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TOTALLY COMPLEX SUBMANIFOLDS OF THE CAYLEY PROJECTIVE PLANE

By

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Abstract. Let K be the sectional curvature of a compact submanifold M of the Cayley projective plane CaP^2. In this paper, we prove that the compact totally complex submanifold M of complex dimension 2 in CaP^2 satisfying $K > (1/8)$ is totally geodesic and $M = CP^2$.

1. Introduction

Let M be an n-dimensional compact Kaehler submanifold of complex projective space $CP^m(1)$. Denote by K the sectional curvature of M. In [6], Ros and Verstraelen showed that if $K > (1/8)$, then M is totally geodesic. The analogous result in the case of totally complex submanifolds of quaternion projective space $HP^m(1)$ was obtained by Xia [7]. In the present paper, we prove the following same type result for totally complex submanifolds of the Cayley projective plane CaP^2.

THEOREM. *Let M be a compact totally complex submanifold of complex dimension 2, immersed in the Cayley projective plane CaP^2. If the sectional curvature K of M satisfying $K > (1/8)$, then M is totally geodesic in CaP^2 and $M^2 = CP^2$.*

2. Cayley projective plane

In this section, we review simply the fundamental results about the Cayley projective plane, for details see [4].

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Let us denote by Ca the Cayley number, it possesses a multiplicative identity 1 and a positive definite bilinear form $\langle \cdot, \cdot \rangle$ with norm $\|a\| = \langle a, a \rangle$, satisfying $\|ab\| = \|a\| \, \|b\|$, for $a, b \in Ca$. Every element $a \in Ca$ can be expressed in the form $a = a_0 1 + a_1$ for $a_0 \in R$ and $\langle a_1, 1 \rangle = 0$. The conjugation map $a \to a^* = a_0 1 - a_1$ is an anti-automorphism $(ab)^* = b^* a^*$.

A canonical basis for Ca is any basis of the form $\{1, e_0, e_1, \dots, e_6\}$ satisfying: (i) $\langle e_i, 1 \rangle = 0$; (ii) $\langle e_i, e_j \rangle = \{0 \text{ for } i \neq j, \text{and } 1 \text{ otherwise}\}$; (iii) $e_i^2 = -1$; $e_i e_j + e_j e_i = 0$ $(i \neq j)$; (iv) $e_i e_{i+1} = e_{i+3}$ for $i \in Z_7$.

Let V be a vector space of real dimension 16 with automorphism group Spin(9). the splitting

$$V = Ca \oplus Ca$$

together with the above canonical basis on each summand, endows V with what we may refer to as a Cayley structure. We know that the Cayley projective plane CaP^2 is the 16-dimensional Riemannian symmetric space whose tangent space admits the Cayley structure pointwise. In the following, let $\{I_0, \dots, I_6\}$ be the Cayley structure on CaP^2.

The curvature tensor $\bar{R}$ of CaP^2 is given in [2] as follows

$$\begin{aligned} \bar{R}((a,b),(c,d))(e,f) = \frac{1}{4}((&4\langle c,e\rangle a - 4\langle a,e\rangle c + (ed)b^* - (eb)d^* \\ &+ (ad - cb)f^*), (4\langle d,f\rangle b - 4\langle b,f\rangle d + a^*(cf) \\ &- c^*(af) - e^*(ad - cb))). \end{aligned} \tag{2.1}$$

On $Ca \oplus Ca$ we have the positive definite bilinear form $\langle \, , \rangle$ given by

$$\langle (a,b),(c,d) \rangle = \langle a,c \rangle + \langle b,d \rangle. \tag{2.2}$$

3. Totally complex submanifolds

Let $V \subset T_x CaP^2$ be a real vector subspace, we say that V is a totally complex subspace if there exists an I such that there exists a basis with $I = I_0$ and (i) $I_0 V \subset V$, and (ii) $I_k V$ is perpendicular to V for $1 \leq k \leq 6$. Clearly, if V is a maximal subspace of this kind then $\dim_R V = 4$.

Let M be a compact Riemannian manifold isometrically immersed in CaP^2 by $j : M \to CaP^2$. Denote by h and A the second fundamental form of j and the Weingarten endomorphism respectively. Then we have

$$\langle h(X,Y), N \rangle = \langle X, A_N(Y) \rangle \quad X, Y \in TM, N \in TM^\perp \tag{3.1}$$

We take $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ to be the Riemannian connections on CaP^2, M and the normal connection on M respectively. The corresponding curvature tensors are denoted by $\bar{R}$, R, and $R^{\perp}$ respectively. The first and second covariant derivatives of h are given by

$$(3.2) \qquad (\overline{\nabla}h)(X, Y; Z) = \nabla_z^{\perp}(h(X, Y)) - h(\nabla_z X, Y) - h(X, \nabla_z Y)$$

$$(3.3) \qquad (\overline{\nabla}^2 h)(X, Y; Z; W) = \nabla_w^{\perp}(\overline{\nabla}h)(X, Y; Z) - (\overline{\nabla}h)(\nabla_w X, Y; Z) - (\overline{\nabla}h)(X, \nabla_w Y; Z) - (\overline{\nabla}h)(X, Y; \nabla_w Z)$$

$$X, Y, Z, W \in TM.$$

The Codazzi equation takes the following form

$$(3.4) \qquad (\overline{\nabla}h)(X_{\tau(1)}, X_{\tau(2)}; X_{\tau(3)}) = (\overline{\nabla}h)(X_1, X_2; X_3),$$

where $\tau(i) \in S_3$ the permutation group and the arguments are in the tangent space of M. Recalling that h and $(\overline{\nabla}h)$ are symmetric, we have the Ricci identity

$$(3.5) \qquad (\overline{\nabla}^2 h)(X, Y; Z; W) - (\overline{\nabla}^2 h)(X, Y; W; Z) = -R^{\perp}(Z, W)h(X, Y) + h(R(Z, W)X, Y) + h(X, R(Z, W)Y).$$

We say that $j : M \to CaP^2$ is a totally complex immersion if $W = j_*(TM)$ is a totally complex subspace for each point of M. Observe that every totally complex submanifold of CaP^2 has a Kaehler structure. We set $I = I_0$, and consequently we have

$$(3.6) \qquad \begin{aligned} &\text{(a)} \quad \overline{\nabla}_X I = 0 \\ &\text{(b)} \quad h(IX, Y) = Ih(X, Y) \\ &\text{(c)} \quad A_{IN} = IA_N = -A_N I \\ &\text{(d)} \quad IR(X, IX)X = R(X, IX)IX \end{aligned}$$

where X, $Y \in T_x M$ and $N \in T_x M^{\perp}$.

Define $f(u) = |h(u, u)|^2$, where $u \in UM$, the unit tangent bundle over M. Assume f attains its maximum at some vector $v \in UM_p$, $p \in M$, then ([5]):

$$(3.7) \qquad A_{h(v,v)}v = |h(v, v)|^2 v.$$

Lemma 3.1. *Let M^n be a compact totally complex submanifold in CaP^2. Assume f attains its maximum at $v \in UM_p$, then*

$$(3.8)\qquad 3|h(v,v)|^2(1-4|h(v,v)|^2)+\sum_{i=1}^{6}\langle h(v,v),I_iv\rangle^2+4|(\overline{\nabla}h)(v,v;v)|^2\le 0.$$

PROOF. Fix v in UM_p. For any $u\in UM_p$, let $r_u(t)$ be the geodesic in M satisfying the initial conditions $r_u(0)=p$, $r'_u(0)=u$. Parallel translating along $r_u(t)$ gives rise to a vector field $V_u(t)$. Put $f_u(t)=f(V_u(t))$, then

$$(3.9)\qquad \frac{d^2}{dt^2}f_u(0)=2\langle(\overline{\nabla}^2h)(v,v;u;u),h(v,v)\rangle+2|(\overline{\nabla}h)(u,v;v)|^2.$$

Using (3.4), (3.5) and (3.6), we have

$$(3.10)\qquad \begin{aligned}\langle(\overline{\nabla}^2h)(v,v;Iv;Iv),h(v,v)\rangle &= \langle\overline{\nabla}^2h)(v,Iv;v;Iv),h(v,v)\rangle\\ &= -\langle(\overline{\nabla}^2h)(v,v;v;v),h(v,v)\rangle\\ &\quad+\langle R^{\perp}(Iv,v)h(Iv,v),h(v,v)\rangle\\ &\quad-2\langle R(Iv,v)Iv,A_{h(v,v)}v\rangle.\end{aligned}$$

From the Ricci equation, (2.1), (2.2) and (3.6), we obtain

$$(3.11)\qquad \begin{aligned}&\langle R^{\perp}(Iv,v)h(Iv,v),h(v,v)\rangle\\ &=\langle\overline{R}(Iv,v)h(Iv,v),h(v,v)\rangle+\langle[A_{h(Iv,v)},A_{h(v,v)}]Iv,v\rangle\\ &=-\frac{1}{2}|h(v,v)|^2-2|A_{h(v,v)}v|^2+\frac{1}{2}\sum_{i=1}^{6}\langle h(v,v),I_iv\rangle^2\end{aligned}$$

Now, by the Gauss equation and using (2.1), (2.2) and (3.6) we have

$$(3.12)\qquad \langle R(Iv,v)Iv,A_{h(v,v)}v\rangle=-|h(v,v)|^2+2|A_{h(v,v)}v|^2.$$

Since f attains its maximum at v, we have

$$(3.13)\qquad \frac{d^2}{dt^2}f_v(0)+\frac{d^2}{dt^2}f_{Iv}(0)\le 0.$$

Combining (3.9)–(3.13) and noticing (3.7), we get (3.8).

4. Proof of the Theorem

We will prove the Theorem by showing that under its assumptions the hypothesis that M is not totally geodesic leads to a contradiction.

From Lemma (3.1) it follows that, by the hypothesis $h \neq 0$.

$$|h(v,v)|^2 \geq \frac{1}{4} \tag{4.1}$$

For any $u \in UM_p$, let $r_u(t)$ be the geodesic in M determined by the initial conditions $r_u(0) = p$ and $r'_u(0) = u$. Parallel translation of v along $r_u(t)$ yields a vector field $V_u(t)$. Then we know that the function f_u defined by $f_u(t) = f(V_u(t))$ attains a maximum at $t = 0$. This implies that

$$\frac{d^2}{dt^2} f_u(0) + \frac{d^2}{dt^2} f_{Iu}(0) \leq 0. \tag{4.2}$$

for all $u \in UM_p$.

By direct computations we have

$$\frac{d^2}{dt^2} f_u(0) = 2\langle (\overline{\nabla}^2 h)(v,v;u;u), h(v,v)\rangle + 2|(\overline{\nabla} h)(u,v;v)|^2 \tag{4.3}$$

Using (3.4), (3.5) and (3.6), we have

$$\begin{aligned} \langle (\overline{\nabla}^2 h)(v,v;Iu;Iu), h(v,v)\rangle &= \langle (\overline{\nabla}^2 h)(v,Iv;u;Iu), h(v,v)\rangle \\ &= -\langle (\overline{\nabla}^2 h)(v,Iv;Iu;u), h(v,v)\rangle \\ &\quad + \langle R^{\perp}(Iu,u)Ih(v,v), h(v,v)\rangle \\ &\quad - 2\langle R(Iu,u)Iv, A_{h(v,v)}v\rangle. \end{aligned} \tag{4.4}$$

From the Ricci equation, (2.1), (2.2), and (3.6), we obtain

$$\begin{aligned} &\langle R^{\perp}(Iu,u)Ih(v,v), h(v,v)\rangle \\ &= \langle \tilde{R}(Iu,u)Ih(v,v), h(v,v)\rangle + \langle [A_{h(Iv,v)}, A_{h(v,v)}]Iu, u\rangle \\ &= -\frac{1}{2}|h(v,v)|^2 - 2|A_{h(v,v)}u|^2 + \frac{1}{2}\sum_{i=1}^{6} \langle h(v,v), I_i u\rangle^2 \end{aligned} \tag{4.5}$$

By the Gauss equation, we get

$$\langle R(Iu,u)Iv, A_{h(v,v)}v\rangle = -|h(v,v)|^2 \langle R(u,Iu)Iv, v\rangle. \tag{4.6}$$

From (4.2)–(4.6), we obtain

$$2|h(v,v)|^2\langle R(u,Iu)Iv,v\rangle - \frac{1}{2}|h(v,v)|^2 - 2|A_{h(v,v)}u|^2 + \sum_{i=1}^{6}\langle h(v,v), I_i u\rangle^2 \leq 0. \tag{4.7}$$

Since $n=2$, we can always choose a unit eigenvector u of $A_{h(v,v)}$ such that $\langle u,v\rangle=\langle u,Iv\rangle=0$, using the equation of Gauss which implies that

$$R(u,v)v=\frac{1}{4}u+A_{h(v,v)}u-A_{h(u,u)}v \tag{4.8}$$

$$R(u,Iv)Iv=\frac{1}{4}u-A_{h(v,v)}u-A_{h(u,u)}v \tag{4.9}$$

we have

$$A_{h(v,v)}u=\frac{1}{2}(R(u,v)v-R(u,Iv)Iv)=\frac{1}{2}(K(u,v)-K(u,Iv))u \tag{4.10}$$

where $K(r,s)$ is the sectional curvature of M at p for the plane spanned by r, $s\in T_pM$. The Bianchi identity shows that

$$\langle R(u,Iu)Iv,v\rangle=K(u,v)+K(u,Iv) \tag{4.11}$$

From (4.7), (4.10) and (4.11) we obtain

$$\begin{aligned}2|h(v,v)|^2(K(u,v)+K(u,Iv))-\frac{1}{2}|h(v,v)|^2-\frac{1}{2}(K(u,v)^2+K(u,Iv)^2\\ -2K(u,v)K(u,Iv))+\sum_{i=1}^{6}\langle h(v,v),I_iu\rangle^2\le 0.\end{aligned} \tag{4.12}$$

or equivalently,

$$aK(u,v)+bK(u,Iv)-\frac{1}{2}|h(v,v)|^2+\sum_{i=1}^{6}\langle h(v,v),I_iu\rangle^2\le 0. \tag{4.13}$$

where

$$a=2|h(v,v)|^2-\frac{1}{2}K(u,v)+\frac{1}{2}K(u,Iv) \tag{4.14}$$

$$b=2|h(v,v)|^2-\frac{1}{2}K(u,Iv)+\frac{1}{2}K(u,v) \tag{4.15}$$

Now, we prove that $a,b>0$. From the equation of Gauss it follows that

$$K(u,v)+K(u,Iv)=\frac{1}{2}-2|h(v,v)|^2\le\frac{1}{2} \tag{4.16}$$

By (4.1) and (4.14), we have

$$1-K(u,v)+K(u,Iv)\le 2a \tag{4.17}$$

From (4.16) and (4.17), we know

$$1 + 2K(u, Iv) \leq 2a + \frac{1}{2} \tag{4.18}$$

Which by the assumption $K > (1/8)$ implies that $a > 0$. In the same way it follows that also $b > 0$. Since a and b are strictly positive and $K > (1/8)$, by (4.13) we get the strictly inequality

$$\frac{1}{8}(a+b) - \frac{1}{2}|h(v,v)|^2 + \sum_{i=1}^{6} \langle h(v,v), I_i u \rangle^2 < 0 \tag{4.19}$$

But from (4.14) and (4.15) it follows that

$$a + b = 4|h(v,v)|^2 \tag{4.20}$$

Which combines with (4.19) yields the desired contradiction. So M is totally geodesic, by the Theorem 2.2 in [4], we known that $M = CP^2$.

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