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SMOOTHLY SYMMETRIZABLE SYSTEMS AND THE REDUCED DIMENSIONS

Dedicated to Professor Kunihiko KAJITANI on his sixtieth birthday

By

Tatsuo NISHITANI and Jean VAILLANT

1. Introduction

In this note we study a first order system

$$L(x, D) = \sum_{j=1}^{n} A_j(x) D_j$$

where $A_1 = I$ is the identity matrix of order m and $A_j(x)$ are $M(m, \boldsymbol{R})$-valued smooth functions, where $M(m, \boldsymbol{R})$ denotes the set of all $m \times m$ real matrices. Let $L(x, \xi)$ be the symbol of $L(x, D)$:

$$L(x, \xi) = \sum_{j=1}^{n} A_j(x) \xi_j.$$

Our main concern is to study when we can symmetrize $L(x, \xi)$ smoothly. We write

$$L(x, \xi) = (\phi_j^i(x, \xi))$$

where $\phi_j^i(x, \xi)$ stands for the (i, j)-th entry of $L(x, \xi)$ which is a linear form in ξ.

DEFINITION 1.1. Let us denote

$$d(L(x, \cdot)) = \dim \operatorname{span}\{\phi_j^i(x, \cdot)\}.$$

We call $d(L(x, \cdot))$ the reduced dimension of L at x. This is equal to the dimension of the linear subspace of $M(m, \boldsymbol{R})$ spanned by $A_1(x), A_2(x), \ldots, A_n(x)$.

DEFINITION 1.2. We say that $A \in M(m, \boldsymbol{R})$ is real diagonalizable if A is diagonalizable and all eigenvalues of A are real.

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We remark that if $L(x,\xi)$ is real diagonalizable for every ξ then $d(L(x,\cdot)) \le m(m+1)/2$. Now we have

THEOREM 1.1. *Assume that $d(L(x_0,\cdot)) \ge m(m+1)/2$ if $m=2$ or $d(L(x_0,\cdot)) \ge m(m+1)/2-1$ if $m \ge 3$ and $L(x,\xi)$ is real diagonalizable for every (x,ξ), x near x_0. Then $L(x,\xi)$ is smoothly symmetrizable, that is there is a smooth $S(x)$ defined near x_0 such that*

$$S(x)^{-1}L(x,\xi)S(x)$$

is symmetric.

REMARK 1.1. Let $n=2$ and assume that $L(x,\xi)$ is "uniformly" diagonalizable. If $m=2$ and $A_2(x)$ is real analytic then L is strongly hyperbolic (see [5]). On the other hand, if $A_2(x)$ depends only on x_1 real analytically then it is proved in [6] that $L(x,\xi)$ is smoothly symmetrizable.

When $m=4$ and $d(L(x_0,\cdot)) = m(m+1)-2$, assuming that $L(x_0,\xi)$ is uniformly diagonalizable, the symmetrizability of $L(x_0,\cdot)$ is discussed in [9]. In [1], [3], [4] the existence of smooth symmetrizer $S(x,\xi)$, depending also on ξ is studied under similar (but microlocalized) assumptions.

The proof of Theorem 1.1 consists in two steps. In the first step, we refer to our previous results in [2], [8]. From the assumption it is clear that

$$d(L(x,\cdot)) \ge \frac{m(m+1)}{2} - 1$$

near x_0 then by Theorem 3.4 in [2], for every x near x_0, there is a $S(x)$ such that $S(x)^{-1}L(x,\xi)S(x)$ is symmetric for every ξ. Since $S(x)$ is free from multiplication by non-zero factor, $S(x)$ may be discontinuous. Let us set

$$H(x) = S(x)\,{}^tS(x)$$

which is of course positive definite and satisfies

$$L(x,\xi)H(x) = H(x)\,{}^tL(x,\xi). \tag{1.1}$$

In the second step we show that there is a scalar $h(x)$, $h(x) \ne 0$ near x_0 such that

$$\tilde{H}(x) = \frac{H(x)}{h(x)}$$

is smooth near x_0 and positive definite. Then taking $\tilde{S}(x)$ as

$$\tilde{S}(x) = (\tilde{H}(x))^{1/2}$$

$\tilde{S}(x)$ is a desired smooth symmetrizer because $\tilde{H}(x)$ verifies (1.1) and hence

$$\tilde{S}(x)^{-1}L(x,\xi)\tilde{S}(x) = \tilde{S}(x)^{t}L(x,\xi)\tilde{S}(x)^{-1}$$

which shows that $\tilde{S}(x)$ symmetrizes $L(x,\xi)$.

REMARK 1.2. Let us take $A_1 = I, A_2, \ldots, A_n$ real symmetric constant matrices which span the space of real symmetric matrices, so that $n = m(m+1)/2$. Set

$$P(\xi) = \sum_{j=1}^{n} A_j \xi_j$$

and take non singular smooth matrix $N(x)$. Then

$$L(x,\xi) = N(x)^{-1}P(\xi)N(x)$$

verifies the assumptions in Theorem 1.1. By Theorem 1.1 there is a smooth $S(x)$ such that $S(x)^{-1}L(x,\xi)S(x)$ is symmetric. We show that this $S(x)$ is not diagonal if ${}^{t}N(x)N(x)$ is not. This means that in Theorem 1.1 we could not choose a diagonal symmetrizer in general (compare with Lemma 2.3 in [2]). Then it is easy to see that

$$P(\xi)N(x)S(x)^{t}S(x)^{t}N(x) = N(x)S(x)^{t}S(x)^{t}N(x)P(\xi).$$

A matrix which commutes with all real symmetric matrices is a scalar matrix and hence

$$N(x)S(x)^{t}S(x)^{t}N(x) = c(x)I$$

with some scalar $c(x)$. Thus we can "recover" ${}^{t}N(x)N(x)$ as

$${}^{t}N(x)N(x) = c(x)[S(x)^{t}S(x)]^{-1}.$$

In particular, $S(x)$ is not diagonal if ${}^{t}N(x)N(x)$ is not as claimed above.

2. Preliminaries

We first note that the diagonalizability of $L(x_0,\xi)$ for every ξ implies that

$$\mathrm{span}\{\phi_j^i(x_0,\cdot) \mid i \geq j\} = \mathrm{span}\{\phi_j^i(x_0,\cdot) \mid i \leq j\}$$

which follows from Lemma 2.2 in [2]. In particular we have

$$d(L(x_0,\cdot)) = \dim \mathrm{span}\{\phi_j^i(x_0,\cdot) \mid i \geq j\} = \dim \mathrm{span}\{\phi_j^i(x_0,\cdot) \mid i \leq j\}.$$

To prove Theorem 1.1, the point is the following two lemmas.

LEMMA 2.1. *Let $m \geq 3$. Assume that $\phi_i^i(x_0, \cdot)$, $i \in \{2, \ldots, m\} = \Delta_1$, $\phi_j^i(x_0, \cdot)$, $i > j$ are linearly independent and $L(x, \xi)$ is real diagonalizable for every (x, ξ), x near x_0. Then there is a smooth $S(x)$ defined near x_0 such that with*

$$S(x)^{-1}L(x, \xi)S(x) = (\tilde{\phi}_j^i(x, \xi))$$

one can express $\tilde{\phi}_q^p(x, \xi)$, $2 \leq p < q$ as

$$\tilde{\phi}_q^p(x, \xi) = \sum_{i>j} C_{qi}^{pj}(x)\tilde{\phi}_j^i(x, \xi)$$

near x_0.

PROOF. Take $\psi_k(\xi)$, $1 \leq k \leq l$ so that $\phi_i^i(x_0, \cdot)$, $i \in \Delta_1$, $\phi_j^i(x_0, \cdot)$, $i > j$ and $\psi_k(\cdot)$, $1 \leq k \leq l$ to be a basis for the space of all linear forms in ξ. Then it is clear that $\phi_{p+1}^p(x, \xi)$ is written as

$$\phi_{p+1}^p(x, \cdot) = \sum_{i \in \Delta_1} C_{p+1i}^{pi}(x)\phi_i^i(x, \cdot) + \sum_{i>j} C_{p+1i}^{pj}(x)\phi_j^i(x, \cdot) + \sum_{k=1}^{l} D_k(x)\psi_k(\cdot) \tag{2.1}$$

with smooth $C_{p+1i}^{pj}(x)$, $D_k(x)$. The same arguments proving Lemma 3.1 in [2], applied with each fixed x, show that $\phi_{p+1}^p(x) = 0$ if

$$\phi_p^p(x, \xi) = \phi_{p+1}^{p+1}(x, \xi), \quad \phi_j^i(x, \xi) = 0 \quad i > j.$$

This implies that $\phi_{p+1}^p(x, \xi)$ can be written as

$$\phi_{p+1}^p(x, \cdot) = a(x)(\phi_p^p(x, \cdot) - \phi_{p+1}^{p+1}(x, \cdot)) + \sum_{i>j} a_{p+1i}^{pj}(x)\phi_j^i(x, \cdot).$$

If $p \geq 2$, by the linearly independence of $\phi_j^i(x_0, \cdot)$ we conclude that

$$\phi_{p+1}^p(x, \cdot) = C_{p+1p+1}^{pp}(x)(\phi_p^p(x, \cdot) - \phi_{p+1}^{p+1}(x, \cdot)) + \sum_{i>j} C_{p+1i}^{pj}(x)\phi_j^i(x, \cdot).$$

Now let us define the matrix

$$T_q^p(c) = I + Q_q^p(c)$$

where every entry of $Q_q^p(c)$ is zero except for the (p, q)-th entry which is c. Considering

$$T_m^{m-1}(C_{mm-1}^{m-1m-1}(x)) \cdots T_2^1(C_{21}^{11}(x))L(x, \xi)T_2^1(-C_{21}^{11}(x)) \cdots T_m^{m-1}(-C_{mm-1}^{m-1m-1}(x))$$

which we denote by $(\tilde{\phi}_j^i(x,\xi))$ provisionary, it is easy to see that

$$\operatorname{span}\{\tilde{\phi}_j^i(x,\cdot) \mid i > j\} = \operatorname{span}\{\phi_j^i(x,\cdot) \mid i > j\}$$

and

$$\tilde{\phi}_{p+1}^p(x,\xi) = 0 \quad \text{if} \quad \phi_j^i(x,\xi) = 0, \quad i > j.$$

Hence one can write

$$\tilde{\phi}_{p+1}^p(x,\xi) = \sum_{i>j} \tilde{C}_{p+1i}^{pj}(x)\tilde{\phi}_j^i(x,\xi).$$

We proceed by induction on $i-j$. Assume that

$$\phi_j^i(x,\xi) = 0 \quad \text{for } i < j \leq i+r \quad \text{if} \quad \phi_j^i(x,\xi) = 0, \quad i > j.$$

Let $q = p + r + 1$. Note that one can write

$$\phi_q^p(x,\cdot) = \sum_{i\in\Delta_1} C_{qi}^{pi}(x)\phi_i^i(x,\cdot) + \sum_{i>j} C_{qi}^{pj}(x)\phi_j^i(x,\cdot) + \sum_{k=1}^{l} D_k(x)\psi_k(\cdot) \tag{2.2}$$

with smooth $C_{qi}^{pj}(x)$, $D_k(x)$. Applying again the same arguments as in the proof of Lemma 3.1 in [2], for each fixed x, we get

$$\phi_q^p(x,\xi) = 0 \quad \text{if} \quad \phi_p^p(x,\xi) = \phi_q^q(x,\xi), \quad \phi_j^i(x,\xi) = 0, \quad i > j.$$

This proves that $D_k(x) = 0$, $1 \leq k \leq l$, $C_{qi}^{pi}(x) = 0$ unless $i = p$ or $i = q$ and $C_{qp}^{pp}(x) = -C_{qq}^{pq}(x)$ in (2.2). Repeating the same arguments as before, one can find a smooth $T(x)$ such that, with

$$T(x)^{-1}L(x,\xi)T(x) = (\tilde{\phi}_j^i(x,\xi))$$

we have

$$\operatorname{span}\{\tilde{\phi}_j^i(x,\cdot) \mid i > j\} = \operatorname{span}\{\phi_j^i(x,\cdot) \mid i > j\}$$

and

$$\tilde{\phi}_q^p(x,\xi) = 0 \quad \text{if} \quad \phi_j^i(x,\xi) = 0, \quad i > j.$$

Note that with

$$T(x)^{-1}(m_j^i)T(x) = (\tilde{m}_j^i)$$

one has $\tilde{m}^i_j = 0$ for $i < j \le i + r$ if $m^i_j = 0$, $i < j \le i + r$. This proves that

$$\tilde{\phi}^i_j(x, \xi) = 0 \quad \text{for } i < j \le i + r + 1 \quad \text{if } \tilde{\phi}^i_j(x, \xi) = 0, \quad i > j.$$

By induction we get the desired assertion. ■

REMARK 2.1. In Lemma 2.1 if $\phi^i_i(x_0, \cdot)$, $i = 1, \dots, m$, $\phi^i_j(x_0, \cdot)$, $i > j$ are linearly independent, one can find a smooth $S(x)$ so that the assertion of the lemma holds for any $\tilde{\phi}^p_q(x, \xi)$, $1 \le p < q$.

We modify this lemma for later use. Let us denote

$$\Delta = \{1, 2, \dots, m\}, \quad I = \{(i, j) \mid i > j, (i, j) \neq (2, 1)\}.$$

LEMMA 2.2. *Let $m \ge 4$. Assume that $\phi^i_i(x_0, \cdot)$, $i \in \Delta$, $\phi^i_j(x_0, \cdot)$, $(i, j) \in I$ are linearly independent and $L(x, \xi)$ is real diagonalizable for every (x, ξ), x near x_0. Then there is a smooth $S(x)$ defined near x_0 such that with*

$$S(x)^{-1} L(x, \xi) S(x) = (\tilde{\phi}^i_j(x, \xi))$$

one can express $\tilde{\phi}^p_q(x, \xi)$, $3 \le p < q$ as

$$\tilde{\phi}^p_q(x, \xi) = \sum_{(i,j) \in I} C^{pj}_{qi}(x) \tilde{\phi}^i_j(x, \xi)$$

near x_0.

PROOF. As in the proof of Lemma 2.1, we take $\psi_k(\xi)$, $1 \le k \le l$ so that $\phi^i_i(x_0, \cdot)$, $i \in \Delta$, $\phi^i_j(x_0, \cdot)$, $(i, j) \in I$, $\psi_k(\cdot)$, $1 \le k \le l$ form a basis of the space of linear forms in ξ. Then one can express $\phi^p_q(x, \cdot)$, $p < q$ as

$$\phi^p_q(x, \cdot) = \sum_{i \in \Delta} C^{pi}_{qi}(x) \phi^i_i(x, \cdot) + \sum_{(i,j) \in I} C^{pj}_{qi}(x) \phi^i_j(x, \cdot) + \sum_{k=1}^{l} D_k(x) \psi_k(\cdot) \tag{2.3}$$

with smooth $C^{pj}_{qi}(x)$, $D_k(x)$. We apply the same arguments as in the proof of Lemma 2.1 to $(m-2) \times (m-2)$ submatrix consisting of the last $m-2$ rows and columns of $L(x, \xi)$. To do so we first show that for $p \ge 3$ we have $\phi^p_{p+1}(x, \xi) = 0$ if

$$\phi^p_p(x, \xi) = \phi^{p+1}_{p+1}(x, \xi), \quad \phi^i_j(x, \xi) = 0, \quad (i, j) \in I. \tag{2.4}$$

Set

$$J(x, \xi) = \{i \mid 3 \le i \le m, i \neq p, i \neq p + 1, \phi^i_i(x, \xi) = \phi^p_p(x, \xi)\}$$

and denote by $i(x,\xi)$ the multiplicity of the eigenvalue 0 of

$$F(x,\xi) = \begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} - \phi_p^p I_2.$$

Assume that ξ satisfies (2.4). Then it is obvious that 0 is an eigenvalue of $L(x,\xi) - \phi_p^p(x,\xi)I$ with multiplicity $\sharp J(x,\xi) + i(x,\xi) + 2$. By the diagonalizability, every $(m - \sharp J(x,\xi) - i(x,\xi) - 1)$-th minor of $L(x,\xi) - \phi_p^p(x,\xi)I$ is zero.

If $i(x,\xi) = 0$ then take $(m - \sharp J(x,\xi) - 1)$-th minor obtained removing the i-th rows and columns for $i \in J(x,\xi)$ and $(p+1)$-th row and p-th column, which is

$$\phi_{p+1}^p(x,\xi) \prod_{i \notin J(x,\xi), i \neq p, i \neq p+1} (\phi_i^i(x,\xi) - \phi_p^p(x,\xi)) \det F(x,\xi)$$

up to the sign. Since this should be zero we get $\phi_{p+1}^p(x,\xi) = 0$ if (2.4) is verified. If $i(x,\xi) = 1$ then some entry of $F(x,\xi)$ is different from zero. We now take $(m - \sharp J(x,\xi) - 2)$-th minor obtained removing the same rows and columns in the case $i(x,\xi) = 0$ and one more row and column (the first or the second) so that a non zero entry of $F(x,\xi)$ consists in the minor. Then the minor is, up to the sign

$$\phi_{p+1}^p(x,\xi) \prod_{i \in J(x,\xi), i \neq p, i \neq p+1} (\phi_i^i(x,\xi) - \phi_p^p(x,\xi)) \times (\text{non zero entry of } F).$$

This shows again $\phi_{p+1}^p(x,\xi) = 0$ if ξ verifies (2.4). Finally if $i(x,\xi) = 2$ then take $(m - \sharp J(x,\xi) - 3)$-th minor obtained removing the same rows and columns as in the first case and two more (the first and the second) rows and columns. The minor is, up to the sign

$$\phi_{p+1}^p(x,\xi) \prod_{i \notin J(x,\xi), i \neq p, i \neq p+1} (\phi_i^i(x,\xi) - \phi_p^p(x,\xi))$$

and hence one obtains $\phi_{p+1}^p(x,\xi) = 0$ for ξ verifying (2.4). These prove the assertion. In particular this shows that $C_{p+1i}^{pi}(x) = 0$ unless $i = p$ or $i = p+1$, $D_k(x) = 0$, $1 \le k \le l$ and $C_{p+1p}^{pp}(x) = -C_{p+1p+1}^{pp+1}(x)$. Then we apply the same arguments as in the proof of Lemma 2.1 to lower $(m-2) \times (m-2)$ submatrix of L. Remarking that $T_q^p(c)$, $3 \le p < q$ leaves the principal 2×2 submatrix invariant, the rest of the proof is a repetition of Lemma 2.1. ■

3. Proof of Theorem

Let $m \geq 3$ and we work near x_0. We divide the cases into two:
(1) $\dim \operatorname{span}\{\phi_j^i(x_0, \cdot) \mid i > j\} = m(m+1)/2 - m$
(2) $\dim \operatorname{span}\{\phi_j^i(x_0, \cdot) \mid i > j\} = m(m+1)/2 - m - 1$

We first study the case (1). Since $\dim \operatorname{span}\{\phi_1^1(x_0, \cdot), \dots, \phi_m^m(x_0, \cdot)\} \geq m-1$ we may suppose that $\phi_2^2(x_0, \cdot), \dots, \phi_m^m(x_0, \cdot)$ are linearly independent considering $N^{-1}L(x, \xi)N$ with suitable constant matrix N if necessary. From Lemma 2.1 there exists a smooth $T(x)$, defined near x_0 such that with

$$T(x)^{-1}L(x, \xi)T(x) = (\tilde{\phi}_j^i(x, \xi))$$

one can express $\tilde{\phi}_q^p(x, \xi)$, $2 \leq p < q$ as

$$\tilde{\phi}_q^p(x, \xi) = \sum_{i>j} \tilde{C}_{qi}^{pj}(x)\tilde{\phi}_j^i(x, \xi) \tag{3.1}$$

near x_0. In what follows we drop the sign tilde. As sketched in Introduction, from Theorems 3.4 and 4.1 in [2], for every fixed x, there is a $S(x)$ such that $S(x)^{-1}L(x, \xi)S(x)$ is symmetric for every ξ. Let us set

$$H(x) = S(x)^t S(x) \tag{3.2}$$

then it is easy to see

$$L(x, \xi)H(x) = H(x)^t L(x, \xi). \tag{3.3}$$

We show that there is a $\tilde{H}(x)$ which is positive definite, smooth near x_0 and verifies (3.3) so that to prove Theorem 1.1 it is enough to take $S(x) = (\tilde{H}(x))^{1/2}$.

Let us write

$$H(x) = \begin{pmatrix} h_1^1 & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where H_{22} is the $(m-1) \times (m-1)$ matrix consisting of the last $m-1$ rows and columns of H. We also write

$$L = \begin{pmatrix} \phi_1^1 & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

where the blocking corresponds to that of $H(x)$. The equation (3.3) yields

$$\begin{cases} h_1^1 L_{21} + L_{22}H_{21} = \phi_1^1 H_{21} + H_{22}{}^t L_{12} \\ L_{21}H_{12} + L_{22}H_{22} = H_{21}{}^t L_{21} + H_{22}{}^t L_{22}. \end{cases} \tag{3.4}$$

Take $\xi = \xi(x)$ so that $L_{21}(x,\xi) = 0$. Then the second equation of (3.4) becomes

$$L_{22}(x,\xi)H_{22}(x) = H_{22}(x)^t L_{22}(x,\xi). \tag{3.5}$$

Take now $\xi = \xi(x)$ such that $\phi_j^i(x,\xi) = 0$ for $i > j$ then in virtue of (3.1), the equation (3.5) will be

$$h_j^i(x)(\phi_i^i(x,\xi) - \phi_j^j(x,\xi)) = 0, \quad i,j \geq 2 \tag{3.6}$$

where we have denoted $H(x) = (h_j^i(x))$. Since $\phi_i^i(x,\cdot)$, $i = 2,\dots,m$ are linearly independent near x_0 one can take $\xi = \xi(x)$ so that $\phi_i^i(x,\xi) - \phi_j^j(x,\xi) \neq 0$ if $i \neq j$, $i,j \geq 2$ and this shows that $h_j^i(x) = 0$ if $i \neq j$, $i,j \geq 2$, that is $H_{22}(x)$ is diagonal. We now observe the last row of (3.5). It yields

$$\phi_i^m(x,\xi)h_i^i(x) = h_m^m(x)\phi_m^i(x,\xi), \quad i = 2,\dots,m. \tag{3.7}$$

Since no $\phi_i^m(x_0,\cdot)$, $i = 2,\dots,m$ is zero, writing

$$\phi_i^m(x,\xi) = \sum_{k=1}^{n} C_{ik}^m(x)\xi_k$$

there is $k(i)$ such that $C_{ik(i)}^m(x_0) \neq 0$ and hence $C_{ik(i)}^m(x) \neq 0$ near x_0. Thus from (3.7) it follows that

$$C_{mk(i)}^i(x)h_m^m(x) = C_{ik(i)}^m(x)h_i^i(x)$$

and hence we have

$$\frac{h_i^i(x)}{h_m^m(x)} = \frac{C_{mk(i)}^i(x)}{C_{ik(i)}^m(x)}$$

which shows that the left-hand side is smooth near x_0. Then we conclude that $H_{22}(x)$ is diagonal and $H_{22}(x)/h_m^m(x)$ is smooth near x_0. Note that

$$h_m^m(x_0) > 0 \tag{3.8}$$

because $H(x_0)$ is positive definite. Dividing $H(x)$ by $h_m^m(x)$ and we denote it by $\tilde{H}(x)$. Note that (3.3) still holds and hence (3.4) does. Moreover by (3.8) $\tilde{H}(x)$ is positive definite. The first equation of (3.4) is

$$\tilde{h}_1^1 L_{21} + (L_{22} - \phi_1^1)\tilde{H}_{21} = \tilde{H}_{22}{}^t L_{12}. \tag{3.9}$$

We take ξ so that $\phi_j^i(x,\xi) = 0$, $i > j$ then the equation is reduced to

$$(\phi_j^j(x,\cdot) - \phi_1^1(x,\cdot))\tilde{h}_1^j(x) = \tilde{h}_j^j(x)\phi_j^1(x,\cdot), \quad j = 2,\dots,m. \tag{3.10}$$

If there is j such that $\phi_j^j(x_0,\cdot)-\phi_1^1(x_0,\cdot)=0$ then $\phi_i^i(x_0,\cdot)-\phi_1^1(x_0,\cdot)$ is not zero for $i\geq 2$, $i\neq j$. This shows that $\tilde{h}_1^i(x)$ is smooth for $i\geq 2$, $i\neq j$ by (3.10). Fixing i, $i\geq 2$, $i\neq j$, we study the $(i,1)$-th and (i,j)-th entries in (3.3). Then we get

$$\phi_1^i(x,\xi)\tilde{h}_1^1(x)+\sum_{k\geq 2,k\neq j}\phi_k^i(x,\xi)\tilde{h}_1^k(x)+\phi_j^i(x,\xi)\tilde{h}_1^j(x) \tag{3.11}$$

$$=\tilde{h}_1^i(x)\phi_1^i(x,\xi)+\tilde{h}_i^i(x)\phi_i^1(x,\xi)$$

$$\phi_1^i(x,\xi)\tilde{h}_j^1(x)+\phi_j^i(x,\xi)\tilde{h}_j^j(x)=\tilde{h}_1^i(x)\phi_1^j(x,\xi)+\tilde{h}_i^i(x)\phi_i^j(x,\xi). \tag{3.12}$$

Since $\tilde{h}_j^j(x)$ $(j\geq 2)$ is smooth and $\phi_1^i(x_0,\cdot)$ is not zero then from (3.12) we conclude that $\tilde{h}_j^1(x)$ is smooth near x_0. Since $\tilde{h}_j^1(x)=\tilde{h}_1^j(x)$ and $\tilde{h}_1^k(x)$, $k\geq 2$, $k\neq j$ are smooth, we then conclude from (3.11) that $\tilde{h}_1^1(x)$ is also smooth near x_0. This proves that $\tilde{H}(x)$ is smooth near x_0. Since $\tilde{H}(x_0)$ is positive definite we get the assertion. When $\phi_i^i(x_0,\cdot)-\phi_1^1(x_0,\cdot)\neq 0$, $i=2,\dots,m$ the proof is an easier repetition.

We next study the case (2). Since $d(L(x_0,\cdot))=m(m+1)/2-1$ we may assume that $\phi_i^i(x_0,\cdot)$, $1\leq i\leq m$, $\phi_j^i(x_0,\cdot)$, $(i,j)\in I$, $I=\{(i,j)\mid i>j,(i,j)\neq(2,1)\}$ are linearly independent, considering $N^{-1}L(x,\xi)N$ with a suitable constant matrix N if necessary. We now apply Lemma 2.2 to conclude that there is a smooth $T(x)$ such that with

$$T(x)^{-1}L(x,\xi)T(x)=(\tilde{\phi}_j^i(x,\xi))$$

one can express $\tilde{\phi}_q^p(x,\xi)$, $3\leq p<q$ as

$$\tilde{\phi}_q^p(x,\cdot)=\sum_{(i,j)\in I}\tilde{C}_{qi}^{pj}(x)\tilde{\phi}_j^i(x,\cdot) \tag{3.13}$$

near x_0. In what follows we drop the sign tilde and let $H(x)$ be as before. Let us write

$$H(x)=\begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix},\quad H_{11}=\begin{pmatrix}h_1^1 & h_2^1\\ h_1^2 & h_2^2\end{pmatrix}$$

where $H_{22}(x)$ is the $(m-2)\times(m-2)$ matrix consisting of the last $m-2$ rows and columns of H. Denote

$$L(x,\xi)=\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}$$

where the blocking corresponds to that of H. From (3.3) one has

$$L_{21}H_{12} + L_{22}H_{22} = H_{21}{}^tL_{21} + H_{22}{}^tL_{22} \tag{3.14}$$

$$L_{21}H_{11} + L_{22}H_{21} = H_{21}{}^tL_{11} + H_{22}{}^tL_{12}. \tag{3.15}$$

By choosing ξ so that $\phi_j^i(x,\xi) = 0$, $(i,j) \in I$, if $m \geq 4$ then thanks to (3.13) the same arguments as before show that $H_{22}(x)$ is diagonal. Observing again the last row of (3.15), we see that $H_{22}(x)/h_m^m(x)$ is smooth near x_0. If $m = 3$ this is obvious. We divide $H(x)$ by $h_m^m(x)$ and denote it by $\tilde{H}(x)$. Note that $\tilde{H}(x_0)$ is positive definite. Take ξ again so that

$$\phi_j^i(x,\xi) = 0, \quad (i,j) \in I.$$

Then (3.15) is written as

$$L_{22}\tilde{H}_{21} - \tilde{H}_{21}{}^tL_{11} = \tilde{H}_{22}{}^tL_{12} = (g_j^i) \tag{3.16}$$

because $L_{21}(x,\xi) = 0$. Note that $L_{22}\tilde{H}_{21} - \tilde{H}_{21}{}^tL_{11}$ is

$$\begin{pmatrix} (\phi_3^3 - \phi_1^1)\tilde{h}_1^3 - \phi_2^1\tilde{h}_2^3 & (\phi_3^3 - \phi_2^2)\tilde{h}_2^3 - \phi_1^2\tilde{h}_1^3 \\ (\phi_4^4 - \phi_1^1)\tilde{h}_1^4 - \phi_2^1\tilde{h}_2^4 & (\phi_4^4 - \phi_2^2)\tilde{h}_2^4 - \phi_1^2\tilde{h}_1^4 \\ \vdots & \vdots \\ (\phi_m^m - \phi_1^1)\tilde{h}_1^m - \phi_2^1\tilde{h}_2^m & (\phi_m^m - \phi_2^2)\tilde{h}_2^m - \phi_1^2\tilde{h}_1^m \end{pmatrix}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\begin{pmatrix} \phi_j^j - \phi_1^1 & -\phi_2^1 \\ -\phi_1^2 & \phi_j^j - \phi_2^2 \end{pmatrix}\begin{pmatrix} \tilde{h}_1^j \\ \tilde{h}_2^j \end{pmatrix} = \begin{pmatrix} g_1^{j-2} \\ g_2^{j-2} \end{pmatrix}, \quad j = 3,\ldots,m. \tag{3.18}$$

Here we remark that there is ξ^0 such that

$$\phi_j^i(x_0,\xi^0) = 0, \quad (i,j) \in I, \quad \det\begin{pmatrix} \phi_j^j - \phi_1^1 & -\phi_2^1 \\ -\phi_1^2 & \phi_j^j - \phi_2^2 \end{pmatrix}(x_0,\xi^0) \neq 0.$$

We prove the assertion by contradiction. Suppose

$$\det\begin{pmatrix} \phi_j^j - \phi_1^1 & -\phi_2^1 \\ -\phi_1^2 & \phi_j^j - \phi_2^2 \end{pmatrix}(x_0,\xi) = 0$$

for every ξ with $\phi_j^i(x_0,\xi) = 0$, $(i,j) \in I$. Since $(\phi_j^j - \phi_1^1)(x_0,\cdot)$, $(\phi_j^j - \phi_2^2)(x_0,\cdot)$, $(j \geq 3)$ are not zero it is clear that one has either

$$(3.19)\qquad \begin{cases} \phi_2^1(x_0,\xi) = \alpha(\phi_j^j - \phi_1^1)(x_0,\xi) \\ \phi_1^2(x_0,\xi) = \beta(\phi_j^j - \phi_2^2)(x_0,\xi) \end{cases}$$

or

$$(3.20)\qquad \begin{cases} \phi_2^1(x_0,\xi) = \alpha(\phi_j^j - \phi_2^2)(x_0,\xi) \\ \phi_1^2(x_0,\xi) = \beta(\phi_j^j - \phi_1^1)(x_0,\xi) \end{cases}$$

for ξ with $\phi_j^i(x_0,\xi) = 0$, $(i,j) \in I$ where α, β are constants such that $\alpha\beta = 1$. On the other hand recall that one can write

$$\phi_1^2(x_0,\cdot) = \sum_{(i,j)\in I} C_i^j \phi_j^i(x_0,\cdot).$$

Now supposing, say (3.19), we get

$$\phi_1^2(x_0,\xi) = \beta(\phi_j^j - \phi_2^2)(x_0,\xi) + \sum_{(i,j)\in I} \tilde{C}_i^j \phi_j^i(x_0,\xi).$$

But this is impossible because $\phi_j^i(x_0,\cdot)$, $(i,j) \in I$, $\phi_i^i(x_0,\cdot)$, $1 \le i \le m$ are linearly independent and $\beta \neq 0$. This proves the assertion. Then from (3.18) we see that $\tilde{h}_1^j(x)$, $\tilde{h}_2^j(x)$ $(j \ge 3)$ are smooth near x_0. We finally study $\tilde{H}_{11}(x)$. We consider the $(1,2)$-th entry of the equation (3.3). Then

$$(3.21)\qquad (\phi_1^1 - \phi_2^2)\tilde{h}_2^1 + \phi_2^1\tilde{h}_2^2 - \phi_1^2\tilde{h}_1^1 = \sum_{k=3}^{m} \tilde{h}_k^1\phi_k^2 - \sum_{k=3}^{m} \phi_k^1\tilde{h}_2^k = g_1.$$

We next take $(3,1)$-th and $(3,2)$-th entries of the equation. These are

$$(3.22)\qquad \begin{cases} \phi_1^3\tilde{h}_1^1 + \phi_2^3\tilde{h}_1^2 = \sum_{k=1}^{m} \tilde{h}_k^3\phi_k^1 - \sum_{k=3}^{m} \phi_k^3\tilde{h}_1^k = g_2 \\ \phi_1^3\tilde{h}_2^1 + \phi_2^3\tilde{h}_2^2 = \sum_{k=1}^{m} \tilde{h}_k^3\phi_k^2 - \sum_{k=3}^{m} \phi_k^3\tilde{h}_2^k = g_3. \end{cases}$$

Then (3.21) and (3.22) give

$$(3.23)\qquad \begin{pmatrix} -\phi_1^2 & \phi_2^1 & \phi_1^1 - \phi_2^2 \\ \phi_1^3 & 0 & \phi_2^3 \\ 0 & \phi_2^3 & \phi_1^3 \end{pmatrix} \begin{pmatrix} \tilde{h}_1^1 \\ \tilde{h}_2^2 \\ \tilde{h}_2^1 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

Note that the right-hand side is known to be smooth. Since the determinant of the coefficient matrix of the equation (3.23) is:

$$d(x,\xi) = \phi_1^3\phi_2^3(\phi_1^1 - \phi_2^2) + (\phi_2^3)^2\phi_1^2 - (\phi_1^3)^2\phi_2^1$$

and $\phi_1^1(x_0,\cdot)$, $\phi_2^2(x_0,\cdot)$, $\phi_1^3(x_0,\cdot)$, $\phi_2^3(x_0,\cdot)$ are linearly independent, it is clear that there is ξ^0 such that $d(x_0,\xi^0)\neq 0$. This proves that $\tilde{h}_1^1(x)$, $\tilde{h}_2^2(x)$, $\tilde{h}_2^1(x)$ are smooth near x_0, that is $\tilde{H}(x)$ is smooth near x_0 and hence the assertion.

We turn to the case $m=2$. In this case the existence of symmetrizer $S(x)$ for each fixed x follows from [7]. Taking Remark 2.1 into account the proof is a repetition of the arguments proving the smoothness of $H_{22}(x)/h_m^m(x)$ in the case (1). ■

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Tatsuo Nishitani* and Jean Vaillant**
*Department of Mathematics,
Graduate School of Science,
Osaka University,
Machikaneyama 1-16,
Toyonaka Osaka 560, Japan
**Université Pierre et Marie Curie,
Mathématiques
BC 172, 4, Place Jussieu 75252 Paris,
France