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ON THE EXISTENCE OF $\mathscr{C}$-CONFORMAL CHANGES OF RIEMANN-FINSLER METRICS

By

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Abstract. The notion of $\mathscr{C}$-conformality was introduced by M. HASHIGUCHI in [1]. He proved for some special Finsler manifolds that the existence of a $\mathscr{C}$-conformal change implies that the manifold is Riemannian (at least locally).

In this note we show that Hashiguchi's result is valid without any extra condition. This means that the existence of a $\mathscr{C}$-conformal change of the metric can be interpreted as a new sufficient condition for a Finsler manifold to be Riemannian.

1. Preliminaries

1.1. Throughout the paper we use the terminology and conventions described in [3]. Now we briefly summarize the basic notations:

(i) M is an $n(\geq 2)$-dimensional, C^∞, connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathscr{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) ι_X, $\mathscr{L}_X$ $(X \in \mathfrak{X}(M))$ and d are the symbols of the *insertion operator*, the *Lie derivative* (with respect to X) and the *exterior derivative*, respectively.

(v) $\mathfrak{X}^v(TM)$ denotes the $C^\infty(TM)$-module of vertical vector fields on TM. $C \in \mathfrak{X}^v(TM)$ is the *Liouville vector field*, J denotes the *vertical endomorphism* (for the definitions see e.g. [2]).

The vertical lift of a function $\alpha \in C^\infty(M)$ and of a vector field $X \in \mathfrak{X}(M)$ is denoted by α^v and X^v, respectively; we recall that α^v is nothing but the function $\alpha \circ \pi$.

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1.2. Horizontal Endomorphisms

DEFINITION. A $C^{\infty}(TM)$-linear map $h : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ (i.e. a $(1,1)$-tensor or a *vector 1-form* on TM) is said to be a *horizontal endomorphism* on M if it satisfies the following conditions:

(HE1) h is smooth over $\mathscr{T}M$;

(HE2) h is a projector, i.e. $h^2 = h$;

(HE3) $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.

Any horizontal endomorphism h determines a vector 1-form, i.e. a mapping $F : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ such that

$$F \circ h = -J, \quad F \circ J = h. \tag{1}$$

F is called the *almost complex structure associated with h*. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to h) is $X^h := FX^v$.

1.3. Finsler Manifolds

DEFINITION. Let a function $E : TM \to \boldsymbol{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold* with *energy function E* if the following conditions are satisfied:

(F0) $\forall v \in \mathscr{T}M : E(v) > 0, \quad E(0) = 0;$

(F1) E is of class C^1 on TM and smooth on $\mathscr{T}M$;

(F2) $CE = 2E$, i.e., E is homogeneous of degree 2;

(F3) the *fundamental form* $\omega := dd_J E$ is symplectic.

The mapping

$$g : \mathfrak{X}^v(\mathscr{T}M) \times \mathfrak{X}^v(\mathscr{T}M) \to C^{\infty}(\mathscr{T}M), \tag{2}$$
$$(JX, JY) \to g(JX, JY) := \omega(JX, Y)$$

is a well-defined, nondegenerate symmetric bilinear form which is called the *Riemann-Finsler metric* of (M, E). If g is positive definite then we speak of a *positive definite Finsler manifold.*

It is well-known that any Finsler manifold has a canonical horizontal endomorphism h, the so-called *Barthel endomorphism*. Using the *prolonged metric*

(3)
$$g_h : \mathfrak{X}(\mathscr{T}M) \times \mathfrak{X}(\mathscr{T}M) \to C^\infty(\mathscr{T}M)$$
$$(X, Y) \to g_h(X, Y) := g(JX, JY) + g(vX, vY),$$

$v := 1 - h$ (the "vertical projector"), the well-known *first Cartan tensor* $\mathscr{C}$ can be defined by the formula

(4)
$$\omega(\mathscr{C}(X, Y), Z) = \tfrac{1}{2}(\mathscr{L}_{JX}J^* g_h)(Y, Z)$$

(J^* is the adjoint operator of J; [2]). It is easy to check that $\mathscr{C}$ has the following properties:

(5) $\mathscr{C}$ is semibasic;

(6) its lowered tensor $\mathscr{C}_\flat$ defined by the formula

$$\mathscr{C}_\flat(X, Y, Z) := g(\mathscr{C}(X, Y), JZ) \text{ is totally symmetric;}$$

(7)
$$\forall X, Y, Z \in \mathfrak{X}(M) : \mathscr{C}_\flat(X^h, Y^h, Z^h) = \tfrac{1}{2} X^v g(Y^v, Z^v).$$

Consider a smooth function $\varphi : TM \to \boldsymbol{R}$. Since the fundamental form ω is symplectic, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathscr{T}M)$ such that

$$\iota_{\operatorname{grad}\varphi}\omega = d\varphi;$$

this vector field is called the *gradient of* φ.

2. An Observation on Homogeneous Functions

Remark 1. Let $k \in \boldsymbol{Z}$. We recall that a function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is called *positive homogeneous* of degree k if for any vector $v \in \boldsymbol{R}^n \backslash \{0\}$ and positive real number t, we have

$$f(tv) = t^k f(v).$$

It is easy to check that if $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is positive homogeneous of degree 0 and continuous at the point $0 \in \boldsymbol{R}^n$ then f is a constant function.

Proposition 1. *Let us select a subspace W of dimension $n - 1$ and a nonzero vector q of $\boldsymbol{R}^n$ $(n \geq 2)$ such that*

$$\boldsymbol{R}^n = W \oplus \{tq \mid t \in \boldsymbol{R}\} =: W \oplus \mathscr{L}(q).$$

Suppose that a function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ has the following properties:
(i) *it is positive homogeneous of degree* 0;
(ii) *it is continuous at the points q, $-q$;*

(iii) *for any point* $a \in W \backslash \{0\}$ *and scalar* $t \in \boldsymbol{R}$

$$f(a+tq) = f(a).$$

Then f *is constant on* $\boldsymbol{R}^n \backslash \{0\}$.

PROOF. Consider the function $f_1 := f \upharpoonright W \backslash \{0\}$. Let $c : \boldsymbol{N} \to W \backslash \{0\}$, $n \to c_n$ be a sequence such that

$$\lim_{n \to \infty} c_n = 0.$$

Then

$$\lim_{n\to\infty} f_1(c_n) = \lim_{n\to\infty} f(c_n) \overset{\text{(iii)}}{=} \lim_{n\to\infty} f(c_n + q) = f(q),$$

since f is continuous at the point $q \in \boldsymbol{R}^n \backslash \{0\}$. This means that $f(q)$ is the limit of the function f_1 at $0 \in W$ and, consequently, the extended function

$$\tilde{f}_1 : W \to \boldsymbol{R}, \quad a \to \tilde{f}_1(a) := \begin{cases} f_1(a) & (a \neq 0) \\ f(q) & (a = 0) \end{cases}$$

is continuous at the point $0 \in W$ and it preserves the homogenity property of the function f. Therefore, by Remark 1, $\tilde{f}_1$ is constant and in any point $a \in W \backslash \{0\}$,

$$f(a) = \tilde{f}_1(a) = \tilde{f}_1(0) = f(q). \tag{8}$$

Using the relation (8), with the choice $b = a + tq$, where $a \in W \backslash \{0\}$, $t \in \boldsymbol{R}$, we have

$$f(b) = f(a+tq) \overset{\text{(iii)}}{=} f(a) \overset{(8)}{=} f(q). \tag{9}$$

To end the proof, it is enough to check that

$$f(q) = f(-q). \tag{10}$$

This is almost trivial:

$$f(q) = \lim_{n\to\infty} f_1(c_n) = \lim_{n\to\infty} f(c_n) \overset{\text{(iii)}}{=} \lim_{n\to\infty} f(c_n - q) = f(-q),$$

since f is continuous at the point $-q \in \boldsymbol{R}^n \backslash \{0\}$. □

3. $\mathscr{C}$-Conformal Changes of Riemann-Finsler Metrics

DEFINITION ([3]). Let (M, E) and $(M, \tilde{E})$ be Finsler manifolds with Riemann-Finsler metrics g and $\tilde{g}$, respectively; g and $\tilde{g}$ are said to be *conformal equivalent* if there exists a positive smooth function $\varphi : TM \to \boldsymbol{R}$ satisfying the condition $\tilde{g} = \varphi g$.

This function φ is called the *scale function* or the *proportionality function.*

LEMMA 1 (Knebelman's observation; [3]). *The scale function between conformally equivalent Riemann-Finsler metrics is a vertical lift, i.e. it can always be written in the form*

$$\varphi = \exp \circ \alpha^v := \exp \circ \alpha \circ \pi,$$

where $\alpha \in C^\infty(M)$.

PROPOSITION 2 and definition ([6]). *If a Finsler manifold* (M, E) *with the Riemann-Finsler metric* g *and a function* $\alpha \in C^\infty(M)$ *are given, then* $\tilde{g} := \varphi g$ $(\varphi = \exp \circ \alpha^v)$ *is the Riemann-Finsler metric of the Finsler manifold* $(M, \tilde{E})$, *where* $\tilde{E} = \varphi E$.

In this case we speak of a conformal change *of the metric* g.

DEFINITION. Consider a Finsler manifold (M, E). A conformal change $\tilde{g} = \varphi g$ $(\varphi = \exp \circ \alpha^v, \alpha \in C^\infty(M))$ is said to be *$\mathscr{C}$-conformal* at a point $p \in M$ if the following conditions are satisfied:

($\mathscr{C}$1) $\qquad (d\alpha)_p \neq 0$, i.e., α is regular at the point p;

($\mathscr{C}$2) $$i_{F \operatorname{grad} \alpha^v} \mathscr{C} = 0,$$

where F is the almost complex structure associated with the Barthel endomorphism of (M, E).

PROPOSITION 3 ([4]). *Let* (M, E) *be a Finsler manifold and* $\alpha \in \mathscr{C}^\infty(M)$. *Then the following assertions are equivalent*:

(i) $i_{F \operatorname{grad} \alpha^v} \mathscr{C} = 0$,

(ii) $\operatorname{grad} \alpha^v$ *is a vertical lift, i.e., there exists a vector field* $X \in \mathfrak{X}(M)$ *such that*

$$\operatorname{grad} \alpha^v = X^v. \tag{11}$$

LEMMA 2 and definition. *Consider a Finsler manifold* (M, E) *and let us suppose that the change* $\tilde{g} = \varphi g$ $(\varphi = \exp \circ \alpha^v, \alpha \in C^\infty(M))$ *is $\mathscr{C}$-conformal at a point* $p \in M$. *Let* $\sigma \in \mathfrak{X}(M)$ *be an arbitrary vector field with the property* $\sigma(p) \neq 0$ *which obviously implies that* σ *is nonvanishing over a connected open neighbourhood* U *of* p. *Then the mapping*

$$\langle\,,\rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^{\infty}(U),$$

$$(Y,Z) \to \langle\,,\rangle(Y,Z) =: \langle Y,Z\rangle := g(Y^v,Z^v)\circ\sigma$$

is a (pseudo-) Riemannian metric. This metric is called the osculating Riemannian metric along σ.

If, in addition, $\operatorname{grad}_U \alpha \in \mathfrak{X}(U)$ *is the gradient of the function* α *with respect to* $\langle\,,\rangle$ *then*

$$(\operatorname{grad}_U \alpha)^v = \operatorname{grad} \alpha^v. \tag{12}$$

PROOF. Let $X \in \mathfrak{X}(M)$ be the vector field determined by the formula (11). Then for any vector field $Y \in \mathfrak{X}(U)$,

$$\begin{aligned}\langle X, Y\rangle := g(X^v, Y^v)\circ\sigma &\overset{(11)}{=} g(\operatorname{grad}\alpha^v, Y^v)\circ\sigma\\ &= \omega(\operatorname{grad}\alpha^v, Y^h)\circ\sigma = (Y^h\alpha^v)\circ\sigma\\ &= (Y\alpha)^v\circ\sigma = (Y\alpha)\circ\pi\circ\sigma = Y\alpha,\end{aligned}$$

hence $X = \operatorname{grad}_U \alpha$, and consequently

$$(\operatorname{grad}_U \alpha)^v = \operatorname{grad}\alpha^v. \qquad \square$$

REMARK 2. In the sequel we shall fix the vector field X determined by the formula (11) as σ in Lemma 2. (Note that the regularity property ($\mathscr{C}1$) implies that $X(p) \neq 0$.)

Therefore, the osculating Riemannian metric $\langle\,,\rangle$ will be considered as a mapping

$$\langle\,,\rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^{\infty}(U),$$

$$(Y,Z) \to \langle\,,\rangle(Y,Z) =: \langle Y,Z\rangle := g(Y^v,Z^v)\circ X,$$

where U is a fixed connected open neighbourhood of the point p such that for any $q \in U$, $X(q) \neq 0$.

PROPOSITION 4. *Consider a Finsler manifold* (M,E) *with the Riemann-Finsler metric* g *and let us suppose that the change* $\tilde{g} = \varphi g$ $(\varphi = \exp\circ\alpha^v, \alpha \in C^{\infty}(M))$ *is* $\mathscr{C}$*-conformal at a point* $p \in M$. *If* $W \subset TpM$ *is a subspace of dimension* $n-1$ *such that* $TpM = W \oplus \mathscr{L}(X(p))$ *then for any tangent vector* $w \in W\setminus\{0\}$ *and* $t \in \boldsymbol{R}$,

$$g(Y^v,Z^v)(w + tX(p)) = g(Y^v,Z^v)(w).$$

Consequently, for any vector fields $Y, Z \in \mathfrak{X}(M)$, *the function* $g(Y^v, Z^v)$ *is constant on* $TpM\backslash\{0\}$.

PROOF. For the sake of brevity, consider the parametric line

$$\ell : t \in \boldsymbol{R} \to \ell(t) := w + tX(p) \in TpM,$$

where $w \in W\backslash\{0\}$ is an arbitrary fixed tangent vector. Now let us define a function Θ as follows:

$$\Theta : t \in \boldsymbol{R} \to \Theta(t) := g(Y^v, Z^v) \circ \ell(t) \in \boldsymbol{R}.$$

If $(\pi^{-1}(U), (x^i, y^i)_{i=1}^n)$ is the chart induced by a chart $(U, (u^i)_{i=1}^n)$ on M then we have

$$\Theta'(t) = \left(\frac{\partial}{\partial x^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (x^i \circ \ell)'(t) + \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (y^i \circ \ell)'(t).$$

Here, for any $i \in \{1, \ldots, n\}$ and $t \in \boldsymbol{R}$,

$$x^i \circ \ell(t) = u^i \circ \pi \circ \ell(t) = u^i \circ \pi(w + tX(p)) = u^i(p),$$

i.e., $x^i \circ \ell$ is constant, and so for any $t \in \boldsymbol{R}$, $(x^i \circ \ell)'(t) = 0$.

On the other hand

$$y^i \circ \ell(t) = y^i(w + tX(p)) = w^i + tX^i(p),$$

therefore

$$\begin{aligned}
\Theta'(t) &= \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (y^i \circ \ell)'(t) = X^i(p) \cdot \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \\
&= X^i \circ \pi(\ell(t)) \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) = (X^v g(Y^v, Z^v)) \circ \ell(t) \\
&\overset{(7)}{=} 2\mathscr{C}_\flat(X^h, Y^h, Z^h) \circ \ell(t) = 2\mathscr{C}_\flat(FX^v, Y^h, Z^h) \circ \ell(t) \\
&\overset{(11)}{=} 2\mathscr{C}_\flat(F \operatorname{grad} \alpha^v, Y^h, Z^h) \circ \ell(t) \overset{(\mathscr{C}2)}{=} 0
\end{aligned}$$

and, consequently, Θ is also a constant function.

Thus for any real number $t \in \boldsymbol{R}$,

$$\Theta(t) = \Theta(0) :\Leftrightarrow g(Y^v, Z^v)(w + tX(p)) = g(Y^v, Z^v)(w).$$

According to Proposition 1, this means that the function $g(Y^v, Z^v)$ is constant on $TpM\backslash\{0\}$, namely for any tangent vector $v \in TpM\backslash\{0\}$,

$$g(Y^v, Z^v)(v) = g(Y^v, Z^v)(X(p)) = \langle Y, Z \rangle(p). \qquad \square$$

REMARK 3. Without loss of generality we can obviously assume that $\alpha(p) = 0$ under a $\mathscr{C}$-conformal change $\tilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v, \alpha \in C^\infty(M)$) at the point p. If, in addition, the Finsler manifold (M, E) is positive definite, then it is natural to consider the tangent space TpN, $N := \alpha^{-1}(0)$, as the subspace W in Proposition 4.

THEOREM 1. *Let (M, E) be a Finsler manifold. If there exists a $\mathscr{C}$-conformal change $\tilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v, \alpha \in C^\infty(M)$) at a point $p \in M$, then (M, E) is locally Riemannian, more precisely,*

$$g(Y^v, Z^v) = \langle Y, Z \rangle \circ \pi = \langle Y, Z \rangle^v,$$

where $\langle\, , \rangle$ is the osculating Riemannian metric defined over U.

PROOF. It is enough to mention that if $\tilde{g} = \varphi g$ is a $\mathscr{C}$-conformal change at the point $p \in M$ then it is also such a change for any point $q \in U$. (Note that the assumption $X(q) \neq 0$ implies the regularity property $(d\alpha)_q \neq 0$ for any point $q \in U$.)

Therefore, the theorem is a direct consequence of Proposition 4. □

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