TSUKUBA J. MATH.
Vol. 23 No. 3 (1999), 495–504

ON HYPERSPACES AND HOMEOMORPHISM GROUPS HOMEOMORPHIC TO PRODUCTS OF ABSORBING SETS AND R^{∞}

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Abstract. Two theorems are proven:

1) For a topological space X the pair $(\exp(X), \exp_{\omega}(X))$ of hyperspaces of compact and finite subsets of X is homeomorphic to $(Q \times \boldsymbol{R}^{\infty}, \sigma \times \boldsymbol{R}^{\infty})$ if and only if X is a direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of strongly countable-dimensional Peano continua such that each X_n is nowhere dense in X_{n+1};
2) The triple $(\mathscr{H}^c(\boldsymbol{R}), \mathscr{H}^c_{LIP}(\boldsymbol{R}), \mathscr{H}^c_{PL}(\boldsymbol{R}))$ of homeomorphism groups of the line, endowed with the Whitney topology, is homeomorphic to $(s \times \boldsymbol{R}^{\infty}, \Sigma \times \boldsymbol{R}^{\infty}, \sigma \times \boldsymbol{R}^{\infty})$.

In last decades two main kinds of infinite-dimensional manifolds were investigated: manifolds modeled on complete or incomplete "nice" subsets of the Hilbert cube Q (like s, Σ, σ), and nonmetrizable manifolds modeled on the k_{ω}-spaces $\boldsymbol{R}^{\infty} = \varinjlim \boldsymbol{R}^n$ or $Q^{\infty} = \varinjlim Q^n$ [Sa_1]. Many works are devoted to detecting such manifolds in "nature" (see the survey [Ca]).

In this paper we would like to turn the reader attention to spaces homeomorphic to products $\Omega \times \boldsymbol{R}^{\infty}$, where Ω is a "nice" subset in the Hilbert cube. It turns out that many natural constructions of topology and analysis supply us with examples of such spaces. To illustrate this statement we consider two construction: hyperspaces of k_{ω}-spaces and homeomorphism groups of the line.

For a topological space X by $\exp(X)$ the hyperspace of all nonempty compact subsets of X, endowed with the Vietoris topology is denoted; $\exp_{\omega}(X)$ is the subspace of $\exp(X)$ consisting of all nonempty finite subsets of X. In [CP] D. W. Curtis and D. S. Patching proved that $\exp(X)$ is homeomorphic to

1991 *Mathematics Subject Classification.* 57N20, 54B20, 54C35, 54H11.
Received July 21, 1998
Revised June 11, 1999

Q^∞, provided X is a direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of Peano continua such that each X_n is nowhere dense in X_{n+1}. Recall that a *Peano continuum* is a connected locally connected metrizable compact space.

THEOREM 1. *For a topological space X the pair* $(\exp(X), \exp_\omega(X))$ *is homeomorphic to* $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$ *if and only if X is a direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of strongly countable-dimensional Peano continua such that each X_n is nowhere dense in X_{n+1}.*

Here $s = (-1,1)^\omega$ is the pseudo-interior of the Hilbert cube $Q = [-1,1]^\omega$, $\Sigma = \{(q_i) \in Q : \sup_i |q_i| < 1\}$ is its radial interior, and $\sigma = \{(q_i) \in \Sigma : q_i = 0$ for almost all $i\}$.

Next, we consider the homeomorphism group $\mathscr{H}(\boldsymbol{R})$ of the real line endowed with the Whitney topology whose neighborhood base at the identity consists of sets $U(\varepsilon) = \{f \in \mathscr{H}(\boldsymbol{R}) : |f - \mathrm{id}| < \varepsilon\}$, where $\varepsilon : \boldsymbol{R} \to \boldsymbol{R}$ runs over all continuous positive functions. Unlike to the compact-open topology (which was considered in [Ya]), the Whitney topology on $\mathscr{H}(\boldsymbol{R})$ is not locally connected. Nonetheless we can consider the connected component $\mathscr{H}^c(\boldsymbol{R})$ of the identity. It turns out that $\mathscr{H}^c(\boldsymbol{R})$ coincides with the set of all homeomorphisms with compact support (that is homeomorphisms which are identity outside of some compact subset of $\boldsymbol{R}$). Let $\mathscr{H}^c_{LIP}(\boldsymbol{R})$ and $H^c_{PL}(\boldsymbol{R})$ be subgroups in $\mathscr{H}^c(\boldsymbol{R})$ consisting of Lipschitz and piece-linear homeomorphisms, respectively.

THEOREM 2. *The triple* $(\mathscr{H}^c(\boldsymbol{R}), \mathscr{H}^c_{LIP}(\boldsymbol{R}), \mathscr{H}^c_{PL}(\boldsymbol{R}))$ *is homeomorphic to* $(s \times \boldsymbol{R}^\infty, \Sigma \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$.

We begin with

Topological characterization of the pair $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$

First, we recall the definition of the direct limit topology. It is defined on a union $X = \bigcup_{n=1}^\infty X_n$ of an incresing sequence $X_1 \subset X_2 \subset \cdots$ of topological spaces and is denoted by $\varinjlim X_n$. A subset $U \subset X$ is open in $\varinjlim X_n$ if the intersection $U \cap X_n$ is open in X_n for every n.

Recall that $\boldsymbol{R}^\infty$ denotes the direct limit of the tower $\boldsymbol{R}^1 \subset \boldsymbol{R}^2 \subset \cdots$, where each $\boldsymbol{R}^n$ is identified with the subset $\boldsymbol{R}^n \times \{0\}$ of $\boldsymbol{R}^{n+1}$. A topological characterization of the space $\boldsymbol{R}^\infty$ was given in [Sa_1]. This characterization implies that $\boldsymbol{R}^\infty$ is homeomorphic to the direct limit $\varinjlim I^n$ of the sequence $I^1 \subset I^2 \subset \cdots$, where $I = [0,1]$ and each cube I^n is identified with the subset $I^n \times \{0\}$ in I^{n+1}.

All maps considered in this paper are continuous. Writing (X, Y) we always understand that Y is a subspace of X. Saying that $f : (K, C) \to (X, Y)$ is an embedding of a pair (K, C) into a pair (X, Y) we understand that $f : K \to X$ is an embedding with $f^{-1}(Y) = C$. Two pairs (X, Y) and (X', Y') are homeomorphic if $h(Y) = Y'$ for some homeomorphism $h : X \to X'$. The symbol "$\cong$" means "is homeomorphic to".

A closed subset A of a topological space X is called a Z-set in X if every map $f : Q \to X$ can be uniformly approximated by maps whose image misses the set A. An embedding $f : A \to X$ is a Z-embedding, provided $f(A)$ is a Z-set in X.

Below $\mathcal{M}_0, \mathcal{M}_1, \mathcal{A}_1$, and $\mathcal{A}_1(s.c.d.)$ denote respectively the classes of metrizable compacta, Polish spaces, metrizable σ-compact spaces, and metrizable σ-compact strongly countable-dimensional spaces. Recall that a space X is strongly countable-dimensional if X can be written as a countable union $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is a closed finite-dimensional subset of X.

By $(\mathcal{M}_0, \mathcal{A}_1(s.c.d.))$ we denote the class of pairs (K, C) such that $\mathcal{A}_1(s.c.d) \ni C \subset K \in \mathcal{M}_0$. Evidently, $(Q, \sigma) \in (\mathcal{M}_0, \mathcal{A}_1(s.c.d.))$. Moreover, this pair has the following

Universal property of (Q, σ). *For every pair $(K, C) \in (\mathcal{M}_0, \mathcal{A}_1(s.c.d.))$ and every closed subset $B \subset K$ every Z-embedding $f : (B, B \cap C) \to (Q, \sigma)$ extends to a Z-embedding $\bar{f} : (K, C) \to (Q, \sigma)$.*

This property of (Q, σ) is well known and can be derived from the strong $(\mathcal{M}_0, \mathcal{A}_1(s.c.d.))$-universality of (Q, σ) (see [BRZ]).

Theorem 3 (characterizing the pair $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$). *For a pair (X, Y) of topological spaces the following conditions are equivalent*:

(1) (X, Y) *is homeomorphic to* $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$;

(2) (X, Y) *satisfies the conditions*:

a) *the space X is a direct limit of metrizable compacta and Y is σ-compact and strongly countable-dimensional*;

b) *for every pair $(K, C) \in (\mathcal{M}_0, \mathcal{A}_1(s.c.d.))$ and a closed subset $B \subset K$ every embedding $f : (B, B \cap C) \to (X, Y)$ extends to an embedding $\bar{f} : (K, C) \to (X, Y)$*;

(3) *X can be written as a direct limit $\varinjlim X_n$ of a sequence $X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup_{n=1}^{\infty} X_n$ and for every $n \in \boldsymbol{N}$ the pair $(X_n, X_n \cap Y)$ is homeomorphic to (Q, σ) and X_n is a Z-set in X_{n+1}.*

PROOF. We will verify the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3) As we mentioned, $\boldsymbol{R}^\infty$ is homeomorphic to the direct limit $\varinjlim I^n$. Then $Q \times \boldsymbol{R}^\infty$ can be identified with the direct limit of the sequence

$$Q \times I \subset Q \times I^2 \subset Q \times I^3 \subset \cdots,$$

where each $Q \times I^n$ is a Z-set in $Q \times I^{n+1}$ and the pair $(Q \times I^n, Q \times I^n \cap \sigma \times \boldsymbol{R}^\infty) = (Q \times I^n, \sigma \times I^n)$ is homeomorphic to (Q, σ). Hence the pair $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$ as well as each its topological copy satisfies the third condition of Theorem 3.

(3) $\Rightarrow$ (2) Suppose $X = \varinjlim X_n$ where for every $n \in \boldsymbol{N}$ the pair $(X_n, X_n \cap Y)$ is homeomorphic to (Q, σ) and X_n is a Z-set in X_{n+1}. We show that the pair (X, Y) satisfies the condition (2). Evidently, X is a direct limit of Hilbert cubes and $Y = \bigcup_{n=1}^\infty (Y \cap X_n)$ is a σ-compact strongly countable-dimensional space. Fix a pair $(K, C) \in (\mathscr{M}_0, \mathscr{A}_1(s.c.d.))$, a closed subset $B \subset K$, and an embedding $f : (B, B \cap C) \to (X, Y)$. Because of the compactness of B, $f(B) \subset X_n$ for some n. Since X_n is a Z-set in X_{n+1}, the map $f : B \to X_n \subset X_{n+1}$ is a Z-embedding. The universal property of $(Q, \sigma) \cong (X_{n+1}, X_{n+1} \cap Y)$ allows us to extend f to a Z-embedding $\bar{f} : (K, C) \to (X_{n+1}, X_{n+1} \cap Y) \subset (X, Y)$. Hence the pair (X, Y) satisfies the condition (2).

(2) $\Rightarrow$ (1) By the standard "back-and-forth" argument (see [Sa$_1$]), it can be shown that a pair (X, Y) satisfying the condition (2) is unique up to homeomorphism. Since the pair $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$ satisfies the condition (2) (the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2)), each pair satisfying the second condition of Theorem 3 is homeomorphic to $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$. □

Proof of Theorem 1

Suppose $X = \varinjlim X_n$ is a direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of strongly countable-dimensional Peano continua such that each X_n is nowhere dense in X_{n+1}. Without loss of generality, each X_n contains more that one point. To show that the pair $(\exp(X), \exp_\omega(X))$ is homeomorphic to $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$ we will verify the third condition of Theorem 3. According to [CP, 2.4], the hyperspace $\exp(X)$ has the direct limit topology with respect to the tower $\exp(X_1) \subset \exp(X_2) \subset \cdots$. Next, by [CN, 5.1], for every $n \geq 1$ the pair $(\exp(X_n), \exp(X_n) \cap \exp_\omega(X)) = (\exp(X_n), \exp_\omega(X_n))$ is homeomorphic to (Q, σ). It is remarked in the proof of [CP, 3.1] that the nowhere density of X_n in X_{n+1} implies that $\exp(X_n)$ is a Z-set in $\exp(X_{n+1})$. Thus, the pair $(\exp(X), \exp_\omega(X))$ satisfies the third equivalent condition of Theorem 3 what allows us to conclude that this pair is homeomorphic to $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$. Hence, the "if" part of Theorem 1 is proven.

To prove the "only if" part, assume that X is such that the pair $(\exp(X), \exp_\omega(X))$ is homeomorphic to $(Q \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$.

Claim. *Every compact subset $C \subset X$ is contained in a Peano continuum $P \subset X$ such that C is nowhere dense in P.*

Let $f : \exp(X) \to Q \times \boldsymbol{R}^\infty$ be a homeomorphism such that $f(\exp_\omega(X)) = \sigma \times \boldsymbol{R}^\infty$. To prove the claim, we shall construct a Peano continuum $M \subset \sigma \times \boldsymbol{R}^\infty$ such that $M \cap f(\exp(C)) = f(C)$ and $f(C)$ is nowhere dense in M. Write $\boldsymbol{R}^\infty = \varinjlim I^n$. Since $\exp(C)$ is compact, $f(\exp(C)) \subset Q \times I^{n-1}$ for some n. Then $f(\exp(C)) \subset Q \times I^{n-1}$ is a Z-set in $Q \times I^n$. It follows from [BRZ, § 1.2. Ex. 4, 12, 13] that the set $(\sigma \times I^n) \backslash f(\exp(C))$ is homotopy dense in $Q \times I^n$, i.e., there exists a homotopy $h : Q \times I^n \times [0,1] \to Q \times I^n$ such that $h(Q \times I^n \times (0,1]) \subset (\sigma \times I^n) \backslash f(\exp(C))$ and $h(x,0) = x$ for every $x \in Q \times I^n$. Let $d \leq 1$ be any admissible metric on $Q \times I^n$ and consider the map $g : Q \times I^n \to Q \times I^n$ defined for $x \in Q \times I^n$ by $g(x) = h(x, \mathrm{dist}(x, f(C))$. Set $M = g(Q \times I^n)$ and remark that $M \subset f(C) \cup \sigma \times I^n \subset \sigma \times \boldsymbol{R}^\infty$ and $M \cap f(\exp(C)) = f(C)$. Next, M is a Peano continuum and $f(C)$ is nowhere dense in M, i.e., M is a required set.

Since f is a homeomorphism, we get $f^{-1}(M)$ is a Peano continuum in $\exp_\omega(X)$ such that $f^{-1}(M) \cap \exp(C) = C$ and C is nowhere dense in $f^{-1}(M)$. Let $P = \cup f^{-1}(M) = \cup\{A : A \in f^{-1}(M)\}$. By [CN, 2.2], the set $P \subset X$ is a Peano continuum. Evidently, $C \subset P$. Furthermore, the set C is nowhere dense in P. Assuming the converse, we would find a nonempty open set $U \subset P$ such that $U \subset C$. Then $\exp(U)$ is an open set in $\exp(P) \supset f^{-1}(M)$ such that $\exp(U) \subset \exp(C)$. Consequently, $\exp(U) \cap f^{-1}(M) = \exp(U) \cap f^{-1}(M) \cap \exp(C) = \exp(U) \cap f^{-1}(M) \cap C = \exp(U) \cap C$ is an open set in $f^{-1}(M)$ contained in C, i.e., C is somewhere dense in $f^{-1}(M)$, a contradiction with the choice of the set M. Thus the claim is proven.

Since X is a closed subspace in $\exp(X)$ and $\exp(X)$ is homeomorphic to $Q \times \boldsymbol{R}^\infty$, we get X is a direct limit of a tower $C_1 \subset C_2 \subset \cdots$ of metrizable compacta. Using Claim, by induction, construct a tower $X_1 \subset X_2 \subset \cdots$ of Peano continua in X such that $C_n \subset X_n$ and each X_n is nowhere dense in X_{n+1}. Obviously, X has the direct limit topology $\varinjlim X_n$. Next, the space X is strongly countable-dimensional, as a subspace of $\exp_\omega(X)$, a topological copy of $\sigma \times \boldsymbol{R}^\infty$. This yields that each Peano continuum X_n is strongly countable-dimensional. □

Characterizing the quadruple $(Q, s, \Sigma, \sigma) \times \boldsymbol{R}^\infty$

To simplify denotations, we denote by $(Q, s, \Sigma, \sigma) \times \boldsymbol{R}^\infty$ the quadruple $(Q \times \boldsymbol{R}^\infty, s \times \boldsymbol{R}^\infty, \Sigma \times \boldsymbol{R}^\infty, \sigma \times \boldsymbol{R}^\infty)$. More generally, for an n-tuple $(X_1, \dots, X_n)$, a space Y, a subset $B \subset X_1$ and a map $f : Z \to X_1$ let $(X_1, \dots, X_n) \times Y = (X_1 \times Y, \dots, X_n \times Y)$, $B \cap (X_1, \dots, X_n) = (B \cap X_1, \dots, B \cap X_n)$, and $f^{-1}(X_1, \dots, X_n) = (f^{-1}(X_1), \dots, f^{-1}(X_n))$. Saying that $f : (X_1, \dots, X_n) \to (Y_1, \dots, Y_n)$ is an embedding of n-tuples we understand that $f : X_1 \to Y_1$ is an embedding with $f^{-1}(Y_i) = X_i$ for all $1 \le i \le n$.

By $(\mathscr{M}_0, \mathscr{M}_1, \mathscr{A}_1, \mathscr{A}_1(s.c.d.))$ we denote the class of quadruples (X, Y, Z, T) such that $T \subset Z \subset Y \subset X$, $X \in \mathscr{M}_0$, $Y \in \mathscr{M}_1$, $Z \in \mathscr{A}_1$, and $T \in \mathscr{A}_1(s.c.d.)$. Evidently, $(Q, s, \Sigma, \sigma) \in (\mathscr{M}_0, \mathscr{M}_1, \mathscr{A}_1, \mathscr{A}_1(s.c.d.))$. Like the pair (Q, σ) the quadruple (Q, s, Σ, σ) has the following

UNIVERSAL PROPERTY OF (Q, s, Σ, σ) (see [CDM, §2]). *For every quadruple $(K, M, A, C) \in (\mathscr{M}_0, \mathscr{M}_1, \mathscr{A}_1, \mathscr{A}_1(s.c.d.))$ and every closed subset $B \subset K$ every Z-embedding $f : B \cap (K, M, A, C) \to (Q, s, \Sigma, \sigma)$ extends to a Z-embedding $\bar{f} : (K, M, A, C) \to (Q, s, \Sigma, \sigma)$.*

Using this property and repeating the argument of Theorem 3, we may prove

THEOREM 4 (characterizing the quadruple $(Q, s, \Sigma, \sigma) \times \boldsymbol{R}^\infty$). *For a quadruple (X, Y, Z, T) of topological spaces the following conditions are equivalent*:

(1) *(X, Y, Z, T) is homeomorphic to $(Q, s, \Sigma, \sigma) \times \boldsymbol{R}^\infty$*;

(2) *(X, Y, Z, T) satisfies the conditions*:

   a) *the space X can be written as a direct limit $\varinjlim X_n$ of a tower $X_1 \subset X_2 \subset \cdots$ such that $X_n \cap (X, Y, Z, T) \in (\mathscr{M}_0, \mathscr{M}_1, \mathscr{A}_1, \mathscr{A}_1(s.c.d.))$ for every n*;

   b) *for every quadruple $(K, M, A, C) \in (\mathscr{M}_0, \mathscr{M}_1, \mathscr{A}_1, \mathscr{A}_1(s.c.d.))$ and a closed subset $B \subset K$ every embedding $f : B \cap (K, M, A, C) \to (X, Y, Z, T)$ extends to an embedding $\bar{f} : (K, M, A, C) \to (X, Y, Z, T)$*;

(3) *X can be written as a direct limit $\varinjlim X_n$ of a sequence $X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup_{n=1}^\infty X_n$ and for every $n \in \boldsymbol{N}$ the quadruple $X_n \cap (X, Y, Z, T)$ is homeomorphic to (Q, s, Σ, σ) and X_n is a Z-set in X_{n+1}.*

To apply Theorem 4 to the proof of Theorem 2 we need a notion of a small box product.

The small box-product $\boxdot_{i \in \mathscr{I}}(X_i, *_i)$ of a family of pointed spaces $(X_i, *_i)$, $i \in \mathscr{I}$, is the subspace

$$\boxdot_{i\in\mathcal{J}}(X_i,*_i) = \{(x_i)_{i\in\mathcal{J}} \in \Box_{i\in\mathcal{J}}X_i : x_i = *_i \text{ for all but finitely many indices } i\}$$

of the box product $\Box_{i\in\mathcal{J}}X_i$. Recall that a base of the box topology on $\Box_{i\in\mathcal{J}}X_i$ consists of products $\prod_{i\in\mathcal{J}} U_i$, where U_i are open sets in X_i.

If the fixed points $*_i$ in X_i are clear from the context (for example, if X_i are groups) we write $\boxdot_{i\in\mathcal{J}}X_i$ is place of $\boxdot_{i\in\mathcal{J}}(X_i,*_i)$. If all X_i are equal to X we use the symbol $\boxdot_{i\in\mathcal{J}}X$ for $\boxdot_{i\in\mathcal{J}}X_i$.

Let Y be a subspace of a space X and $*\in Y\subset X$ be a fixed point. Observe that $\boxdot_{i\in\mathcal{J}}Y$ can be identified with a subspace $\boxdot_{i\in\mathcal{J}}X$, so it is legal to say about the pair $(\boxdot_{i\in\mathcal{J}}X, \boxdot_{i\in\mathcal{J}}Y)$ which will be denoted by $\boxdot_{i\in\mathcal{J}}(X,Y)$. Analogously for a quadruple (X,Y,Z,T) and a fixed point $*\in T\subset Z\subset Y\subset X$ we introduce a quadruple $\boxdot_{i\in\mathcal{J}}(X,Y,Z,T)$.

Proposition. *For any fixed point $*\in\sigma$ the quadruple $\boxdot_{i\in\mathbf{N}}(Q,s,\Sigma,\sigma)$ is homeomorphic to $(Q,s,\Sigma,\sigma)\times\mathbf{R}^\infty$.*

Proof. The principal observation is that the box topology on $\boxdot_{i\in\mathbf{N}}Q$ coincides with the direct limit topology $\varinjlim Q^n$ with respect to the tower $Q^1 \subset Q^2 \subset \cdots$, where each Q^n is identified with the subset $\{(q_i)_{i\in\mathbf{N}} \in \boxdot_{i\in\mathbf{N}}Q : q_i = * \text{ for } i>n\}$ of $\boxdot_{i\in\mathbf{N}}Q$. This fact can be easily proven using the compactness of Q. Evidently, each Q^n is a Z-set in Q^{n+1}. Next, the quadruple $Q^n\cap \boxdot_{i\in\mathbf{N}}(Q,s,\Sigma,\sigma) = (Q^n,s^n,\Sigma^n,\sigma^n)$ is homeomorphic to (Q,s,Σ,σ) (by any coordinate permutating homeomorphism between Q and Q^n). It this setting it is legal to apply Theorem 4 to conclude that the quadruple $\boxdot_{i\in\mathbf{N}}(Q,s,\Sigma,\sigma)$ is homeomorphic to $(Q,s,\Sigma,\sigma)\times\mathbf{R}^\infty$. $\square$

Proof of Theorem 2

First we verify that the connected component of the identity in the group $\mathcal{H}(\mathbf{R})$ coincides with the set $\mathcal{H}^c(\mathbf{R}) = \{f\in\mathcal{H}(\mathbf{R}) : f|_{\mathbf{R}\setminus[-M,M]} \equiv \mathrm{id} \text{ for some } M\geq 0\}$. For this, remark that $\mathcal{H}^c(\mathbf{R})$ is path-connected: elements $f,g\in\mathcal{H}^c(\mathbf{R})$ can be linked by the path $\{(1-t)f+tg\}_{t\in[0,1]}$ in $\mathcal{H}^c(\mathbf{R})$. Now it suffices to show that each element $f\in\mathcal{H}(\mathbf{R})\setminus\mathcal{H}^c(\mathbf{R})$ can be separated from the identity by an open-and-closed neighborhood. Since $f\notin\mathcal{H}^c(\mathbf{R})$, there is a sequence $(x_n)\subset\mathbf{R}$ such that $\lim_{n\to\infty}|x_n|=\infty$ and $f(x_n)\neq x_n$ for every $n\in\mathbf{N}$. Then

$$U(f) = \left\{h\in\mathcal{H}(\mathbf{R}) : \lim_{n\to\infty}\frac{|h(x_n)-f(x_n)|}{|x_n-f(x_n)|} = 0\right\}$$

is the required closed-and-open neighborhood separating f from the identity of $\mathscr{H}(\boldsymbol{R})$.

To prove that the triple $(\mathscr{H}^c(\boldsymbol{R}), \mathscr{H}^c_{LIP}(\boldsymbol{R}), \mathscr{H}^c_{PL}(\boldsymbol{R}))$ is homeomorphic to $(s, \Sigma, \sigma) \times \boldsymbol{R}^\infty$ we represent $\mathscr{H}^c(\boldsymbol{R})$ as the product $PL^c(\boldsymbol{R}; \boldsymbol{Z}) \times \mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z})$ of the closed subspace

$$PL^c(\boldsymbol{R}; \boldsymbol{Z}) = \{f \in \mathscr{H}^c(\boldsymbol{R}) : f \text{ is linear on each interval } [i, i+1],\ i \in \boldsymbol{Z}\}$$

and the closed subgroup

$$\mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z}) = \{h \in \mathscr{H}^c(\boldsymbol{R}) : f(i) = i \text{ for each } i \in \boldsymbol{Z}\}$$

of $\mathscr{H}^c(\boldsymbol{R})$. An obvious homeomorphism F between $PL^c(\boldsymbol{R}; \boldsymbol{Z}) \times \mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z})$ can be defined by $F(g, h) = g \circ h$ for $(g, h) \in PL^c(\boldsymbol{R}; \boldsymbol{Z}) \times \mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z})$. Its inverse F^{-1} acts as $F^{-1}(f) = (g, g^{-1} \circ f)$ for $f \in \mathscr{H}^c(\boldsymbol{R})$, where $g \in PL^c(\boldsymbol{R}; \boldsymbol{Z})$ is a unique map with $g(i) = f(i)$ for all $i \in \boldsymbol{Z}$.

Let $\mathscr{H}^c_{LIP}(\boldsymbol{R}; \boldsymbol{Z}) = \mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z}) \cap \mathscr{H}^c_{LIP}(\boldsymbol{R})$ and $\mathscr{H}^c_{PL}(\boldsymbol{R}; \boldsymbol{Z}) = \mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z}) \cap \mathscr{H}^c_{PL}(\boldsymbol{R})$. Clearly, the homeomorphism F^{-1} maps the triple $(\mathscr{H}^c(\boldsymbol{R}), \mathscr{H}^c_{LIP}(\boldsymbol{R}), \mathscr{H}^c_{PL}(\boldsymbol{R}))$ onto the triple

$$PL^c(\boldsymbol{R}; \boldsymbol{Z}) \times (\mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z}), \mathscr{H}^c_{LIP}(\boldsymbol{R}; \boldsymbol{Z}), \mathscr{H}^c_{PL}(\boldsymbol{R}; \boldsymbol{Z})).$$

Thus, to prove Theorem 2, it suffices to verify that the latter triple in homeomorphic to $\boldsymbol{R}^\infty \times (s, \Sigma, \sigma)$.

Observe that the triple $(\mathscr{H}^c(\boldsymbol{R}; \boldsymbol{Z}), \mathscr{H}^c_{LIP}(\boldsymbol{R}; \boldsymbol{Z}), \mathscr{H}^c_{PL}(\boldsymbol{R}; \boldsymbol{Z}))$ is homeomorphic to the triple of small box-products $\boxdot_{i \in \boldsymbol{Z}}(\mathscr{H}(I), \mathscr{H}_{LIP}(I), \mathscr{H}_{PL}(I))$, where $\mathscr{H}(I)$ is the group of increasing homeomorphisms of the interval $I = [0, 1]$, endowed with the compact-open topology and $\mathscr{H}_{LIP}(I)$, $\mathscr{H}_{PL}(I)$ are subgroups of $\mathscr{H}(I)$ consisting of Lipschitz and piece-linear homeomorphisms, respectively. By [Sa$_2$], the triple $(\mathscr{H}(I), \mathscr{H}_{LIP}(I), \mathscr{H}_{PL}(I))$ is homeomorphic to (s, Σ, σ). Hence, by Proposition, the triple of small box products $\boxdot_{i \in \boldsymbol{Z}}(\mathscr{H}(I), \mathscr{H}_{LIP}(I), \mathscr{H}_{PL}(I))$ is homeomorphic to $\boldsymbol{R}^\infty \times (s, \Sigma, \sigma)$.

Thus to finish the proof it rests to verify that the space $PL^c(\boldsymbol{R}; \boldsymbol{Z})$ is homeomorphic to $\boldsymbol{R}^\infty$. This can be easily done using the Sakai characterization [Sa$_1$] and observing that the topology of the space $PL^c(\boldsymbol{R}; \boldsymbol{Z})$ coincides with the direct limit topology $\varinjlim PL^c_n(\boldsymbol{R}; \boldsymbol{Z})$, where for $n \in \boldsymbol{N}$

$$PL^c_n(\boldsymbol{R}; \boldsymbol{Z}) = \{f \in PL^c(\boldsymbol{R}; \boldsymbol{Z}) : f(x) = x \text{ for } |x| \geq n\}$$

is a closed subspace in $PL^c(\boldsymbol{R}; \boldsymbol{Z})$, which can be naturally identified with a locally compact $(2n-1)$-dimensional convex subset of $\boldsymbol{R}^{\boldsymbol{Z}}$. □

Some questions and comments

Remark that examples of spaces homeomorphic to $\Omega \times \boldsymbol{R}^{\infty}$ already appeared in literature. For example, in [Ma] P. Mankiewicz has proved that each separable non-metrizable strict (LF)-space is homeomorphic either to $\boldsymbol{R}^{\infty}$ or to $s \times \boldsymbol{R}^{\infty}$. In particular for any countable collection of infinite-dimensional separable Fréchet spaces $X_n, n \in \boldsymbol{N}$, the locally convex sum $\bigoplus_{n \in \boldsymbol{N}} X_n$ is homeomorphic to $s \times \boldsymbol{R}^{\infty}$. Since the topology of $\bigoplus_{n \in \boldsymbol{N}} X_n$ coincides with the box topology of $\boxdot_{n \in \boldsymbol{N}} X_n$ (this follows from [Sch, II. §6]), Mankiewicz result implies that $\boxdot_{n \in \boldsymbol{N}} s$ is homeomorphic to $s \times \boldsymbol{R}^{\infty}$ (this is a part of our Proposition). Thus the topological equivalence of $\mathscr{H}^c(\boldsymbol{R})$ and $s \times \boldsymbol{R}^{\infty}$ could be derived from Mankiewicz Theorem [Ma] (we have however chosen a purely topological proof).

Observe that in our cases, to prove that some space X is homeomorphic to a product $\Omega \times \boldsymbol{R}^{\infty}$ we first found a "nice" embedding of X into a k_{ω}-space $\bar{X}$ and then proved that the pair $(\bar{X}, X)$ was homeomorphic to $(Q \times \boldsymbol{R}^{\infty}, \Omega \times \boldsymbol{R}^{\infty})$. However there are cases, when it is not clear how to construct such a "nice" embedding $X \subset \bar{X}$. This rises the following

PROBLEM 1. *Give a topological characterization of spaces $\Omega \times \boldsymbol{R}^{\infty}$. In particular, characterize topologically the spaces $s \times \boldsymbol{R}^{\infty}$ and $\sigma \times \boldsymbol{R}^{\infty}$.*

PROBLEM 2. *Characterize topological spaces X whose hyperspace $\exp_{\omega}(X)$ of finite subsets is homeomorphic to $\sigma \times \boldsymbol{R}^{\infty}$.*

Remark that by methods developed in this paper many results on topological equivalence of hyperspaces can be proven. For example, it can be shown that $\exp(s \times \boldsymbol{R}^{\infty}) \cong s \times \boldsymbol{R}^{\infty}$, $\exp(Q \times \boldsymbol{R}^{\infty}) \cong \Sigma \times \boldsymbol{R}^{\infty}$, $\exp(\sigma \times \boldsymbol{R}^{\infty}) \cong \exp(\sigma) \times \boldsymbol{R}^{\infty}$, etc.

PROBLEM 3. *Find interesting examples of spaces homeomorphic to products $\Omega \times \boldsymbol{R}^{\infty}$, where Ω is a "nice" subset in Q.*

In [Ba] the author has found a natural example of a space homeomorphic to the countable power $(\boldsymbol{R}^{\infty})^{\omega}$ of $\boldsymbol{R}^{\infty}$—this is the space $\mathscr{D}'$ of distributions on an open set in $\boldsymbol{R}^n$.

PROBLEM 4. *Find a natural example of a space homeomorphic to a product $\Omega \times (\boldsymbol{R}^{\infty})^{\omega}$, where $\Omega \subset Q$. In particular, is $\exp(\mathscr{D}')$ homeomorphic to $(\boldsymbol{R}^{\infty})^{\omega}$? Is $\exp_{\omega}(\mathscr{D}')$ homeomorphic to $\sigma \times (\boldsymbol{R}^{\infty})^{\omega}$?*

Acknowledgement

The author expresses his gratitude to Katsuro Sakai for fruitful discussions on the subject of the paper.

References

[Ba] T. Banakh, Topological classification of strong duals to nuclear *(LF)*-spaces, Studia Math. (to appear)

[BRZ] T. Banakh, T. Radul, M. Zarichnyi, Absorbing sets in infinite-dimensional manifolds, VNTL Publishers, Lviv, 1996.

[Ca] R. Cauty, Strongly universality and its applications, Trudy Matem. Inst. Steklov. **212** (1996), 95–122 (Russian); English transl., Proc. Steklov Inst. Math. **212** (1996), 89–114.

[CDM] R. Cauty, T. Dobrowolski, W. Marciszewski, A contribution to the topological classification of the spaces $C_p(X)$, Fund. Math. **142** (1993), 267–301.

[CN] D. W. Curtis, N. T. Nhu, Hyperspaces of finite subsets which are homeomorphic to $\aleph_0$-dimensional linear metric spaces, Top. Appl. **19** (1985), no. 3, 251–260.

[CP] D. W. Curtis, D. S. Patching, Hyperspaces of direct limits of locally compact metric spaces, Top. Appl. **29** (1988), 55–60.

[Ma] P. Mankiewicz, On topological, Lipschitz, and uniform classification of *LF*-spaces, Studia Math. **52** (1974), 109–142.

[Sa_1] K. Sakai, On $\boldsymbol{R}^\infty$-manifolds and Q^∞-manifolds, Topol. Appl. **18** (1984), no. 1, 69–79.

[Sa_2] K. Sakai, An embedding space triple of the unit interval into a graph and its bundle structure, Proc. Amer. Math. Soc. **111** (1991), 1171–1175.

[Sch] H. H. Schaefer, Topological vector spaces, The Macmillan company, NY, London, 1966.

[Ya] T. Yagasaki, Infinite-dimensional manifold triples of homeomorphism groups, Top. Appl. **76** (1997), no. 3, 261–287.

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