shall prove that

LEMMA 6. *B_σ is an operator of type* 1.

PROOF. Set

$$A_\varepsilon\varphi(\xi) := \left(\int K_\varepsilon(\xi,\eta)\varphi_{\bar J}(\eta)\,d\eta\right)\theta_\xi^{\bar J},$$

$$K_\varepsilon(\xi,\eta) := \psi(\xi,\eta)\Phi_{n-2q}^{\varepsilon}(\Theta(\eta,\xi)),$$

$$\Phi_\alpha^\varepsilon := b_\alpha \rho_\varepsilon^{-(n+\alpha)/2}\bar\rho_\varepsilon^{-(n-\alpha)/2}, \quad \rho_\varepsilon(z,t) := \|z\|^2 + \varepsilon^2 - it,$$

for any $\varepsilon > 0$. For the sake of simplicity, we only look at the case $q = 1$. For any $(0,1)$-form ψ on Ω, the Kohn-Rossi laplacian is expressed by

$$\Box_\Omega\psi = \{-h^{\lambda\bar\mu}\nabla_\lambda\nabla_{\bar\mu}\psi_{\bar\alpha} - 2i\nabla_0\psi_{\bar\alpha} + \psi_{\bar\gamma}R_{\bar\alpha}^{\bar\gamma}\}\theta^{\bar\alpha},$$

where $R_{\lambda\bar{\mu}}$ is the *pseudohermitian Ricci tensor* (cf. e.g. [10], p. 193). This may be written

$$(\Box_\Omega \psi)_{\bar{\alpha}} = \mathscr{L}_{n-2}\psi_{\bar{\alpha}} + \sum_{\mu=1}^{n}\left\{\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\alpha}}T_\mu\psi_{\bar{\rho}} + \frac{1}{2}\Gamma^{\bar{\rho}}_{\mu\bar{\mu}}T_{\bar{\rho}}\psi_{\bar{\alpha}} + \Gamma^{\bar{\rho}}_{\mu\bar{\alpha}}T_{\bar{\mu}}\psi_{\bar{\rho}}\right\} + F^{\bar{\gamma}}_{\bar{\alpha}}\psi_{\bar{\gamma}},$$

(compare to (16.1) in [12], p. 494) for some C^∞ functions $F^{\bar{\gamma}}_{\bar{\alpha}}$ (expressed in termes of the Christoffel symbols and their derivatives, and whose precise form is unimportant). We have (by the proof of Prop. 16.5 in [12])

$$\sigma^* B(\sigma^{-1})^*\varphi(\xi) = \varphi(\xi) - \sigma^*\Box_\Omega A(\sigma^{-1})^*\varphi(\xi) = \varphi(\xi) - \sigma^*\lim_{\varepsilon\to 0}\Box_\Omega A_\varepsilon(\sigma^{-1})^*\varphi(\xi)$$

that is

$$B_\sigma\varphi(\xi) = \varphi(\xi) - \lim_{\varepsilon\to 0}\sigma^*\Box_\Omega A_\varepsilon(\sigma^{-1})^*\varphi(\xi)$$

hence it suffices to show that if we let $\varepsilon \to 0$ then $\sigma^*\Box_\Omega A_\varepsilon(\sigma^{-1})^*\varphi$ goes to φ plus an operator of order 1 applied to φ. We have

$$\sigma^*\Box_\Omega A_\varepsilon(\sigma^{-1})^*\varphi(\xi) = \left[\Box_\Omega\left(\left(\int K_\varepsilon(\cdot,\eta)((\sigma^{-1})^*\varphi)_{\bar{\alpha}}(\eta)\,d\eta\right)\theta^{\bar{\alpha}}\right]_{\sigma(\xi)} \circ (d_\xi\sigma)$$

$$= g_\sigma(\xi)^{\bar{\alpha}}_{\bar{\beta}}\left[\mathscr{L}_{n-2}\psi_{\bar{\alpha}} + \sum_\mu\left\{\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\alpha}}T_\mu\psi_{\bar{\rho}} + \frac{1}{2}\Gamma^{\bar{\rho}}_{\mu\bar{\mu}}T_{\bar{\rho}}\psi_{\bar{\alpha}} + \Gamma^{\bar{\rho}}_{\mu\bar{\alpha}}T_{\bar{\mu}}\psi_{\bar{\rho}}\right\} + F^{\bar{\gamma}}_{\bar{\alpha}}\psi_{\bar{\gamma}}\right]_{\sigma(\xi)}\theta^{\bar{\beta}}_\xi$$

where

$$\psi_{\bar{\alpha}}(\xi) := \int K_\varepsilon(\xi,\eta)((\sigma^{-1})^*\varphi)_{\bar{\alpha}}(\eta)\,d\eta.$$

and $\mathscr{L}_{n-2} = -\sum_\alpha T_\alpha T_{\bar{\alpha}} - 2iT$. Therefore, using

$$(T_\mu f)(\sigma(\xi)) = g_{\sigma^{-1}}(\sigma(\xi))^\lambda_\mu T_\lambda(f\circ\sigma)$$

we get

$$\sigma^*\Box_\Omega A_\varepsilon(\sigma^{-1})^*\varphi(\xi) = \left\{A^0_{\varepsilon,\bar{\beta}}\varphi(\xi) + \sum_{\mu=1}^{n}\sum_{i=1}^{3}A^i_{\varepsilon,\mu\bar{\beta}}\varphi(\xi)\right\}\theta^{\bar{\beta}}_\xi$$

$$+ \left(\int g_\sigma(\xi)^{\bar{\alpha}}_{\bar{\beta}}[\mathscr{L}^\zeta_{n-2}K_\varepsilon(\zeta,\eta)]_{\zeta=\sigma(\xi)}g_{\sigma^{-1}}(\eta)^{\bar{\gamma}}_{\bar{\alpha}}\varphi_{\bar{\gamma}}(\sigma^{-1}(\eta))\,d\eta\right)\theta^{\bar{\beta}}_\xi \qquad (26)$$

where

$$A^0_{\varepsilon,\bar\beta}\varphi(\xi) = g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}F^{\bar\gamma}_{\bar\alpha}(\sigma(\xi))\int K_\varepsilon(\sigma(\xi),\eta)g_{\sigma^{-1}}(\eta)^{\bar\rho}_{\bar\gamma}\varphi_{\bar\rho}(\sigma^{-1}(\eta))\,d\eta,$$

$$A^1_{\varepsilon,\mu\bar\beta}\varphi(\xi) = g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}\Gamma^{\bar\rho}_{\mu\bar\alpha}(\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\lambda}_{\mu}\int [T^\xi_\lambda K_\varepsilon(\sigma(\xi),\eta)]g_{\sigma^{-1}}(\eta)^{\bar\gamma}_{\bar\rho}\varphi_{\bar\gamma}(\sigma^{-1}(\eta))\,d\eta,$$

$$A^2_{\varepsilon,\mu\bar\beta}\varphi(\xi) = \frac{1}{2}g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}\Gamma^{\bar\rho}_{\mu\bar\mu}(\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\bar\lambda}_{\bar\rho}\int [T^\xi_{\bar\lambda} K_\varepsilon(\sigma(\xi),\eta)]g_{\sigma^{-1}}(\eta)^{\bar\gamma}_{\bar\alpha}\varphi_{\bar\gamma}(\sigma^{-1}(\eta))\,d\eta,$$

$$A^3_{\varepsilon,\mu\bar\beta}\varphi(\xi) = g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}\Gamma^{\bar\rho}_{\mu\bar\alpha}(\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\bar\lambda}_{\bar\mu}\int [T^\xi_{\bar\lambda} K_\varepsilon(\sigma(\xi),\eta)]g_{\sigma^{-1}}(\eta)^{\bar\gamma}_{\bar\rho}\varphi_{\bar\gamma}(\sigma^{-1}(\eta))\,d\eta.$$

Clearly $A^0_{\varepsilon,\bar\beta}$ gives, in the limit as $\varepsilon \to 0$, an operator of type 2 (and hence of type 1). We claim that $A^i_{\varepsilon,\mu\bar\beta}$ give (as $\varepsilon \to 0$) operators of type 1, as well. For instance, let us look at $A^1_{\varepsilon,\mu\bar\beta}$ (the remaining operators may be treated in a similar manner). Note that

$$\Phi^\varepsilon_\alpha(\Theta(\sigma(\eta),\sigma(\xi))) = a(\sigma)^{-n}\Phi^{\varepsilon/\sqrt{a(\sigma)}}_\alpha(\Theta(\eta,\xi)) \tag{27}$$

Indeed (by Lemma 5)

$$\begin{aligned}\rho_\varepsilon(g_\sigma(\eta)z(\xi), a(\sigma)t(\xi)) &= a(\sigma)\|z(\xi)\|^2 + \varepsilon^2 - ia(\sigma)t(\xi)\\ &= a(\sigma)\rho_{\varepsilon/\sqrt{a(\sigma)}}(z(\xi), t(\xi)).\end{aligned}$$

Consequently

$$K_\varepsilon(\sigma(\xi),\sigma(\eta)) = a(\sigma)^{-n}\psi_\sigma(\xi,\eta)\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(\Theta(\eta,\xi))$$

and a change of variables $\eta' = \sigma^{-1}(\eta)$ leads to

$$\begin{aligned}A^1_{\varepsilon,\mu\bar\beta}\varphi(\xi) &= a(\sigma)^{n+1}g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}\Gamma^{\bar\rho}_{\mu\bar\alpha}(\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\lambda}_{\mu}\\ &\quad\cdot\int T^\xi_\lambda[\psi_\sigma(\xi,\eta)\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(\Theta(\eta,\xi))]g_{\sigma^{-1}}(\sigma(\eta))^{\bar\gamma}_{\bar\rho}\varphi_{\bar\gamma}(\eta)\,d\eta\end{aligned}$$

which goes, as $\varepsilon \to 0$, to

$$\begin{aligned}&a(\sigma)^{n+1}g_\sigma(\xi)^{\bar\alpha}_{\bar\beta}\Gamma^{\bar\rho}_{\mu\bar\alpha}(\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\lambda}_{\mu}\\ &\quad\cdot T^\xi_\lambda\left[\int \psi_\sigma(\xi,\eta)\Psi_{n-2}(\Theta(\eta,\xi))g_{\sigma^{-1}}(\sigma(\eta))^{\bar\gamma}_{\bar\rho}\varphi_{\bar\gamma}(\eta)\,d\eta\right].\end{aligned}$$

As previously shown, $\psi_\sigma(\xi,\eta)\Phi_{n-2}(\Theta(\eta,\xi))$ is a kernel of type 2; yet, by Prop. 15.14 in [12], p. 487, for any operator A of type 2, $T_\lambda A$ is an operator of type 1, hence the claim is proved.

To deal with the last term in (26) we write

$$\mathscr{L}^{\zeta}_{n-2}K_{\varepsilon}(\zeta,\eta) = [\mathscr{L}^{\zeta}_{n-2}\psi(\zeta,\eta)]\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\zeta)) + \psi(\zeta,\eta)\mathscr{L}^{\zeta}_{n-2}[\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\zeta))]$$

$$- \frac{1}{2}\sum_{\alpha=1}^{n}\{[T^{\zeta}_{\alpha}\psi(\zeta,\eta)]T^{\zeta}_{\bar{\alpha}}[\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\zeta))]$$

$$+ [T^{\zeta}_{\bar{\alpha}}\psi(\zeta,\eta)]T^{\zeta}_{\alpha}[\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\zeta))]\} \tag{28}$$

The first term on the right hand side of (28), when substituted into (26), leads (as $\varepsilon \to 0$) to an operator of order 1 applied to φ. We need to recall the notion of *Heisenberg-type order*. A function $f(\xi, y)$ on $\Omega \times \boldsymbol{H}_n$ is of *order* O^k, $k = 1, 2, \ldots$, if $f \in C^{\infty}$ and for any compact set $K \subset \Omega$ there is a constant $C_K > 0$ so that $|f(\xi, y)| \le C_K|y|^k$ (Heisenberg norm). If $(z, t) = \Theta_{\xi}^{-1}$ are pseudohermitian normal coordinates at ξ then (cf. Theor. 4.3 in [14], p. 177, a refinement of Theor. 14.10 and Corollary 14.9 in [12], p. 475)

$$(\Theta_{\xi}^{-1})_{*}T_{\alpha} = \frac{\partial}{\partial z^{\alpha}} + i\bar{z}^{\alpha}\frac{\partial}{\partial t} + O^1\mathscr{E}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) + O^2\mathscr{E}\left(\frac{\partial}{\partial t}\right),$$

where $O^k\mathscr{E}$ denotes an operator involving linear combinations of the indicated derivatives, with O^k coefficients. Similarly, $(\Theta_{\xi}^{-1})_{*}\mathscr{L}_{n-2}$ is the operator $\mathscr{S}_{n-2}$ (given by (22) with $\alpha = n-2$) plus higher (Heisenberg-type) order terms.

Let $\delta(\xi, \eta)$ be the distribution on $\Omega \times \Omega$ defined by

$$\int \delta(\xi,\eta)f(\xi)g(\eta)\, d\xi d\eta = \int f(\xi)g(\xi)\, d\xi.$$

As to the second term in the right hand side of (28), when substituted into (26), it gives an integral operator applied to φ, which goes to φ for $\varepsilon \to 0$, as desired. Indeed

$$\lim_{\varepsilon\to 0}\int g_{\sigma}(\xi)^{\bar{\alpha}}_{\bar{\beta}}\psi(\sigma(\xi),\eta)\mathscr{L}^{\zeta}_{n-2}[\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\zeta))]_{\zeta=\sigma(\xi)}g_{\sigma^{-1}}(\eta)^{\bar{\gamma}}_{\bar{\alpha}}\varphi_{\bar{\gamma}}(\sigma^{-1}(\eta))\, d\eta$$

is, up to higher order terms [leading to first order operators applied to φ (cf. also [12], p. 495)]

$$\int g_{\sigma}(\xi)^{\bar{\alpha}}_{\bar{\beta}}\psi(\sigma(\xi),\eta)[\mathscr{S}_{n-2}\Phi_{n-2}](\Theta(\eta,\sigma(\xi)))g_{\sigma^{-1}}(\eta)^{\bar{\gamma}}_{\bar{\alpha}}\varphi_{\bar{\gamma}}(\sigma^{-1}(\eta))\, d\eta$$

$$= \int g_{\sigma}(\xi)^{\bar{\alpha}}_{\bar{\beta}}\psi(\sigma(\xi),\eta)\delta(\sigma(\xi),\eta)g_{\sigma^{-1}}(\eta)^{\bar{\gamma}}_{\bar{\alpha}}\varphi_{\bar{\gamma}}(\sigma^{-1}(\eta))\, d\eta$$

$$= g_{\sigma}(\xi)^{\bar{\alpha}}_{\bar{\beta}}\psi(\sigma(\xi),\sigma(\xi))g_{\sigma^{-1}}(\sigma(\xi))^{\bar{\gamma}}_{\bar{\alpha}}\varphi_{\bar{\gamma}}(\xi) = \delta^{\gamma}_{\beta}\psi_{\sigma}(\xi,\xi)\varphi_{\bar{\gamma}}(\xi) = \varphi_{\bar{\beta}}(\xi).$$

Q.e.d.. Finally, we deal with the third term in the right hand side of (28) (the fourth term may be dealt with in a similar way). It may be written (at $\zeta = \sigma(\xi)$) as

$$g_{\sigma^{-1}}(\sigma(\xi))^{\beta}_{\alpha} g_{\sigma^{-1}}(\sigma(\xi))^{\bar{\gamma}}_{\bar{\alpha}} T^{\xi}_{\beta}[\psi(\sigma(\xi),\eta)] T^{\xi}_{\bar{\gamma}}[\Phi^{\varepsilon}_{n-2}(\Theta(\eta,\sigma(\xi))]$$

hence the corresponding integral is (after a change of variable)

$$a(\sigma)^{n+1} \sum_{\rho} \int g_{\sigma}(\xi)^{\bar{\alpha}}_{\bar{\beta}} g_{\sigma^{-1}}(\sigma(\xi))^{\lambda}_{\rho} g_{\sigma^{-1}}(\sigma(\xi))^{\bar{\mu}}_{\bar{\rho}} T^{\xi}_{\lambda}[\psi_{\sigma}(\xi,\eta)]$$

$$\cdot T^{\xi}_{\bar{\mu}}[\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(\Theta(\eta,\xi))] g_{\sigma^{-1}}(\sigma(\eta))^{\bar{\gamma}}_{\bar{\alpha}} \varphi_{\bar{\gamma}}(\eta)\, d\eta.$$

Set $\psi_{\lambda,\sigma}(\xi,\eta) := T^{\xi}_{\lambda}[\psi_{\sigma}(\xi,\eta)]$ and note that $\psi_{\lambda,\sigma} \in C^{\infty}_0$ and (as T_{λ} is a differential operator) $Supp(\psi_{\lambda,\sigma}) \subset Supp(\psi_{\sigma}) \subset \{(\xi,\eta) \in D : \rho(\xi,\eta) \le 1\}$. The following result completes the proof

LEMMA 7

$$\int \psi_{\lambda,\sigma}(\xi,\eta) T^{\xi}_{\bar{\mu}}[\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(\Theta(\eta,\xi))] g_{\sigma^{-1}}(\sigma(\eta))^{\bar{\gamma}}_{\bar{\alpha}} \varphi_{\bar{\gamma}}(\eta)\, d\eta \tag{29}$$

goes, as $\varepsilon \to 0$, *to an operator of order* 1 *applied to* φ.

PROOF. The kernel of the operator (29) is

$$T^{\xi}_{\bar{\mu}}[\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(\Theta(\eta,\xi))] = [(d_{\xi}\Theta_{\eta})T_{\bar{\mu},\xi}](\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})$$

$$= \left[L_{\bar{\mu}} + O^1\mathscr{E}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) + O^2\mathscr{E}\left(\frac{\partial}{\partial t}\right)\right]_{\Theta_{\eta}(\xi)} (\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})$$

$$= -2(z^{\mu}\bar{\rho}^{-1}_{\varepsilon/\sqrt{a(\sigma)}}\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})_{\Theta_{\eta}(\xi)} + \sum_{\lambda} O^1(\bar{z}^{\lambda} f_{\varepsilon}\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})(\Theta_{\eta}(\xi))$$

$$+ \sum_{\lambda} O^1(z^{\lambda} f_{\varepsilon}\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})(\Theta_{\eta}(\xi)) + O^2(if_{\varepsilon}\Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})(\Theta_{\eta}(\xi))$$

where

$$f_{\varepsilon} := -\bar{\rho}^{-1}_{\varepsilon/\sqrt{a(\sigma)}} - (n-1)\rho^{-1}_{\varepsilon/\sqrt{a(\sigma)}}.$$

The Heisenberg group carries the contact form

$$\theta_0 = dt + 2\sum_j (x^j\, dy^j - y^j\, dx^j),$$

$z^j = x^j + iy^j$. Let $dV = \theta_0 \wedge (d\theta_0)^n$ be the natural volume form on $\boldsymbol{H}_n$. Set $h := \Theta_\xi^{-1}$. Note that $\Theta(h(u), \xi) = -\Theta_\xi(h(u)) = -u$. Also

$$(h^*\omega)(u) = (1 + O^1)\, dV(u)$$

(cf. again Theor. 4.3 in [14], p. 177). Then

$$\int_\Omega \psi_{\lambda,\sigma}(\xi,\eta)(z^\mu \bar{\rho}^{-1}_{\varepsilon/\sqrt{a(\sigma)}} \Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2})(\Theta(\eta,\xi)) g_{\sigma^{-1}}(\sigma(\eta))^{\bar{\gamma}}_{\bar{\alpha}} \varphi_{\bar{\gamma}}(\eta)\, d\eta$$

$$= \int_{\boldsymbol{H}_n} \psi_{\lambda,\sigma}(\xi, h(u))(z^\mu(u) \bar{\rho}_{\varepsilon/\sqrt{a(\sigma)}}(u)^{-1} \Phi^{\varepsilon/\sqrt{a(\sigma)}}_{n-2}(u)$$

$$\cdot\, g_{\sigma^{-1}}(\sigma(h(u)))\varphi_{\bar{\gamma}}(h(u))(1 + O^1)\, dV(u)$$

$$= \varepsilon^{-2n-2} \int \psi_{\lambda,\sigma}(\xi, h(u)) z^\mu(u) \frac{\Phi^1_{n-2}(\varepsilon^{-1}u)}{\bar{\rho}_1(\varepsilon^{-1}u)}$$

$$\cdot\, g_{\sigma^{-1}}(\sigma(h(u)))\varphi_{\bar{\gamma}}(h(u))(1 + O^1)\, dV(u)$$

where $\varepsilon^{-1}u$ is short for $\delta_{\varepsilon^{-1}}u$. A change of variable $v = \varepsilon^{-1}u$ gives (as $dV(u) = \varepsilon^{2n+2}\, dV(v)$)

$$\varepsilon \int \psi_{\lambda,\sigma}(\xi, h(\varepsilon v)) z^\mu(v) \frac{\Phi^1_{n-2}(v)}{\bar{\rho}_1(v)} \cdot g_{\sigma^{-1}}(\sigma(h(\varepsilon v)))\varphi_{\bar{\gamma}}(h(\varepsilon v))(1 + O^1(\varepsilon v))\, dV(v).$$

The absolute value of this integral may be estimated by above by

$$\varepsilon \sup_{\rho(\xi,\eta)\le 1} [\psi_{\lambda,\sigma}(\xi,\eta) g_{\sigma^{-1}}(\sigma(\eta))\varphi_{\bar{\gamma}}(\eta)] \int_{|v|\le 1} z^\mu(v) \left|\frac{\Phi^1_{n-2}(v)}{\bar{\rho}_1(v)}\right| (1 + \varepsilon|v|)\, dV(v)$$

which goes to zero, as $\varepsilon \to 0$. Moreover, in the limit, the O^1 and O^2 terms are

$$\sum_\lambda O^1(\bar{z}^\lambda f \Phi_{n-2})(\Theta_\eta(\xi)) + \sum_\lambda O^1(z^\lambda f \Phi_{n-2})(\Theta_\eta(\xi)) + O^2(f\Phi_{n-2})(\Theta_\eta(\xi))$$

where $f(z,t) = -[n\|z\|^2 + (n-2)it]/[\|z\|^4 + t^2]$. Note that $|f(y)| \le C_n |y|^{-2}$ hence $O^1 \bar{z}^\lambda f$, $O^1 z^\lambda f$ and $O^2 f$ are bounded. Now, for instance, let us look at $k(y) = (O^1 \bar{z}^\lambda f \Phi_{n-2})(y)$ (the discussion of the remaining terms is similar). First, note that $\bar{z}^\lambda f \Phi_{n-2}$ is homogeneous of degree $-2n-1$, with respect to dilations. The Taylor series expansion (about $0 = \Theta_\eta(\eta)$) of the O^1 coefficients is a sum of homogeneous terms of degree at least 1 (with coefficients depending on η) plus a remainder of arbitrarily high order, hence the 'principal part' of $k(y)$ is homogeneous of degree $-2n$. Therefore $k(\Theta(\eta,\xi))$ is a kernel of type 1. Q.e.d..

To end the proof of Theorem 5, we shall show that $A_\Omega \Box_\Omega - a(G)I$ is an operator of type 1. First, note that A_σ, and then A_Ω, is symmetric. Indeed, for any two $(0,1)$-forms φ and ψ

$$(A_\sigma \varphi, \psi) = a(\sigma)^{2n+1} \int g_\sigma(\xi)^{\bar\alpha}_{\bar\beta} K(\sigma(\xi), \sigma(\eta)) g_{\sigma^{-1}}(\sigma(\eta))^{\bar\gamma}_{\bar\alpha} \varphi_{\bar\gamma}(\eta) \psi^{\bar\beta}(\xi)\, d\eta d\xi.$$

As $\Phi_\alpha(-y) = \overline{\Phi_{\bar\alpha}(y)}$, it follows that $\overline{K(\sigma(\xi), \sigma(\eta))} = K(\sigma(\eta), \sigma(\xi))$. Hence

$$(A^*_\sigma \psi)_{\bar\mu}(\eta) = a(\sigma)^{2n+1} h_{\gamma\bar\mu}(\eta) \int g_{\sigma^{-1}}(\sigma(\eta))^{\gamma}_{\alpha} K(\sigma(\eta), \sigma(\xi)) g_\sigma(\xi)^{\alpha}_{\beta} \psi^{\beta}(\xi)\, d\xi$$

$$= a(\sigma)^{2n} \int g_\sigma(\eta)^{\bar\lambda}_{\bar\mu} h_{\alpha\bar\lambda}(\eta) K(\sigma(\eta), \sigma(\xi)) g_\sigma(\xi)^{\alpha}_{\beta} \psi^{\beta}(\xi)\, d\xi$$

$$= a(\sigma)^{2n+1} \int g_\sigma(\eta)^{\bar\lambda}_{\bar\mu} h_{\alpha\bar\lambda}(\eta) K(\sigma(\eta), \sigma(\xi)) g_{\sigma^{-1}}(\sigma(\xi))^{\bar\gamma}_{\bar\beta} h^{\alpha\bar\beta}(\xi) \psi_{\bar\gamma}(\xi)\, d\xi.$$

Finally (as $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$)

$$(A^*_\sigma \psi)_{\bar\mu} = (A_\sigma \psi)_{\bar\mu},$$

q.e.d.. Moreover, $\Box_\Omega$ is symmetric on compactly supported forms hence

$$A_\Omega \Box_\Omega \psi = a(G)\psi - \frac{1}{|G|} \sum_{\sigma \in G} a(\sigma) B^*_\sigma \psi$$

and the transpose of B_σ (an operator of type 1) is again of type 1.

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