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SINGULAR COMPACTIFICATIONS OF PRODUCT SPACES

By

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Abstract. Assume that both X and Y are non-compact locally compact spaces. Let $\delta(X\times Y)$ be a compactification of $X\times Y$ such that $\delta(X\times Y)\geq\omega X\times\omega Y$, where ωX and ωY are the one-point compactifications of X and Y, respectively. Then J. L. Blasco [2] proved the theorem that $\delta(X\times Y)$ is not a weakly singular compactification of $X\times Y$ if X is pseudcompact. In this paper we give an alternative, simpler proof for the above theorem. Furthermore, in the case X is either a non-separable metrizable space or a separable metrizable space with a non-compact quasi-component space $Q(X)$ and $d(Y)\leq d(X)$, where $d(X)$ is the density of X, for any compact space S we establish a theorem that $X\times Y$ has a singular compactification with S as a remainder if and only if X has a singular compactification with S as a remainder.

1. Introduction

In this paper all topological spaces are locally compact and Hausdorff and all compactifications are Hausdorff. For compactifications αX and γX of X we will write $\alpha X\geq\gamma X$ if there exists a continuous map $f:\alpha X\rightarrow\gamma X$ such that $f\restriction_X$ is an identity on X. If such an f exists which is a homeomorphism we will write $\alpha X\approx\gamma X$ and two compactifications αX and γX are called *equivalent* or αX *is equivalent to* γX. In this paper we will investigate the singular compactifications of product spaces. The concept of singular set of a map was introduced by G. T. Whyburn [23] and [24]. Later it was investigated by G. L. Cain, Jr. [3], [4] and R. F. Dickman, Jr. [13]. Furthermore, [6], [8] and [11] treated singular com-

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pactifications in detail. A compactification αX of X is *singular* (resp. *weakly singular*) if and only if the remainder $\alpha X - X$ is a retract (resp. neighborhood retract) of αX [17]. Note that every singular compactification is weakly singular and not every weakly singular compactification is singular.

The technique of singular compactifications is very important to the theory of Wallman-type compactifications. For example, proving A. K. Steiner and E. F. Steiner's Theorem which is known as a reduction theorem (cf. [22], theorem), we need to construct a singular compactification of a discrete space in order to get a geometrical proof (cf. [7], example 2). Then singular compactifications are interesting ones in their own right.

In 1965, W. W. Comfort [12] asked the question of whether there are two non-empty retractive spaces whose product is also retractive, where a non-compact space X is *retractive* provided that $\beta X - X$ is a retract of the Stone-Čech compactificaiton βX. It is well-known that every retractive space must be locally compact and pseudocompact (cf. [15], theorem 0.1).

Subsequently, W. W. Comfort's question was solved by J. L. Blasco [1]. Let X and Y be non-compact spaces. J. L. Blasco proved that $X \times Y$ is not retractive and then $\beta(X \times Y)$ is not a singular compactification of $X \times Y$ (cf. [1], theorem 1).

Recently, J. L. Blasco extends the above theorem in the following: Let $\delta(X \times Y)$ be a compactification of $X \times Y$ such that $\delta(X \times Y) \geq \omega X \times \omega Y$, where ωX and ωY are the one-point compactifications of X and Y, respectively. If X is pseudocompact, then $\delta(X \times Y)$ is not a weakly singular compactification of $X \times Y$ (cf. [2], corollary 2.4(b)). He uses a certain functional analysis technique to prove this theorem. In section 2, we will give an alternative, simpler proof for the above theorem.

In 1985, T. Kimura [20] gave the necessary and sufficient condition is that a product space $X \times Y$ has an $\aleph_0$-point compactification. Recently, T. Kimura [21] gave the necessary and sufficient conditions on metric spaces X and Y which characterize the product space $X \times Y$ having the set of all compact metric spaces as remainders. This is a partial answer for the problem posed by J. Hatzenbuhler and D. A. Mattson [18]. Here we are interested in the class of singular compactifications. Then considering these aspects, we may ask the following question: Fix a compact space K. Give necessary and sufficient conditions on non-compact spaces X and Y which characterize the product space $X \times Y$ having a singular compactification with K as a remainder. In section 3, in the case X is either a non-separable metrizable space or a separable metrizable space with a non-compact quasi-component space $Q(X)$ and $d(Y) \leq d(X)$, where $d(X)$ is the density of X, for any compact space S we establish a theorem that $X \times Y$ has a singular compactification with S as a remainder if and only if X has a singular compactification with S as a remainder.

For undefined notation and terminology, see [9] or [16].

2. A remark on Blasco's Theorem

In this section we will give an alternative, simpler proof for J. L. Blasco's Theorem [2]. ωX denotes the one-point compactification of a non-compact space X throughout this paper. Let X be a set and κ a cardinal. We will write $[X]^{\kappa}$ for $\{A \subset X : |A| = \kappa\}$. Recall that a space is *pseudocompact* if and only if every sequence of infinitely many non-empty open sets has a cluster point.

At first, we will begin with the following lemma which was proved by G. D. Faulkner [17].

LEMMA 2.1. *Let αX be a compactification of a non-compact space X and γX a compactification of X such that $\gamma X \leq \alpha X$. If αX is singular (resp. weakly singular), then γX is singular (resp. weakly singular).*

In this paper we will write ω_0 for $\{0, 1, \ldots\}$. Now, we will give an alternative, simpler proof for J. L. Blasco's Theorem [2].

THEOREM 2.1. *Let X be a non-compact space, Y a non-compact space and $\delta(X \times Y)$ a compactification of $X \times Y$ with $\delta(X \times Y) \geq \omega X \times \omega Y$. If X is pseudocompact, then $\delta(X \times Y)$ is not a weakly singular compactification of $X \times Y$.*

PROOF. From Lemma 2.1 it is sufficient to show that $\omega X \times \omega Y$ is not a weakly singular compactification of $X \times Y$. We set $Z = X \times Y$ and $\delta Z = \omega X \times \omega Y$. ωX and ωY denote $X \cup \{p_\omega\}$ and $Y \cup \{q_\omega\}$ respectively, where $p_\omega \notin X$ and $q_\omega \notin Y$. Assume that δZ is a weakly singular compactification of Z. Then there exists a compact subset F in Z and a retraction $r : \delta Z - F \to \delta Z - Z$. Without loss of generality, we can assume that $F = F_X \times F_Y$, where F_X and F_Y are compact subsets of X and Y, respectively. Since Z is locally compact, $\delta Z - Z$ is closed in δZ. Let K_X and K_Y be relatively compact open subsets of X and Y respectively such that $K_X \supset F_X$ and $K_Y \supset F_Y$. Take a point $x_0 \in X - \mathrm{cl}_X K_X$. Let U_0' be a compact neighborhood of x_0 such that $U_0' \cap \mathrm{cl}_X K_X = \emptyset$. Since r is continuous, $r^{-1}(U_0' \times \{q_\omega\})$ is neighborhood of (x_0, q_ω). Then there exist compact neighborhoods U_0 of x_0 and B_0 of q_ω respectively such that $r(U_0 \times B_0) \subset U_0' \times \{q_\omega\}$ and $B_0 \cap \mathrm{cl}_Y K_Y = \emptyset$. Since $(Y \cap \mathrm{int}_{\omega Y} B_0) - \mathrm{cl}_Y K_Y \neq \emptyset$, we take a point $y_0 \in (Y \cap \mathrm{int}_{\omega Y} B_0) - \mathrm{cl}_Y K_Y$ such that $r((x_0, y_0)) \in U_0 \times \{q_\omega\}$. Let V_0' be a compact neighborhood of y_0 such that $V_0' \cap \mathrm{cl}_Y K_Y = \emptyset$. Since r is continuous, $r^{-1}(\{p_\omega\} \times V_0')$ is a neighborhood of (p_ω, y_0). Then there exist compact neighborhoods A_0 of p_ω and V_0 of y_0 respectively such that $A_0 \cap (U_0' \cup \mathrm{cl}_X K_X) = \emptyset$, $V_0 \subset V_0'$ and $r(A_0 \times V_0) \subset \{p_\omega\} \times V_0'$. We will define inductively the sequences $\{x_n\}_{n<\omega_0}$, $\{y_n\}_{n<\omega_0}$, $\{A_n\}_{n<\omega_0}$, $\{B_n\}_{n<\omega_0}$, $\{U_n\}_{n<\omega_0}$ $\{V_n\}_{n<\omega_0}$, $\{U_n'\}_{n<\omega_0}$ and $\{V_n'\}_{n<\omega_0}$ with the following properties for all $n < \omega_0$:

(1) U_n and U_n' (resp. V_n and V_n') are compact neighborhoods of x_n (resp. y_n) such that $U_n \subset U_n' \subset X - \mathrm{cl}_X K_X$ (resp. $V_n \subset V_n' \subset Y - \mathrm{cl}_Y K_Y$),

(2) A_n (resp. B_n) is a compact neighborhood of p_ω (resp. q_ω) such that $A_{n+1} \subset A_n$ (resp. $B_{n+1} \subset B_n$),

(3) $A_n \cap (U_n' \cup \mathrm{cl}_X K_X) = \emptyset$ and $U_{n+1}' \subset A_n$,

(4) $B_{n+1} \cap (V_n' \cup \mathrm{cl}_Y K_Y) = \emptyset$ and $V_n' \subset B_n$,

(5) $r(A_n \times V_n) \subset \{p_\omega\} \times V_n'$ and $r(U_n \times B_n) \subset U_n' \times \{q_\omega\}$,

(6) $r((x_{n+1}, y_n)) \in \{p_\omega\} \times V_n$ and $r((x_n, y_n)) \in U_n \times \{q_\omega\}$.

Assume that the construction is made for any $k < n+1$. Then $r(A_n \times V_n) \subset \{p_\omega\} \times V_n'$ by (5). Take a point $x_{n+1} \in X \cap \mathrm{int}_{\omega X} A_n$ such that $r((x_{n+1}, y_n)) \in \{p_\omega\} \times V_n$. Let U_{n+1}' be a compact neighborhood of x_{n+1} with $U_{n+1}' \subset A_n \cap X$. As above, there exist compact neighborhoods B_{n+1} and U_{n+1} of q_ω and x_{n+1} respectively such that $U_{n+1} \subset U_{n+1}'$, $B_{n+1} \subset B_n$, $B_{n+1} \cap V_n' = \emptyset$ and $r(U_{n+1} \times B_{n+1}) \subset U_{n+1}' \times \{q_\omega\}$. Take a point $y_{n+1} \in Y \cap \mathrm{int}_{\omega Y} B_{n+1}$ such that $r((x_{n+1}, y_{n+1})) \in U_{n+1} \times \{q_\omega\}$. Let V_{n+1}' be a compact neighborhood of y_{n+1} with $V_{n+1}' \subset Y \cap B_{n+1}$. Then there exist compact neighborhoods A_{n+1} and V_{n+1} of p_ω and y_{n+1} respectively such that $V_{n+1} \subset V_{n+1}'$, $A_{n+1} \subset A_n$, $A_{n+1} \cap U_{n+1}' = \emptyset$ and $r(A_{n+1} \times V_{n+1}) \subset \{p_\omega\} \times V_{n+1}'$. Now the inductive process is complete.

CLAIM (1). If (u, v) is a cluster point of the sequence $\{(x_{n+1}, y_n)\}_{n<\omega_0}$, then $r((u, v)) = (p_\omega, q_\omega)$.

CLAIM (2). Put $S = \{(u_j, v_j)\}_{j<\omega_0}$, where $u_j \in U_{n_j}$, $v_j \in V_{m_j}$ and $n_j \leq m_j < n_{j+1}$ for any $j < \omega_0$. If (u, v) is a cluster point of the sequence S, then $r((u, v)) = (p_\omega, q_\omega)$.

We will prove the Claim (1). From (6) it follows that the sequence $\{r((x_{n+1}, y_n))\}_{n<\omega_0} \subset \{p_\omega\} \times (Y - K_Y)$. Therefore $r((u, v)) \in \mathrm{cl}_{\delta Z}(\{p_\omega\} \times (Y - K_Y))$. Note that $(u, v) \in \mathrm{cl}_{\delta Z}\{(x_j, y_k) : k \geq j \geq 0\}$. From (2), (4) and (5), $\{r((x_j, y_k)) : k \geq j \geq 0\} \subset (X - K_X) \times \{q_\omega\}$, therefore $r((u, v)) \in \mathrm{cl}_{\delta Z}((X - K_X) \times \{q_\omega\})$. Since $\mathrm{cl}_{\delta Z}((X - K_X) \times \{q_\omega\}) \cap \mathrm{cl}_{\delta Z}(\{p_\omega\} \times (Y - K_Y)) = \{(p_\omega, q_\omega)\}$, we have proved that $r((u, v)) = (p_\omega, q_\omega)$.

Claim (2) can be proved with a similar argument since $r((u_j, v_j)) \in (X - K_X) \times \{q_\omega\}$ for every $j < \omega_0$ and $r((u_k, v_j)) \in \{p_\omega\} \times (Y - K_Y)$ for every $k \geq j \geq 0$. Claims are proved.

Since X is pseudocompact, $\{\mathrm{int}_X U_n : n < \omega_0\}$ is not locally finite. Since X is locally compact, there exist a compact subset K in X and $A \in [\omega_0]^{\omega_0}$ such that $P_n = \mathrm{int}_X(K \cap U_n) \neq \emptyset$ for every $n \in A$. On the other hand, we note that $\{(x_{n+1}, y_n)\}_{n<\omega_0}$ has a cluster point in δZ. If (u, v) is a cluster point of $\{(x_{n+1}, y_n)\}_{n<\omega_0}$, then from Claim (1) $r((u, v)) = (p_\omega, q_\omega)$. Then $(p_\omega, q_\omega) \in \mathrm{cl}_{\delta Z}\{r((x_{n+1}, y_n))\}_{n<\omega_0}$. From (6) and this fact it follows that each neighborhood V of q_ω in ωY there exists a $B(V) \in [\omega_0]^{\omega_0}$ such that $V \cap V_n \neq \emptyset$ for every $n \in B(V)$. Let $n_0 = \min A$. Since $P_{n_0} \neq \emptyset$, we take a point $t_0 \in P_{n_0}$. Then there exists a compact neighborhood Q_0 of q_ω in ωY such that $r(\{t_0\} \times Q_0) \subset P_{n_0} \times \{q_\omega\}$. Since Q_0 is a compact neighborhood of q_ω, we take a number $m_0 \in B(Q_0)$ such that $m_0 \geq n_0$. We can take a point $z_0 \in V_{m_0} \cap Q_0$ since $m_0 \in B(Q_0)$. Continuing by induction, we obtain the sequences $\{t_j\}_{j<\omega_0}$, $\{z_j\}_{j<\omega_0}$, $\{P_{n_j}\}_{j<\omega_0}$, $\{Q_j\}_{j<\omega_0}$ and $\{B(Q_j)\}_{j<\omega_0}$ with the following properties for every $j < \omega_0$:

(1) $t_j \in P_{n_j}$ and $r(\{t_j\} \times Q_j) \subset P_{n_j} \times \{q_\omega\} \subset K \times \{q_\omega\}$,

(2) $z_j \in V_{m_j} \cap Q_j$,

(3) $n_j \leq m_j < n_{j+1}$ where $m_j \in B(Q_j)$ and n_j, $n_{j+1} \in A$.

If (u, v) is a cluster point of the sequence $\{(t_j, z_j)\}_{j<\omega_0}$, from Claim (2) it follows that $r((u, v)) = (p_\omega, q_\omega)$. However, this is impossible since the sequence $\{r((t_j, z_j))\}_{j<\omega_0} \subset K \times \{q_\omega\}$. Thus there exists no retractions $r : \delta Z - F \to \delta Z - Z$. Therefore δZ can not be a weakly singular compactification of Z. Then the proof is complete. □

Let αX be a compactification of X. For an open set U of X, we set $\mathrm{ext}_{\alpha X}\, U = \alpha X - \mathrm{cl}_{\alpha X}(X - U)$.

LEMMA 2.2. *Let X be a non-compact space and Y a non-compact space. If αX and δY are compactifications of X and Y respectively, then $\omega X \times \omega Y \leq \alpha X \times \delta Y$.*

PROOF. Put $\omega X = X \cup \{p_\omega\}$ and $\omega Y = Y \cup \{q_\omega\}$, where we assume that $p_\omega \notin X$ and $q_\omega \notin Y$. We will define a map $\pi : \alpha X \times \delta Y \to \omega X \times \omega Y$ as follows:

$$\pi(z) = \begin{cases} z, & \text{if } z \in X \times Y \\ (p_\omega, y), & \text{if } z = (x, y) \in (\alpha X - X) \times \{y\} \text{ for some } y \in Y \\ (x, q_\omega), & \text{if } z = (x, y) \in \{x\} \times (\delta Y - Y) \text{ for some } x \in X \\ (p_\omega, q_\omega), & \text{if } z \in (\alpha X - X) \times (\delta Y - Y) \end{cases}$$

It is sufficient to show that π is continuous and then the only thing in need of proof is that we have to show the following three cases.

CASE 1. Let U be an open neighborhood of p_ω in ωX and V an open set of Y such that $\mathrm{cl}_Y V$ is compact. Then we will verify that $\pi^{-1}(U \times V) = \mathrm{ext}_{\alpha X}(U \cap X) \times V$. In fact, since $X - (U \cap X)$ is compact in X, $\mathrm{ext}_{\alpha X}(U \cap X) = (U \cap X) \cup (\alpha X - X)$. Then $\pi^{-1}(U \times V) = ((U \cap X) \times V) \cup \bigcup_{y \in V}(\alpha X - X) \times \{y\} = \mathrm{ext}_{\alpha X}(U \cap X) \times V$.

CASE 2. Let U be an open set of X such that $\mathrm{cl}_X U$ is compact and V an open neighborhood of q_ω in ωY. Then mimicking the similar argument of Case 1, we can verify that $\pi^{-1}(U \times V) = U \times \mathrm{ext}_{\delta Y}(V \cap Y)$.

CASE 3. Let U and V be open neighborhoods of p_ω and q_ω in ωX and ωY respectively. Then we will verify that $\pi^{-1}(U \times V) = \mathrm{ext}_{\alpha X}(U \cap X) \times \mathrm{ext}_{\delta Y}(V \cap Y)$. Note that $\pi^{-1}(U \times V) = (U \cap X) \times (V \cap Y) \cup (\alpha X - X) \times (\delta Y - Y) \cup (U \cap X) \times (\delta Y - Y) \cup (\alpha X - X) \times (V \cap Y)$. Since $\mathrm{ext}_{\alpha X}(U \cap X) = (U \cap X) \cup (\alpha X - X)$ and $\mathrm{ext}_{\delta Y}(V \cap Y) = (V \cap Y) \cup (\delta Y - Y)$, $\pi^{-1}(U \times V) = \mathrm{ext}_{\alpha X}(U \cap X) \times \mathrm{ext}_{\delta Y}(V \cap Y)$.

Cases 1, 2 and 3 imply that π is continuous. Hence $\omega X \times \omega Y \leq \alpha X \times \delta Y$. □

From Lemma 2.2 we can get the following corollary:

COROLLARY 2.1. *Let X be a non-compact space and Y a non-compact space. If X is pseudocompact, $\alpha X \times \gamma Y$ is not a weakly singular compactification of $X \times Y$ for any compactifications αX and γY of X and Y respectively.*

The following example shows that the pseudocompactness in Corollary 2.1 can not be dropped.

EXAMPLE 2.1. Let X be the half open interval $[0, 1)$ with a usual topology. Obviously, we note that $\omega X \times \omega X$ is a singular compactification of $X \times X$.

In Corollary 2.1 we note that if X is pseudocompact, $\alpha X \times \gamma Y$ is not a singular compactification of $X \times Y$ for any compactifications αX and γY of X and Y, respectively. Here, the condition of pseudocompactness is not a necessary condition, i.e., there exists a non-pseudocompact space X such that $\alpha X \times \gamma X$ is not a singular compactification of $X \times X$ for any compactifications αX and γX of X.

EXAMPLE 2.2. Let $\boldsymbol{R}$ be the real line with a usual topology. However, $\omega\boldsymbol{R} \times \omega\boldsymbol{R}$ is not a singular compactification of $\boldsymbol{R} \times \boldsymbol{R}$. In fact, it is well-known the fact that there exists no retractions $r : \omega\boldsymbol{R} \times \omega\boldsymbol{R} \to \omega\boldsymbol{R} \times \omega\boldsymbol{R} - \boldsymbol{R} \times \boldsymbol{R}$. From Lemma 2.1 and Lemma 2.2 $\alpha\boldsymbol{R} \times \delta\boldsymbol{R}$ is not a singular compactification of $\boldsymbol{R} \times \boldsymbol{R}$ for any compactifications $\alpha\boldsymbol{R}$ and $\delta\boldsymbol{R}$ of $\boldsymbol{R}$.

The following remark was pointed out by Professor K. Kawamura.

REMARK 2.1. If αX is a compactification of X with closed unit interval $\boldsymbol{I}$ as a remainder, then αX is singular since $\boldsymbol{I}$ is an AR. On the other hand, the converse J. L. Blasco's Theorem cannot hold. In fact, let X be denoted by the half open interval $[0,1)$ with a usual topology and let P be a pseudo-arc (cf. [19]). Recall that pseudo-arc is a hereditarily indecomposable continuum and every continuous image of $\boldsymbol{I}$ into a pseudo-arc is a one point. Fix a point $p \in P$ and put $Y = P - \{p\}$. We can easily verify to see that $\omega X \times \omega Y$ is not a weakly singular compactification of $X \times Y$ and both X and Y is not pseudocompact.

3. Characterization of singular compactifications of product spaces

If one factor is compact, the following proposition holds.

PROPOSITION 3.1. *Let X be a non-compact space and K a compact space. Then αX is a singular compactification of X if and only if $\alpha X \times K$ is a singular compactification of $X \times K$.*

PROOF. Necessity. Since αX is a singular compactification of X, there exists a retraction $r : \alpha X \to \alpha X - X$. Then a map $s : \alpha X \times K \to (\alpha X - X) \times K$ is defined by $s((x,k)) = (r(x),k)$ for $(x,k) \in \alpha X \times K$. Clearly, we note that s is a retraction from $\alpha X \times K$ onto $(\alpha X - X) \times K$. Thus $\alpha X \times K$ is a singular compactification of $X \times K$.

Sufficiency. Since $\alpha X \times K$ is a singular compactification of $X \times K$, there exists a retraction $r : \alpha X \times K \to (\alpha X - X) \times K$. Take a point $k \in K$. Then a map $s : (\alpha X - X) \times K \to (\alpha X - X) \times \{k\}$ is defined by $s((x,y)) = (x,k)$ for $(x,y) \in (\alpha X - X) \times K$. φ denotes $(s \circ r)\restriction_{\alpha X \times \{k\}}$. Then we note that $\varphi : \alpha X \times \{k\} \to (\alpha X - X) \times \{k\}$ is a retraction. Thus αX is a singular compactification of X. □

Let X be a non-compact space, Y a compact space and $f : X \to Y$ a continuous map. Then the *singular set* $S(f)$ of f is the set $\{y \in Y :$ for every open set U of Y containing y, $\mathrm{cl}_X f^{-1}(U)$ is not compact} [5]. We say that f is *singular* if $S(f) = Y$ [17]. If f is singular, then we can construct a *singular compactification* of X as follows: On the set $X \cup Y$, basic neighborhoods of points in X remain the same as in X. Points in Y have neighborhoods of the form $U \cup (f^{-1}(U) - F)$, where U is open in Y and F is compact in X. Then $X \cup Y$ with this topology is a compactification of X, and is denoted by $X \cup_f S(f)$. A compactification αX of X is said to be *singular* if $\alpha X \approx X \cup_f S(f)$ for some singular map f (cf. [11] and [17]); the fundamental idea of this compactification is originated from [10].

A compact space S is called a *singular set* of X if there exists a continuous map $f : X \to S$ such that $S = S(f)$.

Proposition 3.2. *Let X be a non-compact space and S a compact space. Then X has a singular compactification with S as a remainder if and only if S is a singular set of X.*

Proof. Necessity. Suppose that αX is a singular compactification of X with S as a remainder. Note that there exists a retraction $r : \alpha X \to \alpha X - X$ $(= S)$. Put $f = r\restriction_X$. Then we will verify that $S = S(f)$. In fact, take a point $x \in S$ and let U be a neighborhood of x in S. We will show that $\mathrm{cl}_X f^{-1}(U)$ is not compact. Take a net $\{x_\nu\}_{\nu \in N}$ $(\subset X)$ converging to x in αX, where N is a suitable directed set with some order $\leq$. Then we note that there exists a $\nu_0 \in N$ such that $\nu \geq \nu_0$ then $x_\nu \in r^{-1}(U)$. Note that $r(x_\nu) = f(x_\nu) \in U$ for all $\nu \geq \nu_0$. Then $x_\nu \in f^{-1}(U)$ for all $\nu \geq \nu_0$. If $\mathrm{cl}_X f^{-1}(U)$ is compact, then $x \in \mathrm{cl}_X f^{-1}(U)$. This is a contradiction.

Sufficiency. This follows from the definition of singular compactifications. □

From the above proposition we realize that every singular compactification depends on a singular map. The following example shows that there exists singular compactifications αX and γX of X such that αX is not equivalent to γX, even if $\alpha X - X$ is homeomorphic to $\gamma X - X$.

Example 3.1. Let $X_0 = X_1 = [0,1)$ with a usual topology and $X_2 = \omega_0$ with a discrete topology. Then we put $X = \bigoplus_{i<3} X_i$. Put $\alpha_2 X = \omega(X_0 \oplus X_1) \oplus \omega X_2$ and $\gamma_2 X = \omega X_0 \oplus \omega(X_1 \oplus X_2)$. Then $\alpha_2 X$ is not equivalent to $\gamma_2 X$, even if $\alpha_2 X - X$ is homeomorphic to $\gamma_2 X - X$. In fact, denote $\omega(X_0 \oplus X_1) - X_0 \oplus X_1 = \{p_0\}$, $\omega X_2 - X_2 = \{p_1\}$, $\omega X_0 - X_0 = \{q_0\}$ and $\omega(X_1 \oplus X_2) - X_1 \oplus X_2 = \{q_1\}$. Clearly, $\alpha_2 X - X$ is homeomorphic to $\gamma_2 X - X$. Suppose that $\alpha_2 X \approx \gamma_2 X$ and then $\alpha_2 X \geq \gamma_2 X$. Then there exists a continuous map $f : \alpha_2 X \to \gamma_2 X$ such that $f \restriction_X$ is an identity on X. Then we note that either $f(p_0) = q_0$ or $f(p_0) = q_1$ holds. Since neither ωX_0 nor $\omega(X_1 \oplus X_2)$ contains $f(\omega(X_0 \oplus X_1) - K)$ for any compact subset K of $X_0 \oplus X_1$, we can get a contradiction. This implies that $\alpha_2 X \not\geq \gamma_2 X$.

Let $d(X)$ be the density of a space X. The rest of this section $\boldsymbol{D}_\kappa$ is a discrete space with cardinality κ. Proving our main theorem, we will begin with the following lemmas:

LEMMA 3.1. *Let S be a compact space and Y a non-compact space which is a continuous image of a non-compact space X. If Y has a singular compactification with S as a remainder, then X has a singular compactification with S as a remainder.*

PROOF. From Proposition 3.2 S is a singular set of Y. Then there exists a singular map $f : Y \to S$ such that $S = S(f)$. Assume that $g : X \to Y$ is a continuous onto map. Then we will show that $S = S(f \circ g)$. In fact, take a point $x \in S$ and let U be an open neighborhood of x in S. Assume the contrary $\mathrm{cl}_X\, g^{-1}(f^{-1}(U))$ is compact. Since $g(\mathrm{cl}_X\, g^{-1}(f^{-1}(U))) \supset f^{-1}(U)$, we note that $\mathrm{cl}_Y\, f^{-1}(U)$ is compact. This is a contradiction. This implies that $x \in S(f \circ g)$ and then we have shown that $S = S(f \circ g)$. Again from the Proposition 3.2 X has a singular compactification with S as a remainder. □

LEMMA 3.2. *Let X be a non-compact space, Y a space and S a compact space. If X has a singular compactification with S as a remainder, then $X \times Y$ has a singular compactification with S as a remainder.*

PROOF. Assume that X has a singular compactification with S as a remainder. Since X is a continuous image of $X \times Y$, from Lemma 3.1 $X \times Y$ has a singular compactification with S as a remainder. □

From Lemmas 3.1 and 3.2 we will prove the main lemma:

LEMMA 3.3. *Let κ be an infinite cardinal and $X = \bigoplus_{\alpha<\kappa} X_\alpha$ with $X_\alpha \neq \emptyset$ and $d(X_\alpha) \leq \kappa$ for any $\alpha < \kappa$ and Y a space with $d(Y) \leq d(X)$. Then for any compact space S the following conditions are equivalent:*

(1) *$X \times Y$ has a singular compactification with S as a remainder,*

(2) *X has a singular compactification with S as a remainder.*

PROOF. (1) $\Rightarrow$ (2). Assume that $X \times Y$ has a singular compactification with S as a remainder. Since $d(X) = \kappa$, we note that $d(S) \leq \kappa$. Let D be a dense subset of S. Enumerate D as $\{x_\alpha : \alpha < d(S)\}$. Note that $\boldsymbol{D}_\kappa$ can be represented as the infinite disjoint topological sum $\bigoplus_{\alpha<d(S)} D_\alpha$ such that $|D_\alpha| = \kappa$ for every $\alpha < d(S)$. A map $\varphi : \boldsymbol{D}_\kappa \to D$ defined by $\varphi(d) = x_\alpha$ for every $d \in D_\alpha$. Note that φ is continuous and $S(\varphi) = S$. From Proposition 3.2 $\boldsymbol{D}_\kappa$ has a singular compactification with S as a remainder. From Lemma 3.1 there exists a singular compactification of X with S as a remainder, because $\boldsymbol{D}_\kappa$ is a continuous image of X.

(2) $\Rightarrow$ (1). This part of the proof follows from Lemma 3.2. We have thus proved the lemma. □

It is well-known the fact that every non-separable metrizable space can be represented as the infinite disjoint topological sum. We will prove the main theorem in the case X is a non-separable metrizable space:

THEOREM 3.1. *Let X be a non-separable metrizable space and Y a space with $d(Y) \leq d(X)$. Then for any compact space S the following conditions are equivalent*:

(1) *$X \times Y$ has a singular compactification with S as a remainder,*

(2) *X has a singular compactification with S as a remainder,*

(3) *$d(S) \leq d(X)$ holds.*

PROOF. Since X is a non-separable metrizable space, X can be represented as $\bigoplus_{\alpha<\kappa} X_\alpha$, where X_α is σ-compact for every $\alpha < \kappa$ and $\kappa \geq \omega_1$. Without loss of generality, we can assume that $X_\alpha \neq \emptyset$ for any $\alpha < \kappa$. Then from Lemma 3.3 we note that (1) is equivalent to (2). Finally, we will show that (2) is equivalent to (3). Clearly, we note that (2) implies (3). It is sufficient to show that (3) implies (2). Let D be a dense subset of S. Enumerate D as $\{x_\alpha : \alpha < d(S)\}$. Note that $\boldsymbol{D}_\kappa$ can be represented as the infinite disjoint topological sum $\bigoplus_{\alpha<d(S)} D_\alpha$ such that $|D_\alpha| = \kappa$ for every $\alpha < d(S)$. Define a map $\varphi : \boldsymbol{D}_\kappa \to D$ as follows: $\varphi(d) = x_\alpha$ for every $d \in D_\alpha$. Note that $S = S(\varphi)$. From Proposition 3.2 $\boldsymbol{D}_\kappa$ has a singular compactification with S as a remainder and then from Lemma 3.1 X has a singular compactification with S as a remainder, because X can be represented as $\bigoplus_{\alpha<\kappa} X_\alpha$, where X_α is a non-empty σ-compact space for any $\alpha < \kappa$. We have thus proved the theorem. □

Mimicking the proof of Theorem 3.1 we can get the following corollary:

COROLLARY 3.1. *Let X and Y be non-separable metrizable spaces. Then for any compact space S the following conditions are equivalent*:

(1) *$X \times Y$ has a singular compactification with S as a remainder,*

(2) *either X or Y has a singular compactification with S as a remainder,*

(3) *either $d(S) \leq d(X)$ or $d(S) \leq d(Y)$ holds.*

Let $Q(X)$ be the set of all quasi-components of a space X and $p : X \to Q(X)$ the natural projection from X onto $Q(X)$. We given $Q(X)$ the topology generated by $\{\mathscr{C} : \mathscr{C} \subset Q(X) \text{ and } p^{-1}(\mathscr{C}) \text{ is clopen in } X\}$ as a base for open sets. We call the space $Q(X)$ with this topology the quasi-component space of X [14].

T. Kimura [21] proved the following lemma:

LEMMA 3.4 ([21], T. Kimura). *Let X be a separable metrizable space. If the quasi-component space $Q(X)$ is not compact, then X can be represented as the infinite disjoint topological sum.*

From Lemmas 3.3 and 3.4 we can get the main theorem in the case X is a separable metrizable space with a non-compact quasi-component space $Q(X)$:

THEOREM 3.2. *Let X be a separable metrizable space with a non-compact quasi-component space $Q(X)$ and Y a space with $d(Y) \leq d(X)$. Then for any compact space S the following conditions are equivalent*:

(1) *$X \times Y$ has a singular compactification with S as a remainder,*

(2) *X has a singular compactification with S as a remainder.*

From Theorem 3.2 and the similar argument above we can get the following corollary:

COROLLARY 3.2. *Let X and Y be separable metrizable spaces and either a quasi-component space $Q(X)$ or a quasi-component space $Q(Y)$ is not compact. Then for any compact space S the following conditions are equivalent*:

(1) *$X \times Y$ has a singular compactification with S as a remainder,*

(2) *either X or Y has a singular compactification with S as a remainder.*

In Theorem 3.2 the condition that X has a non-compact quasi-component space $Q(X)$ can not be dropped.

EXAMPLE 3.2. Put $X = [0, 1)$ with a usual topology and $Y = [0, 1] \oplus [0, 1)$, where $[0, 1]$ with a usual topology. We note that $X \times Y$ has a singular compactification with D_2 as a remainder. However, X can not have a singular compactification with D_2 as a remainder.

References

[1] Blasco, J. L., Retractive product spaces, Topology. Appl. **37** (1990), 137–140.
[2] ———, On linear extension operator from growths of compactifications of products, Topology. Appl. **71** (1996), 193–200.
[3] Cain, Jr., G. L., Compact and related mappings, Duke. Math. J. **33** (1966), 639–645.
[4] ———, Mappings with prescribed singular sets, Nieuw. Arch. Wisk. (3) **17** (1969), 200–203.
[5] Cain, Jr., G. L., Chandler, R. E. and Faulkner, G. D., Singular sets and remainders, Trans. Amer. Math. Soc. **268** (1981), 161–171.
[6] Caterino, A., Faulkner, G. D. and Vipera, M. C., Two applications of singular sets to the theory of compactifications, Rend. Istit. Math. Univ. Trieste **21** (1989), 248–358.
[7] ———, Construction of compactifications using essential semilattice homomorphism, Proc. Amer. Math. Soc. **116** (1992), 851–860.
[8] Caterino, A. and Vipera, M. C., Singular compactifications and compactification lattices, Rend. Circ. Math. Palermo (2) Suppl. **24** (1990), 299–309.
[9] Chandler, R. E., Hausdorff Compactifications, Marcel Dekker, Inc., New York-Basel, 1976.
[10] Chandler, R. E., Faulkner, G. D., Guglielmi, J. P. and Memory, M., Generalizing the Alexandroff-Urysohn double circumference construction, Proc. Amer. Math. Soc. **83** (1981), 606–608.
[11] Chandler, R. E. and Faulkner, G. D., Singular compactifications: The order structure, Proc. Amer. Math. Soc. **100** (1987), 377–382.
[12] Comfort, W. W., Retractions and other continuous maps from βX onto $\beta X - X$, Trans. Amer. Math. Soc. **114** (1965), 1–9.
[13] Dickman, Jr., R. F., Unified spaces and singular sets for mappings of locally compact spaces, Fund. Math. **62** (1968), 103–123.
[14] Dickman, R. F. and McCoy, R. A., The Freudenthal compactification, Dissert. Math. **262** (1988), 1–35.
[15] van Douwen, E. K., Retractions from βX onto $\beta X - X$, Gen. Top. Appl. **9** (1978), 169–173.
[16] Engelking, R., General Topology, Helderman Verlag, Berlin, 1989.
[17] Faulkner, G. D., Compactifications whose remainders are retracts, Proc. Amer. Math. Soc. **103** (1988), 984–988.
[18] Hatzenbuhler, J. and Mattson, D. A., The Cantor set as a remainder of $X \times Y$, Houston J. Math. **13** (1987), 367–371.
[19] Hocking, J. G. and Young, G. S., Topology, Addison-Wesley Publishing company, inc., 1961.
[20] Kimura, T., $\aleph_0$-point compactifications of locally compact spaces and product spaces, Proc. Amer. Math. Soc. **93** (1985), 164–168.
[21] ———, Product spaces for which all compact metric spaces are remainders, preprint.
[22] Steiner, A. K., and Steiner. E. F., On the reduction of the Wallman compactification Problem to discrete space, Gen. Top. Appl. **7** (1977), 35–37.
[23] Whyburn, G. T., A unified space for mappings, Trans. Amer. Math. Soc. **74** (1953), 344-350.
[24] ———, Compactification of mappings, Math. Ann. **166** (1966), 168–174.

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