CR-SUBMANIFOLDS IN COMPLEX HYPERBOLIC SPACES SATISFYING AN EQUALITY OF CHEN

By

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1. Introduction

Recently, Bang-Yen Chen has introduced new type of Riemannian curvature invariants and obtained sharp inequalities involving these invariants for arbitrary submanifolds in Riemannian and Kaehlerian space forms. It is natural and interesting to investigate and understand submanifolds which satisfy the equality case of this type of inequalities, and such submanifolds have been investigated by many geometers (cf. for instance, [2–6, 8–10, 12–16]). In this paper, we investigate CR-submanifolds of complex hyperbolic spaces which satisfy the equality case of one of Chen's inequalities.

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$(1.1) \qquad \tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$(1.2) \qquad \tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For an integer $k \geq 0$, denote by $\mathscr{S}(n, k)$ the finite set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathscr{S}(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuple $(n_1, \ldots, n_k) \in \mathscr{S}(n)$, Chen's curvature invariant $\delta(n_1, \ldots, n_k)$ introduced in [4, 5, 6] are given by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

Let M be a submanifold in a Kaehler manifold $\tilde{M}$. A subspace $V \subset T_p M$ is called *totally real* if $JV \subset T_p^\perp M$, where $T_p M$ and $T_p^\perp M$ denote the tangent space and the normal space of M at p, respectively. A submanifold M of $\tilde{M}$ is called a CR-submanifold if there exists on M a differentiable holomorphic distribution $\mathscr{D}$ such that its orthogonal complement $\mathscr{D}^\perp \subset TM$ is a totally real distribution [1].

For a $(2n + 1)$-dimensional CR-submanifolds with $2n$-dimensional maximal holomorphic tangent subspace (i.e, $\dim \mathscr{D}^\perp = 1$) in complex hyperbolic m-space $CH^m(-4)$ of constant holomorphic sectional curvature -4, we have the following sharp inequality involving the intrinsic invariant $\delta_k := \delta(2, \ldots, 2)$ (2 appears k times) and the squared mean curvature ([5, 6]):

$$(1.3) \qquad \delta_k \leq \frac{(2n + 1)^2 (2n - k)}{2(2n + 1 - k)} H^2 - 2(n^2 + n - k),$$

where H^2 denotes the squared mean curvature.

Let M be a real $2n$-dimensional Kaehler manifold. For a k-tuple $(2n_1, \ldots, 2n_k) \in \mathscr{S}(2n)$, Chen has also introduced the *complex δ-invariants* $\delta^c(2n_1, \ldots, 2n_k)$ by

$$\delta^c(2n_1, \ldots, 2n_k) = \tau - \inf\{\tau(L_1^c) + \cdots + \tau(L_k^c)\},$$

where $L_1^c, \ldots, L_k^c$ run over all k mutually orthogonal complex subspaces of $T_p M$, $p \in M$, with dimensions $2n_1, \ldots, 2n_k$, respectively.

For $\delta_n^c := \delta^c(2, \ldots, 2)$ (2 appears n times) of a $2n$-dimensional Kaehler submanifold in the complex Euclidean space, we have the following result from [5].

$$(1.4) \qquad \delta_n^c \leq 0.$$

It was proved in [5] that a real hypersurface of a complex hyperbolic $(n + 1)$-space $CH^{n+1}(-4)$ satisfies the equality case of (1.3) if and only if the real hypersurface is an open portion of a tubular hypersurface of radius $r \in R_+$ over a totally geodesic $CH^{n/2}(-4)$ ($n + 1$ is odd, $k = n$) or an open portion of a horosphere in $CH^{n+1}(-4)$.

B. Y. Chen and L. Vrancken has completely classified in [12] 3-dimensional CR-submanifolds of complex hyperbolic spaces which satisfy the equality case of (1.3) for $n = 1$ and $k = 1$.

A submanifold is said to be *linearly full in* $CH^m(-4)$ if it does not lie in any totally geodesic complex hypersurface of $CH^m(-4)$.

In this paper, in case $m > n + 1$, we investigate linearly full $(2n + 1)$-dimensional *CR*-submanifolds with $\dim \mathcal{D}^{\perp} = 1$ in $CH^m(-4)$ which satisfy the equality case of (1.3). Then we obtain $k = n$. We are able to establish the explicit representation of such submanifolds in an anti-de Sitter space-time via Hopf's fibration, in terms of Kaehler submanifolds of the complex Euclidean $(m - 1)$-space C^{m-1} which satisfy the equality case of (1.4). Our result is a generalization of Chen and Vrancken's result with $n = 1$ and $k = 1$ ([12]). In case $m = n + 2$ we completely classify such submanifolds.

In section 2, we provide some fundamental equations on pseudo-Riemannian submanifolds. In section 3, we present our main theorem. In section 4, we present the sharp, general inequalities which relate the Chen invariants to the squared mean curvature for submanifolds in Riemannian and Kaehlerian space forms. In section 5, we present the inequality for $(2n + 1)$-dimensional *CR*-submanifolds with $\dim \mathcal{D}^{\perp} = 1$ in $CH^m(-4)$, and give necessary conditions for the *CR*-submanifolds to satisfy the equality case of the inequality. In the last section, we provide the proof of our main theorem.

2. Preliminaries

Let $\tilde{M}$ be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric $\tilde{g}$. Denote by $\tilde{\nabla}$ the metric connection of $\tilde{M}$ and by $\langle , \rangle$ the inner product induced from the metric $\tilde{g}$. A tangent vector X to $\tilde{M}$ is called space-like (respectively, light-like or time-like) if $\langle X, X \rangle > 0$ or $X = 0$ (respectively, if $\langle X, X \rangle = 0$ and $X \neq 0$ or if $\langle X, X \rangle < 0$).

Let M be an n-dimensional submanifold of $\tilde{M}$. If the metric tensor of $\tilde{M}$ induces a pseudo-Riemannian metric (respectively, Riemannian metric) on M, then M is called a pseudo-Riemannian (respectively, Riemannian) submanifold of $\tilde{M}$. Let ∇ denote the metric connection on M with respect to the induced metric.

For vector fields X, Y tangent to the submanifold, we have the equation of Gauss:

$$(2.1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h is the second fundamental form of M in $\tilde{M}$. The mean curvature vector $\vec{H}$ of the immersion is given by $\vec{H} = 1/n$ trace h. A submanifold is said to be minimal if its mean curvature vector vanishes identically. Denote by D the linear connection induced on the normal bundle $T^{\perp}M$ of M in $\tilde{M}$. For each vector field

ξ normal to M, the Weingarten formula is given by

$$(2.2) \qquad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where A is the shape operator. It is well-known that the second fundamental form and the shape operator are related by $\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M of $\tilde{M}$, respectively, and by R^D the curvature tensor of the normal connection D. Then the equation of Gauss and Ricci are given respectively by

$$(2.3) \qquad R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle h(X, W), h(Y, Z) \rangle$$

$$- \langle h(X, Z), h(Y, W) \rangle,$$

$$(2.4) \qquad R^D(X, Y; \xi, \eta) = \tilde{R}(X, Y; \xi, \eta) + \langle [A_\xi, A_\eta](X), Y \rangle$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, we define the covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^\perp M$ by

$$(2.5) \qquad (\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The equation of Codazzi is given by

$$(2.6) \qquad (\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z).$$

The Riemann curvature tensor of a complex space form $\tilde{M}(4\varepsilon)$ of constant holomorphic sectional curvature 4ε takes the form:

$$(2.7) \quad \tilde{R}(X, Y)Z = \varepsilon\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY$$

$$+ 2\langle X, JY \rangle JZ\},$$

where J denotes the almost complex structure of $\tilde{M}(4\varepsilon)$.

3. Statement of Main Theorem

Consider the complex number $(m + 1)$-space C_1^{m+1} endowed with the pseudo-Euclidean metric g_0 given by (for the details, cf. [11, 17])

$$(3.1) \qquad g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{m} dz_j d\bar{z}_j,$$

where $\bar{z}_k$ denotes the complex conjugate of z_k.

On C_1^{m+1} we define

(3.2) $$\langle z, w \rangle = -z_0 \bar{w}_0 + \sum_{k=1}^{m} z_k \bar{w}_k.$$

Put

(3.3) $$H_1^{2m+1}(-1) = \{ z = (z_0.z_1. \ldots .z_m) \in C_1^{m+1} : \langle z, z \rangle = -1 \},$$

Then $H_1^{2m+1}(-1)$ is a real hypersurface of C^{m+1} whose tangent space at $z \in H_1^{2m+1}(-1)$ is given by $T_z H_1^{2m+1}(-1) = \{ w \in C^{m+1} : \text{Re}\langle z, w \rangle = 0 \}$. It is known that $H_1^{2m+1}(-1)$ together with the induced metric g is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space time.

We put

$$H_1^1 = \{ \lambda \in C : \lambda \bar{\lambda} = 1 \}.$$

Then we have an H_1^1-action on $H_1^{2m+1}(-1)$ given by $z \mapsto \lambda z$. At each point z in $H_1^{2m+1}(-1)$, the vector iz is tangent to the flow of the action. Since $(\ ,\ )$ is Hermitian, we have $\langle iz, iz \rangle = -1$. Note that the orbit is given by $x(t) = e^{it} z$ and $dx(t)/dt = ix(t)$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/_{\sim}$, under the identification induced from the action, is the complex hyperbolic space $CH^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $CH^m(-4)$ is induced from the canonical almost complex structure J on C_1^{m+1}, the multiplication by i, via the totally geodesic fibration:

(3.4) $$\pi : H_1^{2m+1}(-1) \to CH^m(-4).$$

The main result of this paper is the following.

MAIN THEOREM. *Let U be a domain of C^n and $\Psi : U \to C^{m-1}$ be a holomorphic isometric immersion in C^{m-1} satisfying the equality case of* (1.4). *Define* $z : R^2 \times U \to C_1^{m+1}$ *by*

(3.5) $$z(u, t, w_1, \ldots, w_n) = \left(-1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi \right) e^{it}.$$

Then $\langle z, z \rangle = -1$ and the image $z(R^2 \times U)$ in H_1^{2m+1} is invariant under the group action of H_1^1. Moreover the quotient space $z(R^2 \times U)/_{\sim}$ is a $(2n + 1)$-dimensional

CR-submanifold with $\dim \mathscr{D}^\perp = 1$ of $CH^m(-4)$ which satisfies the equality case of (1.3) for $k = n$.

Conversely, in case $m > n + 1$, up to rigid motions of $CH^m(-4)$, every linearly full $(2n + 1)$-dimensional CR-submanifold with $\dim \mathscr{D}^\perp = 1$ of $CH^m(-4)$ satisfying the equality of (1.3) is obtained in such way with $k = n$.

4. Some Inequalities

Let M be a submanifold of an m-dimensional Kaehlerian space form $\tilde{M}^m(4\varepsilon)$ with constant holomorphic sectional curvature 4ε. For any vector X tangent to M we put $JX = PX + FX$, where PX and FX are tangential and normal components of JX, respectively. For a subspace $L \subset T_pM$ of dimension r we put

$$\Psi(L) = \sum_{1 \leq i < j \leq r} \langle Pu_i, u_j \rangle^2.$$

where $\{u_1, \ldots, u_r\}$ is an orthonormal basis of L. $\Psi(L)$ is an well-defined number which is independent of the choice of the orthonormal basis $\{u_1, \ldots, u_r\}$.

For each $(n_1, \ldots, n_k) \in \mathscr{S}(n)$, let $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ denote the constants given by

$$(4.1) \qquad c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)},$$

$$(4.2) \qquad b(n_1, \ldots, n_k) = \frac{1}{2}\left(n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1)\right).$$

We also need the following results from [4, 5]

LEMMA 4.1 (General Inequalities).   Given an n-dimensional submanifold M in a Kaehlerian space form $\tilde{M}^m(4\varepsilon)$, we have

$$(4.3) \qquad \tau - \sum_{i=1}^{k} \tau(L_i) \leq c(n_1, \ldots, n_k)H^2 + b(n_1, \ldots, n_k)\varepsilon + \frac{3}{2}\|P\|^2\varepsilon - 3\varepsilon\sum_{i=1}^{k}\Psi(L_i)$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathscr{S}(n)$. The equality case of inequality (4.3) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $e_1, \ldots, e_{2m}$ at p, such that

(a) $L_j = Span\{e_{n_1 + \cdots + n_{j-1} + 1}, \ldots, e_{n_1 + \cdots + n_j}\}$,

(b) the shape operators of M in $\tilde{M}^m(4\varepsilon)$ at p take the following forms:

$$(4.4) \qquad A_r = \begin{bmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_r & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \mu_r \end{bmatrix},$$

$$r = n + 1, \ldots, 2m,$$

where $A_r := A_{e_r}$ and each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$(4.5) \qquad \text{trace}(A_1^r) = \cdots = \text{trace}(A_k^r) = \mu_r.$$

PROPOSITION 4.2. *Let M be a (real) $2n$-dimensional Kaehler submanifold of a Kaehlerian space form $\tilde{M}^m(4\varepsilon)$. Then, for any k-tuple $(2n_1, \ldots, 2n_k) \in \mathscr{S}(2n)$, the complex δ-invariant $\delta^c(2n_1, \ldots, 2n_k)$ satisfies*

$$(4.6) \qquad \delta^c(2n_1, \ldots, 2n_k) \leq 2\left(n(n+1) - \sum_{j=1}^{k} n_j(n_j + 1) \right)\varepsilon.$$

The equality case of inequality (4.6) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $e_1, \ldots, e_{2m}$ at p, such that $e_1, \ldots, e_{2n}$ are tangent to M and $e_{2l} = Je_{2l-1}$ $(1 \leq l \leq k)$ and the shape operators of M in $\tilde{M}^m(4\varepsilon)$ at p take the following forms:

$$(4.7) \qquad A_r = \begin{bmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$r = 2n + 1, \ldots, 2m,$$

where each A_j^r is a symmetric $(2n_j) \times (2n_j)$ submatrix with zero trace.

By the property of the second fundamental form of a Kaehler submanifold of a Kaehler manifold (cf. [18]) we have the following Proposition.

Proposition 4.3.   *Every Kaehler hypersurface of* $\tilde{M}^{n+1}(4\varepsilon)$ *satisfies*

$$\delta_n^c = 2(n^2 + n - 2k)\varepsilon.$$

Proof.   Let M be a Kaehler hypersurface in $\tilde{M}^{n+1}(4\varepsilon)$. Since $JA_{2n+1} = -A_{2n+1}J$, at each point p of M, we can choose an orthonormal basis $e_1, Je_1, \ldots,$ e_n, Je_n of $T_p(M)$ with respect to which the shape operator A_{2n+1} is of the following form:

$$(4.8) \qquad A_{2n+1} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & -\lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ 0 & & & & -\lambda_n \end{bmatrix},$$

It is well known that the shape operators of M satisfy

$$(4.9) \qquad\qquad A_{Je_{2n+1}} = JA_{2n+1}.$$

From (4.8) and (4.9), we have

$$(4.10) \qquad A_{je_{2n+1}} = \begin{bmatrix} 0 & \lambda_1 & & & 0 \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_n \\ 0 & & & \lambda_n & 0 \end{bmatrix}.$$

It follows from (4.8), (4.10) and Proposition 4.2 that M satisfies the equality case of (4.6) for a *n-tuple* $(2, \ldots, 2) \in \mathscr{S}(2n)$. $\square$

5. Some Lemmas

First we recall the following result on CR-submanifolds from [7].

Lemma 5.1.   *Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Denote by $T^\perp M = J\mathscr{D}^\perp \oplus v$ the orthogonal decomposition of the normal bundle, where $\mathscr{D}^\perp$ is the totally real distribution and v a complex subbundle of $T^\perp M$. We have*

$$(5.1) \qquad\qquad \langle \nabla_U Z, X \rangle = \langle J(A_{JZ}U), X \rangle,$$

$$(5.2) \qquad\qquad A_{JZ}W = A_{JW}Z,$$

$$(5.3) \qquad\qquad A_{J\xi}X = -A_\xi JX,$$

for vector fields Z, W *in* $\mathscr{D}^\perp$, ξ *in* v, U *in* TM *and vector field* X *in the holomorphic distribution* $\mathscr{D}$.

We also need the following result using Lemma 4.1

LEMMA 5.2. *Let* $x : M \to CH^m(-4)$ *be a* $(2n + 1)$*-dimensional CR-submanifold with* $\dim \mathscr{D}^\perp = 1$. *Then*

$$(5.4) \qquad \delta_k \leq \frac{(2n + 1)^2(2n - k)}{2(2n + 1 - k)} H^2 - 2(n^2 + n - 2k)$$

Equality sign of (5.4) *holds for some* k *if and only if, there exists an orthonormal basis* $e_1, \ldots, e_{2m}$ *at* p, *such that* $e_1, \ldots, e_{2n+1}$ *are tangent to* M *and* $e_l = Je_{2l-1}$ $(1 \leq l \leq k)$ *and the shape operators of* M *in* $\tilde{M}^m(4\varepsilon)$ *at* p *take the following forms*:

$$(5.5) \qquad A_r = \begin{bmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_r & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \mu_r \end{bmatrix},$$

$$r = 2n + 2, \ldots, m,$$

where each A_j^r *is a symmetric* 2×2 *submatrix such that*

$$(5.6) \qquad \mathrm{trace}(A_1^r) = \cdots = \mathrm{trace}(A_k^r) = \mu_r.$$

PROOF. For mutually orthonormal plane sections $L_1, \ldots, L_k$, we have

$$\|P\|^2 - 2 \sum_{j=1}^k \Psi(L_j) \geq 2n - 2k,$$

with equality holding if and only if $L_1, \ldots, L_k$ are complex planes. Combining this and (4.3) yields (5.4). $\qquad \square$

We also need the following lemmas.

LEMMA 5.3. *Let* $x : M \to CH^m(-4)$ *be a* $(2n + 1)$*-dimensional CR-submanifold with* $\dim \mathscr{D}^\perp = 1$. *If* M *satisfies the equality case of* (5.4), *then the mean curvature vector* $\vec{H}$ *lies in* $J\mathscr{D}^\perp$.

Proof. If $m = n + 1$, there is nothing to prove. So, we assume $m > n + 1$. Hence, there is a complex subbundle v of the normal bundle perpendicular to $J\mathscr{D}^\perp$ such that $T^\perp M = J\mathscr{D}^\perp \oplus v$. Let $\{e_1, \ldots, e_{2m}\}$ be an orthonormal frame field on M mentioned in Lemma 5.2. Then from (5.3) for a nonzero vector field $\xi \in v$,

$$\langle A_\xi e_1, e_1 \rangle + \langle A_\xi Je_1, Je_1 \rangle = \langle A_{J\xi} Je_1, e_1 \rangle - \langle A_{J\xi} e_1, Je_1 \rangle = 0.$$

Therefore the mean curvature vector $\vec{H}$ lies in $J\mathscr{D}^\perp$.   □

LEMMA 5.4. *Let* $x : M \to CH^m(-4)$ *be a linearly full* $(2n + 1)$-*dimensional CR-submanifold with* $\dim \mathscr{D}^\perp = 1$. *If* $m > n + 1$ *and* M *satisfies the equality case of* (5.4), *then* $k = n$ *and, with respect to some suitable orthonormal frame field* $\{e_1, \ldots, e_{2m}\}$, *the second fundamental form of* M *in* $CH^m(-4)$ *satisfies*

$$h(e_{2r-1}, e_{2r-1}) = Je_{2n+1} + \phi_r \xi_r, \quad h(e_{2r}, e_{2r}) = Je_{2n+1} - \phi_r \xi_r,$$

(5.7) $$h(e_{2r-1}, e_p) = 0, \quad h(e_{2r}, e_q) = 0, \quad h(e_{2r-1}, e_{2r}) = \phi_r J\xi_r,$$

$$h(e_l, e_{2n+1}) = 0, \quad h(e_{2n+1}, e_{2n+1}) = 2Je_{2n+1},$$

where $r = 1, \ldots, n$, $l = 1, \ldots, 2n$ *and* $p, q \notin \{2r - 1, 2r\}$ *and* ϕ_r *are functions and* ξ_r *are in* v.

Proof. Let $\{e_1, \ldots, e_{2m}\}$ be an orthonormal frame field on M mentioned in Lemma 5.2 such that e_{2n+2} is parallel to the mean curvature vector field and $\{e_1, \ldots, e_{2n+1}\}$ diagonalize the shape operator A_{2n+2}. Under the hypothesis, we have $\vec{H} \in J\mathscr{D}^\perp$ according to Lemma 5.3. Without loss of generality we may assume that $Je_{2n+1} = e_{2n+2}$, and moreover lemma 5.2 implies $h(X, e_{2n+1}) \in J\mathscr{D}^\perp$ for any X tangent to M. Hence, using

$$-A_{2n+2}X + D_X(Je_{2n+1}) = \tilde{\nabla}_X(Je_{2n+1}) = J(\nabla_X e_{2n+1}) + Jh(X, e_{2n+1}),$$

we obtain $D_X(Je_{2n+1}) \in J\mathscr{D}^\perp$ for any $X \in TM$. Since $J\mathscr{D}^\perp$ is of rank one and Je_{2n+1} is of unit length, this yields $D(Je_{2n+1}) = 0$. Thus, Je_{2n+1} is a parallel normal vector field. The Coddazi equation yields

(5.8) $$\langle \tilde{R}(X, Y)Z, Je_{2n+1} \rangle = \langle (\nabla_X A_{2n+2})Y - (\nabla_Y A_{2n+2})X, Z \rangle$$

for X, Y, Z tangent to M.
    On the other hand, since

$$R(X, Y)Z = -\langle Y, Z \rangle X + \langle X, Z \rangle Y - \langle JY, Z \rangle JX + \langle JX, Z \rangle JY$$

$$+ 2\langle JX, Y \rangle JZ,$$

(5.8) becomes

(5.9) $\qquad (\nabla_X A_{2n+2})Y - (\nabla_Y A_{2n+2})X = -\langle X, e_{2n+1}\rangle PY + \langle Y, e_{2n+1}\rangle PX$

$$+ 2\langle PX, Y\rangle e_{2n+1}.$$

Next, by differentiating $J(Je_{2n+1}) = -e_{2n+1}$ covariantly and by comparing the tangential and normal parts, we get

(5.10) $\qquad\qquad\qquad\qquad \nabla_X e_{2n+1} = PA_{2n+2}X.$

Further, by differentiating $A_{2n+2}e_{2n+1} = \mu_{2n+2}e_{2n+1}$ covariantly and using (5.10) we obtain

$$(\nabla_X A_{2n+2})(e_{2n+1}) + A_{2n+2}PA_{2n+2}X = (X\mu_{2n+2})e_{2n+1} + \mu_{2n+2}PA_{2n+2}X$$

and hence,

$$\langle(\nabla_X A_{2n+2})Y, e_{2n+1}\rangle + \langle A_{2n+2}PA_{2n+2}X, Y\rangle$$

$$= (X\mu_{2n+2})\langle e_{2n+1}, Y\rangle + \mu_{2n+2}\langle PA_{2n+2}X, Y\rangle.$$

Thus

$$\langle(\nabla_X A_{2n+2})Y, e_{2n+1}\rangle - \langle(\nabla_Y A_{2n+2})X, e_{2n+1}\rangle + 2\langle A_{2n+2}PA_{2n+2}X, Y\rangle$$

$$= (X\mu_{2n+2})\langle e_{2n+1}, Y\rangle - (Y\mu_{2n+2})\langle e_{2n+1}, X\rangle$$

$$+ \mu_{2n+2}\langle PA_{2n+2}X, Y\rangle - \mu_{2n+2}\langle PA_{2n+2}Y, X\rangle.$$

This and (5.9) yield

(5.11) $\qquad 2\langle PX, Y\rangle + 2\langle A_{2n+2}PA_{2n+2}X, Y\rangle$

$$= (X\mu_{2n+2})\langle e_{2n+1}, Y\rangle - (Y\mu_{2n+2})\langle e_{2n+1}, X\rangle$$

$$+ \mu_{2n+2}\langle PA_{2n+2}X, Y\rangle - \mu_{2n+2}\langle PA_{2n+2}Y, X\rangle.$$

We assume that $k < n$. Then we have

$$A_{2n+2}e_{2n-1} = \mu_{2n+2}e_{2n-1}, \qquad A_{2n+2}Je_{2n-1} = \mu_{2n+2}Je_{2n-1}.$$

By choosing $X = e_{2n-1}$, $Y = Je_{2n-1}$, We have $2 + 2\mu_{2n+2}^2 = 2\mu_{2n+2}^2$. This is a contradiction. Therefore, $k = n$.

On the other hand by the equation (2.7), we have

(5.12) $\qquad\qquad \tilde{R}(X, Y; Je_{2n+1}, \xi) = R^D(X, Y; Je_{2n+1}, \xi) = 0$

by virtue of $D(Je_{2n+1}) = 0$. Hence, the equation of Ricci yields $[A_{Je_{2n+1}}, A\xi] = 0$ for any $\xi \in v$. We put $A_{2n+2}e_{2k-1} = \alpha_k e_{2k-1}$, $A_{2n+2}Je_{2k-1} = \beta_k Je_{2k-1}$.

If $\alpha_k \neq \beta_k$, for any k $(k = 1, \ldots, n)$, the shape operators take the following forms:

$$(5.13) \qquad A_r = \begin{bmatrix} \gamma_1^r & & & & & 0 \\ & -\gamma_1^r & & & & \\ & & \ddots & & & \\ & & & \gamma_n^r & & \\ & & & & -\gamma_n^r & \\ 0 & & & & & 0 \end{bmatrix},$$

$$r = 2n + 3, \ldots, 2m.$$

Then, for any $\xi \in v$, (5.3) yields $A_{J\xi}e_{2k-1} = -A_\xi Je_{2k-1}$. Combining this with (5.13) we have $A_{2n+3} = \cdots = A_{2m} = 0$. Since Je_{2n+1} is a parallel normal vector field, this implies that M is contained in a totally geodesic complex hyperbolic space $CH^{n+1}(-4)$ of $CH^m(-4)$. This is a contradiction. Hence, there exist orthonormal vectors $\{e_{2k-1}, Je_{2k-1}\}$ such that $\alpha_k = \beta_k$. Then by choosing $X = e_{2k-1}$, $Y = Je_{2k-1}$, from (5.11) we have

$$2 + 2\alpha_k^2 = 2\alpha_k \mu_{2n+2},$$

$$2\alpha_k = \mu_{2n+2}.$$

Replace e_{2n+1} by $-e_{2n+1}$ if necessary, we have $\alpha_k = 1$, $\mu_{2n+2} = 2$. If there exist orthonormal vectors $\{e_{2j-1}, Je_{2j-1}\}$ such that $\alpha_j \neq \beta_j$, by choosing $X = e_{2j-1}$, $Y = Je_{2j-1}$, we have

$$2 + 2\alpha_j\beta_j = 2(\alpha_j + \beta_j),$$

$$\alpha_j + \beta_j = 2.$$

Hence, we have $\alpha_j = \beta_j = 1$. This is a contradiction. Therefore, for any k $(k = 1, \ldots, n)$, $\alpha_k = \beta_k = 1$ and $\mu_{2n+2} = 2$. $\qquad \square$

6. Proof of the Main Theorem

Let U be a domain of C^n and $\Psi : U \to C^{m-1}$ be a holomorphic isometric immersion in C^{m-1} satisfying the equality case of (1.4). Define $z : R^2 \times U \to C_1^{m+1}$ by

$$(6.1) \qquad z(u, t, w_1, \ldots, w_n) = \left(-1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi\right)e^{it}.$$

Then $\langle z, z \rangle = -1$. Thus, the image $z(\boldsymbol{R}^2 \times U)$ of $\boldsymbol{R}^2 \times U$ under z is contained in the anti-de Sitter space time $H_1^{2m+1}(-1)$.

Let $(w_1, \ldots, w_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$ denote the standard coordinates of $U \subset \boldsymbol{C}^{m-1}$. Then

$$(6.2) \qquad \frac{\partial}{\partial x_r} = \frac{\partial}{\partial w_r} + \frac{\partial}{\partial \bar{w}_r}, \qquad \frac{\partial}{\partial y_r} = i\frac{\partial}{\partial w_r} - i\frac{\partial}{\partial \bar{w}_r},$$

where $r = 1, \ldots, n$.

We obtain from (6.1) and (6.2) that

$$z_u = (i, i, 0)e^{it}, \qquad z_t = iz,$$

$$z_{x_r} = \left(-\frac{1}{2}\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} + \Psi\frac{\partial \bar{\Psi}}{\partial \bar{w}_r} \right), -\frac{1}{2}\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} + \Psi\frac{\partial \bar{\Psi}}{\partial \bar{w}_r} \right), \frac{\partial \Psi}{\partial w_r} \right)e^{it},$$

$$(6.3) \qquad z_{y_r} = \left(-\frac{i}{2}\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} - \Psi\frac{\partial \bar{\Psi}}{\partial \bar{w}_r} \right), -\frac{i}{2}\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} - \Psi\frac{\partial \bar{\Psi}}{\partial \bar{w}_r} \right), i\frac{\partial \Psi}{\partial w_r} \right)e^{it},$$

$$z_{uu} = 0, \qquad z_{ut} = iz_u, \qquad z_{ux_r} = z_{uy_r} = 0,$$

$$z_{tt} = -z, \qquad z_{tx_r} = iz_{x_r}, \qquad z_{ty_r} = iz_{y_r}.$$

Let $E_1', \ldots, E_{2n}'$ be an orthonormal basis of U mentioned in Proposition 4.3 for an n-tuple $(2, \ldots, 2) \in \mathscr{S}(2n)$ such that $E_{2r}' = iE_{2r-1}'$ $(r = 1, \ldots, n)$. Here we put

$$E_k = E_k'e^{it} + (E_k', \Psi)iz_u, \qquad k = 1, \ldots, 2n$$

$$(6.4) \qquad E_{2n+1} = z_t + z_u,$$

$$E_{2n+2} = z_t.$$

Then $E_1, \ldots, E_{2n}$ are orthonormal tangent vector fields such that $E_{2r} = iE_{2r-1}$ $(r = 1, \ldots, n)$ and iE_{2n+1}, iE_{2n+2} are normal vector fields.

If we put

$$(6.5) \qquad E_k' = \sum_{r=1}^{n}(f_r z_{x_r} + g_r z y_r),$$

then from (6.4) we have

$$(6.6) \qquad E_k = \sum_{r=1}^{n}\left\{ f_r z_{x_r} + g_r z_{y_r} + \frac{f_r}{2}i\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} - \Psi\frac{\partial \bar{\Psi}}{\partial w_r} \right)z_u \right.$$

$$\left. - \frac{g_r}{2}\left(\frac{\partial \Psi}{\partial w_r}\bar{\Psi} + \Psi\frac{\partial \bar{\Psi}}{\partial w_r} \right)z_u \right\},$$

where f_r and g_r are functions. By virtue of (6.6), the second fundamental form $\tilde{h}$ of $z(R^2 \times U)$ in C_1^{m+1} satisfies

(6.7) $\tilde{h}(E_k, E_l) = (\tilde{\nabla}_{E_k} E_l)^\perp = -\dfrac{1}{2}(\tilde{\nabla}_{E_k} E_l, (-1, 1, 0, \ldots, 0))iz_u + (\tilde{\nabla}_{E_k'} E_l')^\perp e^{it}$

where $k, l = 1, \ldots, 2n$ and $\{\cdots\}^\perp$ denotes the normal component of $\{\cdots\}$ and $\tilde{\nabla}$ is the standard covariant differential in C_1^{m+1}.

From (6.3), (6.4), (6.6), (6.7) and Proposition 4.3 for an n-tuple $(2, \ldots, 2) \in \mathscr{S}(2n)$ we have

$$\tilde{h}(E_{2r-1}, E_{2r-1}) + \tilde{h}(E_{2r}, E_{2r}) \in Span\{iE_{2n+2} - iE_{2n+1}\},$$

$$\tilde{h}(E_{2r-1}, E_p) \in Span\{iE_{2n+2} - iE_{2n+1}\},$$

(6.8)

$$\tilde{h}(E_{2r}, E_q) \in Span\{iE_{2n+2} - iE_{2n+1}\},$$

$$\tilde{h}(E_l, E_{2n+1}) = 0, \quad \tilde{h}(E_{2r+1}, E_{2n+1}) = 2iE_{2n+1} - iE_{2n+2}$$

where $r = 1, \ldots, n$, $l = 1, \ldots, 2n$ and $p, q \notin \{2r - 1, 2r\}$.

On the other hand, from (6.3)–(6.7) we have

$$\langle \tilde{h}(E_{2r-1}, E_{2r-1}), iE_{2n+1}\rangle = 1, \quad \langle \tilde{h}(E_{2r-1}, E_{2r-1}), iE_{2n+2}\rangle = 1,$$

$$\langle \tilde{h}(E_{2r}, E_{2r}), iE_{2n+1}\rangle = 1, \quad \langle \tilde{h}(E_{2r}, E_{2r}), iE_{2n+2}\rangle = 1,$$

(6.9)

$$\langle \tilde{h}(E_{2r-1}, E_p), iE_{2n+1}\rangle = 0, \quad \langle \tilde{h}(E_{2r-1}, E_p), iE_{2n+2}\rangle = 0,$$

$$\langle \tilde{h}(E_{2r}, E_q), iE_{2n+1}\rangle = 0, \quad \langle \tilde{h}(E_{2r}, E_q), iE_{2n+2}\rangle = 0,$$

where $r = 1, \ldots, n$ and $p, q \notin \{2r - 1, 2r\}$.

Therefore, we have

$$\tilde{h}(E_{2r-1}, E_p) = 0,$$

$$\tilde{h}(E_{2r}, E_q) = 0,$$

(6.10)

$$\tilde{h}(E_{2r-1}, E_{2r-1}) = iE_{2n+1} - iE_{2n+2} + \phi_r \tilde{\xi}_r,$$

$$\tilde{h}(E_{2r}, E_{2r}) = iE_{2n+1} - iE_{2n+2} - \phi_r \tilde{\xi}_r,$$

where $r = 1, \ldots, n$, $p, q \notin \{2r - 1, 2r\}$ and ϕ_r are functions and $\tilde{\xi}_r$ are unit normal vector fields perpendicular to iE_{2n+1}, iE_{2n+2}. Moreover, from (6.7) and (6.10), we have

(6.10') $\tilde{h}(E_{2r-1}, E_{2r}) = \phi_r i\tilde{\xi}_r.$

Since iz is always tangent to $z(\mathbf{R}^2 \times U)$, the image $z(\mathbf{R}^2 \times U)$ in $H_1^{2m+1}(-1)$ is invariant under the group action of H_1^1. Hence, $z(\mathbf{R}^2 \times U)$ is projectable via the Hopf's fibration $\pi : H_1^{2m+1}(-1) \to CH^m(-4)$. It is known that the Hopf fibration π is a Riemannian submersion. The image $\pi(z(\mathbf{R}^2 \times U))$ is a $(2n+1)$-dimensional *CR*-submanifold of $CH^m(-4)$ whose holomorphic distribution $\mathscr{D}$ is spanned by $\pi_*(E_1), \ldots, \pi_*(E_{2n})$ and whose totally real distribution $\mathscr{D}^\perp$ is spanned by $\pi_*(E_{2n+1})$.

It follows from (6.8), (6.10) and (6.10′) that the second fundamental form h of $\pi(z(\mathbf{R}^2 \times U))$ in $CH^m(-4)$ satisfies

$$h(e_{2r-1}, e_{2r-1}) = Je_{2n+1} + \phi_r\xi_r, \quad h(e_{2r}, e_{2r}) = Je_{2n+1} - \phi_r\xi_r,$$

(6.11) $\quad h(e_{2r-1}, e_p) = 0, \quad h(e_{2r}, e_q) = 0, \quad h(e_{2r-1}, e_{2r}) = \phi_r J\xi_r,$

$$h(e_l, e_{2n+1}) = 0, \quad h(e_{2n+1}, e_{2n+1}) = 2Je_{2n+1},$$

where $r = 1, \ldots, n$, $l = 1, \ldots, 2n$ and $p, q \notin \{2r-1, 2r\}$ and $\xi_r = \pi_*(\tilde{\xi}_r)$ are normal vector field perpendicular to Je_{2n+1}, $e_1 = \pi_*(E_1), \ldots, e_{2n} = \pi_*(E_{2n})$. Therefore, by applying Lemma 5.2, we conclude that the $(2n+1)$-dimensional *CR*-submanifold $\pi(z(\mathbf{R}^2 \times U))$ in $CH^m(-4)$ satisfies the equality case of (5.4).

Conversely, we suppose that M is a $(2n+1)$-dimensional *CR*-submanifold of $CH^m(-4)$ with $\dim \mathscr{D}^\perp = 1$ which satisfies the equality case of (5.4). Then, with respect to some suitable orthonormal frame field $\{e_1, \ldots, e_{2m}\}$, the second fundamental form satisfy (5.7).

Let $\hat{M} = \pi^{-1}(M)$ denote the inverse image of M via the Hopf fibration $\pi : H_1^{2m+1} \to CH^m(-4)$. Then $\hat{M}$ is a principal circle bundle over M with time-like totally geodesic fibers. Let $z : \hat{M} \to H_1^{2m+1}(-1) \subset C_1^{m+1}$ denote the immersion of $\hat{M}$ in C_1^{m+1}. Let $\tilde{\nabla}$ and $\hat{\nabla}$ denote the metric connections of C_1^{m+1} and $H_1^{2m+1}(-1)$, respectively. We denote by X^* the horizontal lift of a tangent vector X of $CH^m(-4)$. Then we have (cf. [11, 17])

(6.12) $\qquad \tilde{\nabla}_{X^*} Y^* = (\nabla_X Y)^* + (h(X, Y))^* + \langle JX, Y \rangle V + \langle X, Y \rangle z,$

(6.13) $\qquad\qquad \tilde{\nabla}_{X^*} V = \tilde{\nabla}_V X^* = (JX)^*,$

(6.14) $\qquad\qquad\qquad \tilde{\nabla}_V V = -z,$

for vector fields X, Y tangent to M, where z is the position vector of $\hat{M}$ in C_1^{2m+1} and $V = iz \in T_z H_1^{2m+1}(-1)$.

Let $E_1, \ldots, E_{2n+1}, \xi_r^*$ be the horizontal lifts of $e_1, \ldots, e_{2n+1}, \xi_r$, respectively and let $E_{2n+2} = iz$, and let $\{\omega_i^j\}$ be connection forms of $\hat{M}$. Then, in same say as [12],

from (5.1), (6.12) (6.13) and (6.14), we obtain

$$(6.15) \qquad \tilde{\nabla}_{E_{2r-1}}E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r-1})E_j + iE_{2n+1} + \phi_r\xi_r^* - iE_{2n+2},$$

$$(6.16) \qquad \tilde{\nabla}_{E_{2r-1}}E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r-1})E_j - E_{2n+1} + i\phi_r\xi_r^* + E_{2n+2},$$

$$(6.17) \qquad \tilde{\nabla}_{E_{2r}}E_{2r-1} = \sum_{j=1}^{2n} \omega_{2r-1}^j(E_{2r})E_j + E_{2n+1} + i\phi_r\xi_r^* - E_{2n+2},$$

$$(6.18) \qquad \tilde{\nabla}_{E_{2r}}E_{2r} = \sum_{j=1}^{2n} \omega_{2r}^j(E_{2r})E_j + iE_{2n+1} - \phi_r\xi_r^* - iE_{2n+2},$$

$$(6.19) \qquad \tilde{\nabla}_{E_{2r-1}}E_{2n+1} = E_{2r},$$

$$(6.20) \qquad \tilde{\nabla}_{E_{2r}}E_{2n+1} = -E_{2r-1},$$

$$(6.21) \qquad \tilde{\nabla}_{E_{2n+1}}E_{2n+1} = 2iE_{2n+1} - iE_{2n+2},$$

$$(6.22) \qquad \tilde{\nabla}_{E_{2r-1}}E_{2n+2} = \tilde{\nabla}_{E_{2n+2}}E_{2r-1} = E_{2r},$$

$$(6.23) \qquad \tilde{\nabla}_{E_{2r}}E_{2n+2} = \tilde{\nabla}_{E_{2n+2}}E_{2r} = -E_{2r-1},$$

$$(6.24) \qquad \tilde{\nabla}_{E_{2n+1}}E_{2n+2} = \tilde{\nabla}_{E_{2n+2}}E_{2n+1} = iE_{2n+1},$$

$$(6.25) \qquad \tilde{\nabla}_{E_{2n+2}}E_{2n+2} = iE_{2n+2},$$

$$(6.26) \qquad \tilde{\nabla}_{E_{2r-1}}E_p, \ \tilde{\nabla}_{E_p}E_{2r-1}, \ \tilde{\nabla}_{E_{2r}}E_q, \ \tilde{\nabla}_{E_q}E_{2r} \in Span\{E_1,\ldots,E_{2n}\},$$

where $r = 1,\ldots,n$ and $p,q \notin \{2r-1,2r\}$.

It follows from (6.16), (6.17) and (6.19)–(6.25) that the distribution $\mathscr{D}_1$ spanned by $E_1,\ldots,E_{2n},E_{2n+1} - E_{2n+2}$ is integrable. The distribution $\mathscr{D}_2$ spanned by E_{2n+1} is clearly integrable, since it is of rank one. Hence, there exist coordinates $\{s,t,x_1,y_1,\ldots,x_n,y_n\}$ such that $\partial/\partial s, \partial/\partial x_1,\ldots,\partial/\partial y_n$ are tangent to integral submanifolds of $\mathscr{D}_1, \partial/\partial s = E_{2n+1} - E_{2n+2}$ and $\partial/\partial t = E_{2n+1}$.

Applying (6.19)–(6.25), we get

$$\tilde{\nabla}_{E_1}(E_{2n+1} - E_{2n+2}) = \cdots = \tilde{\nabla}_{E_{2n}}(E_{2n+1} - E_{2n+2})$$

$$= \tilde{\nabla}_{E_{2n+1}-E_{2n+2}}(E_{2n+1} - E_{2n+2}) = 0.$$

Hence, along each integral submanifold of $\mathscr{D}_1$, $Z =: E_{2n+1} - E_{2n+2}$ is a constant

light-like vector in C_1^{m+1}. Moreover, from (6.21) and (6.24), we have $\tilde{\nabla}_{E_{2n+1}} Z = iZ$. Since $E_{2n+1} = \partial/\partial t$, we get $\partial Z/\partial t = iZ$. Solving this differential equation yields

$$Z = e^{it} Z_0 \quad \text{on } \hat{M},$$

where Z_0 is a light-like constant vector. Without loss of generality, we may assume $Z_0 = (i, i, 0, \ldots, 0) \in C_1^{m+1}$.

Let M_1 be an integral submanifold of $\mathcal{D}_1$. Without loss of generality, we may assume that M_1 is defined by $t = 0$. Then in the same way as [12], we can write

$$z(s, 0, w_1, \ldots, w_n) = f(s, w_1, \ldots, w_n)(i, i, 0, \ldots, 0) + c(1, -1, 0, \ldots, 0)$$

$$+ (0, 0, \Psi_1(w_1, \ldots, w_n) \ldots, \Psi_{m-1}(w_1, \ldots w_n)),$$

where c is a constant determined by the initial conditions and $w_r = x_r + iy_r$, and $f, \Psi_1, \ldots, \Psi_{m-1}$ are functions.

Let M_1 be an integral submanifold of $\mathcal{D}_1$ and Let ψ denote the map which is the projection of $z : M_1 \rightarrow C_1^{m+1}$ onto the complex Euclidean $(m - 1)$-subspace C^{m-1} spanned by the last $m - 1$ standard coordinate vectors $\varepsilon_3, \ldots, \varepsilon_{m+1}$ of C_1^{m+1}.

In the same way as [12], we have

(6.27)

$$z(s, t, w_1, \ldots, w_n) = \left(c + \frac{1}{4c}(1 + |\Psi|^2) + i(s + t + k(w_1, \ldots, w_n))\right),$$

$$- c + \frac{1}{4c}(1 + |\Psi|^2) + i(s + t + k(w_1, \ldots, w_n)), \Psi(w_1, \ldots, w_n))e^{it},$$

where $k(w_1, \ldots, w_n)$ is a real valued function. Moreover $\Psi(w_1, \ldots, w_n) : \psi(M_1) \rightarrow C^{m-1}$ is a holomorphic isometric immersion. Since orthonormal tangent vector fields $E_1, \ldots, E_{2n}$ lie in $Span\{z_{x_1}, z_{y_1}, \ldots, z_{x_n}, z_{y_n}, z_s\}$, we have

(6.28) $\qquad \tilde{\nabla}_{E_k} E_l = -\frac{1}{2}(\langle \tilde{\nabla}_{E_k} E_l, (-1, 1, 0, \ldots, 0)\rangle) i z_s + \tilde{\nabla}_{\psi_*(E_k)} \psi_*(E_l),$

where $k, l = 1, \ldots, 2n$.

From (6.15), (6.18), (6.26) and (6.28) we have

$$\tilde{\nabla}_{\psi_*(E_{2r-1})} \psi_*(E_{2r-1}) + \tilde{\nabla}_{\psi_*(E_{2r})} \psi_*(E_{2r}) \in Span\{\psi_*(E_1), \ldots, \psi_*(E_{2n})\},$$

(6.29) $\qquad \tilde{\nabla}_{\psi_*(E_{2r-1})} \psi_*(E_p) \in Span\{\psi_*(E_1), \ldots, \psi_*(E_{2n})\},$

$$\tilde{\nabla}_{\psi_*(E_{2r})} \psi_*(E_q) \in Span\{\psi_*(E_1), \ldots, \psi_*(E_{2n})\},$$

where $r = 1, \ldots, n$ and $p, q \notin \{2r - 1, 2r\}$.

It follows from (6.29) that the second fundamental form h' of $\psi(M_1)$ in C^{m-1} satisfies

$$h'(\psi_*(E_{2k-1}), \psi_*(E_{2k-1})) + h'(\psi_*(E_{2k}), \psi_*(E_{2k})) = 0,$$

(6.30) $$h'(\psi_*(E_{2k-1}), \psi_*(E_p)) = 0,$$

$$h'(\psi_*(E_{2k}), \psi_*(E_q)) = 0,$$

where $r = 1, \ldots, n$ and $p, q \notin \{2r - 1, 2r\}$. Proposition 4.3 and (6.30) implies that $\psi(M_1)$ satisfies the equality case of (1.4). If we regard $s + t + k(w)$ as a new variable and denote it by u, then (6.27) yields

(6.31) $$z(s, t, w_1, \ldots, w_n) = \left(c + \frac{1}{4c}(1 + |\Psi|^2) + ui, \right.$$

$$\left. -c + \frac{1}{4c}(1 + |\Psi|^2) + ui, \Psi(w) \right) e^{it}.$$

By choosing the initial conditions $z(0, 0, 0, \ldots, 0) = (-1, 0, \ldots, 0)$, we obtain from (6.31) that $c = -1/2$. Consequently, we obtain (3.5) from (6.31). This completes the proof of theorem.

Finally, for $m = n + 2$, we have the following corollary to the main theorem using Proposition 4.4.

COROLLARY 6.1.   *Let U be a domain of C^n and $\Psi : U \to C^{n+1}$ be a holomorphic isometric immersion in C^{n+1}. Define $z : R^2 \times U \to C_1^{n+2}$ by*

(6.32) $$z(u, t, w_1, \ldots, w_n) = \left(-1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi \right) e^{it}.$$

Then $\langle z, z \rangle = -1$ and the image $z(R^2 \times U)$ in H_1^{2n+3} is invariant under the group action of H_1^1. Moreover the quotient space $z(R^2 \times U)/_\sim$ is a $(2n+1)$-dimensional CR-submanifold with $\dim \mathscr{D}^\perp = 1$ of $CH^{n+2}(-4)$ which satisfies the equality case of (1.3) for $k = n$.

Conversely, up to rigid motions of $CH^{n+1}(-4)$, every linearly full $(2n+1)$-dimensional CR-submanifold with $\dim \mathscr{D}^\perp = 1$ of $CH^{n+2}(-4)$ satisfying the equality case of (1.3) is obtained in such way with $k = n$.

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