TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 589-609

BRAIDING STRUCTURES OF DOUBLE CROSSPRODUCTS

By

Hui-Xiang CHEN

Introduction

The double crossproduct structure $X \bowtie A$ of two bialgebras (Hopf algebras) X and A is given by Majid [Mj], and Drinfel'd quantum double $D(H)$ is such a double crossproduct. Doi and Takeuchi [DT] studied the double crossproducts determined by a skew pairing. These results have some interesting application in quantum group theory. The concept of a braided bialgebra has been introduced by Larson-Towber [LTo] and Hayashi [H]. The author [C] studied the quasi-triangular structures of bicrossed coproducts. In this paper, we discuss the dual case, and study the braiding structures of double crossproducts. We also discuss the relations between the comodule categories attached to double crossproduct, and construct several (braided) monoidal functors.

1. Preliminaries

Throughout, we work over a fixed field k. Unless otherwise stated, all maps are k-linear. $\mathrm{Hom}(H,k)$ is denoted by H^*. For $f,g \in H^*$, H a bialgebra, $f * g$ is its convolution product [S]. For $\sigma \in (H \otimes H)^*$, we write $\sigma(x \otimes y) = \sigma(x,y)$, $x, y \in H$. We use the sigma notation: for $x \in H$,

$$\Delta(x) = \sum x_1 \otimes x_2, \quad (\Delta \otimes id)\Delta(x) = \sum x_1 \otimes x_2 \otimes x_3, \quad etc.$$

Let (X,A) be a pair of *matched bialgebras* (Hopf algebras), see [K], that is, X is a left A-module coalgebra via $a \rightharpoonup x$, A is a right X-module coalgebra via $a \leftharpoonup x$, such that the following conditions are satisfied:

(M1) $a \rightharpoonup (xy) = \sum (a_1 \rightharpoonup x_1)((a_2 \leftharpoonup x_2) \rightharpoonup y)$,
(M2) $a \rightharpoonup 1 = \varepsilon(a)1$,
(M3) $(ab) \leftharpoonup x = \sum (a \leftharpoonup (b_1 \rightharpoonup x_1))(b_2 \leftharpoonup x_2)$,
(M4) $1 \leftharpoonup x = \varepsilon(x)1$,
(M5) $\sum (a_1 \leftharpoonup x_1) \otimes (a_2 \rightharpoonup x_2) = \sum (a_2 \leftharpoonup x_2) \otimes (a_1 \rightharpoonup x_1)$,

Received May 29, 1997.

for all $a, b \in A$, $x, y \in X$. In this case, Majid's double crossproduct $X \bowtie A$ is defined (see [K, Mj or R]): as coalgebra $X \bowtie A = X \otimes A$, write $x \bowtie a$ for $x \otimes a$, and multiplication is defined by

$$(x \bowtie a)(y \bowtie b) = \sum x(a_1 \rightharpoonup y_1) \bowtie (a_2 \leftharpoonup y_2)b, \quad a, b \in A, x, y \in X.$$

If X and A are Hopf algebras with antipodes S_X and S_A resp., then $X \bowtie A$ is also a Hopf algebra with antipode S given by

$$\begin{aligned} S(x \bowtie a) &= (1 \bowtie S_A(a))(S_X(x) \bowtie 1) \\ &= \sum (S_A(a_2) \rightharpoonup S_X(x_2)) \bowtie (S_A(a_1) \leftharpoonup S_X(x_1)). \end{aligned}$$

A bilinear form $\tau : X \otimes A \to k$ is a *skew pairing* [DT] if the following conditions are satisfied:

(SP1) $\tau(xy, a) = \sum \tau(x, a_1)\tau(y, a_2)$,
(SP2) $\tau(x, ab) = \sum \tau(x_1, b)\tau(x_2, a)$,
(SP3) $\tau(x, 1) = \varepsilon(x)$,
(SP4) $\tau(1, a) = \varepsilon(a)$,

for all $x, y \in X$ and $a, b \in A$. If $\tau : X \otimes A \to k$ is an invertible (in $(X \otimes A)^*$) skew pairing, set

$$a \rightharpoonup x = \sum \tau(x_1, a_1)x_2\tau^{-1}(x_3, a_2),$$

$$a \leftharpoonup x = \sum \tau(x_1, a_1)a_2\tau^{-1}(x_2, a_3),$$

then (X, A) is matched. In this case, the double crossproduct $X \bowtie A$ is denoted by $X \bowtie_\tau A$, and the multiplication is given by

$$(x \bowtie a)(y \bowtie b) = \sum \tau(y_1, a_1)xy_2 \bowtie a_2 b\tau^{-1}(y_3, a_3).$$

In general, let σ be a 2-*cocycle* on a bialgebra H, that is, σ is an invertible bilinear form on H and the following condition is satisfied:

$$\sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2), \quad x, y, z \in H,$$

then the bialgebra H^σ (see [D]) is constructed from H by altering the multiplication as follows and using the same unit, comultiplication, and counit:

$$x \cdot y(\text{in} H^\sigma) = \sum \sigma(x_1, y_1)x_2 y_2 \sigma^{-1}(x_3, y_3), \quad x, y \in H.$$

If τ is an invertible skew pairing on $X \otimes A$, then $[\tau]$ is a 2-cocycle on the tensor product bialgebra $X \otimes A$ defined by [DT]:

$$[\tau](x \otimes a, y \otimes b) = \varepsilon(x)\varepsilon(b)\tau(y, a), \quad x, y \in X, a, b \in A,$$

and $X \bowtie_\tau A = (X \otimes A)^{[\tau]}$.

A bilinear form $\tau \in (X \otimes A)^*$ is a *paring* if τ satisfies (SP1), (SP3), (SP4) and the following condition (P2):

(P2) $\tau(x, ab) = \sum \tau(x_1, a)\tau(x_2, b)$.

A *braided* (or *co-quasitriangular*) *bialgebra* means a pair (H, σ) where H is a bialgebra and σ is an invertible skew pairing on $H \otimes H$ such that

$$\sum \sigma(x_1, y_1)x_2 y_2 = \sum y_1 x_1 \sigma(x_2, y_2)$$

for all $x, y \in H$. In this case, σ is called a *braiding* of H.

Let $X \bowtie A$ be a double crossproduct, then $i_X : X \to X \bowtie A$, $x \mapsto x \bowtie 1$ and $i_A : A \to X \bowtie A$, $a \mapsto 1 \bowtie a$ are injective bialgebra morphism.

PROPOSITION 1.1. *Let $\alpha : X \to H$, $\beta : A \to H$ be bialgebra maps such that*

$$\beta(a)\alpha(x) = \sum \alpha(a_1 \rightharpoonup x_1)\beta(a_2 \leftharpoonup x_2), \quad x \in X, a \in A,$$

then there exists a unique bialgebra map $F : X \bowtie A \to H$ such that $F \circ i_X = \alpha$ and $F \circ i_A = \beta$.

PROOF. Set $F : X \bowtie A \to H$, $F(x \bowtie a) = \alpha(x)\beta(a)$, i.e. $F = m_H(\alpha \otimes \beta) : X \bowtie A \xrightarrow{\alpha \otimes \beta} H \otimes H \xrightarrow{m_H} H$, where m_H is the multiplication map of H which is a coalgebra map. Since both α and β are coalgebra maps, so is F. Clearly, $F(1 \bowtie 1) = 1$.

$$\begin{aligned} F((x \bowtie a)(y \bowtie b)) &= \sum F(x(a_1 \rightharpoonup y_1) \bowtie (a_2 \leftharpoonup y_2)b) \\ &= \sum \alpha(x(a_1 \rightharpoonup y_1))\beta((a_2 \leftharpoonup y_2)b) \\ &= \sum \alpha(x)\alpha(a_1 \rightharpoonup y_1)\beta(a_2 \leftharpoonup y_2)\beta(b) \\ &= \alpha(x)\beta(a)\alpha(y)\beta(b) = F(x \bowtie a)F(y \bowtie b), \end{aligned}$$

henceby, F is a bialgebra map. Clearly $F \circ i_X = \alpha$ and $F \circ i_A = \beta$. Since $X \bowtie A$ is generated by $i_X(X) = X \bowtie 1$ and $i_A(A) = 1 \bowtie A$ as an algebra, F is unique.

PROPOSITION 1.2. *Let $\alpha : X \to H$, $\beta : A \to H$ be bialgebra maps, then the map*

$$F : X \bowtie A \to H, \quad F(x \bowtie a) = \alpha(x)\beta(a)$$

is a bialgebra map if and only if

$$\beta(a)\alpha(x) = \sum \alpha(a_1 \rightharpoonup x_1)\beta(a_2 \leftharpoonup x_2), \quad \textit{for all } x \in X, a \in A.$$

PROOF. Easy.

PROPOSITION 1.3. *Let $X \bowtie A$ be a double crossproduct. If $X \bowtie A$ has a braiding, then so do X and A, and there is an invertible skew pairing τ on $X \otimes A$ such that*

$$a \rightharpoonup x = \sum \tau(x_1, a_1)x_2\tau^{-1}(x_3, a_2),$$

$$a \leftharpoonup x = \sum \tau(x_1, a_1)a_2\tau^{-1}(x_2, a_3),$$

and consequently, $X \bowtie A = X \bowtie_\tau A$.

PROOF. Suppose $\tilde{\sigma}$ is a braiding of the bialgebra $X \bowtie A$, i.e. $\tilde{\sigma}$ is an invertible skew pairing on $(X \bowtie A) \otimes (X \bowtie A)$ and

$$\sum \tilde{\sigma}(x_1 \bowtie a_1, y_1 \bowtie b_1)(x_2 \bowtie a_2)(y_2 \bowtie b_2)$$
$$= \sum (y_1 \bowtie b_1)(x_1 \bowtie a_1)\tilde{\sigma}(x_2 \bowtie a_2, y_2 \bowtie b_2) \qquad (*)$$

Set $\sigma(x, y) = \tilde{\sigma}(x \bowtie 1, y \bowtie 1)$, $\eta(a, b) = \tilde{\sigma}(1 \bowtie a, 1 \bowtie b)$, $\tau(x, b) = \tilde{\sigma}(x \bowtie 1, 1 \bowtie b)$, $x, y \in X$, $a, b \in A$, then by using injective bialgebra maps $i_X : X \to X \bowtie A$ and $i_A : A \to X \bowtie A$, one can easily check that (X, σ) and (A, η) are braided bialgebras, and that τ is an invertible skew pairing on $X \otimes A$. By $(*)$, we have

$$\sum \tilde{\sigma}(x_1 \bowtie 1, 1 \bowtie b_1)(x_2 \bowtie 1)(1 \bowtie b_2)$$
$$= \sum (1 \bowtie b_1)(x_1 \bowtie 1)\tilde{\sigma}(x_2 \bowtie 1, 1 \bowtie b_2),$$

i.e., $\sum \tau(x_1, b_1)(x_2 \bowtie b_2) = \sum (1 \bowtie b_1)(x_1 \bowtie 1)\tau(x_2, b_2)$. It follows that

$$\sum (b_1 \rightharpoonup x_1) \bowtie (b_2 \leftharpoonup x_2) = \sum \tau(x_1, b_1)(x_2 \bowtie b_2)\tau^{-1}(x_3, b_3).$$

By applying $id \otimes \varepsilon$ and $\varepsilon \otimes id$ to the both sides of the equation respectively, we get

$$b \rightharpoonup x = \sum \tau(x_1, b_1)x_2\tau^{-1}(x_3, b_2)$$

and

$$b \leftharpoonup x = \sum \tau(x_1, b_1) b_2 \tau^{-1}(x_2, b_3).$$

REMARK 1.4. Let A be a Hopf algebra with bijective antipode S, then the finite dual A° of A is also a Hopf algebra with bijective antipode S°, and $(A^\circ)^{cop}$ is a Hopf algebra with antipode $s = (S^\circ)^{-1}$. Set

$$\sigma : (A^\circ)^{cop} \otimes A \to k, \quad f \otimes a \mapsto \langle f, a \rangle,$$

then a straightforward verification shows that σ is an invertible skew pairing on $(A^\circ)^{cop} \otimes A$, and the double crossproduct $(A^\circ)^{cop} \bowtie_\sigma A$ is a Hopf algebra. If A is a finite dimensional Hopf algebra, then $(A^\circ)^{cop} = (A^*)^{cop}$ and $(A^*)^{cop} \bowtie_\sigma A = D(A)$ the Drinfel'd double (see [DT, remark 2.3]).

PROPOSITION 1.5. *Let A be a Hopf algebra with bijective antipode, X be a bialgebra and τ be an invertible skew pairing on $X \otimes A$. Then the map*

$$F : X \bowtie_\tau A \to (A^\circ)^{cop} \bowtie_\sigma A, \quad x \bowtie a \mapsto \tau(x, -) \bowtie a$$

is a bialgebra map, where $(A^\circ)^{cop} \bowtie_\sigma A$ is defined as in remark 1.4. And F is injective if and only if τ is left non-degenerate, i.e. $\tau(x, A) = 0$ implies $x = 0$.

PROOF. By [DT, p. 5719], the maps

$$\tau_L : X \to (A^\circ)^{cop}, \quad x \mapsto \tau(x, -)$$

is a bialgebra map. By the commutative diagram

$$\begin{array}{ccc} X \otimes A & \xrightarrow{\tau_L \otimes id} & (A^\circ)^{cop} \otimes A \\ & {\scriptstyle \tau} \searrow \quad \swarrow {\scriptstyle \sigma} & \\ & k, & \end{array}$$

the proposition follows immediately.

PROPOSITION 1.6. *Let X and A be finite dimensional Hopf algebra, τ be an invertible skew pairing on $X \otimes A$. Then*

(1) *If τ is non-degenerate, then $X \bowtie_\tau A$ and $D(A)$ are isomorphic as Hopf algebras.*

(2) *Set $I = \ker \tau_R$, where $\tau_R : A \to (X^*)^{op}$, $a \mapsto \tau(-, a)$ is a Hopf algebra map* ([DT, p. 5719]), *then there exists a Hopf algebra surjection from $X \bowtie_\tau A$ to Drinfel'd double $D(A/I)$.*

PROOF. By [DT, p. 5719], the maps

$$\tau_L X \to (A^*)^{cop}, \quad x \mapsto \tau(x, -),$$

and

$$\tau_R : A \to (X^*)^{op}, \quad a \mapsto \tau(-, a)$$

are bialgebra maps, consequently, τ_L and τ_R are Hopf algebra maps.

(1) If τ is non-degenerate, then τ_L and τ_R are injective, henceby $\dim X \leq \dim A^* = \dim A$ and $\dim A \leq \dim X^* = \dim X$. It follows that τ_L is bijective. Thus by proposition 1.5,

$$F : X \bowtie_\tau A \to D(A) = (A^*)^{cop} \bowtie_\sigma A, \quad x \bowtie a \mapsto \tau_L(x, -) \bowtie a$$

is a Hopf algebra isomorphism.

(2) Set $J = ker\tau_L$, then J is a Hopf ideal of X, I is a Hopf ideal of A. Let $\pi_X : X \to \bar{X} = X/J$, $x \mapsto \bar{x}$ and $\pi_A : A \to \bar{A} = A/I$, $a \mapsto \bar{a}$ be the natural Hopf projection, then

$$\pi = \pi_X \otimes \pi_A : X \otimes A \to \bar{X} \otimes \bar{A}, \quad x \otimes a \mapsto \bar{x} \otimes \bar{a}$$

is a Hopf algebra surjection. Since $\ker \pi = J \otimes A + X \otimes I \subseteq \ker \tau$, there exists a uinque bilinear form $\bar{\tau}$ on $\bar{X} \otimes \bar{A}$ such that $\bar{\tau}(\bar{x}, \bar{a}) = \tau(x, a)$ for all $x \in X, a \in A$. It is clear that $\bar{\tau}$ is an invertible skew pairing on $\bar{X} \otimes \bar{A}$, and that $\pi : X \bowtie_\tau A \to \bar{X} \bowtie_{\bar{\tau}} \bar{A}$, $x \bowtie a \mapsto \bar{x} \bowtie \bar{a}$ is a Hopf algebra surjection. By (1) $\bar{X} \bowtie_{\bar{\tau}} \bar{A} \cong D(\bar{A})$, it follows that

$$X \bowtie_\tau A \to D(A/I), \quad x \bowtie a \mapsto \bar{\tau}(\bar{x}, -) \bowtie \bar{a}$$

is a Hopf algebra surjection.

2. Braidings of $X \bowtie_\tau A$

Throughout this section, unless otherwise stated, let τ be a fixed invertible skew pairing on $X \otimes A$, $X \bowtie_\tau A$ be the double crossproduct determined by τ. Then $X \bowtie_\tau A = (X \otimes A)^{[\tau]}$ as in section 1.

For an algebra B, let $Z(B)$ denote the center of B, $U(B)$ denote the group of units in B, and $UZ(B) = U(Z(B))$.

LEMMA 2.1. *Let v be a bilinear form on $X \otimes A$, then the following three conditions are equivalent*:

(1) v *is central, i.e.* $v \in Z((X \otimes A)^*)$,
(2) $\sum v(x_1, a_1)x_2 \otimes a_2 = \sum x_1 \otimes a_1 v(x_2, a_2)$, $x \in X, a \in A$,
(3) $\sum v(x_1, a)x_2 = \sum v(x_2, a)x_1$ *and* $\sum v(x, a_1)a_2 = \sum v(x, a_2)a_1$, $x \in X, a \in A$.

PROOF. Easy.

Let $\mathcal{G}(X, A)$ denote the set of all central invertible pairing on $X \otimes A$, that is

$$\mathcal{G}(X, A) = \{v \in UZ((X \otimes A)^*) \mid v \text{ is a pairing}\}.$$

LEMMA 2.2. (1) *If* $v \in \mathcal{G}(X, A)$, *then* v *is also a skew pairing.*
(2) $\mathcal{G}(X, A)$ *is an abelian multiplicative subgroup of* $(X \otimes A)^*$.

PROOF. Easy.

LEMMA 2.3. *Let* τ' *be a bilinear form on* $X \otimes A$. *Then* τ' *is an invertible skew pairing on* $X \otimes A$ *such that* $X \bowtie_{\tau'} A = X \bowtie_{\tau} A$ *if and only if* $\tau' = \tau * v$ *for some* $v \in \mathcal{G}(X, A)$. *In particular,* τ' *is an invertible skew pairing on* $X \otimes A$ *such that* $X \bowtie_{\tau'} A = X \otimes A$ *if and only if* $\tau' \in \mathcal{G}(X, A)$.

PROOF. If τ' is an invertible skew pairing on $X \otimes A$ such that $X \bowtie_{\tau'} A = X \bowtie_{\tau} A$, then

$$\begin{aligned}(1 \bowtie a)(x \bowtie 1) &= \sum \tau(x_1, a_1)x_2 \bowtie a_2\tau^{-1}(x_3, a_3) \\ &= \sum \tau'(x_1, a_1)x_2 \bowtie a_2\tau'^{-1}(x_3, a_3).\end{aligned}$$

Set $v = \tau^{-1} * \tau'$, then

$$\sum v(x_1, a_1)x_2 \otimes a_2 = \sum x_1 \otimes a_1 v(x_2, a_2),$$

and hence τ is central by lemma 2.1. Clearly, v is invertible, and so $v \in UZ((X \otimes A)^*)$. Now we will check that v is a pairing. In fact, for any $x, y \in X$, $a, b \in A$,

$$\begin{aligned}v(xy, a) &= \sum \tau^{-1}(x_1y_1, a_1)\tau'(x_2y_2, a_2) \\ &= \sum \tau^{-1}(y_1, a_1)\tau^{-1}(x_1, a_2)\tau'(x_2, a_3)\tau'(y_2, a_4) \quad \text{(By [DT, SP1'], SP1)} \\ &= \sum \tau^{-1}(y_1, a_1)v(x, a_2)\tau'(y_2, a_3) \\ &= \sum v(x, a_1)\tau^{-1}(y_1, a_2)\tau'(y_2, a_3) \quad \text{(By lemma 2.1)} \\ &= \sum v(x, a_1)v(y, a_2),\end{aligned}$$

and similarly, $v(x,ab) = \sum v(x_1,a)v(x_2,b)$. Clearly, $v(x,1) = \varepsilon(x)$ and $v(1,a) = \varepsilon(a)$. Thereby, $v \in \mathscr{G}(X,A)$ and $\tau' = \tau * v$.

Conversely, if $\tau' = \tau * v$ for some $v \in \mathscr{G}(X,A)$, then τ' is invertible. For any $x, y \in X$, $a, b \in A$,

$$\begin{aligned}\tau'(xy,a) &= \sum \tau(x_1y_1,a_1)v(x_2y_2,a_2)\\ &= \sum \tau(x_1,a_1)\tau(y_1,a_2)v(x_2,a_3)v(y_2,a_4)\\ &= \sum \tau(x_1,a_1)v(x_2,a_2)\tau(y_1,a_3)v(y_2,a_4)\\ &= \sum \tau'(x,a_1)\tau'(y,a_2),\end{aligned}$$

and similarly, $\tau'(x,ab) = \sum \tau'(x_1,b)\tau'(x_2,a)$, and clearly, $\tau'(x,1) = \varepsilon(x)$ and $\tau'(1,a) = \varepsilon(a)$. Thus we have proved that τ' is an invertible skew pairing on $X \otimes A$.

By $\tau'^{-1} = v^{-1} * \tau^{-1}$ and lemma 2.1, we have

$$\sum v(y_1,a_1)y_2 \otimes a_2 v^{-1}(y_3,a_3) = y \otimes a,$$

and therefore,

$$\begin{aligned}&\sum \tau'(y_1,a_1)xy_2 \otimes a_2 b\tau'^{-1}(y_3,a_3)\\ &\quad = \sum \tau(y_1,a_1)v(y_2,a_2)xy_3 \otimes a_3 bv^{-1}(y_4,a_4)\tau^{-1}(y_5,a_5)\\ &\quad = \sum \tau(y_1,a_1)xy_2 \otimes a_2 b\tau^{-1}(y_3,a_3).\end{aligned}$$

This shows that $X \bowtie_{\tau'} A = X \bowtie_{\tau} A$.

PROPOSITION 2.4. *Let (X,σ) and (A,η) be braided bialgebras. Define a bilinear form $[\sigma,\eta]$ on $(X \bowtie_\tau A) \otimes (X \bowtie_\tau A)$ by*

$$[\sigma,\eta](x \bowtie a, y \bowtie b) = \sum \tau(x_1,b_1)\sigma(x_2,y_1)\eta(a_1,b_2)\tau^{-1}(y_2,a_2).$$

Then $(X \bowtie_\tau A, [\sigma,\eta])$ is a braided bialgebra.

PROOF. Since (X,σ) and (A,η) are braided bialgebras, $(X \otimes A, \sigma \otimes \eta)$ is also braided, where

$$(\sigma \otimes \eta)(x \otimes a, y \otimes b) = \sigma(x,y)\eta(a,b), \quad x, y \in X, a, b \in A.$$

It follows by [T, lemma 1.3] that $(X \bowtie_\tau A, (\sigma \otimes \eta)^{[\tau]})$ is braided. By a simple computation, one get $(\sigma \otimes \eta)^{[\tau]} = [\sigma,\eta]$, and so the proposition follows.

COROLLARY 2.5. *Let $X \bowtie A$ be a double crossproduct. Then $X \bowtie A$ has a braiding if and only if both X and A have braiding, and there is an invertible skew pairing τ on $X \otimes A$ such that $X \bowtie A = X \bowtie_\tau A$.*

PROOF. It follows by proposition 1.3 and proposition 2.4.

COROLLARY 2.6. (1) *$X \bowtie_\tau A$ has a braiding if and only if X and A have braiding.*

(2) *For $\sigma \in (X \bowtie X)^*$, $\eta \in (A \otimes A)^*$ with $\sigma(1,1) = 1 = \eta(1,1)$, define $[\sigma, \eta] \in ((X \bowtie_\tau A) \otimes (X \bowtie_\tau A))^*$ as in proposition 2.4. then $(X \bowtie_\tau A, [\sigma, \eta])$ is a braided bialgebra if and only if (X, σ) and (A, η) are braided bialgebras.*

PROOF. It follows from proposition 1.3 and proposition 2.4.

COROLLERY 2.7. *Let H be a finite dimensional Hopf algebra. If H and H^* have braiding, then Drinfel'd double $D(H)$ also has a braiding.*

PROPOSITION 2.8. *$\mathcal{G}(X \bowtie_\tau A, X \bowtie_\tau A)$ is isomorphic to $\mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$ as a group. In particular, $\mathcal{G}(X \otimes A, X \otimes A) = \mathcal{G}(X \bowtie_\tau A, X \bowtie_\tau A)$.*

PROOF. For any $(\nu, \sigma, \eta, \nu') \in \mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$, define $\Phi(\nu, \sigma, \eta, \nu')$ by

$$\Phi(\nu, \sigma, \eta, \nu')(x \bowtie a, y \bowtie b) = \sum \nu(x_1, b_1)\sigma(x_2, y_1)\eta(a_1, b_2)\nu'(y_2, a_2).$$

Then a straightforward verification shows that Φ is a group morphism from $\mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$ to $\mathcal{G}(X \bowtie_\tau A, X \bowtie_\tau A)$. On the other hand, for any $\bar{\sigma} \in \mathcal{G}(X \bowtie_\tau A, X \bowtie_\tau A)$, set

$$\nu(\bar{\sigma})(x, b) = \bar{\sigma}(x \bowtie 1, 1 \bowtie b),$$

$$\sigma(\bar{\sigma})(x, y) = \bar{\sigma}(x \bowtie 1, y \bowtie 1),$$

$$\eta(\bar{\sigma})(a, b) = \bar{\sigma}(1 \bowtie a, 1 \bowtie b),$$

$$\nu'(\bar{\sigma})(y, a) = \bar{\sigma}(1 \bowtie a, y \bowtie 1),$$

for all $x, y \in X$, $a, b \in A$. It is clear that $(\nu(\bar{\sigma}), \sigma(\bar{\sigma}), \eta(\bar{\sigma}), \nu'(\bar{\sigma}))$ is contained in $\mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$. Henceby, the map

$$\theta : \mathcal{G}(X \bowtie_\tau A, X \bowtie_\tau A) \to \mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$$

given by $\theta(\bar{\sigma}) = (\nu(\bar{\sigma}), \sigma(\bar{\sigma}), \eta(\bar{\sigma}), \nu'(\bar{\sigma}))$ is well-defined. One can easily check that $\theta \circ \Phi = id$ and $\Phi \circ \theta = id$, and so Φ is a group isomorphism.

Let $\tau = \varepsilon \otimes \varepsilon$ in the above argument, we get the same isomorphism Φ from $\mathcal{G}(X, A) \times \mathcal{G}(X, X) \times \mathcal{G}(A, A) \times \mathcal{G}(X, A)$ to $\mathcal{G}(X \otimes A, X \otimes A)$.

LEMMA 2.9. *If (H, σ) is a braided bialgebra, $\nu \in \mathcal{G}(H, H)$, then $(H, \sigma * \nu)$ is also braided.*

PROOF. By lemma 2.3, $\sigma * \nu$ is an invertible skew pairing on $H \otimes H$ and $H \bowtie_{\sigma * \nu} H = H \bowtie_{\sigma} H$. Thus it follows from [DT, prop. 3.1] that $(H, \sigma * \nu)$ is a braided bialgebra.

THEOREM 2.10. *Let (X, σ) and (A, η) be braided bialgebras, ν and ν' be in $\mathcal{G}(X, A)$. Define a bilinear form $[\sigma, \eta, \nu, \nu']$ on $(X \bowtie_{\tau} A) \otimes (X \bowtie_{\tau} A)$ by*

$$[\sigma, \eta, \nu, \nu'](x \bowtie a, y \bowtie b)$$
$$= \sum \tau(x_1, b_1)\sigma(x_2, y_1)\eta(a_1, b_2)\tau^{-1}(y_2, a_2)\nu(x_3, b_3)\nu'(y_3, a_3).$$

Then $(X \bowtie_{\tau} A, [\sigma, \eta, \nu, \nu'])$ is a braided bialgebra and any braiding of $X \bowtie_{\tau} A$ has this form.

PROOF. By proposition 2.4, $(X \bowtie_{\tau} A, [\sigma, \eta])$ is a braided bialgebra. By proposition 2.8, $\Phi(\nu, 1, 1, \nu') \in \mathcal{G}(X \bowtie_{\tau} A, X \bowtie_{\tau} A)$, and

$$\Phi(\nu, 1, 1, \nu')(x \bowtie a, y \bowtie b) = \nu(x, b)\nu'(y, a).$$

Since $[\sigma, \eta, \nu, \nu'] = [\sigma, \eta] * \Phi(\nu, 1, 1, \nu')$, it follows from lemma 2.9 that $(X \bowtie_{\tau} A, [\sigma, \eta, \nu, \nu'])$ is a braided bialgebra.

Conversely, suppose that $(X \bowtie_{\tau} A, \tilde{\sigma})$ is a braided bialgebra. By [T, lemma 1.3], $\tilde{\sigma} = \bar{\sigma}^{[\tau]}$ for some braiding $\bar{\sigma}$ on $X \otimes A$. Set

$$\sigma(x, y) = \bar{\sigma}(x \otimes 1, y \otimes 1),$$
$$\eta(a, b) = \bar{\sigma}(1 \otimes a, 1 \otimes b),$$
$$\nu(x, b) = \bar{\sigma}(x \otimes 1, 1 \otimes b),$$
$$\nu'(y, a) = \bar{\sigma}(1 \otimes a, y \otimes 1),$$

for all $x, y \in X$, $a, b \in A$. Then (X, σ) and (A, η) are braided bialgebras. One can easily check that $\nu, \nu' \in \mathcal{G}(X, A)$.

Since $(X \otimes A, \bar{\sigma})$ is a braided bialgebra,

$$\begin{aligned}\bar{\sigma}(x \otimes a, y \otimes b) &= \bar{\sigma}((x \otimes 1)(1 \otimes a), y \otimes b) \\ &= \sum \bar{\sigma}(x \otimes 1, y_1 \otimes b_1)\bar{\sigma}(1 \otimes a, y_2 \otimes b_2) \\ &= \sum \bar{\sigma}(x_1 \otimes 1, 1 \otimes b_1)\bar{\sigma}(x_2 \otimes 1, y_1 \otimes 1) \\ &\quad \times \bar{\sigma}(1 \otimes a_1, 1 \otimes b_2)\bar{\sigma}(1 \otimes a_2, y_2 \otimes 1) \\ &= \sum \nu(x_1, b_1)\sigma(x_2, y_1)\eta(a_1, b_2)\nu'(y_2, a_2) \\ &= \sum \sigma(x_1, y_1)\eta(a_1, b_1)\nu(x_2, b_2)\nu'(y_2, a_2).\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\sigma}(x \bowtie a, y \bowtie b) &= \bar{\sigma}^{[\tau]}(x \bowtie a, y \bowtie b) \\ &= \sum [\tau](y_1 \otimes b_1, x_1 \otimes a_1)\bar{\sigma}(x_2 \otimes a_2, y_2 \otimes b_2)[\tau]^{-1}(x_3 \otimes a_3, y_3 \otimes b_3) \\ &= \sum \tau(x_1, b_1)\sigma(x_2, y_1)\eta(a_1, b_2)\nu(x_3, b_3)\nu'(y_2, a_2)\tau^{-1}(y_3, a_3) \\ &= [\sigma, \eta, \nu, \nu'](x \bowtie a, y \bowtie b),\end{aligned}$$

and therefore $\tilde{\sigma} = [\sigma, \eta, \nu, \nu']$.

For a bialgebra H, let $\mathscr{B}(H)$ denote the set of all braidings of H.

PROPOSITION 2.11. *There exists a bijective correspondence between* $\mathscr{B}(X \bowtie_\tau A)$ *and the set direct product* $\mathscr{B}(X) \times \mathscr{B}(A) \times \mathscr{G}(X, A) \times \mathscr{G}(X, A)$ *given by*

$$\mathscr{B}(X) \times \mathscr{B}(A) \times \mathscr{G}(X, A) \times \mathscr{G}(X, A) \to \mathscr{B}(X \bowtie_\tau A)$$

$$(\sigma, \eta, \nu, \nu') \mapsto [\sigma, \eta, \nu, \nu'],$$

where $[\sigma, \eta, \nu, \nu']$ *is defined as in theorem* 2.10.

PROOF. By theorem 2.10, the above map is surjective. One can easily check that if $[\sigma, \eta, \nu, \nu'] = [\sigma_1, \eta_1, \nu_1, \nu_1']$ then $\sigma = \sigma_1, \eta = \eta_1, \nu = \nu_1, \nu' = \nu_1'$. It follows that the map given in the proposition is bijective.

THEOREM 2.12. *Let* σ *be an invertible skew pairing on* $X \otimes X$, $[\sigma] = [\sigma, \sigma]$ *be a bilinear form on* $(X \bowtie_\sigma X) \otimes (X \bowtie_\sigma X)$ *defined as in proposition* 2.4, *that is*

$$[\sigma](x \bowtie a, y \bowtie b) = \sum \sigma(x_1, b_1)\sigma(x_2, y_1)\sigma(a_1, b_2)\sigma^{-1}(y_2, a_2), \quad a, b, x, y \in X.$$

Then the following three conditions are equivalent:

(1) (X, σ) *is a co-triangular bialgebra.*

(2) $(X \bowtie_\sigma X, [\sigma])$ *is a co-triangular bialgebra.*

(3) *The multiplication*

$$X \bowtie_\sigma X \to X, \quad x \bowtie a \mapsto xa$$

is a bialgebra morphism, and $[\sigma](x \bowtie a, y \bowtie b) = \sigma(xa, yb)$.

PROOF. (1) $\Leftrightarrow$ (2). If (X, σ) is co-triangular, then $(X \bowtie_\sigma X, [\sigma])$ is a braided bialgebra by proposition 2.4, and $\sigma^{-1}(x, y) = \sigma(y, x)$. Henceby,

$$\begin{aligned}
&[\sigma]^{-1}(x \bowtie a, y \bowtie b) \\
&\quad = \sum \sigma(y_1, a_1)\sigma^{-1}(a_2, b_1)\sigma^{-1}(x_1, y_2)\sigma^{-1}(x_2, b_2) \\
&\quad = \sum \sigma(y_1, a_1)\sigma(b_1, a_2)\sigma(y_2, x_1)\sigma^{-1}(x_2, b_2) \\
&\quad = [\sigma](y \bowtie b, x \bowtie a),
\end{aligned}$$

it follows that $(X \bowtie_\sigma X, [\sigma])$ is a co-triangular bialgebra.

Conversely, suppose $(X \bowtie_\sigma X, [\sigma])$ is a co-triangular bialgebra. Since $X \to X \bowtie_\sigma X, x \mapsto x \bowtie 1$ is a bialgebra injection and $\sigma(x, y) = [\sigma](x \bowtie 1, y \bowtie 1)$, (X, σ) is also a co-triangular bialgebra.

(1) $\Leftrightarrow$ (3). By [DT, prop. 3.1], (X, σ) is a braided bialgebra if and only if the multiplication

$$X \bowtie_\sigma X \to X, \quad x \bowtie a \mapsto xa$$

is a bialgebra map. In this case, since

$$\begin{aligned}
\sigma(xa, yb) &= \sum \sigma(x, y_1 b_1)\sigma(a, y_2 b_2) \\
&= \sum \sigma(x_1, b_1)\sigma(x_2, y_1)\sigma(a_1, b_2)\sigma(a_2, y_2),
\end{aligned}$$

it follows that $[\sigma](x \bowtie a, y \bowtie b) = \sigma(xa, yb)$ holds for all $x, y, a, b \in X$ if and only if $\sigma^{-1}(x, y) = \sigma(y, x)$ holds for all $x, y \in X$.

3. Monoidal Functors Attached to $X \bowtie_\tau A$

Throughout this section, let $H = X \bowtie_\tau A$ be a double crossproduct determined by a fixed invertible skew pairing τ on $X \otimes A$.

For any coalgebra C, let $\boldsymbol{M}^C$ denote the category of right C-comodules. For $M \in \boldsymbol{M}^C$, $\rho_M : M \to M \otimes C$ denotes the comodule structure map, and write

$\rho_M(m) = \sum m_0 \otimes m_1$ as usual. If $f : C \to D$ is a coalgebra morphism and $M \in \boldsymbol{M}^C$, then M is a right D-comodule with the comodule structure map given by $(id \otimes f)\rho_M$. If C is a bialgebra, then $\boldsymbol{M}^C$ is a monoidal category (or tensor category) with the usual tensor product, the unit k and the usual associativity and unit constraints. Furthermore, if (C, σ) is a braided bialgebra, then $\boldsymbol{M}^C$ is a braided monoidal category with the braiding given by

$$t_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto \sum \sigma(m_1, n_1) n_0 \otimes m_0,$$

where $M, N \in \boldsymbol{M}^C$, $m \in M$ and $n \in N$. The reader is directed to [K or M] for detailes of these concepts, basic results, and (braided) monoidal functors also.

Now we have three monoidal categories $\boldsymbol{M}^X$, $\boldsymbol{M}^A$ and $\boldsymbol{M}^H$. Since $\pi_X : H \to X$, $x \bowtie a \mapsto x\varepsilon(a)$ and $\pi_A : H \to A$, $x \bowtie a \mapsto \varepsilon(x)a$ are coalgebra morphisms, any $M \in \boldsymbol{M}^H$ becomes a right X-comodule and also a right A-comodule as before, denote them by M_X and M_A respectively, that is, $\rho_{M_X} = (id \otimes \pi_X)\rho_M$ and $\rho_{M_A} = (id \otimes \pi_A)\rho_M$.

Define $F_X : \boldsymbol{M}^H \to \boldsymbol{M}^X$ and $F_A : \boldsymbol{M}^H \to \boldsymbol{M}^A$ by

$$F_X(M) = M_X, F_X(f) = f, \quad \text{and} \quad F_A(M) = M_A, F_A(f) = f,$$

for all $M \in Ob(\boldsymbol{M}^H)$, $f \in Hom(\boldsymbol{M}^H)$, then F_X is a functor from $\boldsymbol{M}^H$ to $\boldsymbol{M}^X$, and F_A is a functor from M^H to $\boldsymbol{M}^A$.

LEMMA 3.1. *For any $h, z \in H$, we have:*
(1) $\sum \pi_X(h_1 z_1)[\tau](h_2, z_2) = \sum [\tau](h_1, z_1)\pi_X(h_2)\pi_X(z_2)$.
(2) $\sum \pi_A(h_1 z_1)[\tau](h_2, z_2) = \sum [\tau](h_1, z_1)\pi_A(h_2)\pi_A(z_2)$.

PROOF. It follows by a straightforward verification.

LEMMA 3.2. *Let M, N be right H-comodules, then*

(1) $$\phi_X(M, N) : F_X(M) \otimes F_X(N) \to F_X(M \otimes N), \quad m \otimes n \mapsto \sum m_0 \otimes n_0[\tau](m_1, n_1)$$

is a right X-comodule isomorphism.

(2) $$\phi_A(M, N) : F_A(M) \otimes F_A(N) \to F_A(M \otimes N), \quad m \otimes n \mapsto \sum m_0 \otimes n_0[\tau](m_1, n_1)$$

is a right A-comodule isomorphism.

PROOF. (1) It is clear that ϕ_X is a bijective linear map. Note that $F_X(M) \otimes F_X(N) = M_X \otimes N_X$, $F_X(M \otimes N) = (M \otimes N)_X$, and their comodule structures are different,

$$\rho_{M_X \otimes N_X}(m \otimes n) = \sum m_0 \otimes n_0 \otimes \pi_X(m_1)\pi_X(n_1)$$

and

$$\rho_{(M\otimes N)_X}(m \otimes n) = \sum m_0 \otimes n_0 \otimes \pi_X(m_1 n_1)$$

for all $m \in M$, $n \in N$. In order to show that $\phi_X = \phi_X(M,N)$ is a right X-comodule morphism, we need to prove $\rho_{(M\otimes N)_X}\phi_X = (\phi_X \otimes id_X)\rho_{M_X\otimes N_X}$. In fact, let $m \in M$, $n \in N$, then

$$\begin{aligned}
&\rho_{(M\otimes N)_X}\phi_X(m \otimes n) \\
&\quad = \rho_{(M\otimes N)_X}\Big(\sum m_0 \otimes n_0[\tau](m_1, n_1)\Big) \\
&\quad = \sum m_0 \otimes n_0 \otimes \pi_X(m_1 n_1)[\tau](m_2, n_2),
\end{aligned}$$

and

$$\begin{aligned}
&(\phi_X \otimes id_X)\rho_{M_X\otimes N_X}(m \otimes n) \\
&\quad = (\phi_X \otimes id_X)\Big(\sum m_0 \otimes n_0 \otimes \pi_X(m_1)\pi_X(n_1)\Big) \\
&\quad = \sum m_0 \otimes n_0 \otimes [\tau](m_1, n_1)\pi_X(m_2)\pi_X(n_2).
\end{aligned}$$

It follows by lemma 3.1 (1) that $\rho_{(M\otimes N)_X}\phi_X(m \otimes n) = (\phi_X \otimes id_X)\rho_{M_X\otimes N_X}(m \otimes n)$, and so $\rho_{(M\otimes N)_X}\phi_X = (\phi_X \otimes id_X)\rho_{M_X\otimes N_X}$.

(2) It follows by a similar verification by lemma 3.1 (2). Note that $\phi_X(M,N) = \phi_A(M,N)$ as linear maps.

THEOREM 3.3. (1) *The triple (F_X, id_k, ϕ_X) is a monoidal functor from the monoidal category $\boldsymbol{M}^H$ to $\boldsymbol{M}^X$.*

(2) *The triple (F_A, id_k, ϕ_A) is a monoidal functor from the monoidal category $\boldsymbol{M}^H$ to $\boldsymbol{M}^A$.*

PROOF. (1) By lemma 3.2, $\phi_X(M,N)$ is an isomorphism in $\boldsymbol{M}^X$, henceby one can easily check that $\{\phi_X(M,N)\}$ is a family of natural isomorphism indexed by all couples (M,N) of objects of $\boldsymbol{M}^H$. By the definition of a monoidal functor (see [K, p. 287]), we need to show that the following three diagrams commute:

$$\begin{array}{ccc}
(M_X \otimes N_X) \otimes W_X & \xrightarrow{a} & M_X \otimes (N_X \otimes W_X) \\
\downarrow{\scriptstyle \phi_X(M,N) \otimes id_{W_X}} & & \downarrow{\scriptstyle id_{M_X} \otimes \phi_X(N,W)} \\
(M \otimes N)_X \otimes W_X & & M_X \otimes (N \otimes W)_X \\
\downarrow{\scriptstyle \phi_X(M \otimes N, W)} & & \downarrow{\scriptstyle \phi_X(M, N \otimes W)} \\
((M \otimes N) \otimes W)_X & \xrightarrow{F_X(a)=a} & (M \otimes (N \otimes W))_X
\end{array} \tag{3.1}$$

$$\begin{array}{ccc}
k \otimes M_X & \xrightarrow{l_{M_X}} & M_X \\
\downarrow{\scriptstyle id_k \otimes id_{M_X}} & & \uparrow{\scriptstyle F_X(l_M)=l_M} \\
k \otimes M_X & \xrightarrow{\phi_X(k,M)} & (k \otimes M)_X
\end{array} \tag{3.2}$$

and

$$\begin{array}{ccc}
M_X \otimes k & \xrightarrow{r_{M_X}} & M_X \\
\downarrow{\scriptstyle id_{M_X} \otimes id} & & \uparrow{\scriptstyle F_X(r_M)=r_M} \\
M_X \otimes k & \xrightarrow{\phi_X(M,k)} & (M \otimes k)_X
\end{array} \tag{3.3}$$

for all $M, N, W \in \boldsymbol{M}^H$, where a is the canonical associativity constraint, l and r are natural isomorphisms. By [DT, lemma 1.4], we know $\phi_X(k, M) = id$ and $\phi_X(M, k) = id$. So the commutativities of diagrams (3.2) and (3.3) follow immediately.

As to (3.1), let $m \in M$, $n \in N$, $w \in W$, then

$$\begin{aligned}
&\phi_X(M, N \otimes W)(id \otimes \phi_X(N, W))a((m \otimes n) \otimes w) \\
&\quad = \phi_X(M, N \otimes W)(id \otimes \phi_X(N, W))(m \otimes (n \otimes w)) \\
&\quad = \phi_X(M, N \otimes W)\Big(\sum m \otimes (n_0 \otimes w_0)[\tau](n_1, w_1)\Big) \\
&\quad = \sum m_0 \otimes (n_0 \otimes w_0)[\tau](m_1, n_1 w_1)[\tau](n_2, w_2),
\end{aligned}$$

and

$$\begin{aligned}
&a\phi_X(M \otimes N, W)(\phi_X(M, N) \otimes id)((m \otimes n) \otimes w) \\
&\quad = a\phi_X(M \otimes N, W)\Big(\sum (m_0 \otimes n_0)[\tau](m_1, n_1) \otimes w\Big) \\
&\quad = \sum m_0 \otimes (n_0 \otimes w_0)[\tau](m_1 n_1, w_1)[\tau](m_2, n_2).
\end{aligned}$$

Since $[\tau]$ is a 2-cocycle on $X \otimes A$ and $X \bowtie_\tau A = (X \otimes A)^{[\tau]}$, $[\tau]^{-1}$ is a 2-cocycle on $X \bowtie_\tau A$. By $[\tau] = ([\tau]^{-1})^{-1}$, it follows from [D, theorem 1.6 (a1)] that the diagram (3.1) is commutative.

(2) Since $\phi_A(M,N) = \Phi_X(M,N)$ as linear maps, (2) follows immediately from (1).

For any monoidal categories $(\mathscr{C}, \otimes, I, a, l, r)$ and $(\mathscr{C}', \otimes', I', a', l', r')$, the product category $\mathscr{C} \times \mathscr{C}'$ is also a monoidal category with the tensor product, unit, associativity and unit constraints as follows:

$$(U,U') \otimes (V,V') = (U \otimes V, U' \otimes' V'),$$
$$(f,f') \otimes (g,g') = (f \otimes g, f' \otimes' g'),$$
$$a = (a,a'), \quad I = (I,I'), \quad l = (l,l') \quad \text{and} \quad r = (r,r').$$

If $(F,\phi_0,\phi_2)((F',\phi_0',\phi_2'))$ is a monoidal functor from monoidal category $\mathscr{C}(\mathscr{C}')$ to $\mathscr{D}(\mathscr{D}')$, then $((F,F'),(\phi_0,\phi_0'),(\phi_2,\phi_2'))$ is a monoidal functor from $\mathscr{C} \times \mathscr{C}'$ to $\mathscr{D} \times \mathscr{D}'$. Thus by theorme 3.3 we get a monoidal functor $((F_X,F_A),(id,id),(\phi_X,\phi_A))$ from $\boldsymbol{M}^H \times \boldsymbol{M}^H$ to $\boldsymbol{M}^X \times \boldsymbol{M}^A$. On the other hand, the diagonal functor $D : \boldsymbol{M}^H \to \boldsymbol{M}^H \times \boldsymbol{M}^H$, where $D(M) = (M,M)$ and $D(f) = (f,f)$, is a strict monoidal functor. By composing these two functors we get:

PROPOSITION 3.4. *Let* $F : \boldsymbol{M}^H \to \boldsymbol{M}^X \times \boldsymbol{M}^A, F(M) = (M_X, M_A)$, $F(f) = (f,f)$, $\phi_0 = (id,id) : (k,k) \to F(k) = (k,k)$,

$$\phi_2(M,N) = (\phi_X(M,N), \phi_A(M,N)) : (M_X \otimes N_X, M_A \otimes N_A) \to ((M \otimes N)_X, (M \otimes N)_A),$$

where $M,N \in Ob(\boldsymbol{M}^H)$, $f \in Hom(\boldsymbol{M}^H)$. *Then* (F,ϕ_0,ϕ_2) *is a monoidal functor from* $\boldsymbol{M}^H$ *to* $\boldsymbol{M}^X \times \boldsymbol{M}^A$.

For $U \in \boldsymbol{M}^X$, $V \in \boldsymbol{M}^A$, define

$$\rho_{U \otimes V} : U \otimes V \to (U \otimes V) \otimes H, \quad u \otimes v \mapsto \sum (u_0 \otimes v_0) \otimes (u_1 \bowtie v_1),$$

then $U \otimes V$ is a right H-comodule, i.e., $U \otimes V \in \boldsymbol{M}^H$. If $f : U \to U'$ and $g : V \to V'$ are morphisms in $\boldsymbol{M}^H$ and $\boldsymbol{M}^A$ resp., then $f \otimes g : U \otimes V \to U' \otimes V'$ is a morphism in $\boldsymbol{M}^H$. Thus we get a functor

$$G : \boldsymbol{M}^X \times \boldsymbol{M}^A \to \boldsymbol{M}^H$$
$$G(U,V) = U \otimes V, G(f,g) = f \otimes g,$$

where $(U,V) \in Ob(\boldsymbol{M}^X \times \boldsymbol{M}^A)$, $(f,g) \in Hom(\boldsymbol{M}^X \times \boldsymbol{M}^A)$.

LEMMA 3.5. *For any objects* $(U,V),(U',V')$ *of* $\boldsymbol{M}^X \times \boldsymbol{M}^A$, *define*

$$\psi_2((U,V),(U',V')) : G(U,V) \otimes G(U',V') \to G((U,V) \otimes (U',V'))$$
$$(u \otimes v) \otimes (u' \otimes v') \mapsto \sum (u \otimes u'_0) \otimes (v_0 \otimes v')\tau^{-1}(u'_1, v_1).$$

Then $\{\psi_2((U,V),(U',V'))\}$ *is a family of natural isomorphisms indexed by all couples* $((U,V),(U',V'))$ *of objects of* $\boldsymbol{M}^X \times \boldsymbol{M}^A$.

PROOF. Write $\psi_2 = \psi_2((U,V),(U',V'))$, $\rho_l = \rho_{G(U,V)\otimes G(U',V,)}$ and $\rho_r = \rho_{G((U,V)\otimes(U',V'))}$ resp., then by definition,

$$G(U,V) \otimes G(U',V') = (U \otimes V) \otimes (U' \otimes V'),$$
$$G((U,V) \otimes (U',V')) = G(U \otimes U', V \otimes V') = (U \otimes U') \otimes (V \otimes V'),$$
$$\rho_l((u \otimes v) \otimes (u' \otimes v')) = \sum (u_0 \otimes v_0) \otimes (u'_0 \otimes v'_0) \otimes (u_1 \bowtie v_1)(u'_1 \bowtie v'_1)$$

and

$$\rho_r((u \otimes u') \otimes (u \otimes v')) = \sum (u_0 \otimes u'_0) \otimes (v_0 \otimes v'_0) \otimes (u_1 u'_1 \bowtie v_1 v'_1)$$

for all $u \in U$, $u' \in U'$, $v \in V$, $v' \in V'$. Therefore,

$$\rho_r\psi_2((u \otimes v) \otimes (u' \otimes v'))$$
$$= \rho_r\Big(\sum (u \otimes u'_0) \otimes (v_0 \otimes v')\tau^{-1}(u'_1, v_1)\Big)$$
$$= \sum (u_0 \otimes u'_0) \otimes (v_0 \otimes v'_0) \otimes (u_1 u'_1 \bowtie v_1 v'_1)\tau^{-1}(u'_2, v_2),$$

and

$$(\psi_2 \otimes id_H)\rho_l((u \otimes v) \otimes (u' \otimes v'))$$
$$= (\psi_2 \otimes id_H)\Big(\sum (u_0 \otimes v_0) \otimes (u'_0 \otimes v'_0) \otimes (u_1 \bowtie v_1)(u'_1 \bowtie v'_1)\Big)$$
$$= \sum (u_0 \otimes u'_0) \otimes (v_0 \otimes v'_0)\tau^{-1}(u'_1, v_1) \otimes (u_1 \bowtie v_2)(u'_2 \bowtie v'_1)$$
$$= \sum (u_0 \otimes u'_0) \otimes (v_0 \otimes v'_0)\tau^{-1}(u'_1, v_1) \otimes \tau(u'_2, v_2)(u_1 u'_3 \bowtie v_3 v'_1)\tau^{-1}(u'_4, v_4)$$
$$= \sum (u_0 \otimes u'_0) \otimes (v_0 \otimes v'_0) \otimes (u_1 u'_1 \bowtie v_1 v'_1)\tau^{-1}(u'_2, v_2),$$

and so $\rho_r\psi_2 = (\psi_2 \otimes id_H)\rho_l$, that is, ψ_2 is a morphism in $\boldsymbol{M}^H$. The rest is obvious.

THEOREM 3.6. (G, ψ_0, ψ_2) is a monoidal functor from the monoidal category $\boldsymbol{M}^X \times \boldsymbol{M}^A$ to $\boldsymbol{M}^H$, where G and ψ_2 are given above, $\psi_0 : k \to G(k,k) = k \otimes k$ is the natural isomorphism.

PROOF. We need to show that the following three diagrams commute:

$$\begin{array}{ccc}
(G(U,V)\otimes G(U',V'))\otimes G(U'',V'') & \xrightarrow{a} & G(U,V)\otimes(G(U',V')\otimes G(U'',V'')) \\
\downarrow{\scriptstyle \psi_2((U,V),(U',V'))\otimes id_{G(U'',V'')}} & & \downarrow{\scriptstyle id_{G(U,V)}\otimes\psi_2((U',V'),(U'',V''))} \\
G((U,V)\otimes G(U',V'))\otimes G(U'',V'') & & G(U,V)\otimes G((U',V')\otimes(U'',V'')) \\
\downarrow{\scriptstyle \psi_2((U,V)\otimes(U',V'),(U'',V''))} & & \downarrow{\scriptstyle \psi_2((U,V),(U',V')\otimes(U'',V''))} \\
G(((U,V)\otimes(U',V'))\otimes(U'',V'')) & \xrightarrow{G(a)} & G((U,V)\otimes((U',V')\otimes(U'',V')))
\end{array} \tag{3.4}$$

$$\begin{array}{ccc}
k\otimes G(U,V) & \xrightarrow{l_{G(U,V)}} & G(U,V) \\
\downarrow{\scriptstyle \psi_0\otimes id_{G(U,V)}} & & \uparrow{\scriptstyle G(l_{(U,V)})} \\
G(k,k)\otimes G(U,V) & \xrightarrow{\psi_2((k,k),(U,V))} & G((k,k)\otimes(U,V))
\end{array} \tag{3.5}$$

$$\begin{array}{ccc}
G(U,V)\otimes k & \xrightarrow{r_{G(U,V)}} & G(U,V) \\
\downarrow{\scriptstyle id_{G(U,V)}\otimes\psi_0} & & \uparrow{\scriptstyle G(r_{(U,V)})} \\
G(U,V)\otimes G(k,k) & \xrightarrow{\psi_2((U,V),(k,k))} & G((U,V)\otimes(k,k)).
\end{array} \tag{3.6}$$

By definition, we rewrite the three diagrams as follows:

$$\begin{array}{ccc}
((U\otimes V)\otimes(U'\otimes V'))\otimes(U''\otimes V'') & \xrightarrow{a} & (U\otimes V)\otimes((U'\otimes V')\otimes(U''\otimes V'')) \\
\downarrow{\scriptstyle \psi_2\otimes id_{U''\otimes V''}} & & \downarrow{\scriptstyle id_{U\otimes V}\otimes\psi_2} \\
((U\otimes U')\otimes(V\otimes V'))\otimes(U''\otimes V'') & & (U\otimes V)\otimes((U'\otimes U'')\otimes(V'\otimes V'')) \\
\downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_2} \\
((U\otimes U')\otimes U'')\otimes((V\otimes V')\otimes V'') & \xrightarrow{a\otimes a} & (U\otimes(U'\otimes U''))\otimes(V\otimes(V'\otimes V''))
\end{array} \tag{3.4}$$

$$\begin{array}{ccc}
k\otimes(U\otimes V) & \xrightarrow{l_{U\otimes V}} & U\otimes V \\
\downarrow{\scriptstyle \psi_0\otimes id_{U\otimes V}} & & \uparrow{\scriptstyle l_U\otimes l_V} \\
(k\otimes k)\otimes(U\otimes V) & \xrightarrow{\psi_2} & (k\otimes U)\otimes(k\otimes V)
\end{array} \tag{3.5}$$

$$\begin{array}{ccc} (U \otimes V) \otimes k & \xrightarrow{r_{U\otimes V}} & U \otimes V \\ \downarrow{\scriptstyle id_{U\otimes V} \otimes \psi_0} & & \uparrow{\scriptstyle r_U \otimes r_V} \\ (U \otimes V) \otimes (k \otimes k) & \xrightarrow{\psi_2} & (U \otimes k) \otimes (V \otimes k). \end{array} \tag{3.6}$$

By [DT, lemma 1.4], one can easily check that (3.5) and (3.6) are commutative.

As to (3.4), let $u \in U$, $v \in V$, $u' \in U'$, $v' \in V'$, $u'' \in U''$ and $v'' \in V''$, then

$$\begin{aligned} &(a \otimes a)\psi_2(\psi_2 \otimes id_{U''\otimes V''})(((u \otimes v) \otimes (u' \otimes v')) \otimes (u'' \otimes v'')) \\ &\quad = (a \otimes a)\psi_2\Big(\sum((u \otimes u'_0) \otimes (v_0 \otimes v')\tau^{-1}(u'_1, v_1)) \otimes (u'' \otimes v'')\Big) \\ &\quad = (a \otimes a)\Big(\sum((u \otimes u'_0) \otimes u''_0) \otimes ((v_0 \otimes v'_0) \otimes v'')\tau^{-1}(u''_1, v_1v'_1)\tau^{-1}(u'_1, v_2)\Big) \\ &\quad = \sum(u \otimes (u'_0 \otimes u''_0)) \otimes (v_0 \otimes (v'_0 \otimes v''))\tau^{-1}(u''_1, v_1v'_1)\tau^{-1}(u'_1, v_2) \\ &\quad = \sum(u \otimes (u'_0 \otimes u''_0)) \otimes (v_0 \otimes (v'_0 \otimes v''))\tau^{-1}(u''_1, v_1)\tau^{-1}(u''_2, v'_1) \\ &\qquad \tau^{-1}(u'_1, v_2), \quad \text{(By [DT, SP2'])} \end{aligned}$$

and

$$\begin{aligned} &\psi_2(id_{U\otimes V} \otimes \psi_2)a(((u \otimes v) \otimes (u' \otimes v')) \otimes (u'' \otimes v'')) \\ &\quad = \psi_2(id_{U\otimes V} \otimes \psi_2)((u \otimes v) \otimes ((u' \otimes v') \otimes (u'' \otimes v''))) \\ &\quad = \psi_2\Big((u \otimes v) \otimes \Big(\sum(u' \otimes u''_0) \otimes (v'_0 \otimes v'')\tau^{-1}(u''_1, v'_1)\Big)\Big) \\ &\quad = \sum(u \otimes (u'_0 \otimes u''_0)) \otimes (v_0 \otimes (v'_0 \otimes v''))\tau^{-1}(u'_1u''_1, v_1)\tau^{-1}(u''_2, v'_1) \\ &\quad = \sum(u \otimes (u'_0 \otimes u''_0)) \otimes (v_0 \otimes (v'_0 \otimes v''))\tau^{-1}(u''_1, v_1)\tau^{-1}(u'_1, v_2) \\ &\qquad \tau^{-1}(u''_2, v'_1). \quad \text{(By [DT, SP1'])} \end{aligned}$$

It follows that the diagram (3.4) commutes. So (G, ψ_0, ψ_2) is a monoidal functor from $\boldsymbol{M}^X \times \boldsymbol{M}^A$ to $\boldsymbol{M}^H$.

Now assume that (X, σ) and (A, η) are braided bialgebras then $(H, [\sigma, \eta])$ is also a braided bialgebra by proposition 2.4. Thus $\boldsymbol{M}^X$, $\boldsymbol{M}^A$ and $\boldsymbol{M}^H$ are braided monoidal categories with braidings given by

$$t_{U,U'} : U \otimes U' \to U' \otimes U, u \otimes u' \mapsto \sum \sigma(u_1, u'_1)u'_0 \otimes u_0,$$

$$t_{V,V'} : V \otimes V' \to V' \otimes V, v \otimes v' \mapsto \sum \eta(v_1, v'_1)v'_0 \otimes v_0,$$

and

$$t_{M,N} : M \otimes N \to N \otimes M, m \otimes n \mapsto \sum [\sigma, \eta](m_1, n_1)n_0 \otimes m_0,$$

respectively, where $U, U' \in \boldsymbol{M}^X$, $V, V' \in \boldsymbol{M}^A$ and $M, N \in \boldsymbol{M}^H$. Henceby, $\boldsymbol{M}^X \times \boldsymbol{M}^A$ is also a braided monoidal category with braiding $t_{(U,V),(U',V')} = (t_{U,U'}, t_{V,V'})$. In this case we have:

THEOREM 3.7. *(G, ψ_0, ψ_2) is a braided monoidal functor from $\boldsymbol{M}^X \times \boldsymbol{M}^A$ to $\boldsymbol{M}^H$.*

PROOF. By definition (see [K, p. 327]), we only need to show the diagram

$$\begin{array}{ccc} G(U,V) \otimes G(U',V') & \xrightarrow{\psi_2((U,V),(U',V'))} & G((U,V) \otimes (U',V')) \\ \Big\downarrow{\scriptstyle t_{G((U,V),G(U',V')}} & & \Big\downarrow{\scriptstyle G(t_{(U,V),(U',V')})} \\ G(U',V') \otimes G(U,V) & \xrightarrow{\psi_2((U',V'),(U,V))} & G((U',V') \otimes (U,V)) \end{array} \tag{3.7}$$

commutes. Let us rewrite (3.7) as follows:

$$\begin{array}{ccc} (U \otimes V) \otimes (U' \otimes V') & \xrightarrow{\psi_2} & (U \otimes U') \otimes (V \otimes V') \\ \Big\downarrow{\scriptstyle t_{U\otimes V, U'\otimes V'}} & & \Big\downarrow{\scriptstyle t_{U,U'} \otimes t_{V,V'}} \\ (U' \otimes V') \otimes (U \otimes V) & \xrightarrow{\psi_2} & (U' \otimes U) \otimes (V' \otimes V). \end{array} \tag{3.7}$$

For any $u \in U$, $v \in V$, $u' \in U'$, $v' \in V'$, we have

$$\begin{aligned} &(t_{U,U'} \otimes t_{V,V'})\psi_2((u \otimes v) \otimes (u' \otimes v')) \\ &\quad = (t_{U,U'} \otimes t_{V,V'})\Big(\sum (u \otimes u'_0) \otimes (v_0 \otimes v')\tau^{-1}(u'_1, v_1)\Big) \\ &\quad = \sum \sigma(u_1, u'_1)(u'_0 \otimes u_0) \otimes \eta(v_1, v'_1)(v'_0 \otimes v_0)\tau^{-1}(u'_2, v_2) \\ &\quad = \sum (u'_0 \otimes u_0) \otimes (v'_0 \otimes v_0)\sigma(u_1, u'_1)\eta(v_1, v'_1)\tau^{-1}(u'_2, v_2), \end{aligned}$$

and

$$\begin{aligned} &\psi_2(t_{U\otimes V, U'\otimes V'}((u \otimes v) \otimes (u' \otimes v')) \\ &\quad = \psi_2\Big(\sum [\sigma, \eta](u_1 \bowtie v_1, u'_1 \bowtie v'_1)(u'_0 \otimes v'_0) \otimes (u_0 \otimes v_0)\Big) \\ &\quad = \sum (u'_0 \otimes u_0) \otimes (v'_0 \otimes v_0)\tau^{-1}(u_1, v'_1)[\sigma, \eta](u_2 \bowtie v_1, u'_1 \bowtie v'_2) \\ &\quad = \sum (u'_0 \otimes u_0) \otimes (v'_0 \otimes v_0)\tau^{-1}(u_1, v'_1)\tau(u_2, v'_2)\sigma(u_3, u'_1)\eta(v_1, v'_3)\tau^{-1}(u'_2, v_2) \\ &\quad = \sum (u'_0 \otimes u_0) \otimes (v'_0 \otimes v_0)\sigma(u_1, u'_1)\eta(v_1, v'_1)\tau^{-1}(u'_2, v_2). \end{aligned}$$

Completing the proof.

ACKNOWLEDGMENT. The author would like to show his gratefulness to Prof. M. Takeuchi for his reading the first version of the paper and useful suggestions, which led the shortened proofs of proposition 2.4, theorem 2.10 and theorem 3.3.

References

[D] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra **21**, 1731–1749 (1993).
[DT] Y. Doi & M. Takeuchi, Multiplication alteration by two-cocycles-the quantum version, Comm. Algebra **22**(14), 5715–5732 (1994).
[H] T. Hayashi, Quantum groups and quantum determinants, J. Algebra **152**, 146–165 (1992).
[K] C. Kassel, Quantum groups, Springer-Verlag, New York (1995).
[LT] R. Larson & J. Towber, Two dual classes of bialgebras related to the concepts of "quantum group" and "quantum Lie algebras", Comm. Algebra **19**, 3295–3345 (1991).
[M] S. Montgomery, Hopf algebras and their actions on rings, CBMS, Series in Math. **82**, AMS (1993).
[Mj] S. Majid, Quasitriangular Hopf algebras and Yang-Baxter equations, Internat. J. Modern Physics **A5**, 1–91 (1990).
[R] D. E. Radford, Minimal quasitriangular Hopf algebras, J. Algebra **157**, 285–315 (1993).
[S] M. E. Sweedler, Hopf Algebras, Benjamin, New York (1969).
[T] M. Takeuchi, Cocycle deformations of coordinate rings of quantum matrices, J. Algebra **189**, 23–33 (1997).
[C] H. X. Chen, Quasitriangular structures of bicrossed coproducts, J. Algebra **204**, 504–531 (1998).

Department of Mathematics
Teacher's College,
Yangzhou University
Yangzhou Jiangsu 225002,
China.

Present address:
Institute of Mathematics
Fudan University
Shanghai 200433, China.