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ON EQUATIONS OF THE TYPE $Au = g(x, u, Du)$ WITH DEGENERATE AND NONLINEAR BOUNDARY CONDITIONS

By

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1. Introduction and Main Results

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Let

$$Au(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second order elliptic differential operator with real C^∞ functions a_{ij}, c on $\overline{\Omega}$ satisfying the following properties:

(p1) $a_{ij}(x) = a_{ji}(x),\ i, j = 1, \ldots, n, x \in \overline{\Omega}$.

(p2) There exists a positive constant C_0 such that for all $x \in \overline{\Omega}$ and all $\xi \in \boldsymbol{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C_0|\xi|^2.$$

(p3) $c(x) \geq 0$ in $\overline{\Omega}$.

Let Du be the gradient of u. We consider the following class of *degenerate* boundary value problems for semilinear second order elliptic differential operators

$$\text{(P)} \qquad Au = g(x, u, Du) \text{ in } \Omega, \quad Bu = a\frac{\partial u}{\partial \nu} + bu = \varphi \text{ on } \partial\Omega$$

in the framework of Sobolev spaces $W_p^2(\Omega)$ with $p > n$, where B is a degenerate boundary operator. Let us remark that

$$W_p^2(\Omega) \hookrightarrow C^1(\overline{\Omega}) \quad \text{if } p > n,$$

where $\hookrightarrow$ denotes the continuous embedding. Here:

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(p4) a and b are real-valued C^∞ functions defined on $\partial\Omega$.

(p5) $\partial/\partial\nu = \sum_{i,j=1}^n a_{ij}n_j(\partial/\partial x_i)$ is the conormal derivative corresponding with the operator A, where $n = (n_1, \dots, n_n)$ is the unit exterior normal to the boundary $\partial\Omega$.

Note that (P) is *nondegenerate* (or *coercive*) if and only if either $a \neq 0$ on $\partial\Omega$ or $a \equiv 0$ and $b \neq 0$ on $\partial\Omega$. If $a \equiv 1$ and $b \equiv 0$, then we have the Neumann problem. The case when $a \equiv 0$ and $b \equiv 1$ hold coincides with the Dirichlet problem. Furthermore, if $a(x') \neq 0$ on $\partial\Omega$, then we get the third boundary problem (or Robin problem). We remark that the so-called Lopatinskij-Shapiro complementary condition does not hold at the points $x' \in \partial\Omega$ with $a(x') = 0$. The main theorem for elliptic boundary value problems, see Wloka [23, Hauptsatz 13.1], implies that the ellipticity of a differential operator and the Lopatinskij-Shapiro condition are equivalent to the Fredholm property of a boundary value problem if one uses spaces of Besov type $B_{p,p}^{s-1/p}(\partial\Omega)$ for the description of the boundary operator B. To overcome these difficulties one introduces a subspace of $B_{p,p}^{1-1/p}(\partial\Omega)$ which is associated to our degenerate boundary operator. For more details, we refer to Taira [19] and Runst [14].

We make the following three assumptions (H1)–(H3):

(H1) $a(x') \geq 0$ and $b(x') \geq 0$ on $\partial\Omega$.

(H2) $b(x') > 0$ on $\Sigma = \{x' \in \partial\Omega : a(x') = 0\}$.

(H3) $c(x) \geq 0$ in Ω, $c \not\equiv 0$ in Ω.

Problem (P) with homogeneous nondegenerate boundary conditions has been studied by Amann [2], Amann and Crandall [3] and Kazdan and Kramer [6]. In these papers, it is assumed that the nonlinear function $g(x, \xi, \eta)$ is continuous with respect to all of its variables and grows at most quadratically in η, i.e., one assumes that there exists a nonnegative and increasing function $d : [0, \infty) \to \boldsymbol{R}$ such that the Bernstein condition (cf. Bernstein [4], Nagumo [10]) holds:

$$(1) \qquad |g(x, \xi, \eta)| \leq d(|\xi|)(1 + |\eta|^2) \quad \text{for all } (x, \xi, \eta) \in \overline{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n.$$

In this case, condition (1) is sufficient to obtain an a priori estimate of $\|u|L_\infty\|$ which generates an a priori estimate of $\|Du|L_\infty\|$, and finally an estimate of $\|u|W_p^2\|$ for the solution u. We remark that if g grows faster than quadratically in η, then Serrin [18] has proved that there are smooth data for which the Dirichlet boundary value problem has no solution. Furthermore, we refer also to the counterexamples given in Section 4.

In the above-mentioned papers, it was shown the existence of solutions in the Sobolev space $W_p^2(\Omega), p > n$, provided that suitable sub- and supersolutions are known. In [6], the authors have given conditions of Landesman–Lazer type which imply the existence of sub- and supersolutions for a semilinear elliptic boundary problem of type (P) under homogeneous Dirichlet boundary condition. In a paper by Inkmann [5], existence and multiplicity results for (P) with nonlinear boundary condition

$$Bu = \frac{\partial u}{\partial v} = f(x, u) \quad \text{on } \partial\Omega$$

have been proved.

A survey of existence and multiplicity results for nonlinear coupled systems of the type (P) with inhomogeneous degenerate and nonlinear boundary conditions may be found in Schmitt [17] in the framework of Hölder space $C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. In this paper, the conditions imposed on the function $g(x, \xi, \eta)$ (which also satisfies (1)) are of geometrical nature, in the sense that a nonempty bounded open convex set M in ξ-space exists such that the vector field $g(x, \cdot, \eta)$ is always outwardly directed on ∂M for all $x \in \overline{\Omega}$ and certain values of η. We refer to [17, Theorem 2.3.]. In this paper, it was shown that from these conditions one can derive sub- and supersolution type results.

In a recent paper by Taira [21], the author extended the results obtained by Amann and Crandall [3] to the case of degenerate boundary conditions. He used essentially the same approach as that of Amann and Crandall [3] to prove existence and uniqueness theorems in the framework of Sobolev spaces $W_p^2(\Omega)$, $p > n$.

On the other side, Pohozaev [11] considered the nonlinear problem

$$\Delta u = g(x, u, Du) \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega. \tag{2}$$

He extended the condition (1) in the following way: Assume that the nonlinear function g belongs only to $L_p(\Omega)$ for any arbitrarily fixed $u \in W_p^2(\Omega)$, $p > n$. In this case, condition (1) is not longer sufficient for the proof of an a priori estimate of $\|Du|L_\infty\|$ for the solution u of (2), if one has an a priori estimate of $\|u|L_\infty\|$. It was shown there that the following assumptions on g imply an a priori estimate of $\|Du|L_\infty\|$ for the solution u of (2) from an a priori estimate of $\|u|L_\infty\|$:

(H4) Let $g(x, \xi, \eta) : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$ satisfy the Caratheodory condition: g is measurable with respect to x for all $(\xi, \eta) \in \boldsymbol{R} \times \boldsymbol{R}^n$ and continuous with respect to (ξ, η) for almost all $x \in \Omega$.

(H5) Let the growth condition

$$|g(x,\xi,\eta)| < b(x,\xi)(1+|\eta|^{\mu}) \tag{3}$$

be fulfiled with $\mu = 2 - n/p$ for almost all $x \in \Omega$ and all $(\xi,\eta) \in \boldsymbol{R} \times \boldsymbol{R}^n$, where the function $b(x,\xi)$ satisfies also the Caratheodory condition, and such that for any fixed $c > 0$

$$\sup_{|\xi|<c} b(\cdot,\xi) \in L_p(\Omega).$$

Note that (3) coincides with (1) in the special case $p = \infty$.

One of the aims of this paper is to prove that one can extend the results of [11] to degenerate and nonlinear boundary operators. With respect to the mentioned papers of Amann and Crandall [3], Taira [21], etc., we suppose weaker smoothness assumptions on the function g in the sense of [11], and we can also consider inhomogeneous and nonlinear boundary conditions. The investigations of such problems are motivated by nonlinear diffusion processes, see e.g. Keller [7] and Schmitt [17].

We use the following notations.

A function $u \in W_p^2(\Omega)$, $p > n$, is a *solution* of (P) if

$$Au = g(x,u,Du) \text{ a.e. in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega.$$

A function $u_+ \in W_p^2(\Omega)$ is called a *supersolution* of problem (P) if it satisfies the condition

$$Au_+ \geq g(x,u_+,Du_+) \text{ a.e. in } \Omega, \quad Bu_+ = \varphi \text{ on } \partial\Omega.$$

Similarly, a function $u_- \in W_p^2(\Omega)$ is called a *subsolution* of problem (P) if it satisfies the condition

$$Au_- \leq g(x,u_-,Du_-) \text{ a.e. in } \Omega, \quad Bu_- = \varphi \text{ on } \partial\Omega.$$

Now we can formulate our existence result. We remark that the definition of the function spaces of type $B_{p,p}^{*,1-1/p}(\partial\Omega)$ will be given in the next section.

Theorem 1. *Suppose that* (H1)–(H3) *are fulfiled, and that g satisfies the conditions* (H4), (H5), *and a Lipschitz condition given by*

(H6) *it holds*

$$|g(x,\xi,\eta_1) - g(x,\xi,\eta_2)| \leq b_1(x,\xi,\eta_1,\eta_2) \cdot |\eta_1 - \eta_2| \tag{4}$$

for almost all $x \in \Omega$ *and all* $(\xi, \eta_1, \eta_2) \in \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n$, *where the function* $b_1(x, \xi, \eta_1, \eta_2)$ *satisfies the Caratheodory condition, and for any fixed* $c > 0$

$$\sup\{b_1(\cdot, \xi, \eta_1, \eta_2) : |\xi| < c, |\eta_1| < c, |\eta_2 < c\} \in L_p(\Omega),$$

for some $p > n$.

Let $\varphi \in B_{p,p}^{*,1-1/p}(\partial\Omega)$. *If there exist a subsolution* u_- *and a supersolution* u_+ *in* $W_p^2(\Omega)$ *of* (P) *with* $u_- \leq u_+$ *in* $\overline{\Omega}$, *then* (P) *has a solution* $u \in W_p^2(\Omega)$ *such that* $u_- \leq u \leq u_+$ *in* $\overline{\Omega}$.

The following uniqueness theorem is a generalization and an improvement of [21, Theorem 2].

THEOREM 2. *Suppose that* (H1)–(H5) *are satisfied for some* $p > n$. *Let* $\varphi \in B_{p,p}^{*,1-1/p}(\partial\Omega)$. *If* $g(x, \xi, \eta)$ *is strictly decreasing with respect to* ξ *for almost all* $x \in \Omega$ *and all* $\eta \in \boldsymbol{R}^n$, *then* (P) *has at most one solution* $u \in W_p^2(\Omega)$.

Finally, we apply Theorem 1 and Theorem 2 to prove the existence and the uniqueness of the solution for special classes of nonlinear boundary conditions.

THEOREM 3. *Let all assumption of Theorem* 1 *be satisfied. Further, let* $\gamma(x', \xi) : \partial\Omega \times \boldsymbol{R} \to \boldsymbol{R}$ *be a smooth function. Assume that*

$$\frac{\partial \gamma}{\partial \xi}(x', \xi) \geq 0 \quad \text{for all } (x', \xi) \in \partial\Omega \times \boldsymbol{R} \tag{5}$$

and

$$\gamma(x', 0) = 0 \quad \text{for all } x' \in \partial\Omega. \tag{6}$$

(a) *If there exist a subsolution* u_- *and a supersolution* u_+ *in* $W_p^2(\Omega)$ *of problem*

$$Au = g(x, u, Du) \text{ in } \Omega, \quad B_\gamma u = \frac{\partial u}{\partial \nu} + \gamma(x', u) = \varphi \text{ on } \partial\Omega \tag{7}$$

with $u_- \leq u_+$ *in* $\overline{\Omega}$, *then* (7) *has a solution* $u \in W_p^2(\Omega)$ *such that* $u_- \leq u \leq u_+$ *in* $\overline{\Omega}$.

(b) *If* $g(x, \xi, \eta)$ *is strictly decreasing with respect to* ξ *for almost all* $x \in \Omega$ *and all* $\eta \in \boldsymbol{R}^n$, *then* (7) *has at most one solution* $u \in W_p^2(\Omega)$.

The paper is organized in the following way. In Section 2, an existence and uniqueness theorem for the corresponding linearized boundary value problem is given. The next section deals with an a priori estimate $\|u| W_p^2\|$ for the solution u of (P). In Section 4, we prove Theorems 1–3. These results follow from a priori estimates, a generalization of Aleksandrov's maximum principle, see Aleksandrov [1], to degenerate boundary conditions and Leray-Schauder degree arguments.

2. Linear Theory

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded and smooth domain with boundary $\partial\Omega$. Furthermore, let (A, B) have the same meaning as before. At first we consider the corresponding linearized boundary problem

$$Au = f \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega \tag{1}$$

in the framework of *Sobolev spaces*.

Suppose that $1 < p < \infty$. If $k = 1, 2, \ldots$, then the Sobolev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) = \left\{ u \in L_p(\Omega) : \|u|W_p^k\| = \sum_{|\alpha| \le k} \|D^\alpha u|L_p(\Omega)\| < \infty \right\}.$$

Let $B_{p,p}^{k-1/p}(\partial\Omega)$ be the *Besov space* of all boundary function φ of functions $u \in W_p^k(\Omega)$ (in the sense of traces) equipped with the norm

$$\|\varphi|B_{p,p}^{k-1/p}\| = \inf\{\|u|W_p^k\| : u \in W_p^k(\Omega) \text{ with } u_{|\partial\Omega} = \varphi\}.$$

We introduce a subspace of $B_{p,p}^{1-1/p}(\partial\Omega)$ which is associated to our boundary operator B: Let

$$B_{p,p}^{*,1-1/p}(\partial\Omega) = \{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in B_{p,p}^{1-1/p}(\partial\Omega), \varphi_2 \in B_{p,p}^{2-1/p}(\partial\Omega)\},$$

and the norm is given by

$$\|\varphi|B_{p,p}^{*,1-1/p}\| = \inf\{\|\varphi_1|B_{p,p}^{1-1/p}\| + \|\varphi_2|B_{p,p}^{2-1/p}\| : \varphi = a\varphi_1 + b\varphi_2\}.$$

Remark. It is not hard to check that $B_{p,p}^{*,1-1/p}(\partial\Omega)$ becomes a Banach space. We remark that $B_{p,p}^{*,1-1/p}(\partial\Omega) = B_{p,p}^{2-1/p}(\partial\Omega)$ if $a \equiv 0$ on $\partial\Omega$ and $B_{p,p}^{*,1-1/p}(\partial\Omega) = B_{p,p}^{1-1/p}(\partial\Omega)$ if $a > 0$ on $\partial\Omega$. In this sense, the space $B_{p,p}^{*,1-1/p}(\partial\Omega)$ can be considered as a interpolation space with respect to our boundary operator B.

Now the following existence and uniqueness result for problem (1) holds (see [19, Theorem 1], [14]):

Proposition 1. *Let* (H1)–(H3) *be satisfied. Then the map*

$$(A, B) : W_p^2(\Omega) \to L_p(\Omega) \times B_{p,p}^{*,1-1/p}(\partial\Omega)$$

is an algebraic and topological isomorphism for all p, $1 < p < \infty$.

We remark that this result was proved in [14] in the more general framework of the two scales of function spaces of Besov–Triebel–Lizorkin type.

For our further investigations, the following maximum principle due to Aleksandrov [1, Theorem 2] will become important. We formulate a special result which is sufficient for our considerations. Hereby the notation $\geq$ is used in the sense of distributions, i.e., a distribution $f \in D'(\Omega)$ is said to be non-negative ($f \geq 0$) if and only if $f(\varphi) \geq 0$ for any test function $\varphi \in D(\Omega)$ with $\varphi \geq 0$.

Lemma 1. *Let $a_1, \dots, a_n$ and a_0 be function in $L_n(\Omega)$ such that $a_0 \geq 0$ in Ω. Suppose that $u \in W_n^2(\Omega)$ satisfies*

$$Au + \sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + a_0 u \leq 0 \quad a.e. \ in \ \Omega.$$

Then u does not take its positive maximum in Ω if it is not a constant function.

Using this lemma we are able to show the following assertion. The subset Σ has the same meaning as in (H2).

Lemma 2. *Let $a_1, \dots, a_n$ and a_0 be functions in $L_n(\Omega)$ such that $a_0 \geq 0$ in Ω. If a function $u \in W_p^2(\Omega)$, $p > n$, satisfies*

$$Au + \sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + a_0 u \geq 0 \quad a.e. \ in \ \Omega,$$

then $u \geq 0$ in Ω.

Further, if $u \not\equiv 0$ in Ω satisfies the boundary conditions

$$Bu = a \frac{\partial u}{\partial \nu} + bu = 0 \quad on \ \partial\Omega,$$

then it holds

$$u > 0 \ in \ \overline{\Omega} \backslash \Sigma, \quad u = 0 \ on \ \Sigma \ and \ \frac{\partial u}{\partial \nu} < 0 \ on \ \Sigma.$$

Proof. Note that an application of Lemma 1 to $-u$ shows

$$u \geq 0 \quad \text{in } \Omega.$$

Now we assume that $u \not\equiv 0$. If there is a point $x_0 \in \Omega$ with $u(x_0) = 0$, then Lemma 1, see also [1, Theorem 2(I)], imply that $u \equiv 0$ in Ω. Hence we obtain $u > 0$ in Ω.

If there is a point $y_0 \in \partial\Omega$ with $u(y_0) = 0$, then it follows from the boundary point lemma (see [12, Section 2.3, Theorem 8]) that $\partial u/\partial v(y_0) < 0$. Furthermore, we have

$$Bu(y_0) = a(y_0)\frac{\partial u}{\partial v}(y_0) = 0.$$

This implies that $y_0 \in \Sigma$.

Conversely, if $y_0 \in \Sigma$, then we have by (H2) that $b(y_0) > 0$. Therefore,

$$Bu(y_0) = b(y_0)u(y_0) = 0$$

implies $u(y_0) = 0$. We have proved $u(y_0) = 0$ if and only if $y_0 \in \Sigma$, and $\partial u/\partial v(y_0) < 0$ on Σ. The proof is finished. ∎

Further, we apply the following mapping property of the nonlinear Nemytskij operator

$$T_\gamma(u) : x \mapsto \gamma(x, u(x)),$$

see [16, Section 5.3, Subsection 5.5.2]. We remark that for $p > n$ we have the continuous embedding

$$B^{1-1/p}_{p,p}(\partial\Omega) \hookrightarrow C(\partial\Omega).$$

PROPOSITION 2. *Let $\gamma(x', \xi)$ be a smooth function with respect to $x' \in \partial\Omega$ and $\xi \in \boldsymbol{R}$. Let $p > n$ and $u \in B^{1-1/p}_{p,p}(\partial\Omega)$. Then there exists a constant $c_\gamma > 0$, independent of u, such that*

$$\|\gamma(\cdot, u)|B^{1-1/p}_{p,p}(\partial\Omega)\| \le c_\gamma \|u|B^{1-1/p}_{p,p}(\partial\Omega)\|(1 + \|u|L_\infty(\partial\Omega)\|). \tag{2}$$

Furthermore, the map $u \to \gamma(\cdot, u(\cdot))$ is continuous from $B^{1-1/p}_{p,p}(\partial\Omega)$ into $B^{1-1/p}_{p,p}(\partial\Omega)$.

We emphasize that the above results hold also under weaker smoothness conditions on the coefficients a_{ij} of the differential operator A and on the coefficients a and b of the boundary operator B, respectively. For example, Lemma 2 holds for $a_{ij} \in C^1(\overline{\Omega})$, $a \in C^{1+\alpha}(\partial\Omega)$ and $b \in C^{2+\alpha}(\partial\Omega)$, $\alpha > 0$.

Furthermore, for the proof of Proposition 1 one uses mapping properties of pseudo-differential operators which remain true if a and b are sufficiently smooth, see for example Marschall [9]. Finally, we remark that Proposition 2 is true if the function γ belongs to the Lipschitz space Lip μ, $\mu > 2$, see e.g. [13] and [16, Subsection 5.3.4, Theorem 2; Subsection 5.5.2, Theorem 3].

3. A Priori Estimate

Recall that $g(x, \xi, \eta)$ satisfies the Caratheodory condition, see (H4), and the growth condition (H5). Let $p > n$. Hence the nonlinear Nemytskij operator

$$T_g(u) : x \mapsto g(x, u(x), Du(x))$$

is defined on $W_p^2(\Omega)$, $p > n$, and is a continuous operator from $W_p^2(\Omega)$ into $L_p(\Omega)$, see Krasnoselskij [8]. We start with the a priori estimate for the solutions of (P). Let $b(x, \xi)$ have the same meaning as in (H5). Hence $b(x, \xi) > 0$ for almost all $(x, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^n$.

PROPOSITION. *Let* (H1)–(H5) *be satisfied for some* $p > n$. *Let*

$$b_M(x) = \sup\{b(x, \xi) : |\xi| \leq M\}.$$

Then there exists a function $\psi : [0, \infty] \times [0, \infty] \times [0, \infty] \to [0, \infty]$, *bounded on every compact set, such that for any solution* $u \in W_p^2(\Omega)$, $p > n$, *of the problem* (P) *it holds that*

$$\|u|W_p^2\| \leq \psi(M, \|b_M|L_p\|, \|\varphi|B_{p,p}^{*,1-1/p}\|), \tag{1}$$

provided $\|u|L_\infty\| \leq M$.

PROOF. Let $u \in W_p^2(\Omega)$ be a solution of (P) and $\mu = 2 - n/p$, see (H5). Then it holds

$$Au = g(x, u, Du) = \frac{g(x, u, Du)}{1 + |Du|^\mu}(1 + |Du|^\mu).$$

Hence the function u satisfies

$$Au + b_M(x)u = g_1(x)|Du|^\mu + g_0(x) \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega, \tag{2}$$

where

$$b_M(x) > 0, \quad g_1(x) = \frac{g(x, u, Du)}{1 + |Du(x)|^\mu}, \quad g_0(x) = g_1(x) + b_M(x)u(x).$$

The following method of using a parameter t has been applied by many authors. We refer to Amann and Crandall [3] and Taira [21]. We use arguments going back to Pohozaev [11].

Let $t \in [0,1]$. Then we consider the family of parameterized problems

$$(\mathrm{Pt}) \quad Au + b_M(x)u = g_1(x)|Du|^\mu + tg_0(x) \text{ in } \Omega, \quad Bu = a\frac{\partial u}{\partial \nu} + bu = t\varphi \text{ on } \partial\Omega$$

in the framework of Sobolev spaces $W_p^2(\Omega)$ with $p > n$.

We shall show that for fixed $t \in [0,1]$ problem $(\mathrm{P_t})$ has at most one solution. This implies that the solution of $(\mathrm{P_t})$ with $t = 1$ coincides with the solution of the original problem (P).

Let $0 \le t_1 \le t_2 \le 1$. We assume that $v \in W_p^2(\Omega)$ is a solution of $(\mathrm{P_{t_1}})$ and $z \in W_p^2(\Omega)$ is a solution of $(\mathrm{P_{t_2}})$. Hence $w = z - v$ solves

$$(3) \qquad Aw + b_M(x)w = g_1(x)\sum_{i=1}^n h_i(x)\frac{\partial w}{\partial x_i} + (t_2 - t_1)g_0(x) \text{ in } \Omega, \quad Bw = (t_2 - t_1)\varphi \text{ on } \partial\Omega.$$

The function h_i, $i = 1, \dots, n$, are given by

$$(4) \qquad h_i(x) = \int_0^1 H_i(x,\tau)\,d\tau$$

with

$$(5) \qquad H_i(x,\tau) = \mu\left[\sum_{k=1}^n\left(\tau\frac{\partial w}{\partial x_k} + \frac{\partial v}{\partial x_k}\right)^2\right]^{-1+\mu/2}\left(\tau\frac{\partial w}{\partial x_i} + \frac{\partial v}{\partial x_i}\right)(x)$$

if

$$(6) \qquad \sum_{k=1}^n\left(\tau\frac{\partial w}{\partial x_k} + \frac{\partial v}{\partial x_k}\right)^2(x) \neq 0,$$

and

$$(7) \qquad H_i(x,\tau) = 0$$

otherwise.

Now we set $K = (t_2 - t_1)(1 + \|u|L_\infty\|)$.

LEMMA 1. *Let* $0 \le t_1 \le t_2 \le 1$ *be fixed. Then*

$$(8) \qquad \|w|L_\infty\| \le (t_2 - t_1)(1 + \|u|L_\infty\|).$$

PROOF. Note that $w + K$ is a solution of

$$A(w+K) + b_M(x)(w+K) = g_1(x)\sum_{i=1}^{n} h_i(x)\frac{\partial(w+K)}{\partial x_i}$$
$$+ (t_2 - t_1)g_0(x) + b_M(x)K + c(x)K \quad \text{in } \Omega,$$
$$B(w+K) = (t_2 - t_1)\varphi + bK \quad \text{on } \partial\Omega.$$

Now we apply the arguments of Pohozaev [11]. By our assumptions, we have $c(x) \geq 0$, $b_M(x) > 0$, $b_M \in L_p(\Omega)$, $g_1 \in L_{np/(p-n)}(\Omega)$ and $h_i \in L_p(\Omega)$, $i = 1, \dots, n$, with $p > n$. Hence the assumptions of Lemma 2 in Section 2 are satisfied. Furthermore, it holds by the definition of b_M, g_0 and (H5) that

$$(t_2 - t_1)g_0(x) + b_M(x)K + c(x)K \geq 0 \quad \text{in } \Omega.$$

Hence Lemma 2 in Section 2 yields $w + K \geq 0$ in Ω. Similarly one shows $w \leq K$ in Ω. The proof of Lemma 1 is finished. ∎

In order to continue the proof of our proposition we need the uniqueness of the solution of $(\mathrm{P_t})$. This is an easy consequence of Lemma 1, see (8) with $t_1 = t_2$.

LEMMA 2. *Let $t \in [0,1]$ be fixed, and let $p > n$. Then $(\mathrm{P_t})$ has at most one solution $u \in W_p^2(\Omega)$.*

CONTINUATION OF THE PROOF OF PROPOSITION. Let $w = z - v$ be defined as in Lemma 1. Hence w is a solution of

$$Aw + b_M(x)w = g_1(x)(|Dz|^\mu - |Dv|^\mu) + (t_2 - t_1)g_0(x) \quad \text{in } \Omega, \tag{9}$$
$$Bw = (t_2 - t_1)\varphi \quad \text{on } \partial\Omega.$$

Now Proposition 1 in Section 2 yields

$$\|w|W_p^2\| \leq C(\|Aw + b_M w|L_p\| + \|\varphi|B_{p,p}^{*,1-1/p}\|). \tag{10}$$

Note that the constant C depends only on A, Ω, n, p and $\|b_M|L_p\|$.

Let $1/2 < \theta = 1/\mu < 1$. An application of the Gagliardo–Nirenberg inequality, see Zeidler [24, Appendix (54b)] and [16, Subsection 5.2.5], yields

$$\|Dw|L_\infty\| \leq C_1\|w|W_p^2\|^\theta \|w|L_\infty\|^{1-\theta} \tag{11}$$

holds for $w \in W_p^2(\Omega)$ with $p > n$. Here C_1 depends only on Ω, n and p.

On the other hand, (9) gives

$$\|Aw + b_M(x)w|L_p\| \le 2^{\mu-1}\|g_1|L_p\|\,\|Dw|L_\infty\|^\mu + 2^\mu\|g_1|L_p\|\,\|Dv|L_\infty\|^\mu + (t_2 - t_1)\|g_0|L_p\| \tag{12}$$

and

$$\|Bw|B^{*,1-1/p}_{p,p}\| \le (t_2 - t_1)\|\varphi|B^{*,1-1/p}_{p,p}\| \tag{13}$$

for all $0 \le t_1 < t_2 \le 1$. New (8) and (10)–(13) show

$$\|w|W^2_p\| \le 2C(\|g_0|L_p\| + \|\varphi|B^{*,1-1/p}_{p,p}\|) + 2^{\mu+1}C\|g_1|L_p\|\,\|Dv|L_\infty\|^\mu \tag{14}$$

if

$$0 < t_2 - t_1 \le h, \tag{15}$$

where

$$h = h(A, \Omega, n, p, \|b_M|L_p\|) = (2C_1)^{-\mu/(\mu-1)}(C\|g_1|L_p\|)^{-1/(\mu-1)}(1 + M)^{-1}$$

is independent of t_1 and t_2.

Let $k = 1, 2, \dots,$ be fixed, and $t_1 = t^{(k-1)}$ and $t_2 = t^{(k)}$. Now let $v = v^{(k-1)}$ and $z = v^{(k)}$ be the solution in $W^2_p(\Omega)$ of $(\mathrm{P}_{t^{(k-1)}})$ and $(\mathrm{P}_{t^{(k)}})$, respectively. Then we obtain from (14) that

$$\|v^{(k)}|W^2_p\| \le 2C(\|g_0|L_p\| + \|\varphi|B^{*,1-1/p}_{p,p}\|) + \|v^{(k-1)}|W^2_p\| + 2^{\mu+1}C\|g_1|L_p\|\,\|Dv^{(k-1)}|L_\infty\|^\mu \tag{16}$$

for all $t^{(k)}$, $t^{(k+1)}$ in $[0,1]$ with $0 < t^{(k)} - t^{(k-1)} \le h$. Now we put $t^{(0)} = 0$. Then $v^{(0)} \equiv 0$ is the unique solution of $(\mathrm{P}_{t^{(0)}})$. The continuous embedding $W^2_p(\Omega) \hookrightarrow C^1(\overline{\Omega})$ for $p > n$ yields

$$\|Dv^{(k-1)}|L_\infty\| \le c\|v^{(k-1)}|W^2_p\|.$$

Now our proposition follows from (16) by a finite iteration procedure. ∎

4. Proof of the Main Results

Let (H1)–(H6) be satisfied. We prove the existence of the solution of (P) under the assumption that there exist a subsolution u_- and a supersolution u_+ of (P) in $W^2_p(\Omega)$, $p > n$, with $u_- \le u_+$ in $\overline{\Omega}$. We start with the following lemma which is important for our further considerations.

PROPOSITION. *We suppose that the real function* $G_0(x, \xi, \eta)$ *defined on* $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$ *satisfies the Caratheodory condition* (H4) *and*

$$\sup_{(\xi,\eta) \in \boldsymbol{R} \times \boldsymbol{R}^n} |G_0(\cdot, \xi, \eta)| \in L_p(\Omega)$$

with $p > n$. *Then the boundary value problem*

$$Au = G_0(x, u, Du) \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega,$$

where $\varphi \in B^{*,1-1/p}_{p,p}(\partial\Omega)$, *has a solution* $u \in W^2_p(\Omega)$.

PROOF. Proposition 1 in Section 2 shows that for any $v \in C^1(\overline{\Omega})$ the semilinear boundary problem

$$Au = G_0(x, v, Dv) \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega$$

has a unique solution $u \in W^2_p(\Omega)$ with $p > n$. Therefore, we can introduce a continuous linear operator S such that for any $v \in C^1(\overline{\Omega})$ the function $u = Sv \in W^2_p(\Omega)$ is the unique solution of the above problem. The assumptions on G_0 yield the existence of a constant c such that the a priori estimate

$$\|u| W^2_p(\Omega)\| \le c$$

holds for all solution u. The operator

$$S : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$$

is compact. Now Leray–Schauder degree arguments, see [16, Chapter 6], prove the existence of a solution of our problem in $C^1(\overline{\Omega})$ which belongs by definition of S to $W^2_p(\Omega)$. The proof is finished. ■

PROOF OF THEOREM 1. Let T be the truncating operator defined for $u \in W^2_p(\Omega)$, $p > n$, by

$$Tu(x) = \begin{cases} u_+(x) & \text{if } u(x) > u_+(x), \\ u(x) & \text{if } u_-(x) \le u(x) \le u_+(x), \\ u_-(x) & \text{if } u(x) < u_-(x). \end{cases} \tag{1}$$

Then we consider the modified boundary value problem

$$Au = g(x, Tu, Du) \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega. \tag{2}$$

The following lemma shows the connection between the solvability of (P) and (2).

LEMMA 1. *Let all assumptions of Theorem* 1 *be satisfied. If* $u \in W_p^2(\Omega)$ *is a solution of* (2), *then we have*

$$u_- \leq u \leq u_+ \quad in\ \bar{\Omega}. \tag{3}$$

PROOF OF LEMMA 1. We consider $w = u_+ - u$. Then

$$Aw \geq g(x, u_+, Du_+) - g(x, Tu, Du) \text{ in } \Omega, \quad Bw = 0 \text{ on } \partial\Omega. \tag{4}$$

Assume that the contrary to (3) holds. Then there exists a set $\Omega_w \subset \Omega$ given by

$$\Omega_w = \{x \in \Omega : w(x) < 0\}$$

with nonzero Lebesgue measure, and (1) implies

$$Aw \geq g(x, u_+, Du_+) - g(x, u_+, Du) \quad \text{in } \Omega_w.$$

Now assumption (H6) yields

$$Aw \geq -b_1((x, u_+(x), Du_+(x), Du(x))|Dw| \quad \text{in } \Omega_w.$$

Note that $b_1 \in L_p(\Omega)$ with $p > n$. Therefore the term on the right side is a non-positive function in $L_p(\Omega)$. Hence Lemma 1 in Section 2 shows that w attains a strictly negative minimum on $\partial\Omega$, a contradiction to the boundary condition in (4). This proves that $u \leq u_+$ holds in Ω. The other case can be shown analogously. The proof of Lemma 1 is finished. ■

CONTINUATION OF THE PROOF OF THEOREM 1. If

$$M = \max\left\{\max_{x \in \bar{\Omega}} u_+(x), -\min_{x \in \bar{\Omega}} u_-(x)\right\}$$

then Lemma 1, Proposition in Section 3 and the embedding $W_p^2(\Omega) \hookrightarrow C^1(\bar{\Omega})$ for $p > n$ show

$$\begin{aligned} \max_{x \in \bar{\Omega}} |Du(x)| &\leq \|u|C^1\| \leq c\|u|W_p^2\| \\ &\leq c\psi(M, \|b_M|L_p\|, \|\varphi|B_{p,p}^{*,1-1/p}\|) = M_1, \end{aligned} \tag{5}$$

where c is independent of $u \in W_p^2(\Omega)$

Now we put

$$M_2 = \max\left\{M_1, \max_{x \in \bar{\Omega}} |Du_+(x)|, \max_{x \in \bar{\Omega}} |Du_-(x)|,\right\},$$

and

$$G_1(x, \xi, \eta) = \begin{cases} g(x, \xi, \eta) & \text{if } |\eta| \le M_2, \\ g\left(x, \xi, M_2 \frac{\eta}{|\eta|}\right) & \text{if } |\eta| > M_2. \end{cases}$$

We consider the solvability of the semilinear boundary value problem

$$(6) \qquad Au = G_1(x, Tu, Du) \text{ in } \Omega, \quad Bu = \varphi \text{ in } \partial\Omega.$$

The assumptions (H4)–(H6) show that $G_1(x, \xi, \eta)$ satisfies the Caratheodory condition, (H6), and the following growth condition

$$|G_1(x, \xi, \eta)| \le \begin{cases} b(x, \xi)(1 + |\eta|^\mu) & \text{if } |\eta| \le M_2, \\ b(x, \xi)(1 + M_2^\mu) & \text{if } |\eta| > M_2. \end{cases}$$

Furthermore, the function

$$G(x, \xi, \eta) = \begin{cases} G_1(x, u_+(x), \eta) & \text{if } \xi > u_+(x), \\ G_1(x, \xi, \eta) & \text{if } u_+(x) \le \xi \le u_+(x), \\ G_1(x, u_-(x), \eta) & \text{if } \xi < u_-(x) \end{cases}$$

satisfies (H4),

$$(7) \qquad \sup_{(\xi, \eta) \in \boldsymbol{R} \times \boldsymbol{R}^n} |G(\cdot, \xi, \eta)| \in L_p(\Omega)$$

with $p > n$, and

$$G(x, u(x), Du(x)) = G_1(x, Tu(x), Du(x)).$$

Consequently, we can apply our proposition to the problem (6). Therefore we obtain that (6) has a solution $u \in W_p^2(\Omega)$, where $p > n$.

Note that u_- is a subsolution and u_+ is a supersolution of

$$(8) \qquad Au = G_1(x, u, Du) \text{ in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega.$$

Then Lemma 1 shows that

$$u_- \le u \le u_+ \quad \text{in } \overline{\Omega}.$$

Consequently, we have $Tu(x) = u(x)$. Hence the solution u of (6) is also a solution of (8). Finally, Proposition in Section 3, (5) and the construction of $G_1 = g$ for $|\eta| \le M_1 \le M_2$ prove that u is a solution of (P), too. Hence Theorem 1 is shown. ∎

Our next step is to prove the uniqueness of the solution of (P) if g is strictly decreasing.

PROOF OF THEOREM 2. Assume that $u, v \in W_p^2(\Omega)$ are two solutions of problem (P), i.e., we have

$$Au = g(x, u, Du) \text{ a.e. in } \Omega, \quad Bu = \varphi \text{ on } \partial\Omega,$$

$$Av = g(x, v, Dv) \text{ a.e. in } \Omega, \quad Bv = \varphi \text{ on } \partial\Omega.$$

We put $w(x) = u(x) - v(x)$. Assume to the contrary that

$$\max_{x \in \bar{\Omega}} w(x) = M > 0.$$

If there is a point $x_0 \in \Omega$ with $w(x_0) = M$, then Lemma 2 in Section 2 yields

$$w(x) = u(x) - v(x) \equiv M \quad \text{in } \Omega.$$

Hence we have $Du = Dv$ in Ω. Now the assumption that $g(x, \xi, \eta)$ is strictly decreasing with respect to ξ for almost all $x \in \Omega$ and all $\eta \in \boldsymbol{R}^n$, and (H3) imply

$$0 = Aw - g(x, u, Du) + g(x, v, Dv) = cM - (g(x, u, Du) - g(x, v, Du)) > 0$$

in Ω. We obtain a contradiction.

Now we assume that there is a point $x_0' \in \partial\Omega$ such that

$$w(x_0') = M > 0.$$

Then an application of Lemma 2 in Section 2 gives a contradiction. The other case can be handled similarly. The proof of Theorem 2 is complete. ■

We give an easy example for the application of Theorem 1. Let $p > n$. In the following corollary we consider functions of the type

$$u = u_0 + t\varphi_1, \quad t \in \boldsymbol{R}.$$

Here $u_0 \in W_p^2(\Omega)$ is the unique solution of

$$Au_0 = 0 \text{ in } \Omega, \quad Bu_0 = \varphi \text{ on } \partial\Omega,$$

where $\varphi \in B_{p,p}^{*,1-1/p}(\partial\Omega)$. Furthermore, $\varphi_1 \in C^\infty(\bar{\Omega})$ is the unique eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problem

$$Au = \lambda u \text{ in } \partial\Omega, \quad Bu = 0 \text{ on } \partial\Omega$$

which satisfies $\varphi_1 > 0$ in $\bar{\Omega} \backslash \Sigma$ and $\|\varphi_1 | L_\infty\| = 1$. For the existence and further remarks, we refer to Taira [20] and [15].

COROLLARY 1. *Let all assumptions of Theorem* 1 *be satisfied for some* $p > n$. *Suppose that there exist real numbers* t_+ *and* t_-, $t_+ \geq t_-$, *such that*

$$g(x, u_0 + t_+\varphi_1, Du_0 + t_+D\varphi_1) - t_+\lambda_1\varphi_1 \leq 0 \quad in\ \Omega$$

and

$$g(x, u_0 + t_-\varphi_1, Du_0 + t_-D\varphi_1) - t_-\lambda_1\varphi_1 \geq 0 \quad in\ \Omega.$$

Then the boundary value problem (P) *has a solution* $u \in W_p^2(\Omega)$ *satisfying*

$$u_0 + t_-\varphi_1 \leq u \leq u_0 + t_+\varphi_1 \quad in\ \Omega.$$

PROOF. We apply Theorem 1 with the supersolution $u_+ = u_0 + t_+\varphi_1$ and the subsolution $u_- = t_0 + t_-\varphi_1$. ■

REMARK 1. Let us consider the semilinear elliptic boundary value problem

(9) $$\Delta u = g(x, u, Du) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In Amann and Crandall [3] and Kazdan and Kramer [6], it was shown that Proposition in Section 3 holds if (H4) and the Bernstein condition, i.e., (H5) with $\mu = 2$ and $p = \infty$, are satisfied. It means that we have

(10) $$|g(x, \xi, \eta)| \leq b(x, \xi)(1 + |\eta|^2)$$

and

$$\sup_{|\xi| \leq c} b(\cdot, \xi) \in L_\infty(\Omega).$$

An example of Nagumo [10] shows that the growth condition of g with respect to η is needed for the solvability of (9).

Let $\varepsilon > 0$ and $h > 0$. Consider the semilinear boundary value problem

(11) $$u'' = u(1 + (u')^2)^{1+\varepsilon} \text{ in } (0, 1), \quad u(0) = 0, \ u(1) = h.$$

In Schmitt [17, p. 269], it was shown that (11) has no solution if

$$h \geq \frac{(b-a)^{\varepsilon/(1+\varepsilon)}}{\gamma(\varepsilon)},$$

where $\gamma(\varepsilon)$ is a constant which depends only on ε.

REMARK 2. Pohozaev considered the solvability of (9) under the the more general condition (H4) and (H5). The following example in [11] shows that the index $\mu = 2 - n/p$ is not improvable in the sense that (H4) and (H5) does not, in general, imply Proposition in Section 3 if $\mu > 2 - n/p$. Note that $\mu = 2 - n/p$ coincides with the Bernstein condition in the case $p = \infty$.

Let $0 < v < 1$, $0 < \varepsilon < 1$, $\mu > 2 - 1/p$, $1 < p < \infty$ and $b(x) = v^{1-\mu}(v-1)(x+\varepsilon)^{v-2-\mu(v-1)}$. Consider the boundary problem

$$(12) \qquad u'' = b(x)|u'|^{\mu} \text{ in } (0,1), \quad u(0) = 0, \quad u(1) = (1+\varepsilon)^{v} - \varepsilon^{v}.$$

For fixed parameters the solution of (12) is given by

$$u(x) = (x+\varepsilon)^{v} - \varepsilon^{v}.$$

Then for any given $1 < p < \infty$ and $\mu > 2 - 1/p$ there exist constants c_1, c_2 and c_3 such that

$$(13) \qquad \|u|L_\infty\| \le c_1, \quad \|b|L_p\| \le c_2, \quad \|\varphi|B^{2-1/p}_{p,p}\| \le c_3$$

uniformly with respect to $\varepsilon \in (0,1)$.

On the other hand, it was shown that

$$(14) \qquad \|u'|L_\infty\| \to \infty \quad \text{if } \varepsilon \downarrow 0.$$

Therefore (14) implies

$$\|u|W^2_p\| \to \infty \quad \text{if } \varepsilon \downarrow 0.$$

Now we prove Theorem 3, i.e., we investigate the solvability of the following semilinear elliptic boundary value problem, where the boundary operator is nonlinear.

$$(15) \qquad Au = g(x,u,Du) \text{ in } \Omega, \quad B_\gamma u = \frac{\partial u}{\partial v} + \gamma(x',u) = \varphi \text{ on } \partial\Omega.$$

PROOF OF THEOREM 3. We give only an outline of the proof, where we indicate the differences to the proof of Theorem 1 and Theorem 2, respectively. At first we show the corresponding estimate with respect to our boundary condition B_γ in (15), see Proposition in Section 3, where we had in (1) of Section 3

$$\|Bu|B^{*,1-1/p}_{p,p}(\partial\Omega)\| = \|\varphi|B^{*,1-1/p}_{p,p}(\partial\Omega)\|.$$

In our case, B_γ can be written as a nonlinear perturbation of the Neumann boundary condition. Recall that we have $u \in C^1(\overline{\Omega})$ with $\|u|L_\infty\| \le M$. Then it holds

$$(16)\quad \|u|B^{2-1/p}_{p,p}(\partial\Omega)\|$$
$$\le c'\left(\left\|\frac{\partial u}{\partial \nu}\Big|B^{1-1/p}_{p,p}(\partial\Omega)\right\| + \|u|B^{1-1/p}_{p,p}(\partial\Omega)\|\right)$$
$$\le c'(\|u|B^{1-1/p}_{p,p}(\partial\Omega)\|) + \|\gamma(\cdot,u)|B^{1-1/p}_{p,p}(\partial\Omega)\| + \|\varphi|B^{1-1/p}_{p,p}(\partial\Omega)\|)$$
$$\le c''(\|u|B^{1-1/p}_{p,p}(\partial\Omega)\| + M + \|u|B^{1-\varepsilon}_{p,p}(\partial\Omega)\| + \|\varphi|B^{1-1/p}_{p,p}(\partial\Omega)\|)$$
$$\le c_1\|\varphi|B^{1-1/p}_{p,p}(\partial\Omega)\| + c_2\|u|B^{1-\varepsilon}_{p,p}(\partial\Omega)\|,$$

where c_1 and c_2 are independent of u and φ. Here $0 < \varepsilon < 1/p$ is small enough such that $1 - \varepsilon > (n-1)/p$. We applied the results from Section 2, especially Proposition 2, i.e.,

$$\|\gamma(\cdot,u)|B^{1-1/p}_{p,p}(\partial\Omega)\| \le c_\gamma(M + \|u|B^{1-\varepsilon}_{p,p}(\partial\Omega)\|).$$

If the dimension n satisfies $n \ge 2$, then we have the continuous embedding $L_\infty(\partial\Omega) \hookrightarrow B^0_{p,p}(\partial\Omega)$. Applying this embedding and the known interpolation property

$$\|v|B^{\theta s_0+(1-\theta)s_1}_{p,p}\| \le C\|v|B^{s_0}_{p,p}\|^\theta \|v|B^{s_1}_{p,p}\|^{1-\theta}, \quad 0 < \theta < 1,$$

we obtain

$$(17)\qquad \|u|B^{1-\varepsilon}_{p,p}(\partial\Omega)\| \le c(1 + \|u|B^{2-1/p}_{p,p}(\partial\Omega)\|^\theta)$$

for $s_0 = 2 - 1/p$, $s_1 = 0$ and $0 < \theta = (1-\varepsilon)/(2-1/p) < 1$. Here c is independent of u. From (16) and (17) it follows

$$\|u|B^{2-1/p}_{p,p}(\partial\Omega)\| \le C_1(\|\varphi|B^{1-1/p}_{p,p}(\partial\Omega)\| + 1 + \|u|B^{2-1/p}_{p,p}(\partial\Omega)\|^\theta).$$

Note that $0 < \theta < 1$ holds. Hence there exists a positive number K such that

$$\|u|B^{2-1/p}_{p,p}(\partial\Omega)\| \le K$$

holds. This and (17) imply the existence of a positive number K_1 with

$$\|u|B^{1-\varepsilon}_{p,p}(\partial\Omega)\| \le K_1.$$

Using the estimates in (16) we get finally

$$\left\| \frac{\partial u}{\partial \nu} \middle| B^{1-1/p}_{p,p}(\partial\Omega) \right\| \le c_1' \|\varphi | B^{1-1/p}_{p,p}(\partial\Omega)\| + c_2' \|u | B^{1-\varepsilon}_{p,p}(\partial\Omega)\| \tag{18}$$

$$\le \tilde{c}(1 + \|\varphi | B^{1-1/p}_{p,p}(\partial\Omega)\|), \tag{19}$$

where $\tilde{c} = \tilde{c}(n, p, \Omega)$ is independent of u. Now we can apply Proposition 1 in Section 2. (In the special case $n = 1$, one uses that $L_\infty(\partial\Omega) \hookrightarrow B^0_{p,\infty}(\partial\Omega)$ and a corresponding interpolation property. For details and the definition of the classical Besov spaces, we refer e.g. to [16, Subsections 6.3.2 and 2.1.2].)

In our subsequent considerations in the proof of Theorem 1, we proved estimates of the difference of two solutions $w = z - v$. Here we replace, for example, the boundary condition $Bw = (t_2 - t_1)\varphi$ on $\partial\Omega$, see (9) in Section 3, by

$$B_\gamma w = \frac{\partial w}{\partial \nu} + \gamma(x', z(x')) - \gamma(x', v(x')) = \frac{\partial w}{\partial \nu} + a(x')w = (t_2 - t_1)\varphi,$$

where

$$a(x') = \frac{\partial \gamma}{\partial \xi}(x', \xi(x')) \ge 0 \quad \text{on } \partial\Omega,$$

i.e., we have a Robin boundary condition. Let M have the meaning as before. Then we define

$$\gamma_M(x') = \sup\left\{ \frac{\partial \gamma}{\partial x'}(x', \xi) + \frac{\partial \gamma}{\partial \xi}(x', \xi) : x' \in \partial\Omega, |\xi| \le M \right\}.$$

Therefore an application of Proposition 1 in Section 2 shows that we can estimate the family of Robin boundary operators by

$$C(\max\{1, \|\gamma_M | C^2\|\}) \|\varphi | B^{1-1/p}_{p,p} \partial\Omega\|, \tag{20}$$

where $C = C(n, \Omega, p)$ is independent of φ. Finally, by our assumptions on g, it follows that $v^{(0)} = 0$ is the unique solution of $(\mathrm{P}_{t^{(0)}})$. ■

We finish our consideration with an application of the a priori estimate in Section 3.

COROLLARY 2. *Let all assumptions of Proposition in Section* 3 *be satisfied. Suppose that* $\gamma(x', \xi) : \partial\Omega \times \boldsymbol{R} \to \boldsymbol{R}$ *is a smooth function. Let there exist real numbers* α *and* β, $\alpha < 0 < \beta$, *such*

$$g(x, \alpha, 0) > 0 > g(x, \beta, 0) \quad \textit{for all } x \in \overline{\Omega}, \tag{21}$$

$$b(x')\alpha \le \gamma(x', \alpha), \quad b(x')\beta \ge \gamma(x', \beta) \quad \textit{on } \partial\Omega. \tag{22}$$

Then

$$Au = g(x, u, Du) \text{ in } \Omega, \quad Bu = \gamma(x', u) \text{ on } \partial\Omega \tag{23}$$

has a solution $u \in W_p^2(\Omega)$ *whose range is contained in* $[\alpha, \beta]$.

PROOF. Proposition 1 in Section 2 shows that for $v \in C^1(\overline{\Omega})$ the boundary value problem

$$Au = g(x, v, Dv) \text{ in } \Omega, \quad Bu = \gamma(x', v) \text{ on } \partial\Omega \tag{24}$$

has a unique solution $u \in W_p^2(\Omega)$, $p > n$, denoted by $u = Sv$. We remark that the smoothness of g implies that u belongs to $C^2(\overline{\Omega})$. The operator

$$S : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$$

is completely continuous. Now we apply Leray–Schauder degree arguments. Let M be given by

$$M = \max\{|\alpha|, |\beta|\}.$$

Then M_1 has the same meaning as in (5). Now we define the set

$$K = \{u \in C^1(\overline{\Omega}) : \alpha < u < \beta, |Du| < M_1 + 1 \text{ in } \overline{\Omega}\}. \tag{25}$$

Let $0 < t < 1$ such that there exists $u \in \partial K$ with $u = tS(u)$. Hence $u \in C^1(\overline{\Omega})$ satisfies

$$Au = tg(x, u, Du) \text{ in } \Omega, \quad Bu = t\gamma(x, u) \text{ on } \partial\Omega.$$

By our a priori estimate we derive $|Du| \leq M_1$. Hence there exists a point $x_0 \in \overline{\Omega}$ such that either $u(x_0) = \alpha$ or $u(x_0) = \beta$ holds. Assume the first case, i.e., we have $u(x_0) = \alpha$. (The other case can be considered similarly.)

If $x_0 \in \Omega$, then it holds $Du(x_0) = 0$. Therefore, by the properties of A we obtain in analogy to [16, p. 499] that

$$Au(x_0) = A\alpha \leq c(x_0)\alpha \leq 0.$$

This implies a contradiction to (21).

If $x_0 \in \partial\Omega$, then the function $v(x) = \alpha - u(x)$ attains its maximum at x_0. Therefore,

$$\frac{\partial u(x_0)}{\partial \nu} = -\frac{\partial v(x_0)}{\partial \nu} \leq 0.$$

Assume that $a(x_0) = 0$. This implies that $b(x_0) > 0$ and $b(x_0)\alpha = t\gamma(x_0, \alpha)$. Since $\alpha < 0$, we get a contradiction to (22). Thus $a(x_0) > 0$ and

$$a(x_0)\frac{\partial u}{\partial \nu}(x_0) = t\gamma(x_0, \alpha) - b(x_0)\alpha.$$

Hence we get $b(x_0) \geq t\gamma(x_0, \alpha)$.

CASE 1. We assume that $b(x_0) > 0$. Then we deduce that

$$b(x_0)t\alpha > b(x_0)\alpha \geq t\gamma(x_0, \alpha),$$

i.e., we get with $b(x_0)\alpha > \gamma(x_0, \alpha)$ a contradiction to (22). Therefore, we have shown that for all $t \in (0, 1)$ the equation $u = tS(u)$ has no solution $u \in \partial K$. Now we can apply usual Leray–Schauder degree arguments to prove the existence of a solution $u \in K$ under the assumption that $b(x_0) \neq 0$.

CASE 2. Assume that $b(x_0) = 0$. Let $0 < \varepsilon < 1$. Now we replace $b(x')$ by $b(x') + \varepsilon$. Because of the fact that $b \geq 0$ on $\partial\Omega$ we get that the new boundary operator B satisfies all assumptions of the first case. This implies that we find solutions $u_\varepsilon \in K$. Now limiting and compactness arguments show that there exists a subsequence of $\{u_\varepsilon\} \in C^1(\overline{\Omega})$ (denoted also by $\{u_\varepsilon\}$) such that

$$u_\varepsilon \to u \quad \text{for } \varepsilon \to 0$$

in $C^1(\overline{\Omega})$. Further, u is a solution of the origin problem and satisfies all desired properties. ∎

REMARK 3. Our corollary is a generalization of Pohozaev [11, Theorem 5.1] to nonlinear boundary conditions. Furthermore, we extend the results due to Schmitt [17, Theorem 6.1, Theorem 6.2], where boundary value problems of the type

$$Au = g(x, u) \text{ in } \Omega, \quad Bu = \gamma(x', u) \text{ on } \partial\Omega$$

and

$$u'' = g(x, u, u') \text{ in } (a, b), \quad \gamma_1(u(a), u'(a)) = 0 = \gamma_2(u(b), u'(b)),$$

respectively, were considered. Hereby the nonlinear function g satisfies the Nagumo condition (10).

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References

[1] Aleksandrov, A. D., Uniqueness conditions and estimates for the solution of the Dirichlet problem, Vestnik Leningrad. Univ. **13** (1963), 5–29; English transl., Amer. Math. Soc. Transl. (1968) **68**.

[2] Amann, H., Fixed point equations and nonlinear eigenvalue problems, SIAM Rev. **18** (1976), 620–709.

[3] Amann, H. and Crandall, M. G., On some existence theorems for semi-linear elliptic equations, Indiana Univ. Math. J. **27** (1978), 779–790.

[4] Bernstein, S., Sur certaines equations differentielles ordinaires du second ordre, C. R. Acad. Sci. Paris **138** (1904), 950–951.

[5] Inkmann, F., Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions, Indiana Univ. Math. J. **31** (1982), 213–221.

[6] Kazdan, J. L. and Kramer, R. J., Invariant criteria for existence of solutions to second-order quasi-linear equations, Comm. Pure Appl. Math. **31** (1978), 619–645.

[7] Keller, H. B., Elliptic boundary value problems suggested by nonlinear diffusion processes, Arch. Rational Mech. Anal. **5** (1969), 363–381.

[8] Krasnoselskij, M. A., Topological methods in the theory of nonlinear integral equations (Russian), Moscow, 1956; English transl., MacMillan, New York, 1964.

[9] Marschall, J., Pseudo-differential operators with nonregular symbols of the class $S^m_{\rho,\delta}$, Comm. Part. Diff. Equations **12** (1987), 921–965.

[10] Nagumo, M., On principally linear elliptic differential operators of second order. Osaka Math. J. **6** (1954), 207–229.

[11] Pohozaev, S. I., On equations of the form $\Delta u = f(x, u, Du)$, Math. USSR Sbornik **41** (1982), 269–280.

[12] Protter, M. H. and Weinberger, H. F., Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, N.J., 1967.

[13] Runst, T., Mapping properties of non-linear operators in spaces of Triebel–Lizorkin and Besov type, Analysis Math. **12** (1986), 313–346.

[14] Runst, T., Degenerate boundary value problems in Sobolev spaces of fractional order, Forschungserg. FSU Jena, Math/Inf/98/03.

[15] Runst, T., On the existence of solutions of nonlinear boundary value problems at resonance in Sobolev spaces of fractional order, Hiroshima Math. J. **29** (1999), 315–325.

[16] Runst, T. and Sickel, W., Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, (de Gruyter Series in Nonlinear Analysis and Applications, 3) de Gruyter, Berlin New York, 1996.

[17] Schmitt, K., Boundary value problems for quasilinear second order elliptic equations, Nonlinear Analysis **2** (1978), 263–309.

[18] Serrin, J., The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. Royal Soc. London **264** (1969), 413–469.
[19] Taira, K., Analytic semigroups and semilinear initial boundary value problems, (London Mathematical Society Lecture Notes Series, 223) Cambridge University Press, London New York, 1995.
[20] Taira, K., Bifurcation for nonlinear elliptic boundary value problems I, Collectanea Mathematica **47** (1996), 207–229.
[21] Taira, K., Existence and uniqueness theorems for semilinear elliptic boundary value problems I, Advances in Diff. Equations **2** (1997), 509–534.
[22] Triebel, H., Theory of function spaces, Birkhäuser, Basel Boston Stuttgart, 1983; and Akad. Verlagsges. Geest & Portig, Leipzig, 1983.
[23] Wloka, J., Partielle Differentialgleichungen, Teubner, Stuttgart, 1982.
[24] Zeidler, E., Nonlinear functional analysis and its application. II/B: Nonlinear monotone operators, Springer, New York, 1990.

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