TSUKUBA J. MATH.
Vol. 23 No. 2 (1999), 279-291

REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE SATISFYING A POINTWISE NULLITY CONDITION

By

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Abstract. In this paper, we give a classification of real hypersurfaces of a complex projective space CP^n satisfying a pointwise nullity condition for the structure vector field ξ i.e., $R(X,Y)\xi=k\{\eta(Y)X-\eta(X)Y\}$, k is a function, and further we prove a local structure theorem of real hypersurfaces of CP^n which satisfies $R(X,A\xi)\xi=k\{\eta(A\xi)X-\eta(X)A\xi\}$. The motivation of the present paper is a well-known fact that CP^n does not admit a real hypersurface of constant curvature.

0. Introduction

Let $CP^n=(CP^n,J,\tilde{g})$ be an n-dimensional complex projective space with Fubinistudy metric $\tilde{g}$ of constant holomorphic sectional curvature 4, and let M be an orientable real hypersurface of CP^n and N be a unit normal vector field on M. Then M has an almost contact metric structure (ϕ,ξ,η,g) induced from the Kählerian structure $(J,\tilde{g})$ of CP^n (see Section 1). One of the typical examples of M is a geodesic hypersphere. R. Takagi ([8]) classified homogeneous hypersurfaces of CP^n into six types. T. E. Cecil and P. J. Ryan ([1]) extensively investigated hypersurfaces which are realized as tubes of constant radius r over a complex submanifold of CP^n on which the structure vector field ξ is a principal curvature vector field with principal curvature $\alpha_1=2\cot 2r$ and corresponding focal map $\varphi_r:M\to CP^n$ (defined by $\varphi_r(p)=\exp_p(rN)$) has constant rank. We denote by ∇ the Levi-Civita connection with respect to g. The curvature tensor

(*) was partially supported by BSRI 98-1425 and TGRC-KOSEF, (**) was partially supported by BSRI 98-1404 and TGRC-KOSEF.
1991 Mathematics Subject Classification: 53B25, 53C15
Key words and phrases. real hypersurfaces, complex projective space, the k-nullity distribution.
Received May 11, 1998.
Revised October 26, 1998.

field R on M is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ where X and Y are vector fields on M. It is well-known that $\boldsymbol{CP}^n$ does not admit a real hypersurface with constant sectional curvature (cf. [2]).

On the other hand, S. Tanno ([10]) defined for $k \in \boldsymbol{R}$ the k-nullity distribution $N(k)$ of a Riemannian manifold by $N(k) : p \to N_p(k) = \{z \in T_pM : R(x, y)z = k(g(y, z)x - g(x, z)y)$ for any $x, y \in T_pM\}$. If $T_pM = N_p(k)$ for any point $p \in M$, then we see that M is of constant curvature k. In the present paper, we consider a real hypersurface of $\boldsymbol{CP}^n$ whose structure vector field ξ satisfies a pointwise nullity condition, namely, in Section 2, we give a classification of a real hypersurface M of $\boldsymbol{CP}^n$ which satisfies $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$, where k is a function on M. Moreover in Section 3, we investigate a real hypersurface of $\boldsymbol{CP}^n$ which satisfies $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$, where k is a function on M. In Section 4, we determine real hypersurfaces of $\boldsymbol{CP}^n$ which satisfies $A^2\xi = \lambda A\xi$ and $(\phi \cdot R)(X, A\xi)\xi = 0$, where $\phi \cdot R$ means ϕ operates on R as a derivation. In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

The author thanks to the referee for useful comments and advices on preparing the revised version.

1. Preliminaries

At first, we review the fundamental facts on a real hypersurface of $\boldsymbol{CP}^n$. Let M be a real hypersurface of $\boldsymbol{CP}^n$ and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric of $\boldsymbol{CP}^n$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{1.1}$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\begin{gathered} \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{gathered} \tag{1.2}$$

From (1.2), we get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{1.3}$$

From the fact $\tilde{\nabla} J = 0$ and (1.1), making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Since the ambient space is of constant holomorphic sectional curvature 4, we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{1.7}$$

From (1.6), using (1.2), (1.3), then the Ricci tensor S is given by

$$SX = (2n + 1)X - 3\eta(X)\xi + hAX - A^2X, \tag{1.8}$$

where $h =$ the trace of A. We recall the following

Proposition 1 ([6]). *If ξ is a principal curvature vector field, then the corresponding principal curvature α_1 is constant.*

Proposition 2 ([1]). *Let M be a real hypersurface of CP^n on which ξ is principal with principal curvature $\alpha_1 = 2 \cot 2r$ and the focal map φ_r has constant rank on M. Then the following hold:*

(i) *M lies on a tube (in the direction of $C = \gamma'(r)$ where $\gamma(r) = \exp_p(rN)$ and p is a base point of the normal vector N) of radius r over a certain Kählerian submanifold in CP^n.*

(ii) *Let $\cot\theta$ be a principal curvature of the shape operator A_C at $q = \gamma(r)$ of the Kählerian submanifold. Then the real hypersurface M has a principal curvature $\cot(r - \theta)$ at $p = \gamma(0)$.*

Theorem 1 ([7]). *Let M be a real hypersurface of CP^n. Then the followings are equivalent:*

(i) *M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic* CP^k $(0 \leq k \leq n-1)$, *where* $0 < r < \pi/2$

(ii) $\phi A = A\phi$.

A ruled real hypersurface of CP^n is defined by a foliated one by complex hyperplanes CP^{n-1} and its shape operator is written down in [3]. Namely,

$$
\begin{aligned}
A\xi &= \alpha_1\xi + \mu W \quad (\mu \neq 0), \\
AW &= \mu\xi, \\
AZ &= 0
\end{aligned}
\tag{1.9}
$$

for any $Z \perp \xi$, W, where W is unit vector orthogonal to ξ, α_1 and μ are functions on M. For more details about a ruled real hypersurface of CP^n, we refer to [4]. The ϕ-holomorphic sectional curvature is defined by a sectional curvature of $span\{X, \phi X\}$. In [3] it was proved that

THEOREM 2. *Let M be a real hypersurface of* $CP^n (n \geq 3)$ *with constant* ϕ-*holomorphic sectional curvature. Then M is locally congruent to the following spaces*:

(1) *a geodesic hypersphere* (*that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane* CP^{n-1}, *where* $0 < r < \pi/2$);

(2) *a ruled real hypersurface*;

(3) *a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane* CP^{n-1} *as a ruled hypersurface*.

We define a vector field U on M by $U = \nabla_\xi \xi$ and denote $\alpha_m = \eta(A^m\xi)$. Then from (1.2) and (1.5) we easily observe that

$$
\begin{aligned}
&g(U, \xi) = 0, \quad g(U, A\xi) = 0, \\
&\|U\|^2 = g(U, U) = \alpha_2 - \alpha_1^2.
\end{aligned}
\tag{1.10}
$$

From (1.2), (1.5) and (1.10) we have at once

LEMMA 1. *Let M be a real hypersurface of* CP^n. *Then* ξ *is a principal curvature vector field if and only if M satisfies* $\alpha_2 - \alpha_1^2 = 0$.

Now we recall that ([10]) the k-nullity distribution of a Riemannian manifold, for a real number k, is a distribution

$$N(k) : p \to N_p(k) = \{z \in T_pM : R(x, y)z = k\{g(y, z)x - g(x, z)y\}$$
$$\text{for any } x, y \in T_pM\}.$$

If $T_pM = N_p(k)$ for any point $p \in M$, then we see that M is of constant curvature k. In Section 2, we consider a pointwise nullity condition for the structure vector field ξ.

2. Real Hypersurfaces Satisfying a Pointwise Nullity Condition

In this section, we give a classification of a real hypersurface whose structure vector field ξ satisfying

$$R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\} \tag{2.1}$$

for a function k, where X, Y are any vector fields tangent to M. First we prove

Lemma 2. *Let M be a real hypersurface of CP^n. If M satisfies* (2.1), *then ξ is principal.*

Proof. From (1.6) and (2.1) we have

$$(k-1)\{\eta(Y)X - \eta(X)Y\} = \eta(AY)AX - \eta(AX)AY \tag{2.2}$$

for any vector field X and Y. We may put

$$A\xi = \alpha_1\xi + Z$$

where Z is orthogonal to ξ. For any vector field X orthogonal to ξ, let X_1 be the component of AX orthogonal to ξ, that is, $X_1 = AX - g(AX, \xi)\xi$. Putting $Y = \xi$ in (2.2), then for X orthogonal to ξ we have

$$(k-1)X = \alpha_1 X_1 - g(X, Z)Z. \tag{2.3}$$

First we consider where $\alpha_1 = 0$. Then by taking $X(\neq 0)$ orthogonal to Z in (2.3), we have $k = 1$, and hence, we have $Z = 0$ (by putting $X = Z$), that is, ξ is principal. If there exist a point p such that $\alpha_1(p) \neq 0$, by the continuity of α_1 we see that $\alpha_1 \neq 0$ sufficiently small neighborhood of p. Next we discuss on the neighborhood. If we put $X = Z$, then we see that *span*$\{\xi, Z\}$ is A-invariant. Here, if we put $Y = Z$ in (2.2) and we take $X(\neq 0)$ orthogonal to ξ and Z, then we have

$$\eta(AZ)AX = \eta(AX)AZ. \tag{2.4}$$

Since $span\{\xi, Z\}$ is A-invariant, from (2.4) we have

$$g(Z, Z)X = 0,$$

and hence $Z = 0$, that is, ξ is principal. At last, we conclude that ξ is principal on M. (Q.E.D.)

Since $A\xi = \alpha_1\xi$, taking account of Proposition 1 we may set $\alpha_1 = 2\cot 2r$ for some constant $0 < r < \pi/2$. Thus we have

THEOREM 3. *Under the same assumption as that of Lemma 2 and in addition that $n \geq 3$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of the following spaces:*

(1) *a geodesic hypersphere (that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane $\boldsymbol{C}P^{n-1}$ $(0 < r < \pi/2)$;*

(2) *a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic $\boldsymbol{C}P^l$ $(1 \leq l \leq n-2)$;*

(3) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kählerian submanifold with non-zero principal curvatures $\neq \pm 1$.*

PROOF. It follows from $A\xi = \alpha_1\xi$ and (2.2) that

$$(k-1)\{\eta(Y)X - \eta(X)Y\} = \alpha_1\{\eta(Y)AX - \eta(X)AY\}. \tag{2.8}$$

Since α_1 is constant (by Proposition 1) we divide our arguments into two cases, (i) $\alpha_1 = 0$, (ii) $\alpha_1 \neq 0$:

(i) $\alpha_1 = 0$. From (2.8) we see that $k = 1$, and from (1.6) we see that M satisfies $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$. Since the rank of corresponding focal map $\varphi_{\pi/4}$ is constant, by virtue of Proposition 2 we see that M is locally congruent to (2) or (3).

(ii) $\alpha_1 \neq 0$. Assume $Y = \xi$, $X \perp \xi$ in (2.8). Then we get

$$AX = (k-1)/\alpha_1 \cdot X \tag{2.9}$$

for any vector field X orthogonal to ξ, hence from (2.9) we see that M has at most two distinct principal curvatures. So, Theorem 3 in [1] implies that M is locally congruent to a geodesic hypersphere. (Q.E.D.)

REMARK 1. In the case (3) in Theorem 3, the condition "Kählerian submanifold with principal curvatures $\neq \pm 1$" is necessary. In general, Proposition 2 (ii) shows that the point $p(= \gamma(0))$ is a singular point of M when $r = \theta$.

REMARK 2. In particular, for $k \in \boldsymbol{R}$, if ξ belongs to the k-nullity distribution, then in the case (ii) in the proof of Theorem 3, from (2.9) by using the result in [8], we conclude that M is locally congruent to geodesic hypersphere when the dimension $n=2$, and thus we have same result as Theorem 3 when $n \geq 2$.

We denote $h^{(m)} = \operatorname{trace} A^m$, then in particular $h^{(1)} = h$ in (1.8). We also prove

PROPOSITION 3. *Let M be a real hypersurface of $\boldsymbol{C}P^n$. Then M always satisfies*

$$H_1^2 \leq 2(n-1)H_2,$$

where we put $H_m = h^{(m)}\alpha_m - \alpha_{2m}$. If the equality holds, then ξ is principal ($\alpha_1 = 2\cot 2r$). *Moreover, if we suppose that $n \geq 3$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of* (1), (2), (3) *in Theorem* 3.

PROOF. We put

$$T(X,Y) = R(X,Y)\xi - k\{\eta(Y)X - \eta(X)Y\}$$

for any vector fields X and Y on M, where k is a function. Then T is a (1,2)-tensor field on M. We calculate $\|T\|^2$, then we have

$$\begin{aligned} (2.10)\qquad \|T\|^2 &= \sum_{i,j} g(R(e_i,e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\}, R(e_i,e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\}) \\ &= \|R(\cdot,\cdot)\xi\|^2 - 4k\eta(S\xi) + 4(n-1)k^2, \end{aligned}$$

where $\{e_i\}$ $(i = 1, 2, \dots, 2n-1)$ is an orthonormal basis of the tangent space. From (1.6) and (1.8) a direct calculation yields

$$(2.11)\qquad \|R(\cdot,\cdot)\xi\|^2 = 4(n-1) + 4H_1 + 2H_2,$$

$$(2.12)\qquad \eta(S\xi) = (2n-2) + H_1.$$

From (2.10), (2.11) and (2.12) we have

$$(2.13)\qquad \|T\|^2 = 4(n-1)(1-k)^2 + 4H_1(1-k) + 2H_2 \geq 0.$$

Since (2.13) holds for any k at any point on M, we see that

$$(2.14)\qquad H_1^2 \leq 2(n-1)H_2.$$

Further we see that the equality holds in (2.14) if and only if $\|T\|^2 = 0$. Thus by using Theorem 3, we have our conclusion. (Q.E.D.)

3. Real Hypersurfaces of CP^n Satisfying $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$

In [2] we investigate a real hypersurface of CP^n which satisfies $R(X,\xi)\xi = k\{X - \eta(X)\xi\}$, where k is a function on M. In this section, we prove

THEOREM 4. *Let M be a real hypersurface of CP^n $(n \geq 3)$. Suppose that M satisfies*

$$R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}, \tag{3.1}$$

where k is a function on M. If ξ is principal with the associated principal curvature $\alpha_1 = 2\cot 2r$ and the rank of corresponding focal map φ_r is constant, then M is locally congruent to one of the following spaces:

(1) *a geodesic hypersphere;*

(2) *a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^l $(1 \leq l \leq n-2)$;*

(3) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kählerian submanifold with non-zero principal curvatures $\neq \pm 1$.*

PROOF. From (1.6) and (3.1) we have

$$(k-1)\{\eta(X)A\xi - \alpha_1 X\} = \eta(AX)A^2\xi - \alpha_2 AX. \tag{3.2}$$

Taking the transpose of A, then we have

$$(k-1)\{\eta(AX)\xi - \alpha_1 X\} = \eta(A^2X)A\xi - \alpha_2 AX, \tag{3.3}$$

for any vector field X on M. Since ξ is principal, that is, $A\xi = \alpha_1\xi$, for any vector field Y orthogonal to ξ (3.3) yields

$$\alpha_2 AY = (k-1)\alpha_1 Y. \tag{3.4}$$

Since $\alpha_2 = \alpha_1^2$ is constant (cf. Proposition 1), we divide our arguments into two cases, (i) $\alpha_2 = 0$, (ii) $\alpha_2 \neq 0$:

(i) $\alpha_2 = 0$. We see that $A\xi = 0$ and M satisfies $R(X,\xi)A\xi = k\{\eta(A\xi)X - \eta(AX)\xi\} = 0$. Since the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant, by the same arguments in the proof of the Theorem 3 in Section 2, we see that M is locally congruent to (2) or (3).

(ii) $\alpha_2 \neq 0$. From (3.4) we see that M has at most two distinct principal curvatures. So, Theorem 3 in [1] implies that M is locally congruent to a geodesic hypersphere of CP^n. (Q.E.D.)

Here, we consider the case that ξ is not principal and M satisfies (3.1). Then we may assume that

$$A\xi = \alpha_1\xi + \mu W, \quad \mu \neq 0 \tag{3.5}$$

and

$$AW = \mu\xi + \nu W + \delta Z_1, \tag{3.6}$$

where $Z_1 \perp \xi, W$, W is a unit vector orthogonal to ξ, and μ, ν, δ are functions on M. Then from (3.3) we have

$$\alpha_2 AW = \{\alpha_1\mu(\alpha_1 + \nu) - \mu(k-1)\}\xi + \{\mu^2(\alpha_1 + \nu) + \alpha_1(k-1)\}W. \tag{3.7}$$

So from (3.6) and (3.7) we get

$$\alpha_2\mu = \alpha_1\mu(\alpha_1 + \nu) - \mu(k-1), \quad \alpha_2\nu = \mu^2(\alpha_1 + \nu) + \alpha_1(k-1) \text{ and } \alpha_2\delta = 0. \tag{3.8}$$

Further from (3.2) we have

$$\alpha_2 AZ = \alpha_1(k-1)Z. \tag{3.9}$$

for any vector field Z orthogonal to ξ and W. Therefore from (3.5), (3.7), (3.8) and (3.9) we have

$$A\xi = \alpha_1\xi + \mu W$$

$$AW = \mu\xi + \nu W$$

$$AZ = \alpha_1/\alpha_2 \cdot (k-1)Z,$$

$\alpha_2 = \alpha_1(\alpha_1 + \nu) - (k-1)$ and $\alpha_2\nu = \mu^2(\alpha_1 + \nu) + \alpha_1(k-1)$ for any $Z \perp \xi, W$, where W is a unit vector orthogonal to $\xi, \mu(\neq 0), \alpha_2$ and ν are functions on M.

Let M be a real hypersurface of $\boldsymbol{C}P^n$ which satisfies $R(X, A\xi)\xi = \eta(A\xi)X - \eta(X)A\xi$, i.e., $k = 1$. Then from (3.3) it follows that

$$\alpha_2 AX = \eta(A^2X)A\xi \tag{3.10}$$

for any vector field X on M. If there exist a point p in M such that $\alpha_2(p) \neq 0$, then (3.10) implies that the rank of A at p is at most 1. However it is seen (cf. [11]) that the point p is geodesic. So it is contradictory to the assumption that $\alpha_2(p) \neq 0$. Thus $\alpha_2 = 0$ on M. Therefore by Lemma 1, we see that $A\xi = 0$ on M.

REMARK 3. The above arguments together with (1.9) and (20) in [3] imply that neither ruled real hypersurface nor the case (3) in Theorem 3 satisfy the condition (3.1).

It is well-known that a geodesic hypersphere in CP^n is η-umbilical, that is, $A = aI + b\eta \otimes \xi$, where a, b are constants (cf. [1], [9], etc.). Thus, due to Theorems 2, 4 and Remark 3, we characterize a geodesic hypersphere of CP^n by following

Theorem 5. *Let M be a real hypersurface of CP^n $(n \geq 3)$. Then M is of constant ϕ-holomorphic sectional curvature and M satisfies $R(X, A\xi)\xi = k\{\eta(A\xi)X - \eta(X)A\xi\}$, where k is a constant along M if and only if M is locally congruent to a geodesic hypersphere.*

Remark 4. The above Theorem 5 is a slight improvement of Theorem 4 in [2].

4. Real Hypersurfaces of CP^n Satisfying $\phi \cdot R = 0$

In [6], Y. Maeda investigated a real hypersurface M of CP^n which satisfies

$$(C_1) \qquad A\xi = \alpha_1 \xi,$$

$$(C_2) \qquad \phi \cdot R = 0,$$

where $\cdot$ means that a (1,1)-tensor field ϕ operates on R as a derivation, i.e., for any vector fields X, Y and Z on M

$$(\phi \cdot R)(X, Y)Z = \phi R(X, Y)Z - R(\phi X, Y)Z - R(X, \phi Y)Z - R(X, Y)\phi Z.$$

Under the conditions $(C_1), (C_2)$ and $n \geq 3$, he proved that M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic CP^k $(0 \leq k \leq n-1)$, where $0 < r < \pi/2$ (Theorem 5.4 in [6]).

In this section, we consider the following two conditions (4.1) and (4.2) weaker than (C_1) and (C_2), respectively:

$$(4.1) \qquad A^2\xi = \lambda A\xi,$$

$$(4.2) \qquad (\phi \cdot R)(X, A\xi)\xi = 0$$

for a function λ and for any vector field X on M. We prove

Theorem 6. *Let M be a real hypersurface of CP^n, and suppose that M satisfies* (4.1) *and* (4.2)*. Then ξ is a principal curvature vector field on M. Further assume that $\alpha_1 = 2\cot 2r$ and the rank of the focal map φ_r is constant, then M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius r over totally geodesic CP^k $(0 \leq k \leq n-1)$, where $0 < r < \pi/2$, or a non-homogeneous tube of radius $\pi/4$ of the case* (3) *in Theorem* 4.

Proof. From the assumption (4.2), we get

$$\phi R(X, A\xi)\xi - R(\phi X, A\xi)\xi - R(X, U)\xi = 0. \tag{4.3}$$

From (1.6), (4.1) and (4.3), we have

$$\alpha_2(\phi A - A\phi)X - \lambda g(X, U)A\xi - \lambda g(X, A\xi)U + g(X, A\xi)AU = 0. \tag{4.4}$$

If we put $X = \xi$ in (4.4), then, since $\alpha_2 = \lambda\alpha_1$, we have

$$\alpha_1 AU = 0. \tag{4.5}$$

If there exists a point $p \in M$ such that $\alpha_1(p) = 0$, then we see that $\alpha_2 = 0$, and hence by Lemma 1, we have ξ is principal at p. So, from now we discuss on open subset where $\alpha_1 \neq 0$. Then from (4.5) it follows that

$$AU = 0. \tag{4.6}$$

With (4.6) we easily obtain

$$g((\nabla_X A)\xi, \xi) = d\alpha_1(X),$$

where d denotes the exterior differential. Since $U = \phi A\xi$, from (1.4), (1.7) and (4.6) we have

$$\nabla_\xi U = \alpha_1 A\xi - \alpha_2\xi + \phi\nabla\alpha_1, \tag{4.7}$$

where $\nabla\alpha_1$ denotes the gradient vector field of α_1. Differentiating (4.6) covariantly along M, then by using (1.7) and (4.7) we have

$$(\nabla_U A)\xi = -\phi U - \alpha_1 A^2\xi + \alpha_2 A\xi - A\phi\nabla\alpha_1. \tag{4.8}$$

Also, if we differentiate $A^2\xi = \lambda A\xi$ covariantly along M, then together with (1.5) we have

$$\begin{aligned} g(A\xi, (\nabla_X A)Y) + g((\nabla_X A)\xi, AY) + g(\phi AX, A^2 Y) \\ = d\lambda(X)g(A\xi, Y) + \lambda g((\nabla_X A)\xi, Y) + \lambda g(\phi AX, AY). \end{aligned} \tag{4.9}$$

From (1.7) and (4.9) we have

$$\begin{aligned} &\eta(X)g(A\xi, \phi Y) - \eta(Y)g(A\xi, \phi X) - 2\alpha g(\phi X, Y) \\ &\quad + g((\nabla_X A)\xi, AY) - g((\nabla_Y A)\xi, AX) + g(\phi AX, A^2 Y) - g(\phi AY, A^2 X) \\ &= d\lambda(X)g(A\xi, Y) - d\lambda(Y)g(A\xi, X) + \lambda g((\nabla_X A)\xi, Y) \\ &\quad - \lambda g((\nabla_Y A)\xi, X) + 2\lambda g(\phi AX, AY) \end{aligned}$$

for any vector fields X and Y on M. We put $X = U$ and making use of (1.7), (4.6) and (4.8), then we have

$$g((\nabla_U A)\xi, AY) = 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y). \tag{4.10}$$

Thus, from (4.8) and (4.10) we have

$$\begin{aligned} 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y) \\ = -g(\phi U, AY) - \alpha_1 g(A^2\xi, AY) + \alpha_2 g(A\xi, AY) + d\alpha_1(\phi A^2 Y). \end{aligned} \tag{4.11}$$

Putting $Y = \xi$ in (4.11), then together with (4.1) we get

$$\alpha_1 d\lambda(U) - \lambda d\alpha_1(U) = 2(\alpha_2 - \alpha_1^2).$$

Further we put $Y = A\xi$ in (4.11), then we get

$$\lambda\{\alpha_1 d\lambda(U) - \lambda d\alpha_1(U)\} = (\alpha_2 - \alpha_1^2)(3\alpha_1 - \lambda).$$

Thus, we have $\alpha_2 - \alpha_1^2 = \alpha_1(\lambda - \alpha_1) = 0$, from which using Lemma 1 we see that $A\xi = \alpha_1\xi$ on M. From (4.4) and Lemma 1, it follows that

$$\alpha_1(\phi A - A\phi)X = 0.$$

Since α_1 is constant, by a similar way as in the proof of Theorem 4 and using Theorem 1, we have our assertions. (Q.E.D.)

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