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THE UPPER BOUNDS FOR EIGENVALUES OF DIRAC OPERATORS

By

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Abstract. Let D be a Dirac operator on a compact oriented Riemannian manifold M of dimension $2m$. The operator D can be one of the four classical elliptic operators that arise from geometry, or one of the twisted operators of these four operators. Let λ_k^2 be the k-th nonzero eigenvalue of the operator D^2 counting with multiplicity. We show that

$$\lambda_k^2 \leq c(2m)\max\left\{\left(\frac{N(a)}{V(M)}\right)^{1/m}, \left(\frac{2^{m-1}(m_0+k-1)-2^{-m}m_0+1}{|k_0|V(M)}\right)^{1/m}\right\},$$

where $N(a)$ is an integer determined by the geometry of M, m_0 the dimension of the kernel of D^2 and k_0 an integer defined by the operator D. These results, in case M being a surface, give a partial answer to a conjecture of Yau.

1. Introduction

Let M be a compact Riemannian spin manifold, with a twisted classical Dirac operator denoted by D_V. Vafa and Witten proved in [10] that there exist universal upper bounds for the eigenvalues of operator D_V. By the methods of Vafa and Witten, Baum [3] obtained the upper bounds for the eigenvalues of the classical Dirac operator in geometrical terms of M. Bunke [4], Glazebrook and Kamber [7] made also the related works.

By using the motheds developed by Vafa, Witten and Baum, we prove that there exist universal upper bounds for generalized Dirac operators. Our main

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results are Theorems 3.1 and 3.6. We also improve the estimates of the upper bounds given by Baum. The Dirac operators discussed here can be one of the four classical elliptic operators that arise from geometry, or one of the twisted operators of these four operators. To prove Theorem 3.1 we use Clifford bundles $C\ell(S^{2m})$ and $C\ell(v)$. In §2, we study the properties of these bundles, espectially we compute the Chern character $ch(C\ell^{\pm}(S^{2m}))$.

It is well known that the Hodge-de Rham operator $D = d + \delta$ acting on the space of differential forms is a Dirac operator. In [12, Problem 79], Yau asked how to estimate the first nonzero eigenvalue of Laplacian of D in terms of computable geometry quantities. For these problems, we show in §4 that the upper bounds of the first eigenvalue of Laplacian D^2 can be estimated by the geometry and topology of M. For the case $\dim M = 2m$ and k sufficiently large, the k-th nonzero eigenvalue λ_k^2 of D^2 counting with multiplicity can be bounded by

$$\lambda_k^2 \leq c(2m)\left(\frac{m_0 + k - 1}{2V(M)}\right)^{1/m},$$

where m_0 is the sum of Betti numbers of M. In [12, Problem 71], Yau asked the validity of the following inequality for M being a surface

$$\frac{\lambda_k^2}{k} \leq \frac{c(g+1)}{area(M)},$$

where c is a universal constant, g the genus of M and λ_k^2 the k-th eigenvalue of Laplacian acting on functions. This inequality is valid for the case $k = 1$ (see Hersch [8], Yang and Yau [11]). We shall show that, in this case, the spectrum of Laplacian D^2 acting on functions is the same as that of D^2 acting on differential forms. Therefore Theorems 3.1 and 3.6 give a partial answer to this problem.

All manifolds considered in this paper are compact, oriented and without boundary. The names of elliptic operators used in this paper follows from Gilkey [6] and Lawson and Michelsohn [9].

2. Clifford Bundles on S^{2m}

Let v be the unit outward norm on the unit sphere S^{2m}. Let $C\ell(S^{2m})$ and $C\ell(v)$ be the associated Clifford bundles of TS^{2m} and the normal bundle on S^{2m} respectively, $\mathbf{C\ell}(S^{2m}) = C\ell(S^{2m}) \otimes \mathbf{C}$, $\mathbf{C\ell}(v) = C\ell(v) \otimes \mathbf{C}$. Let $\varphi_1, \dots, \varphi_{2m}$ be an oriented local orthonormal basis of TS^{2m} and $\mathbf{C\ell}(S^{2m}) = \mathbf{C\ell}^+(S^{2m}) \oplus \mathbf{C\ell}^-(S^{2m})$ a decomposition, $\mathbf{C\ell}^{\pm}(S^{2m}) = (1 \pm \omega_c) \cdot \mathbf{C\ell}(S^{2m})$, where $\omega_c = (\sqrt{-1})^m \varphi_1 \cdots \varphi_{2m}$ and the notation $\cdot$ stands for the Clifford multiplication.

LEMMA 2.1. *Let* ∇ *be a covariant derivative on* $\boldsymbol{C\ell}(S^{2m})$ *determined by the Levi-Civita connection on* TS^{2m}. *Then for any* $X \in \Gamma(TS^{2m})$ *and* $\psi \in \Gamma(\boldsymbol{C\ell}^{\pm}(S^{2m}))$, *we have*

$$\nabla_X \psi = X\psi - \tfrac{1}{2}[v \cdot X \cdot \psi - \psi \cdot v \cdot X].$$

PROOF. Let $(x^1, \dots, x^{2m+1})$ be the Euclidean coordinates on $\boldsymbol{R}^{2m+1}$ and $\overline{\nabla}$ the flat connection on $\boldsymbol{R}^{2m+1}$. For any vector fields $\overline{X} = \sum_i \overline{X}^i(\partial/\partial x^i)$, $\overline{Y} = \sum_i \overline{Y}^j(\partial/\partial x^j)$ on $\boldsymbol{R}^{2m+1}$, we have

$$\overline{\nabla}_{\overline{X}} \overline{Y} = \overline{X}\overline{Y} = \sum \overline{X}^i \frac{\partial \overline{Y}^j}{\partial x^i} \frac{\partial}{\partial x^j}.$$

If $X, Y \in \Gamma(TS^{2m})$, the Levi-Civita connection ∇ on S^{2m} is defined by

$$\nabla_X Y = XY - \langle XY, v \rangle v.$$

By $\langle XY, v \rangle = X\langle Y, v \rangle - \langle Y, Xv \rangle = -\langle X, Y \rangle$, we have

$$\nabla_X Y = XY + \langle X, Y \rangle v.$$

Let $\psi = \varphi_{i_1} \cdots \varphi_{i_k}$ be a local section of $\boldsymbol{C\ell}(S^{2m})$. From

$$\begin{aligned} v \cdot X \cdot \varphi_{i_1} \cdots \varphi_{i_k} &= -2\langle X, \varphi_{i_1} \rangle v\varphi_{i_2} \cdots \varphi_{i_k} + \varphi_{i_1} \cdot v \cdot X\varphi_{i_2} \cdots \varphi_{i_k} \\ &= -2\sum_{j=1}^{k} \varphi_{i_1} \cdots \langle X, \varphi_{i_j} \rangle v \cdots \varphi_{i_k} + \varphi_{i_1} \cdots \varphi_{i_k} \cdot v \cdot X, \end{aligned}$$

we have

$$\begin{aligned} \nabla_X \psi &= \sum \varphi_{i_1} \cdots \nabla_X \varphi_{i_j} \cdots \varphi_{i_k} \\ &= X\psi - \tfrac{1}{2}[v \cdot X \cdot \psi - \psi \cdot v \cdot X]. \end{aligned}$$

Since $\nabla_X \omega_{\mathbf{c}} = 0$, the covariant derivative ∇_X preserves the decomposition $\boldsymbol{C\ell}(S^{2m}) = \boldsymbol{C\ell}^+(S^{2m}) \oplus \boldsymbol{C\ell}^-(S^{2m})$. The lemma is proved. □

PROPOSITION 2.2. *The Chern character of* $\boldsymbol{C\ell}^{\pm}(S^{2m})$ *is given by*

$$ch(\boldsymbol{C\ell}^{\pm}(S^{2m})) = 2^m(2^{m-1}\alpha_0 \pm \alpha_{2m}),$$

where α_i *are the generators of* $H^i(S^{2m}; \boldsymbol{Z})$.

LEMMA 2.3. *Let* $\boldsymbol{C\ell}_{2m} = \boldsymbol{C\ell}_{2m}^+ \oplus \boldsymbol{C\ell}_{2m}^-$ *be the decomposition defined as usually. For any* $\varphi = e_{i_1} \cdots e_{i_{2k}}$, $\psi = e_{j_1} \cdots e_{j_l}$, $i_1 < \cdots < i_{2k}$, $j_1 < \cdots < j_l$, *we define* *a map* $F_{\varphi,\psi}: C\ell_{2m}^{\pm} \to C\ell_{2m}^{\pm}$, $F_{\varphi,\psi}(\xi) = \varphi \cdot \xi \cdot \psi$, *where* $e_1, \dots, e_{2m}$ *is an oriented orthonormal basis of* $\boldsymbol{R}^{2m}$. *Then we have*

$$tr(F_{\varphi,\psi}|_{C\ell_{2m}^{\pm}}) = \begin{cases} 2^{2m-1}, & \varphi = \psi = 1; \\ \pm(-\sqrt{-1})^m 2^{2m-1}, & \varphi = e_1 e_2 \cdots e_{2m},\ \psi = 1; \\ 0, & \textit{otherwise}. \end{cases}$$

This lemma can be proved easily. We prove Proposition 2.2 only.

PROOF. Let $\omega^1, \dots, \omega^{2m}$ be a dual basis of $\varphi_1, \dots, \varphi_{2m}$ and $R = 1/8 \sum \cdot R_{ijkl}\omega^i \wedge \omega^j \otimes \varphi_k \varphi_l$ be an operator acting on $\Gamma(C\ell^{\pm}(S^{2m}))$, where $R_{ijkl} = \delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}$ are the components of curvature tensor on sphere S^{2m}. Since

$$R \cdot \varphi_p = \tfrac{1}{2} \sum R_{ijpl}\omega^i \wedge \omega^j \otimes \varphi_l + \varphi_p \cdot R,$$

the curvature operator on the Clifford bundle $C\ell^{\pm}(S^{2m})$ is given by

$$R - \bar{R} : \Gamma(C\ell^{\pm}(S^{2m})) \to A^2(S^{2m}) \otimes \Gamma(C\ell^{\pm}(S^{2m})),$$

$$(R - \bar{R})\xi = R \cdot \xi - \xi \cdot R, \quad \xi \in \Gamma(C\ell^{\pm}(S^{2m})).$$

Then the Chern character of $C\ell^{\pm}(S^{2m})$ is defined by the closed form

$$ch(C\ell^{\pm}(S^{2m})) = tr\left\{ \exp\left[\frac{\sqrt{-1}}{2\pi}(R - \bar{R}) \right] \Big|_{C\ell^{\pm}(S^{2m})} \right\}.$$

By Lemma 2.3, we need only to compute

$$\left(\frac{\sqrt{-1}}{2\pi} \right)^m \frac{1}{m!} R^m = \frac{(-\sqrt{-1})^m}{(8\pi)^m m!} \sum \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{2m}} \otimes \varphi_{i_1}\varphi_{i_2} \cdots \varphi_{i_{2m}}$$

$$= \frac{(-\sqrt{-1})^m (2m)!}{(8\pi)^m m!} \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{2m} \otimes \varphi_1 \varphi_2 \cdots \varphi_{2m}.$$

We obtain

$$ch(C\ell^{\pm}(S^{2m})) = 2^m(2^{m-1}\alpha_0 \pm \alpha_{2m}),$$

where $\alpha_{2m} = (-1)^m((2m)!/2 \cdot m!\pi^m 4^m)\omega^1 \wedge \cdots \wedge \omega^{2m}$ is a generator of $H^{2m}(S^{2m}; \boldsymbol{Z})$. □

3. Upper Bounds for Eigenvalues of Dirac Operators

Let M be a compact oriented Riemannian manifold of dimension $2m$ and $S = S^+ \oplus S^-$ be a bundle of left modules over $C\ell(M)$. Let $D : \Gamma(S) \to \Gamma(S)$ be a generalized Dirac operator on M. The operator D is selfadjoint and $D^0 = D : \Gamma(S^+) \to \Gamma(S^-)$, $D^{0*} = D : \Gamma(S^-) \to \Gamma(S^+)$. Let $e_1, \dots, e_{2m}$ be a local orthonormal basis on M and ∇^S be a covariant derivative on S, then we have (see [9], II. §5)

$$D = \sum_{i=1}^{2m} e_i \nabla^S_{e_i}.$$

The spectrum of the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is symmetric to zero. If λ^2 is an eigenvalue of D^2, $\pm\lambda$ are the eigenvalues of D. Let $0 = \lambda_0^2 < \lambda_1^2 < \cdots$ denote the eigenvalues of D^2 and m_j be the multiplicity of λ_j^2. If $j > 0$, m_j is even.

Let $f : M \to S^{2m}$ be a smooth map. Define

$$\|df\|_x = \sup_{v \in T_x M} \frac{|df(v)|}{|v|}, \quad x \in M; \quad \|df\|_\infty = \max_{x \in M} \|df\|_x,$$

where df is the tangent map of f. One can show that the norm of the cotangent map f_x^* of f equals $\|df\|_x$.

The following theorem is a generalization of [3] and [10].

THEOREM 3.1. *Let S be a Dirac bundle over M and $D : \Gamma(S) \to \Gamma(S)$ be a Dirac operator with the index* $\mathrm{ind}(D^0) = \int_M F$, *where $F = k_0 + \cdots$ is a characteristic form on M, $k_0 \in \mathbf{C}$. We assume that $k_0 \neq 0$. Let $f : M \to S^{2m}$ be a smooth map with degree* $\deg(f)$ *which satisfies* $|k_0| \deg(f) > 2^{m-1}(m_0 + \cdots + m_{k-1}) - 2^{-m} m_0$. *Then the k-th eigenvalue of D is bounded by*

$$|\lambda_k| \leq \sqrt{m(m+1)} \|df\|_\infty.$$

The proof of Theorem 3.1 is similar to that of Baum [3].

Let ∇^f be the covariant derivative on the induced bundle $f^*[C\ell^\pm(S^{2m}) \cdot C\ell(v)]$ defined by

$$\nabla^f_X(f^*\tau) = f^*(\nabla_{df(X)}\tau), \quad X \in \Gamma(TM), \quad \tau \in \Gamma[C\ell^\pm(S^{2m}) \cdot C\ell(v)],$$

where ∇ is the covariant derivative on $C\ell^\pm(S^{2m}) \cdot C\ell(v)$ defined in §2 and $\nabla v = 0$. Let ∇^0 be the covariant derivative on $f^*[C\ell(S^{2m}) \cdot C\ell(v)]$ defined by the trivial connection on $TS^{2m} \oplus v \cong S^{2m} \times \mathbf{R}^{2m+1}$. If there is no danger of confusion we omit the symbol f^*. Define operators $D_f^\pm$ and D_0 as follows:

$$D_f^\pm : \Gamma(S \otimes C\ell^\pm(S^{2m}) \cdot C\ell(v)) \to \Gamma(S \otimes C\ell^\pm(S^{2m}) \cdot C\ell(v)),$$

$$D_f^\pm = \sum e_j [\nabla^S_{e_j} \otimes 1 + 1 \otimes \nabla^f_{e_j}];$$

$$D_0 : \Gamma(S \otimes \boldsymbol{C}\ell(S^{2m}) \cdot \boldsymbol{C}\ell(v)) \to \Gamma(S \otimes \boldsymbol{C}\ell(S^{2m}) \cdot \boldsymbol{C}\ell(v)),$$

$$D_0 = \sum e_j [\nabla^S_{e_j} \otimes 1 + 1 \otimes \nabla^0_{e_j}].$$

The twisted Dirac operators D_0 and $D_f^{\pm}$ are essentially selfadjoint. Denote $D_f = D_f^+ \oplus D_f^-$ and $L_f = D_f - D_0$. The proof of Theorem 3.1 is based on the comparison of the spectrum of the twisted operators D_f and D_0. Since $S \otimes f^*[\boldsymbol{C}\ell(S^{2m}) \cdot \boldsymbol{C}\ell(v)] \cong S \oplus \cdots \oplus S = 2^{2m+1} S$, for each j, the eigenvalue λ_j^2 of D^2 is also the eigenvalue of D_0^2 with multiplicity $2^{2m+1} m_j$. By the perturbation theory, to prove Theorem 3.1, we need only to estimate the norm of L_f and the dimension of $\ker D_f$.

We first calculate the norm of L_f. Let $e_1, \ldots, e_{2m}$ and $\varphi_1, \ldots, \varphi_{2m}$ be lacal orthonormal frame fields on M and S^{2m} respectively, $df(e_i) = \sum a_{ik} \varphi_k$.

LEMMA 3.2. *The operator L_f is a selfadjoint morphism in the bundle $S \otimes \boldsymbol{C}\ell(S^{2m}) \cdot \boldsymbol{C}\ell(v)$ which satisfies*:

$$L_f(\psi \otimes \varphi_{i_1} \cdots \varphi_{i_k}) = \sum_j \sum_{r=1}^{k} e_j \psi \otimes \varphi_{i_1} \cdots \varphi_{i_{r-1}} a_{j i_r} v \cdots \varphi_{i_k},$$

$$L_f(\psi \otimes \varphi_{i_1} \cdots \varphi_{i_k} v) = -\sum e_j \psi \otimes \varphi_{i_1} \cdots \varphi_{i_k} \sum_{l \neq i_1, \ldots, i_k} a_{jl} \varphi_l,$$

$$\psi \in \Gamma(S), \quad i_1 < \cdots < i_k.$$

PROOF. By Lemma 2.1 and $\nabla^0_{e_j} v = f^*(df(e_j))$, $\nabla^f_{e_j} v = 0$, for any $\psi \in \Gamma(S)$, $\varphi \in \Gamma(\boldsymbol{C}\ell(S^{2m}))$, $a, b \in \boldsymbol{C}$, we have

$$L_f(\psi \otimes f^*(\varphi \otimes (a + bv))$$
$$= \sum e_j \psi \otimes f^* \left\{ -\frac{1}{2} [v \cdot df(e_j) \cdot \varphi - \varphi \cdot v \cdot df(e_j)](a + bv) - b\varphi \cdot df(e_j) \right\}.$$

Then

$$L_f(\psi \otimes \varphi_{i_1} \cdots \varphi_{i_k}) = \sum_{j,r} e_j \psi \otimes \varphi_{i_1} \cdots \langle df(e_j), \varphi_{i_r} \rangle v \cdots \varphi_{i_k}.$$

The second equation of the lemma follows from

$$L_f(\psi \otimes \varphi_{i_1} \cdots \varphi_{i_k} v) = \sum e_j \psi \otimes \varphi_{i_1} \cdots \langle df(e_j), \varphi_{i_r} \rangle v \cdots \varphi_{i_k} v$$
$$-\sum e_j \psi \otimes \varphi_{i_1} \cdots \varphi_{i_k} \cdot df(e_j),$$

and

$$df(e_j) = \sum_r a_{ji_r}\varphi_{i_r} + \sum_{l \neq i_1,\dots,i_k} a_{jl}\varphi_l, \quad v \cdot v = \varphi_{i_r} \cdot \varphi_{i_r} = -1. \qquad \square$$

Denote $\langle\, , \rangle$ the inner product on $S \otimes C\ell(S^{2m}) \cdot C\ell(v)$. Define the norm of L_f by

$$\|L_f\|_x = \max_{\xi} \frac{\|L_f\xi\|_x}{\|\xi\|_x}, \quad \xi \in [S \otimes C\ell(S^{2m}) \cdot C\ell(v)]_x, \quad x \in M.$$

Lemma 3.3. $\|L_f\|_x \leq \sqrt{m(m+1)}\|df\|_x$.

Proof. We need only to prove the following two cases:

Case 1. Let $\xi = \sum_{i_1<\cdots<i_k} \psi_{i_1\cdots i_k} \otimes \varphi_{i_1} \cdots \varphi_{i_k}$, $\psi_{i_1\cdots i_k} \in S_x$, $k = 1, 2, \dots, 2m$, then $\|\xi\|^2 = \sum_{i_1<\cdots<i_k} \|\psi_{i_1\cdots i_k}\|^2$. From Lemma 3.2, we have

$$\langle L_f\xi, L_f\xi\rangle = \sum_{j,t}\sum_{r,s} \langle e_j\psi_{i_1\cdots i_k}, e_t\psi_{j_1\cdots j_k}\rangle$$
$$\cdot \langle \varphi_{i_1} \cdots a_{ji_r}v \cdots \varphi_{i_k}, \varphi_{j_1} \cdots a_{tj_s}v \cdots \varphi_{j_k}\rangle.$$

It is easy to see that

$$\begin{aligned}
&\langle \varphi_{i_1} \cdots a_{ji_r}v \cdots \varphi_{i_k}, \varphi_{j_1} \cdots a_{tj_s}v \cdots \varphi_{j_k}\rangle \\
&= (-1)^{r+s} < \varphi_{i_1} \cdots \hat{\varphi}_{i_r} \cdots \varphi_{i_k}, \varphi_{j_1} \cdots \hat{\varphi}_{j_s} \cdots \varphi_{j_k}\rangle a_{ji_r}a_{tj_s} \\
&= \begin{cases} a_{ji_r}a_{ti_r}, & r = s, i_1 = j_1, \dots, i_k = j_k; \\ \pm a_{ji_r}a_{tj_s}, & \{i_1, \dots, \hat{i}_r, \dots, i_k\} = \{j_1, \dots, \hat{j}_s, \dots, j_k\}, \ j_s \neq i_1, \dots, i_k; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence

$$\langle L_f\xi, L_f\xi\rangle = \sum_{j,t}\sum_r \langle e_j\psi_{i_1\cdots i_k}, e_t\psi_{i_1\cdots i_k}\rangle a_{ji_r}a_{ti_r}$$
$$+ \sum_{j,t,r}\sum_{l \neq i_1,\dots,i_k} \pm \langle e_j\psi_{i_1\cdots i_k}, e_t\psi_{i_1\cdots \hat{i}_r \cdots i_k l}\rangle a_{ji_r}a_{tl}.$$

Using the fact that the Clifford multiplication by unit vectors of T_xM on S_x preserves the inner product of S_x, then we have

$$\sum\Big\langle\sum a_{ji_r}e_j\psi_{i_1\cdots i_k},\sum a_{ji_r}e_j\psi_{i_1\cdots i_k}\Big\rangle=\sum\|\psi_{i_1\cdots i_k}\|^2\cdot\Big|\sum a_{ji_r}e_j\Big|^2$$
$$=\sum_{i_1<\cdots<i_k}\|\psi_{i_1\cdots i_k}\|^2\sum_{j,r}a_{ji_r}^2$$

and

$$\Big|\sum\pm\Big\langle\sum a_{ji_r}e_j\psi_{i_1\cdots i_k},\sum a_{tl}e_t\psi_{i_1\cdots\hat{i}_r\cdots i_k l}\Big\rangle\Big|$$
$$\leq\frac{1}{2}\sum\left[\Big\|\sum a_{ji_r}e_j\psi_{i_1\cdots i_k}\Big\|^2+\Big\|\sum a_{tl}e_t\psi_{i_1\cdots\hat{i}_r\cdots i_k l}\Big\|^2\right]$$
$$=\frac{1}{2}(2m-k)\sum\|\psi_{i_1\cdots i_k}\|^2\sum a_{ji_r}^2+\frac{1}{2}(2m-k)\sum\|\psi_{i_1\cdots\hat{i}_r\cdots i_k l}\|^2\sum a_{tl}^2$$
$$=(2m-k)\sum\|\psi_{i_1\cdots i_k}\|^2\sum_{j,r}a_{ji_r}^2.$$

We have

$$\|L_f\xi\|_x^2\leq(k+(2m-k)k)\|\xi\|_x^2\|df\|_x^2.$$

It is easy to see that

$$\max_{1\leq k\leq 2m}\{k+(2m-k)k\}\leq m(m+1).$$

Then

$$\|L_f\xi\|_x\leq\sqrt{m(m+1)}\|\xi\|_x\|df\|_x.$$

CASE 2. Let $\eta=\sum\psi_{i_1\cdots i_k}\otimes\varphi_{i_1}\cdots\varphi_{i_k}v$, then

$$\langle L_f\eta,L_f\eta\rangle=\sum\langle e_j\psi_{i_1\cdots i_k},e_t\psi_{j_1\cdots j_k}\rangle$$
$$\cdot\langle\varphi_{i_1}\cdots\varphi_{i_k}\sum_{l\neq i_1,\ldots,i_k}a_{jl}\varphi_l,\varphi_{j_1}\cdots\varphi_{j_k}\sum_{p\neq j_1,\ldots,j_k}a_{tp}\varphi_p\rangle.$$

Similar to the Case 1, we have

$$\Big\langle\varphi_{i_1}\cdots\varphi_{i_k}\sum a_{jl}\varphi_l,\varphi_{j_1}\cdots\varphi_{j_k}\sum a_{tp}\varphi_p\Big\rangle$$
$$=\begin{cases}\sum\limits_{l\neq i_1,\ldots,i_k}a_{jl}a_{tl}, & i_1=j_1,\ldots,i_k=j_k;\\ \sum\limits_{r,l}\pm a_{ti_r}a_{jl}, & \{i_1,\ldots,i_k,l\}=\{j_1,\ldots,j_k,p\},\ p=i_1,\ldots,i_k;\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\langle L_f\eta, L_f\eta\rangle \le (2m-k+k(2m-k))\|\eta\|_x^2\|df\|_x^2.$$

In this case we also have

$$\|L_f\eta\|_x^2 \le m(m+1)\|\eta\|_x^2\|df\|_x^2.$$

The lemma is proved. □

Define $\|L_f\|^2 = \sup_{\xi} \frac{\int_M \|L_f\xi\|^2}{\int_M \|\xi\|^2}$, $\xi \in \Gamma(S \otimes C\ell(S^{2m}) \cdot C\ell(v))$. Then

$$\|L_f\| \le \sqrt{m(m+1)}\|df\|_\infty.$$

Since $D_0 = D_f - L_f$, by the perturbation theory, in the interval $[-\|L_f\|, \|L_f\|]$, there are at least $\dim(\ker D_f)$ eigenvalues of D_0. Now we estimate the dimension of $\ker D_f$ and complete the proof of Theorem 3.1. By Atiyah-Singer index theorem, the indices of the operators

$$D^{0\pm} = D_f^{\pm} : \Gamma(S^+ \otimes C\ell^{\pm}(S^{2m}) \cdot C\ell(v)) \to \Gamma(S^- \otimes C\ell^{\pm}(S^{2m}) \cdot C\ell(v))$$

are

$$\mathrm{ind}(D_f^{0\pm}) = 2^{2m}(m_0^+ - m_0^-) \pm (-1)^m 2^{m+1} k_0 \deg(f),$$

where $m_0^+ = \dim\ker(D|_{S^+})$, $m_0^- = \dim\ker(D|_{S^-})$ and $m_0 = m_0^+ + m_0^-$. Assuming $(-1)^m k_0 = |k_0|$, it is easy to see that

$$\dim\ker(D_f^+|_{S^+C\ell^+(S^{2m})C\ell(v)}) \ge 2^{m+1}|k_0|\deg(f) + 2^{2m}(m_0^+ - m_0^-) + 2m_0^-,$$

$$\dim\ker(D_f^+|_{S^-C\ell^+(S^{2m})C\ell(v)}) \ge 2m_0^-,$$

$$\dim\ker(D_f^-|_{S^+C\ell^-(S^{2m})C\ell(v)}) \ge 2m_0^+,$$

$$\dim\ker(D_f^-|_{S^-C\ell^-(S^{2m})C\ell(v)}) \ge 2^{m+1}|k_0|\deg(f) - 2^{2m}(m_0^+ - m_0^-) + 2m_0^+.$$

Hence

$$\dim\ker(D_f) \ge 2^{m+2}|k_0|\deg(f) + 4m_0.$$

For the case of $(-1)^m k_0 = -|k_0|$, one can get the same inequality.

We have by assumption

$$\dim(\ker D_f) > 2^{2m+1}(m_0 + \cdots + m_{k-1}).$$

Therefore in the interval $[-\|L_f\|, \|L_f\|]$ there are at least $1 + 2^{2m+1} \cdot (m_0 + \cdots + m_{k-1})$ eigenvalues of D_0. But, as mentioned above, D_0 has the same eigenvalues as D, and the number of the eigenvalues $\pm\lambda_j$, $0 \leq j \leq k-1$, of D_0 with their multiplicities is just $2^{2m+1}(m_0 + \cdots + m_{k-1})$. Hence, the eigenvalues $\pm\lambda_k$ of D_0 lie in the interval $[-\|L_f\|, \|L_f\|]$. This proves Theorem 3.1. □

By [6] and [9], we know that the four classical elliptic operators (such as the Hodge-de Rham, Signature, classical Dirac operators and the Dolbeault operators on Kaehler manifolds) and their twisted operators are all generalized Dirac operators. From Theorem 3.1, we have

COROLLARY 3.4. *Let D_V be a twisted operator of one of the four classical operators and $f : M \to S^{2m}$ be a smooth map with degree $|k_0| \deg(f) > 2^{m-1}(m_0 + \cdots + m_{k-1}) - 2^{-m} m_0$, then the k-th eigenvalue $\pm\lambda_k$ of D_V is bounded by*

$$|\lambda_k| \leq \sqrt{m(m+1)}\|df\|_\infty,$$

where $k_0 = 2^m \operatorname{rank} V$, for the case D^0 is the Signature or Hodge-de Rham operator on M; $k_0 = \operatorname{rank} V$, for the case D^0 is the classical Dirac operator on spin manifold M or Dolbeault operator on kaehler manifold M.

PROOF. Let $D_V : \Gamma(S \otimes V) \to \Gamma(S \otimes V)$ be a twisted operator of $D : \Gamma(S) \to \Gamma(S)$ mentioned above. Then the index of D^0 and D_V^0 can be represented by $\operatorname{ind}(D^0) = \int_M F$ and $\operatorname{ind}(D_V^0) = \int_M F \cdot ch(V) = \int_M k_0 + \cdots$ respectively. If D_V^0 is not the twisted Hodge-de Rham operator, k_0 is a nonzero integer. Then the corollary follows from Theorem 3.1. For the Hodge-de Rham operator the number k_0 is zero. But the spectrum of the twisted Hodge-de Rham operator and the corresponding twisted Signature operator are the same. □

REMARK. The Dolbeault operator D on an almost complex manifold is not the Dirac operator in the sense of [9]. Theorem 3.1 and Corollary 3.4 still holds for such an operator. In fact, Theorem 3.1 holds for the operators which satisfy the conditions of [9, II. §5] but (5.4) in p. 114.

EXAMPLE 1. If M is the unit sphere S^{2m} and $f : S^{2m} \to S^{2m}$ be the identity mapping, then $\|df\|_\infty = 1$. Let $D : S \to S$ be the classical Dirac operator on the spinor bundle. We have $m_0 = 0$ and $k_0 = 1$. Hence the first norzero eigenvalue λ_1^2

of D^2 is bounded by

$$\lambda_1^2 \leq m(m+1).$$

C. Bar [2] showed that $\lambda_1^2 = m^2$ in this case.

When $D : A^+(S^{2m}) \to A^-(S^{2m})$ is the Signature operator on S^{2m}, we have $m_0 = 2$ and $k_0 = 2^m$. Then the first norzero eigenvalue λ_1^2 of Laplacian of Signature operator is also bounded by

$$\lambda_1^2 \leq m(m+1).$$

In order to give the estimates of the upper bounds of λ_k^2 in geometrical terms of M, we set

$V(M)$, the volume of M,

$\iota(M)$, the injective radius of M,

K_1, the upper bound of the sectional curvature of M,

$(2m-1)K_0$, the lower bound of Ricci curvature of M,

$V(K_0, r)$, the volume of the geodesic balls of radius r in space form of constant curvature K_0.

LEMMA 3.5. *Let $N(r)$ be the maximal number of pairwise disjoint geodesic disks in M all having radius $r < \iota(M)$. Then*

$$\frac{V(M)}{V(K_0, 2r)} \leq N(r) \leq \frac{V(M)}{V(K_1, r)}.$$

PROOF. Cf. p. 78 in [5]. □

Let $a > 0$ be the largest number such that $a^2 K_1 \leq \pi^2$, $a^2|K_0| \leq \pi^2$, $a \leq \iota(M)$. Let N_k be the integer part of $2^{m-1}(m_0 + \cdots + m_{k-1}) - 2^{-m} m_0 + 1$.

THEOREM 3.6. *Let D be a Dirac operator satisfying the conditions of Theorem 3.1. For any integer k, we have, if $N_k < (|k_0| V(M)/V(K_0, 2a))$,*

$$\lambda_k^2 \leq c(2m)\left(\frac{N(a)}{V(M)}\right)^{1/m};$$

if $N_k \geq (|k_0| V(M)/V(K_0, 2a))$,

$$\lambda_k^2 \leq c(2m)\left(\frac{N_k}{|k_0| V(M)}\right)^{1/m},$$

where $c(2m)$ is a constant.

PROOF. It is easy to see that $|k_0|V(M)/V(K_0, 2r)$ is a continuous function of $r > 0$. If $N_k \geq (|k_0|V(M)/V(K_0, 2a))$, we can choose a real number $r > 0$ such that

$$\frac{|k_0|V(M)}{V(K_0, 2r)} \leq N_k \leq |k_0|N(r).$$

Then there exist $N(r)$ pairwise disjoint geodesic disks B_j of M all having the radius r. Define a map $f : M \to S^{2m}$ which maps each B_j onto S^{2m} with degree 1. Then

$$|k_0| \deg(f) = |k_0|N(r) \geq N_k > 2^{m-1}(m_0 + \cdots + m_{k-1}) - 2^{-m}m_0.$$

We can apply Theorem 3.1. By assumption, $K_1 r^2 \leq \pi^2$, we can claim (cf. the proof of Proposition 1 in §3 of [3])

$$\|df\|_\infty^2 \leq \frac{\pi^2}{r^2}.$$

Obviously

$$\frac{\pi^2}{r^2} \leq \pi^2 \left(\frac{V(K_0, 2r)}{r^{2m}}\right)^{1/m} \left(\frac{N_k}{|k_0|V(M)}\right)^{1/m}.$$

Since $r^2|K_0| \leq \pi^2$, $V(K_0, 2r)/r^{2m}$ is bounded above.

The case of $N_k < (|k_0|V(M)/V(K_0, 2a)) < |k_0|N(a)$ can be proved as follows.

In this case, there exists $N(a)$ pairwise disjoint geodesic disks B_j of M all having the radius a. The map $f : M \to S^{2m}$ is defined as above. In this case, we also have

$$|k_0| \deg(f) > 2^{m-1}(m_0 + \cdots + m_{k-1}) - 2^{-m}m_0.$$

By assumption, $K_1 a^2 \leq \pi^2$, we also have

$$\|df\|_\infty^2 \leq \frac{\pi^2}{a^2}.$$

Similarly

$$\frac{\pi^2}{a^2} \leq \pi^2 \left(\frac{V(K_0, 2a)}{a^{2m}}\right)^{1/m} \left(\frac{N(a)}{V(M)}\right)^{1/m}.$$

Set $c(2m) = \sup_{r^2|K_0| \leq \pi^2} \left\{ m(m+1)\pi^2 \left(\frac{V(K_0,2r)}{r^{2m}}\right)^{1/m} \right\}$. This proves the theorem. □

4. Laplacian on Forms

Let M be a compact Riemannian manifold of dimension $2m$ and D^2 be the Laplacian of $D = d + \delta$ acting on the differential forms. As shown in the proof of Corollary 3.4, we may consider that $k_0 = 2^m$. From Theorem 3.1 we have

$$\lambda_k^2 \leq m(m+1) \inf_f \|df\|_\infty^2,$$

where $f : M \to S^{2m}$ may be any smooth map with degree $\deg(f) > (1/2) \cdot (m_0 + \cdots + m_{k-1}) - 2^{-2m} m_0$. Notice that $m_0 = \dim(\ker D)$ is the sum of Betti numbers of M which is nonzero. Then $2^{m-1}(m_0 + \cdots + m_{k-1}) \geq N_k$, where N_k is defined in §3. From Theorem 3.6, we have

1) if $N_k < (2^m V(M)/V(K_0, 2a))$, then

$$\lambda_k^2 \leq c(2m)\left(\frac{N(a)}{V(M)}\right)^{1/m};$$

2) if $N_k \geq (2^m V(M)/V(K_0, 2a))$,

$$\lambda_k^2 \leq c(2m)\left(\frac{N_k}{2^m V(M)}\right)^{1/m} \leq c(2m)\left(\frac{m_0 + \cdots + m_{k-1}}{2V(M)}\right)^{1/m}.$$

For the odd dimensional manifold, we have the following theorem.

THEOREM 4.1. *Let M be a compact oriented Riemannian manifold of dimension $2m-1$, λ_1^2 be the first nonzero eigenvalue of Laplacian D^2. Then*

$$\lambda_1^2 \leq 2m(2m-1) \inf_f \|df\|_\infty^2,$$

where $f : M \to S^{2m-1}$ is any smooth map with $\deg(f) > (\sqrt{2^{4m-3} - 1}/2^{2m-1}) m_0$.

PROOF. Let $\tilde{M} = M \times M$ be Riemannian product of M with itself. Then λ_1^2 is also the first nonzero eigenvalue of Laplacian on $\tilde{M}$. Let $f : M \to S^{2m-1}$ be a map with degree $\deg(f) > (\sqrt{2^{4m-3} - 1}/2^{2m-1}) m_0$. We shall show that there exists a smooth map $g : S^{2m-1} \times S^{2m-1} \to S^{4m-2}$ of degree 1 such that for any $p \in \tilde{M}$,

$$\|d(g \circ (f, f))\|_p^2 \leq \|d(f, f)\|_p^2 \leq \|df\|_\infty^2.$$

The degree of the map $g \circ (f, f) : \tilde{M} \to S^{4m-2}$ satisfies

$$\deg(g \circ (f, f)) = \deg^2(f) > \left(\frac{1}{2} - \frac{1}{2^{4m-2}}\right) m_0^2,$$

where m_0^2 is the sum of Betti numbers of manifold $\tilde{M}$. Then Theorem 4.1 follows from Theorem 3.6. The map g can be constructed as follows.

Let (a, θ) be the polar coordinates on $B^n = \{x \in \boldsymbol{R}^n \mid |x| \leq \pi\}$, $a \in S^{n-1}$, $0 \leq \theta \leq \pi$, $n = 2m - 1$. Define

$$\exp_1 : B^n \times B^n \to S^n \times S^n,$$

$$\exp_1((a, \theta), (a', \theta')) = ((a \sin\theta, \cos\theta), (a' \sin\theta', \cos\theta')).$$

Then the standard metric on $S^n \times S^n$ can be represented by

$$\begin{aligned} ds_1^2 &= \sin^2\theta\, da^2 + d\theta^2 + \sin^2\theta'\, da'^2 + d\theta'^2 \\ &= \sin^2\theta\, da^2 + \sin^2\theta'\, da'^2 + \frac{(\theta\, d\theta' - \theta'\, d\theta)^2}{\theta^2 + \theta'^2} + \frac{(\theta\, d\theta + \theta'\, d\theta')^2}{\theta^2 + \theta'^2}. \end{aligned}$$

On the other hand, set $B^{2n} = \{x \in \boldsymbol{R}^{2n} \mid |x| \leq \pi\} \subset B^n \times B^n$, the exponential map $\exp_2 : B^{2n} \subset TS^{2n} \to S^{2n}$ can be written as

$$\exp_2(a'', \theta'') = (a'' \sin\theta'', \cos\theta'').$$

The coordinates on B^{2n} and $B^n \times B^n$ are related by

$$a'' = \left(\frac{\theta}{\sqrt{\theta^2 + \theta'^2}} a, \frac{\theta'}{\sqrt{\theta^2 + \theta'^2}} a'\right), \quad \theta'' = \sqrt{\theta^2 + \theta'^2}.$$

Hence the metric of S^{2n} can be represented by

$$\begin{aligned} ds_2^2 &= \sin^2\theta''\, da''^2 + d\theta''^2 \\ &= \sin^2\sqrt{\theta^2 + \theta'^2}\left(\frac{\theta^2}{\theta^2 + \theta'^2} da^2 + \frac{\theta'^2}{\theta^2 + \theta'^2} da'^2 + \frac{(\theta\, d\theta' - \theta'\, d\theta)^2}{(\theta^2 + \theta'^2)^2}\right) \\ &\quad + \frac{(\theta\, d\theta + \theta'\, d\theta')^2}{\theta^2 + \theta'^2}. \end{aligned}$$

Define a map $\bar{g} : S^n \times S^n \to S^{2n}$ by $\bar{g} = \exp_2 \cdot \exp_1^{-1}$, where $\exp_2$ maps points of $B^n \times B^n - B^{2n}$ to $(0, \ldots, 0, -1)$. When $0 < x < \pi$, $\sin x / x$ is a decreasing function. Hence

$$\frac{\sin\sqrt{\theta^2 + \theta'^2}}{\sqrt{\theta^2 + \theta'^2}} \cdot \frac{\theta}{\sin\theta} \leq 1.$$

Therefore there are orthonormal bases of $T_x(S^n \times S^n)$ and $T_{\bar{g}(x)} S^{2n}$ respectively,

$x \in S^n \times S^n$. With these bases the matrix of tangent map $d\bar{g}_x$ is diagonal whose elements are all equal to or less than 1. The map $\bar{g}$ may not be smooth on the boundary of B^{2n}. Using the map $\bar{g}$, we can construct a smooth map $g : S^n \times S^n \to S^{2n}$ with requaired properties. □

Notice that we can not use $\tilde{M} = M \times S^1$ to estimate the first nonzero eigenvalue of D^2 on M.

The proof of the following corollary is similar to that of Theorem 3.6.

COROLLARY 4.2. *The first nonzero eigenvalue of Laplacian on an odd dimensional manifold M is bounded by*

$$\lambda_1^2 \leq c(2m-1)\max\left\{ \left(\frac{N(a)}{V(M)}\right)^{2/(2m-1)}, \left(\frac{P_1}{2^{2m-1}V(M)}\right)^{2/(2m-1)} \right\},$$

where $c(2m-1)$ is a constant and P_1 the integer part of $\sqrt{2^{4m-3}-1}m_0 + 1$.

Finally we consider the eigenvalue problem on surfaces. Let M be an oriented Riemannian surface with genus g, then $m_0 = 2(1+g)$ and $\chi(M) = 2(1-g)$ is the Euler-Poincare number of M.

LEMMA 4.3. *The number $\lambda^2 \neq 0$ is an eigenvalue of D^2 acting on differential forms with multiplicity n, if and only if λ^2 is an eigenvalue of D^2 acting on functions with multiplicity $n/4$.*

PROOF. Let $\lambda^2(A^i(M)) = \{\xi \in L^2(A^i(M)) \mid D^2\xi = \lambda^2\xi\}$, $i = 0, 1, 2$, be eigenspaces of D^2. The maps

$$\star : A^0(M) \to A^2(M),$$

$$d + \delta : A^0(M) \oplus A^2(M) \to A^1(M)$$

induce isomorphisms between $\lambda^2(A^0(M))$ and $\lambda^2(A^2(M))$; between $\lambda^2(A^0(M) \oplus A^2(M))$ and $\lambda^2(A^1(M))$ respectively. The lemma has been proved. □

THEOREM 4.4. *Denote λ_k^2 the k-th nonzero eigenvalue of D^2 acting on functions counting with multiplicity. Then*

$$\lambda_k^2 \leq 2 \inf_f \|df\|_\infty^2,$$

where $f : M \to S^2$ is any smooth map with $\deg(f) > (1/2)(g+1) + 2k - 2$.

This theorem follows from Lemma 4.3 and Theorem 3.1. From Theorem 3.6, we have the following

THEOREM 4.5. *If* $g + 4k - 2 < (2V(M)/V(K_0, 2a))$, *we have*

$$\lambda_k^2 \le c(2) \cdot \frac{N(a)}{V(M)};$$

otherwise

$$\lambda_k^2 \le c(2) \cdot \frac{g + 4k - 2}{2V(M)}.$$

EXAMPLE 2. Let M be the sphere S^2 with standard metric. From Example 1, we have

$$\lambda_1^2 \le 2.$$

As is well known, the first nonzero eigenvalue on S^2 is 2. The estimate is sharp.

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