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PINKHAM-DEMAZURE CONSTRUCTION FOR TWO DIMENSIONAL CYCLIC QUOTIENT SINGULARITIES

By

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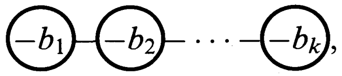
Abstract. The affine ring of any cyclic quotient singularity has countably many natural graded ring structures. In two dimensional case, we construct every such graded ring in terms of $\boldsymbol{Q}$-coefficient divisor on $\boldsymbol{P}^1$ which is naturally associated with a resolution of the singularity.

Introduction

Let n, q be relatively prime integers satisfying $1 \leqq q < n$. Let G be the group generated by $[e_n, e_n^q] \left(= \begin{pmatrix} e_n & 0 \\ 0 & e_n^q \end{pmatrix} \right) \in GL(2, \boldsymbol{C})$, where $e_n = exp(2\pi i/n)$. We consider the usual action of G on $\boldsymbol{C}^2$ and the cyclic quotient singularity $(\boldsymbol{C}^2/G, \{0\})$, and it is denoted by $C_{n,q}$ as in [5]. The weighted dual graph (= w.d.graph) of the exceptional set of the minimal resolution of $C_{n,q}$ is given as follows:

(0.1) 

where $\dfrac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}$ $(:= [[b_1, \dots, b_k]])$ and $b_i \geqq 2$.

If u, v are variables on $\boldsymbol{C}^2$, the affine ring R of $C_{n,q}$ is the invariant subring by the action of G on $\boldsymbol{C}[u, v]$. Let r, s be relatively prime positive integers. If we give the degree of u, v as $deg(u) = r$ and $deg(v) = s$ respectively, we get a graded

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ring structure $R_{r,s}$ of $C_{n,q}$. Varying (r,s) we obtain countably many graded ring structures on R.

Every cyclic quotient singularity has a C^*-action. The affine ring of normal surface singularity with C^*-action has a graded structure. For such any singularity but not cyclic quotient singularity, H. Pinkham [4] gave a nice construction for the affine graded ring in terms of a Q-(coefficient) divisor on a curve. The construction is associated to the minimal good resolution of the singularity. Such construction was generalised by Demazure [1] in more general situation (also see [7]), so let's call it Pinkham-Demazure construction. In this paper we consider Pinkham-Demazure construction for $R_{r,s}$ for $C_{n,q}$. By considering a suitable resolution of $C_{n,q}$ which is not necessarily minimal good, we show that $R_{r,s}$ is represented in terms of a Q-divisor on P^1 which is associated with the resolution.

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1. Preliminaries

In this paper we put $\{a\} = min\{n \in \boldsymbol{Z} \mid n \geqq a\}$ for any real number a. Now let r, s be relatively prime positive integers.

DEFINITION 1.1. Let $\boldsymbol{N}$ be the set of non-negative integers. We define a subset $\Gamma \subseteq \boldsymbol{N} \times \boldsymbol{N}$, the degree of elements of Γ and two positive integers μ and λ as follows:

$$\Gamma = \{(i,j) \in \boldsymbol{N} \times \boldsymbol{N} \mid i + qj \equiv 0(n)\},$$

$$deg(i,j) = ri + sj \quad \text{for} \quad (i,j) \in \Gamma,$$

$$\mu = g.c.d.(n, qr - s) \quad \text{and} \quad \lambda = \frac{n}{\mu}.$$

Let q' be the integer with $qq' \equiv 1(n)$ and $0 < q' < n$. We put $qq' = nq'' + 1$. Then $(s - rq)q' = q's - r - nq''r$ and $(r - q's)q = qr - s - nq''s$. From this we can see $\mu = g.c.d.(n, q's - r)$.

LEMMA 1.2. *$\mu = max\{c \in \boldsymbol{N} \mid deg(i,j)$ is divisible by c for any $(i,j) \in \Gamma\}$.*

PROOF. For any $(i,j) \in \Gamma$, we put $i + qj = na$ for $a \in \boldsymbol{Z}$. Since $deg(i,j) = ri + sj = nra + (s - rq)j$, we have $\mu \mid deg(i,j)$ (i.e., $deg(i,j)$ is divisible by μ). Now we assume that $c \mid deg(i,j)$ for any element $(i,j) \in \Gamma$. Since $(n - q, 1) \in \Gamma$,

$c \,|deg(n-q,1) = nr+s-rq$. Since $(n,0) \in \Gamma$, $c \,|\, nr$. Therefore we have $c \,|\, g.c.d.(nr, qr-s) = g.c.d.(n, qr-s) = \mu$, since $g.c.d.(r, qr-s) = 1$. Q.E.D.

Let R be the invariant subring of $\boldsymbol{C}[u,v]$ by the usual action of a cyclic group $G = \langle [e_n, e_n^q] \rangle$. We define the degree for any element of R as $deg(u^i v^j) = ri+sj$. If we put $R = \bigoplus_{k=0}^{\infty} R_k$ and $R_k = \bigoplus_{\substack{ri+sj=k \\ i+qj \equiv 0(n)}} \boldsymbol{C}u^i v^j$, then $R_k = 0$ for $k \not\equiv 0(\mu)$. Henceforth we may only consider the following graded ring:

$$R_{r,s} = \bigoplus_{k=0}^{\infty} R_k, \quad \text{where } R_k = \bigoplus_{\substack{ri+sj=\mu k \\ i+qj \equiv 0(n)}} \boldsymbol{C}u^i v^j.$$

We call it *normalized graded ring of* $C_{n,q}$ *with weight* (r,s).

Let q', q'' be integers as in above.

LEMMA 1.3. *Let* $b_1, \ldots, b_k$ *be as in* (0.1). *For any real number* a *we have*

$$[[b_1, \ldots, b_{k-1}, b_k + a]] = \frac{n+q'a}{q+q''a}.$$

PROOF. We prove this by the induction on k. We assume that $[[b_2, \ldots, b_k]] = \dfrac{N}{Q}$, where N and Q are relatively prime integers with $0 < Q < N$. Let Q' be the integer satisfying $QQ' \equiv 1(N)$ and $0 < Q < N$. Also let Q'' be integer with $QQ' = NQ''+1$. Since $\dfrac{n}{q} = b_1 - Q/N = (b_1 N - Q)/N$, we can easily see that $n = b_1 N - Q$ and $q = N$. Further we can see that $q' = b_1 Q' - Q''$ and $q'' = Q'$. In fact, we have $N(b_1 Q' - Q'') \equiv 1(n)$ and $0 < b_1 Q' - Q'' = (nQ'+1)/q < n$, so $q' = b_1 Q' - Q''$. We can see $q'' = Q'$ similarly. From the assumption of induction, we have $[[b_2, \ldots, b_k + a]] = (N + Q'a)/(Q + Q''a)$ and so $[[b_1, \ldots, b_k + a]] = (b_1 N - Q + (b_1 Q' - Q'')a)/(N + Q'a) = (n + q'a)/(q + q''a)$. Q.E.D.

2. Pinkham-Demazure Construction

In this section we prove our main result in this paper. Let q', r, s, μ and λ be non-negative integers as in section 1. From the definition of μ and λ, we have $g.c.d.((qr-s)/\mu, \lambda) = 1$. Let α_1, α_2 be integers defined by following congruences:

$$\alpha_1 \left(\frac{qr-s}{\mu} \right) \equiv 1(\lambda r) \quad \text{and} \quad 0 < \alpha_1 < \lambda r,$$

$$\alpha_2\left(\frac{q's-r}{\mu}\right) \equiv 1(\lambda s) \quad \text{and} \quad 0<\alpha_2<\lambda s.$$

Further let $b = \dfrac{\mu}{\lambda rs} + \dfrac{\alpha_1}{\lambda r} + \dfrac{\alpha_2}{\lambda s}$.

LEMMA 2.1. *b is an integer.*

PROOF. We have $\alpha_1 s = \alpha_1 qr - \mu + Anr$ and $\alpha_2 r = \alpha_2 q's - \mu + Bns$ for some $A, B \in \boldsymbol{Z}$. Then

$$b = \frac{\alpha_1 q + \alpha_2 + An}{\lambda s} = \frac{\alpha_2 q' + \alpha_1 + Bn}{\lambda r}.$$

Then we can easily check $b\lambda \in \boldsymbol{Z}$. Another hand we have $b\lambda s \equiv \alpha_1 q + \alpha_2 (n)$ and $b\lambda r \equiv \alpha_1 + \alpha_2 q'(n)$. Then $b\lambda rq \equiv \alpha_1 q + \alpha_2(n)$ since $qq' \equiv 1(n)$. Hence $b\lambda(qr-s) \equiv 0(n)$ and so $bTn = b\lambda\mu T \equiv 0(n)$, where we put $T = (qr-s)/\mu$. Therefore $bT \in \boldsymbol{Z}$. Hence $b \in \boldsymbol{Z}$ because of $b\lambda$, $bT \in \boldsymbol{Z}$ and $g.c.d.(\lambda, T) = 1$. Q.E.D.

DEFINITION 2.2. Let define a $\boldsymbol{Q}$-divisor $D_{r,s}$ and a $\boldsymbol{Z}$-divisor $D_{r,s}^{(k)}$ on $\boldsymbol{P}^1$ as follows:

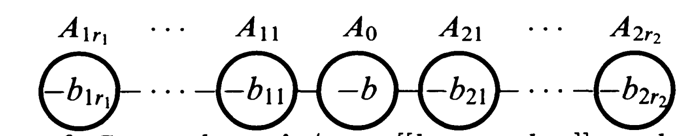
$$D_{r,s} = bP_0 - \frac{\alpha_1}{\lambda r}P_1 - \frac{\alpha_2}{\lambda s}P_2 \quad \text{and} \quad D_{r,s}^{(k)} = kbP_0 - \left\{\frac{\alpha_1 k}{\lambda r}\right\}P_1 - \left\{\frac{\alpha_2 k}{\lambda s}\right\}P_2$$

for any $k \in \boldsymbol{N}_0$, where $P_0, P_1, P_2 \in \boldsymbol{P}^1$. Further, define a graded ring as follows (see [1], [4]):

$$R(D_{r,s}) = \bigoplus_{k=0}^{\infty} H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(D_{r,s}^{(k)})) \cdot t^k.$$

THEOREM 2.3. (i) *The normalized graded ring $R_{r,s}$ of $C_{n,q}$ is isomorphic to* $R(D_{r,s})$.

(ii) *The configuration*



is the w.d.graph for a resolution of $C_{n,q}$, where $\lambda r/\alpha_1 = [[b_{11}, \dots, b_{1r_1}]]$ and $\lambda s/\alpha_2 = [[b_{21}, \dots, b_{2r_2}]]$.

PROOF. (i) Let $\pi : \boldsymbol{C}[x,y] \to \boldsymbol{C}[u,v]$ be a map defined by $u = x^r$ and $v = y^s$. Then

$$R_k \ (= (R_{r,s})_k) = \bigoplus_{\substack{ri+sj=k\mu \\ i+qj\equiv 0(n)}} \boldsymbol{C}u^i v^j \simeq \bigoplus_{\substack{ri+sj=k\mu \\ i+qj\equiv 0(n)}} \boldsymbol{C}x^{ri}y^{sj}.$$

Let $g_0 = [e_r, 1]$, $g_1 = [1, e_s]$, $g_2 = [e_{nrs}^s, e_{nrs}^{qr}]$ and $G' = \langle g_0, g_1, g_2 \rangle$. If we put $i' = ri$ and $j' = sj$, then

$$(2.1) \qquad R_k = \left(\bigoplus_{\substack{i'+j'=k\mu \\ i'/r+qj'/s \equiv 0(n)}} \boldsymbol{C} x^{i'} y^{j'} \right)^{\langle g_0, g_1 \rangle} = \boldsymbol{C}[x, y]_{k\mu}^{G'},$$

where $\boldsymbol{C}[x, y]_{k\mu}$ is the homogeneous submodule of $\boldsymbol{C}[x, y]$ of degree $k\mu$ with respect to $deg(x) = deg(y) = 1$.

Let $\sigma : M = U_0 \cup U_1 \to \boldsymbol{C}^2$ be the monoidal transformation centered at the origin and let (u_i, v_i) the coordinate of U_i ($\simeq \boldsymbol{C}^2$), so $\sigma(u_0, v_0) = (u_0, u_0 v_0) = (x, y)$, $\sigma(u_1, v_1) = (u_1 v_1, u_1) = (x, y)$, and $C := \{u_0 = u_1 = 0\} \simeq \boldsymbol{P}^1$ and M is the total space of a line bundle with degree -1. We have $\boldsymbol{C}[x, y]_{k\mu} \simeq H^0(C, \mathcal{O}_C(k\mu Q)) \cdot t^{k\mu}$, where $Q \in C$. Then

$$(2.2) \qquad R_k \simeq H^0(C, \mathcal{O}_C(k\mu Q))^{G''} \cdot t^{k\mu},$$

where G'' is the group obtained by the lifting of the action of G' through σ. The generators g_0, g_1 and g_2 can be lifted onto U_i $(i = 0, 1)$ such that $g_j \circ \sigma = \sigma \circ \tilde{g}_j$ $(j = 0, 1, 2)$ (see [2]), and they are given as follows:

$$\tilde{g}_0 = [e_r, e_r^{-1}], \quad \tilde{g}_1 = [1, e_s] \quad \text{and} \quad \tilde{g}_2 = [e_{nrs}^s, e_{nrs}^{qr-s}] \quad \text{on } U_0,$$

$$\tilde{g}_0 = [1, e_r], \quad \tilde{g}_1 = [e_s, e_s^{-1}] \quad \text{and} \quad \tilde{g}_2 = [e_{nrs}^{qr}, e_{nrs}^{s-qr}] \quad \text{on } U_1.$$

We have $\tilde{g}_2^{\lambda r} \tilde{g}_1^{(s-qr)/\mu} = [e_\mu, 1]$ on U_0, and $\tilde{g}_2^{\lambda s} \tilde{g}_0^{(qr-s)/\mu} = [e_\mu^q, 1]$ on U_1. Since $(\mu, q) = 1$, $G'' = \langle \tilde{g}_0, \tilde{g}_1, \tilde{g}_2 \rangle$ contains $[e_\mu, 1]$ on U_0 and U_1. Then the cyclic group $H = \langle [e_\mu, 1] \rangle$ ($\simeq \boldsymbol{Z}_\mu$) acts naturally on the fiber coordinates of the line bundle $M \to C$ and acts trivially on C. We put $\bar{M} = M/H$. Then $\bar{M} = \bar{U}_0 \cup \bar{U}_1 \supset \bar{C} = C/H \simeq \boldsymbol{P}^1$ and $\bar{C}^2 = -\mu$, where $\bar{U}_i = U_i/H \simeq \boldsymbol{C}^2$. Further let $\bar{G} = G''/H$ and $\bar{\bar{M}} = \bar{M}/\bar{G}$. Then we have the following diagram:

$$\begin{array}{ccc}
C \subset M & \xrightarrow{\ \sigma\ } & \boldsymbol{C}^2 \\
p_1 \big\downarrow /H & & \big\downarrow /\langle g_0, g_1 \rangle \\
\bar{C} \subset \bar{M} & & \boldsymbol{C}^2 \\
p_2 \big\downarrow /\bar{G} & & \big\downarrow /G \\
\bar{\bar{M}} & \xrightarrow{\ \pi_1\ } & X
\end{array}$$

From (2.2), we have

$$(2.3) \qquad R_k \simeq H^0(\bar{C}, \mathcal{O}_{\bar{C}}(k\mu\bar{Q}))^{\bar{G}} \cdot \bar{t}^k, \quad \text{where } \bar{Q} = p_1(Q) \text{ and } \bar{t} = t^\nu.$$

Now we compute the cyclic quotient singularities on $\bar{\bar{M}}$. On $\bar{U}_0$, we have $\bar{G} = \langle [e_r^{\mu}, e_r^{-1}], [1, e_s], [e_{\lambda r}, e_{\lambda rs}^{(qr-s)/\mu}] \rangle$ on $\bar{U}_0$. Taking the quotient by $[1, e_s]$ we have the following:

$$\bar{U}_0/\bar{G} \simeq \boldsymbol{C}^2/\langle [e_r^{\mu}, e_r^{-s}], [e_{\lambda r}, e_{\lambda r}^{(qr-s)/\mu}] \rangle = \boldsymbol{C}^2/\langle [e_{\lambda r}, e_{\lambda r}^{(qr-s)/\mu}] \rangle = \boldsymbol{C}^2/\langle [e_{\lambda r}^{\alpha_1}, e_{\lambda r}] \rangle.$$

Then we have $\bar{U}_0/\bar{G} \simeq C_{\lambda r, \alpha_1}$, and $\bar{U}_1/\bar{G} \simeq C_{\lambda s, \alpha_2}$ by the same way.

Let $\bar{\bar{C}} = p_2(\bar{C})$, then $p_2 : \bar{C} \to \bar{\bar{C}}$ is a Galois covering of the degree λrs with two ramification points Q_1, Q_2 of the ramification order λrs. Let $P_1 = p_2(Q_1)$ and $P_2 = p_2(Q_2)$. Then $(\bar{\bar{M}}, P_1) \simeq C_{\lambda r, \alpha_1}$ and $(\bar{\bar{M}}, P_2) \simeq C_{\lambda s, \alpha_2}$. Let $\pi_2 : \tilde{X} \to \bar{\bar{M}}$ be the minimal resolution of the singularities P_1, P_2 on $\bar{\bar{M}}$. Let $\tilde{C}$ be the proper transform of $\bar{\bar{C}}$ through π_2. Then, from Orlik and Wagreich's result [3: Theorem 4.3] we have

$$-\tilde{C}^2 = \frac{\mu}{\lambda rs} + \frac{\alpha_1}{\lambda r} + \frac{\alpha_2}{\lambda s} = b.$$

Let $P_0 \in \bar{\bar{C}}$ $(\neq P_1, P_2)$ and let $p_2^{-1}(P_0) = \{Q_{0,1}, \dots, Q_{0,\lambda rs}\}$. If we put $D' = b\sum_{i=1}^{\lambda rs} Q_{0,i} - \alpha_1 s Q_1 - \alpha_2 r Q_2$, then D' is $\bar{G}$-invariant. Since $deg\, D' = \mu$, $D' \sim \mu \bar{Q}$ on $\bar{C} = \boldsymbol{P}^1$ (linearly equivalent). Therefore, from (2.3) we have

$$R_k \simeq H^0(\bar{C}, \mathcal{O}_{\bar{C}}(kD'))^{\bar{G}} \cdot t^k.$$

Further, by virtue of Pinkham's result [4: Lemma 5.2.], we have

$$R_k \simeq H^0(\bar{\bar{C}}, \mathcal{O}_{\bar{\bar{C}}}(D_{r,s}^{(k)})) \cdot t^k.$$

This completes the proof of (i).

(ii) we need only to show the following equality:

$$[[b_{1r_1}, \dots, b_{11}, b, b_{21}, \dots, b_{2r_2}]] = \frac{n}{q}. \tag{2.4}$$

From the assumption and the definition of b,

$$\text{the left side of (2.4)} = \left[\left[b_{1r_1}, \dots, b_{11}, b - \frac{\alpha_2}{\lambda s}\right]\right] = \left[\left[b_{1r_1}, \dots, b_{11} - \frac{\lambda rs}{\mu + \alpha_1 s}\right]\right].$$

Since $nr \mid \alpha_1(qr - s) - \mu$ from the definition of α_1. Then $\beta = (\alpha_1(qr-s) - \mu)/nr$ is an integer with $\beta < (\alpha_1 q)/n < \alpha_1$, and $\lambda r\beta + 1 = \alpha_1((qr-s)/\mu) \equiv 1(\lambda r)$. Therefore if we put $a = (-\lambda rs)/(\mu + \alpha_1 s)$, then the left side of (2.4) $= [[b_{1r_1}, \dots, b_{11} + a]] = (\lambda r + \alpha_1 a)/((qr-s)/\mu + \beta a)$ from Lemma 1.3. We can easily check that $(\lambda r + \alpha_1 a)/((qr-s)/\mu + \beta a) = \dfrac{n}{q}$. Q.E.D.

Therefore, if we consider that A_0 is the central curve of the w.d.graph in (ii), we can construct the graded ring from the w.d.graph of (ii) for $C_{n,q}$ by a similar way as in [4].

EXAMPLE 2.4. Let consider $C_{6,5}$, so the configuration (of the minimal resolution) is given by

A_1 A_2 A_3 A_4 A_5
○-○-○-○-○, where ○ means (−2).

(i) Assume $(r,s)=(1,2)$. Then $\mu=3$, $\lambda=2$, $\alpha_1=1$ and $\alpha_1=3$. Then $b=3/4+1/2+3/4=2$. Therefore, by considering that A_2 is the central curve and $D_{1,2}=2P_0+1/2P_1+3/4P_2$, we can construct the normalized graded ring of $C_{6,5}$ with degree $(1,2)$ as Pinkham-Demazure construction.

(ii) Assume $(r,s)=(7,11)$. Then $\mu=6$, $\lambda=1$, $\alpha_1=2$ and $\alpha_1=7$. Then $b=6/77+2/7+7/11=1$. We consider the following iterating monoidal transformation

○-○-○-○-○ ← ○-(−3)-(−1)-(−3)-○-○
← ○-(−4)-(−1)-○-(−3)-○-○.

By taking the exceptional curve of first kind as the central curve and $D_{7,11}=P_0+2/7P_1+7/11P_2$, we can construct the normalized graded ring with degree $(7,11)$.

As an application of Theorem 2.3 we compute the Poincaré series $P_{r,s}(t)$ $(=\sum_{k=0}^{\infty}(dim_{\mathbf{C}}(R_{r,s})_k)\cdot t^k)$ of the graded ring $R_{r,s}$.

COROLLARY 2.5. (i) $P_{r,s}(t)=\dfrac{1}{(1-t)(1-t^{\lambda rs})}\sum_{k=0}^{\lambda rs}(C_k-C_{k-1})\cdot t^k$, *where we put* $C_k=\begin{cases}0 & (k=-1)\\ deg\ D_{r,s}^{(k)}+1 & (0\leqq k<\lambda rs).\\ \mu & (k=\lambda rs)\end{cases}$

(ii) *The cyclic order* n *of* $C_{n,q}$ *is determined from* $P_{r,s}(t)$.

PROOF. (i) If $b\geqq 2$, then $deg\ D_{r,s}^{(k)}\geqq 0$ for any $k\geqq 0$. Further if $b=1$, then $\lambda rs=\mu+\alpha_1 s+\alpha_2 r$. Therefore, for any $k\geqq 0$

$$deg\ D_{r,s}^{(k)}=k-\left\{\frac{\alpha_1 k}{\lambda r}\right\}-\left\{\frac{\alpha_2 k}{\lambda s}\right\}=\left[\frac{\alpha_2 k}{\lambda s}+\frac{\mu k}{\lambda rs}\right]-\left\{\frac{\alpha_2 k}{\lambda s}\right\}\geqq -1,$$

where [ ] is the Gauss symbol. Then, by Riemann-Roch theorem on $\boldsymbol{P}^1$ we have $dim_{\boldsymbol{C}}\, H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}}(D_{r,s}^{(k)})) = deg\, D_{r,s}^{(k)} + 1$. For any non-negative integer k, we put $a_k = deg\, D_{r,s}^{(k)} + 1$. Since $a_{\lambda rs+k} = a_k + \mu$ for any k, we have

$$P_{r,s}(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\lambda rs-1} (a_k + j\mu) \cdot t^{\lambda rsj+k} = \sum_{j=0}^{\infty} t^{\lambda rsj} \sum_{k=0}^{\lambda rs-1} a_k t^k + \mu \sum_{j=0}^{\infty} j t^{\lambda rsj} \sum_{k=0}^{\lambda rs-1} t^k$$

$$= \frac{1}{1-t^{\lambda rs}} \sum_{k=0}^{\lambda rs-1} a_k t^k + \frac{\mu t^{\lambda rs}}{(1-t^{\lambda rs})^2} \cdot \frac{1-t^{\lambda rs}}{1-t} = \frac{1}{1-t^{\lambda rs}} \left\{ \sum_{k=0}^{\lambda rs-1} a_k t^k + \frac{\mu t^{\lambda rs}}{1-t} \right\}$$

$$= \frac{1}{(1-t)(1-t^{\lambda rs})} \sum_{k=0}^{\lambda rs} (C_k - C_{k-1}) \cdot t^k.$$

(ii) We can easily check the following:

$$\sum_{k=1}^{d-1} \left\{ \frac{ek}{d} \right\} = \frac{(d-1)(e+1)}{2}, \tag{2.5}$$

where d, e are relatively prime positive integers with $1 \leqq e < d$. Now let

$$P_1(t) = (1-t)^2 P_{r,s}(t) \quad \text{and} \quad Q(t) = \sum_{k=0}^{\lambda rs} (C_k - C_{k-1}) \cdot t^k.$$

Since $P_1(t) = Q(t)/(1 + t + \cdots + t^{\lambda rs-1})$, we have $P_1(1) = \mu/(\lambda rs)$ and

$$P_1'(1) = \left.\frac{dP_1(t)}{dt}\right|_{t=1} = \frac{1}{(\lambda rs)^2} \left(\lambda rs Q'(1) - \frac{\lambda rs(\lambda rs - 1)Q(1)}{2} \right).$$

Further we can easily see that $Q(1) = C_{\lambda rs} = \mu$ and $Q'(1) = -\sum_{k=0}^{\lambda rs-1} C_k + \lambda \mu rs$. Using (2.5), $\sum_{k=0}^{\lambda rs-1} C_k = (\lambda \mu rs - \mu + r + s)/2$. Then $P_1'(1) = (2\mu - r - s)/2(\lambda rs)$. Therefore

$$\lambda = \frac{r+s}{2rs(P_1(1) - P_1'(1))}. \tag{2.6}$$

Further we can easily see the following:

$$\mu = \lim_{t \to 1} (1-t)(1-t^{\lambda rs}) P_{r,s}(t). \tag{2.7}$$

Hence the value $n = \lambda \mu$ is determined from $P_{r,s}(t)$. Q.E.D.

However we can not expect that $P_{r,s}(t)$ determines the singularity. For example, let's consider the normalized graded rings of $C_{25,6}$ and $C_{25,16}$ with degree $(1,1)$. For these two cases, we have $\lambda = \mu = 5$. Further both Poincaré serieses are equal, and they are given by $\frac{1}{(1-t)(1-t^5)}\{1+t^2+t^3+t^4+t^5\}$.

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