ANNIHILATOR CHARACTERIZATIONS OF DISTRIBUTIVITY, MODULARITY AND SEMIMODULARITY

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The concept of geodetic annihilators was introduced in [3]. This concept is based on the graph theoretic properties of the Hasse diagram of a finite lattice. The result of [3, Thm. 3] shows that every geodetic annihilator in a finite semimodular lattice is an intersection of prime geodetic annihilators. This and appropriate additional conditions of geodetic annihilators characterize semi-modularity, modularity and distributivity in finite lattices. The graphs of these lattices are also characterized.

1. Characterizations of finite lattices

Mandelker introduced in [2] the concept of an annihilator in lattices: an annihilator $\langle a, b \rangle$ of a with respect to b is the set $\{x \mid x \wedge a \leq b\}$. The dual annihilator $\langle a, b \rangle_d$ is the set $\{y \mid y \vee a \geq b\}$. *We shall consider in this paper finite lattices only.*

Let L be a lattice. We denote the undirected Hasse diagram graph of a lattice L by G_L and call it briefly the graph of the lattice L. The distance $d(a, b)$ between two elements (vertices) a and b in a graph is the length of the shortest $a - b$ path. In graph theory, a shortest path is frequently called a geodesic. We call a set $\langle a, b \rangle_g$ of a lattice L, $a, b \in L$, a geodetic annihilator, briefly a g-annihilator, if $\langle a, b \rangle_g = \{x \mid b$ is on an $x - a$ geodesic in $G_L\}$. A set $B \subset L$ is order convex, if for any two elements $b, c \in B$ with $b \leq c$ every element x satisfying the relation $b \leq x \leq c$ belongs to B. A set B of vertices in a graph is distance convex, if for any two vertices $b, c \in B$ every vertex on any $b - c$ geodesic belongs to B. We first briefly recall some results proved in [3], which are necessary for obtaining results of this note.

Lemma 1 ([3, Thm. 1]). *In a finite distributive lattice L the equality* $\langle a,b \rangle_g = \langle a,b \rangle \cap \langle a,b \rangle_d$ *holds for every pair* $a,b \in L$.

Mandelker proved that a lattice is distributive if and only if the annihilator $\langle a,b \rangle$ is an ideal for any two elements a and b [2, Thm. 1]. The dual of this result holds also, and so every g-annihilator in a distributive lattice is, as the intersection of an ideal and a dual ideal, an order convex sublattice. As known, an order convex sublattice of a distributive lattice is also distance convex.

As usual, we say that an element a covers an element b, in symbols $a \succ b$, if $a > b$ and if $a \geq c \geq b$ implies $a = c$ or $b = c$. A g-annihilator $\langle a,b \rangle_g$ is called prime, if $\langle a,b \rangle_g = L$ or

$$\langle a,b \rangle_g \cap \langle b,a \rangle_g = \varnothing \quad \text{and} \quad \langle a,b \rangle_g \cup \langle b,a \rangle_g = L.$$

As one can easily show, (see e.g. the proof of Theorem 2 in [3]), the relation $\langle a,b \rangle_g \cap \langle b,a \rangle_g = \varnothing$ holds in every graph (and thus in every lattice, too) when $a \neq b$. Thus the condition $\langle a,b \rangle_g \cap \langle b,a \rangle_g = \varnothing$ can be replaced by the condition $a \neq b$.

A g-annihilator $\langle a,b \rangle_g$ with $a \neq b$ is prime only if $a \succ b$ or $b \succ a$. Indeed, if neither $a \succ b$ nor $b \succ a$ holds, there is an $a - b$ geodesic containing an element $x \neq a,b$, and by the definition of a g-annihilator, x belongs neither to $\langle a,b \rangle_g$ nor to $\langle b,a \rangle_g$. Then $\langle a,b \rangle_g \cup \langle b,a \rangle_g \neq L$, and thus $\langle a,b \rangle_g$ cannot be prime. It has been proved in [3, Thm. 3]: "In a finite lattice L satisfying the Jordan-Hölder condition, every g-annihilator is an intersection of prime g-annihilators". This result implies the following lemma.

Lemma 2. *In a finite semimodular lattice L, every g-annihilator $\langle a,b \rangle_g$ is an intersection of prime g-annihilators.*

Now we can prove the first characterization.

Theorem 3. *A finite lattice L is semimodular if and only if the following conditions hold*

   (i) *every g-annihilator in L is an intersection of prime g-annihilators;*

   (ii) *if $b \prec a$, then $b \vee x \in \langle a,b \rangle_g$ for every $x \in \langle a,b \rangle_g$.*

Proof. Assume first that L is semimodular. The property (i) follows from Lemma 2, and so it remains to prove (ii). Let $b \prec a$ and $x \in \langle a,b \rangle_g$. We prove the assertion by induction on the distance $d(x,b)$. If $d(x,b) = 0$ or 1, then either $x = x \vee b$ or $b = x \vee b$, whence an $x - b$ geodesic goes over $x \vee b$, and thus $x \vee b \in \langle a,b \rangle_g$. Assume now that one $y - b$ geodesic goes over $y \vee b$ for all

elements $y \in \langle a,b \rangle_g$ with $d(y,b) < n$, and let $d(x,b) = n$. Let $x, y_1, y_2, \ldots,$ y_{n-1}, b, a be an $x - a$ geodesic, where, let us say, $y_k = y_1 \vee b$ and $y_1 \prec y_2 \prec, \ldots, \prec y_k = y_1 \vee b$. If $x \prec y_1$, then $x \vee b = y_1 \vee b$, and $x \vee b \in \langle a,b \rangle_g$. If $x \succ y_1$, then $x \vee y_2 \succ x, y_2$, and by the same reason we obtain: $x \prec x \vee y_2 \prec x \vee y_3 \prec \cdots \prec x \vee y_k \succ y_k$. Now $x \vee y_k = x \vee y_1 \vee b = x \vee b$, and the $x - y_1 - y_2 - \cdots - y_k$ geodesic has the same length as the $x - x \vee y_2 - x \vee y_3 - \cdots - x \vee y_k - y_k$ geodesic. Hence $x \vee b \in \langle a,b \rangle_g$, and (ii) follows.

Assume conversely that the properties (i) and (ii) hold in L. We shall prove the semimodularity of L by induction on the length of the longest $0 - 1$ chain in L. If the longest $0 - 1$ chain is of length one, then L is clearly semimodular. Assume that L is semimodular if the lenght of the longest $0 - 1$ chain is less than n, and let L be a lattice where the length of this longest chain is n. Assume that $e, f \succ e \wedge f$. The intersection property of (i) goes down to the order convex sublattices of L as well as the property (ii). Hence, if $e \wedge f \neq 0$ or $e \vee f \neq 1$, the length of the longest $0 - 1$ chain in $[e \wedge f]$ (or in $[e \vee f]$) is less than n, and the relation $e \vee f \succ e, f$ follows from the induction assumption. Thus we assume that $0 = e \wedge f \prec e, f$ and $e \vee f = 1$. We may also assume that an $x - y$ geodesic with x from $0 - e - 1$ chain and y from $0 - f - 1$ chain always goes either over the element 1 or the element 0. Choose now two elements a_0 and b_0 as follows: $a_0 \succ b_0 = e$. If $a_0 = 1$, choose $a_0 \succ b_0 = f$, and if also in this case $a_0 = 1$, the relation $e, f \prec e \vee f$ follows. So we may assume that $1 > a_0 \succ b_0 = e$. Now clearly $f \in \langle a_0, b_0 \rangle_g$, and by (ii), $f \vee b_0 = 1 \in \langle a_0, b_0 \rangle_g$. If there is not a $1 - a_0$ geodesic over b_0, we have a contradiction, and if there is a $1 - a_0$ geodesic over b_0, choose new elements a_1 and b_1 as follows: $a_1 \succ b_1 = a_0 \succ b_0 = e \succ 0$. By continuing the choosing process, we certainly obtain a contradiction (at least in the case $1 = a_j \succ b_j = a_{j-1} \succ b_{j-1} \succ b_{j-2} \succ \cdots \succ b_0 = e$), where $1 \in \langle a_j, b_j \rangle_g$ by the condition (ii) and $1 \in \langle b_j, a_j \rangle_g$ by the definition of a g-annihilator. Thus this case is absurd, and the semimodularity of L follows.

By dualizing the condition (ii), a characterization of the dual semimodularity is obtained. A finite lattice is modular if and only if it is semimodular and dually semimodular. Thus Theorem 3 and its dual imply the next characterization.

THEOREM 4.   *A finite lattice L is modular if and only if the following conditions hold*

   (i) *every g-annihilator in L is an intersection of prime g-annihilators;*
   (ii) *if $b \prec a$ then $b \vee x$, $b \wedge x \in \langle a,b \rangle_g$ for every $x \in \langle a,b \rangle_g$.*

Finally we characterize the distributivity.

Theorem 5.   *A finite lattice L is distributive if and only if the following conditions hold*

(i)   *every g-annihilator in L is an intersection of prime g-annihilators;*

(ii)   *every g-annihilator* $\langle a,b\rangle_g$ *is an order convex sublattice of L*

Proof.   Let L be distributive. The property (i) follows from Lemma 2, and Lemma 1 and the considerations immediately after it prove the validity of (ii).

Assume conversely that (i) and (ii) hold for L. By Theorem 4, L is at least modular, and as one can prove, if a finite lattice is modular and nondistributive, it contains the least modular and nondistributive lattice, the diamond M_3, as a sublattice. Let $M_3 = \{a,b,c,d,e\}$ with three noncomparable elements b,c and d and with the order relations: $a < b < e$, $a < c < e$ and $a < d < e$. Because of the distance properties of a finite modular lattice, $\langle b,e\rangle_g \supset \{e,c,d\}$, and $a = c \wedge d \notin \langle b,e\rangle_g$, which contradicts (ii). Thus L cannot contain M_3 as a sublattice, and the distributivity of L follows.

As the proof above shows, the condition can be reduced to the form "every g-annihilator $\langle a,b\rangle_g$ is a sublattice of L".

2.   The graphs of finite lattices

The definitions of a g-annihilator and a prime g-annihilator do not depend on lattice operations, and so we can speak about g-annihilators and prime g-annihilators in finite undirected graphs, too.

A median m for vertices x,y and z is a vertex satisfying the equations $d(x,y) = d(x,m) + d(m,y)$, $d(x,z) = d(x,m) + d(m,z)$ and $d(y,z) = d(y,m) + d(m,z)$. As well known [1], the median of any three elements x, y and z exists, is unique and is equal to $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ in a finite distributive lattice. In a finite modular lattice a median exists for all three elements but it need not be unique. In a dually semimodular finite lattice there are triples having no median but the median $m(x,y,0)$ of x,y and 0 exists for all x and y, it is unique and equal to $x \wedge y$. Dually, in a finite semimodular lattice the unique median $m(x,y,1)$ exists for all pairs x and y and is equal to $x \vee y$. Now we can present our first characterization for graphs.

Theorem 6.   *A finite undirected graph G is isomorphic to the graph G_L of a dually semimodular finite lattice L if and only if the following conditions hold*

(i)   *every g-annihilator in G is an intersection of prime g-annihilators;*

(ii)   *there are two vertices p and q in G such that for no vertex $x \neq p,q$ the vertex p lies on an $x - q$ geodesic and the vertex q lies on an $x - p$ geodesic;*

(iii) *a unique median* $m(x, y, p)$ *exists for all pairs* x *and* y *of vertices in* G, *and if* $x, y \in \langle p, z \rangle_g$, *then also* $(x, y, p) \in \langle p, z \rangle_g$;

(iv) *if* a *and* b *are adjacent and* $b \in \langle p, a \rangle_g$, *then the median* $m(x, b, p)$ *belongs to* $\langle a, b \rangle_g$ *for all* $x \in \langle a, b \rangle_g$.

PROOF. Assume that G satisfies the properties (i)–(iv). We order the vertices of G as follows: $x \leq y \Leftrightarrow \langle p, x \rangle_g \supset \langle p, y \rangle_g$. According to the set theoretical inclusion, the order defined above is a partial order with p as the least element. The definition implies that $x \leq y \Leftrightarrow y \in \langle p, x \rangle_g$. Assume that two vertices x and y of G have a common lower bound b, which implies that $x, y \in \langle p, b \rangle_g$. Now (iii) implies that $m(x, y, p) \in \langle p, b \rangle_g$, and thus $b \leq m(x, y, p)$. Clearly, $m(x, y, p) \leq x, y$. Because G is finite, the considerations above imply that $m(x, y, p)$ is the unique maximal lower bound for x and y, and so the order defined above is a meetsemilattice order. The condition (ii) and the definition of the order relation imply that $x \leq q$ for all vertices x of G, whence q is the greatest element, and so the vertices of G constitute a lattice L. The condition (iv) says that if a covers b, then $x \wedge b \in \langle a, b \rangle_g$ for all $x \in \langle a, b \rangle_g$. This, (i) and the dual of Theorem 3 imply that L is dually semimodular.

If x and y are adjacent in G, then there is a median $m(x, y, p)$ which lies on an $x - y$ geodesic, and thus either $x = m(x, y, p) = x \wedge y$ or $y = m(x, y, p) = x \wedge y$. Assume that $x > y = x \wedge y$. The element x covers y, because otherwise there is a vertex z such that one $x - y - p$ geodesic is also and $x - z - y - p$ geodesic, which is absurd when x and y are adjacent. So we can conclude that x covers y or y covers x if and only if x and y are adjacent in G. Therefore the graph G and the graph of the lattice L are isomorphic, and the first part of the theorem follows.

The converse proof is obtained easily by putting $p = 0$, $q = 1$, and by applying the dual of Theorem 3. This completes the proof.

Theorem 6, its dual and Theorem 4 imply the following characterization of the graphs of finite modular lattices.

THEOREM 7. *A finite undirected graph is isomorphic to the graph* G_L *of a finite modular lattice if and only if the following conditions hold*

(i) *every g-annihilator in* G *is an intersection of prime g-annihilators*;

(ii) *there are two vertices* p *and* q *in* G *such that for no vertex* $x \neq p, q$, *the vertex* p *lies on an* $x - q$ *geodesic and the vertex* q *lies on an* $x - p$ *geodesic*;

(iii) *unique medians* $m(x, y, p)$ *and* $m(x, y, q)$ *exist for all pairs* x *and* y *of vertices in* G, *and if* $x, y \in \langle p, z \rangle_g$, *then also* $m(x, y, p) \in \langle p, z \rangle_g$;

(iv) *if a and b are adjacent and* $b \in \langle p, a \rangle_g$, *then the medians* $m(x, b, p)$ *and* $m(x, b, q)$ *belong to* $\langle a, b \rangle_g$ *for all* $x \in \langle a, b \rangle_g$.

In a finite distributive lattice L, every g-annihilator is a distance convex set, and this property implies that $x \wedge b$, $x \vee b \in \langle a, b \rangle_g$ for all elements $x \in \langle a, b \rangle_g$. On the other hand, a modular lattice, where every g-annihilator is a distance convex set, is distributive. Hence we can substitute the condition (iv) of Theorem 7 by the condition of distance convexity and obtain the following characterization for the graphs of finite distributive lattices.

THEOREM 8. *A finite undirected graph* G *is isomorphic to the graph* G_L *of a finite distributive lattice* L *if and only if the following conditions hold*

(i) *every g-annihilator in* G *is an intersection of prime g-annihilators*;

(ii) *there are two vertices* p *and* q *in* G *such that for no vertex* $x \neq p, q$, *the vertex* p *lies on an* $x - q$ *geodesic and the vertex* q *lies on an* $x - p$ *geodesic*;

(iii) *unique medians* $m(x, y, p)$ *and* $m(x, y, q)$ *exist for all pairs* x *and* y *of vertices in* G;

(iv) *every g-annihilator* $\langle a, b \rangle_g$ *in* G *is a distance convex set.*

References

[1]   Bandelt, H. J. and Hedlikova, J., Median algebras, Discrete Math. **45** (1983), 1–30.
[2]   Mandelker, M., Relative annihilators in lattices, Duke Math. J. **40** (1970), 377–386.
[3]   Nieminen, J., The Jordan-Hölder chain condition and annihilators in finite lattices. Tsukuba J. Math. **14** (1990), 405–411.

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