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ON SPACES WITH LINEARLY HOMEOMORPHIC FUNCTION SPACES IN THE COMPACT OPEN TOPOLOGY

Dedicated to Professor Akihiro Okuyama on his sixtieth birthday

By

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1. Introduction

For a space X, let $C(X)$ be the linear space of all real-valued continuous functions on X, and let $C_0(X)$ (resp. $C_p(X)$) denote the linear topological space $C(X)$ with the compact-open (resp. pointwise convergence) topology. We say that spaces X and Y are *l_0-equivalent* (resp. *l_p-equivalent*) if $C_0(X)$ and $C_0(Y)$ (resp. $C_p(X)$ and $C_p(Y)$) are linearly homeomorphic. For an ordinal number α, let $X^{(\alpha)}$ be the α-th derived set of a space X, where $X^{(0)} = X$. Recall from [3] that an ordinal α is *prime* if it satisfies the following condition: If $\alpha = \beta + \gamma$, then $\gamma = 0$ or $\gamma = \alpha$. Note that 0 and 1 are only finite prime ordinals. For $\alpha \geq \omega$, α is prime if and only if there is an ordinal $\mu \geq 1$ such that $\alpha = \omega^{\mu}$ (cf. [3, Theorem 2.1.21]). Thus, $\omega, \omega^2, \omega^3, \dots$ and the first uncountable ordinal ω_1 are prime. The purpose of this paper is to improve some theorems in Baars and de Groot [3] by proving the following theorem:

THEOREM 1. *Let X and Y be l_0-equivalent metric spaces. For each prime ordinal $\alpha \leq \omega_1$, we have:*
(a) *$X^{(\alpha)} = \varnothing$ if and only if $Y^{(\alpha)} = \varnothing$,*
(b) *$X^{(\alpha)}$ is compact if and only if $Y^{(\alpha)}$ is compact,*
(c) *$X^{(\alpha)}$ is locally compact if and only if $Y^{(\alpha)}$ is locally compact.*

Baars and de Groot proved (a), (b) and (c) in Theorem 1 for $\alpha = 0, 1$ under the additional assumption that X and Y are 0-dimensional and separable ([3, Theorems 4.5.2 and 4.5.3]). For l_p-equivalent metric spaces X and Y, they proved (a) for each prime $\alpha \leq \omega_1$ ([3, Theorems 4.1.15 and 4.1.17]), and proved (b) and

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(c) for each prime $\alpha < \omega_1$ assuming that X and Y are 0-dimensional and separable in addition ([3, Corollary 4.1.14]). Arhangel'skiĭ proved in [1, Corollary 5] that l_p-equivalent paracompact spaces are l_0-equivalent (cf. also [3, Corollary 1.2.21]). Thus, we have the following corollary from Theorem 1.

COROLLARY 1. *Let X and Y be l_p-equivalent metric spaces. Then the statements* (a), (b) *and* (c) *in Theorem* 1 *hold for each prime ordinal* $\alpha \le \omega_1$.

A space X is called *scattered* if there is an ordinal α such that $X^{(\alpha)} = \emptyset$. Baars and de Groot proved in [3, Corollary 4.1.16] that for l_p-equivalent separable metric spaces X and Y, if X is scattered, then so is Y. It is well known that $X^{(\omega_1)} = \emptyset$ for every scattered, locally separable, metric space X. Thus, we have:

COROLLARY 2. *Let X and Y be l_0- or l_p-equivalent, locally separable, metric spaces. If X is scattered, then so is Y.*

In Section 2, we consider a support of a linear map $\varphi : C_0(X) \to C_0(Y)$ and give some lemmas. In Section 3, we prove Theorem 1 and, answering [3, Question 3, p. 37], we give an example of l_p- and l_0-equivalent, first countable spaces X and Y such that X is locally compact, but Y is not.

The terminology and notation will be used as in [3]. In particular, for $f \in C(X)$, $S \subseteq X$ and $\varepsilon > 0$, we write $\langle f, S, \varepsilon \rangle = \{g \in C(X) : |f(x) - g(x)| < \varepsilon$ for each $x \in S\}$. The family $\{\langle f, K, \varepsilon \rangle : f \in C(X), K \in \mathscr{K}(X)$ and $\varepsilon > 0\}$ is a base for $C_0(X)$, where $\mathscr{K}(X)$ is the family of all compact sets of X. The constant function on X taking value 0 is denoted simply by the same symbol 0. As usual, we identify an ordinal number and the space of all smaller ordinal numbers with the order topology. By a space we mean a completely regular T_1-space.

2. Supports of a linear map

Throughout this section, let $\varphi : C(X) \to C(Y)$ be a linear map and let $y \in Y$ be fixed. Arhangel'skiĭ [1] defined the *support* of y with respect to φ to be the set, denoted by $\mathrm{supp}(y)$, of all $x \in X$ such that for every neighborhood U of x, there is $f \in C(X)$ such that $f|_{X \setminus U} = 0$ and $\varphi(f)(y) \neq 0$. The supports played an important rule in [1] and [3]. However, some authors use the term *support* of y to call a set $S \subseteq X$ such that

$$(\forall f \in C(X))(f|_S = 0 \Rightarrow \varphi(f)(y) = 0), \tag{1}$$

and some other authors also use it for a set $S \subseteq X$ such that

$$(2) \qquad (\forall f \in C(X))(S \subseteq \mathrm{int}_X Z(f) \Rightarrow \varphi(f)(y) = 0),$$

where $Z(f) = \{x : f(x) = 0\}$. We first clarify the relation between supp(y) and sets satisfying the conditions (1) and (2), and then prove some lemmas which will be used in the proof of Theorem 1. Let $\mathscr{S}(y)$ be the family of all closed sets in X satisfying (1). Since $X \in \mathscr{S}(y)$, $\mathscr{S}(y) \neq \varnothing$. By the definition of supp(y), we have:

LEMMA 1. *supp*$(y) = \bigcap \{S : S \in \mathscr{S}(y)\}$.

REMARK 1. The set $\mathscr{S}(y)$ need not be a closed filter on X. For example, consider a space X which has disjoint closed sets F_1 and F_2 such that $\mathrm{cl}_{vX} F_1 \cap \mathrm{cl}_{vX} F_2 \neq \varnothing$, where vX is the Hewitt real compactification of X (e.g., the Tychonoff Plank T and its top edge and right edge [4, 8.20]). Pick a point y from the intersection and let $\varphi : C(X) \to C(vX)$ be the linear map which carries f to the continuous extension. Then, since $F_1, F_2 \in \mathscr{S}(y)$, $\mathscr{S}(y)$ fails to have the finite intersection property.

Let $\mathscr{Z}(X)$ be the family of all zero-sets in X and put $\mathscr{Z}(y) = \mathscr{S}(y) \cap \mathscr{Z}(X)$. A *z-filter* on X is the intersection of a filter on X and $\mathscr{Z}(X)$ (cf. [4]).

LEMMA 2. *Assume that there is* $f_0 \in C(X)$ *such that* $\varphi(f_0)(y) \neq 0$. *Then,* $\mathscr{Z}(y)$ *is a z-filter on* X.

PROOF. Since $f_0|_{\varnothing} = 0$ and $\varphi(f_0)(y) \neq 0$, $\varnothing \notin \mathscr{Z}(y)$. Clearly, if $Z_1 \in \mathscr{Z}(y)$ and $Z_1 \subseteq Z_2 \in \mathscr{Z}(X)$, then $Z_2 \in \mathscr{Z}(y)$. Suppose that $Z_1 \cap Z_2 \notin \mathscr{Z}(y)$ for some $Z_1, Z_2 \in \mathscr{Z}(y)$. Then, there is $g \in C(X)$ such that $g|_{Z_1 \cap Z_2} = 0$ and $\varphi(g)(y) \neq 0$. Since $Z_1, Z_2 \in \mathscr{Z}(X)$, we can write $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$. Define a function h by $h(x) = g(x)|f_1(x)|/(|f_1(x)| + |f_2(x)|)$ for $x \in X \backslash (Z_1 \cap Z_2)$ and $h(x) = 0$ for $x \in Z_1 \cap Z_2$. Since $|h| \leq |g|$ and $h|_{Z_1 \cap Z_2} = 0$, $h \in C(X)$. Since $h|_{Z_1} = 0$, $\varphi(h)(y) = 0$. On the other hand, since $h|_{Z_2} = g|_{Z_2}$, $\varphi(h)(y) = \varphi(g)(y) \neq 0$. This contradiction completes the proof. □

By Lemma 2, $\bigcap \{\mathrm{cl}_{\beta X} Z : Z \in \mathscr{Z}(y)\} \neq \varnothing$, where βX is the Čech-Stone compactification of X. Since $\mathscr{Z}(\beta X)$ is a base for the closed sets in βX,

$$(3) \qquad \bigcap \{\mathrm{cl}_{\beta X} S : S \in \mathscr{S}(y)\} = \bigcap \{\mathrm{cl}_{\beta X} Z : Z \in \mathscr{Z}(y)\}.$$

Thus, we have the following lemma:

LEMMA 3. *Assume that there is* $f_0 \in C(X)$ *such that* $\varphi(f_0)(y) \neq 0$. *Then,* $\bigcap \{\mathrm{cl}_{\beta X} S : S \in \mathscr{S}(y)\} \neq \varnothing$.

REMARK 2. In view of Remark 1, the reader might ask if $\bigcap \{\mathrm{cl}_{\upsilon X} S : S \in \mathscr{S}(y)\} \neq \varnothing$ or not. We show that the intersection can be empty. Let $\boldsymbol{N}$ be the discrete space of positive integers. For each $m, n \in \boldsymbol{N}$, define $e_n(m) = 1$ if $m = n$, $e_n(m) = 0$ otherwise, and let $e_0 \in C(\boldsymbol{N})$ be the constant function taking value 1. Since $A = \{e_n : n \in \boldsymbol{N} \cup \{0\}\}$ is linearly independent, there is a Hamel base B for $C(\boldsymbol{N})$ with $A \subseteq B$. For each $f \in C(\boldsymbol{N})$, there is a unique function $\alpha_f : B \to \boldsymbol{R}$ such that $f = \sum_{b \in B} \alpha_f(b)b$. Define $\varphi(f) = \alpha_f(e_0)$ for $f \in C(\boldsymbol{N})$. Then, $\varphi : C(\boldsymbol{N}) \to \boldsymbol{R}$ $(=C(\{y\}))$ is a linear map and $\varphi(e_0) = 1$. If $f|_{\boldsymbol{N}\setminus\{n\}} = 0$ for some $n \in \boldsymbol{N}$, then $\varphi(f) = 0$, because f is expressed as a scalar multiple of e_n. Hence, $\boldsymbol{N}\setminus\{n\} \in \mathscr{S}(y)$ for each $n \in \boldsymbol{N}$. Since $\upsilon\boldsymbol{N} = \boldsymbol{N}$, this implies that $\bigcap \{\mathrm{cl}_{\upsilon \boldsymbol{N}} S : S \in \mathscr{S}(y)\} = \varnothing$.

LEMMA 4. *Assume that there is* $f_0 \in C(X)$ *such that* $\varphi(f_0)(y) \neq 0$ *and that* $\mathscr{S}(y)$ *contains a compact set K. Then,* supp(y) *is nonempty compact and satisfies the condition* (2).

PROOF. By Lemma 1 and (3),

$$\mathrm{supp}(y) = \bigcap \{S \cap K : S \in \mathscr{S}(y)\}$$

$$= \bigcap \{\mathrm{cl}_{\beta X} S : S \in \mathscr{S}(y)\} \tag{4}$$

$$= \bigcap \{\mathrm{cl}_{\beta X} Z : Z \in \mathscr{Z}(y)\}. \tag{5}$$

By (4) and Lemma 3, supp(y) is nonempty compact. Next, suppose that supp$(y) \subseteq \mathrm{int}_X Z(f)$. Then, there is an open set U in βX with $U \cap X = \mathrm{int}_X Z(f)$. By (5) and Lemma 2, there is $Z \in \mathscr{Z}(y)$ such that $\mathrm{cl}_{\beta X} Z \subseteq U$, and hence, $Z \subseteq Z(f)$. Since Z satisfies (1), $\varphi(f)(y) = 0$. Thus, supp(y) satisfies (2). □

Let $\pi_y : C(Y) \to \boldsymbol{R}$ be the y-th projection, i.e., $\pi_y(f) = f(y)$ for each $f \in C(Y)$.

LEMMA 5. *Assume that* $\pi_y \circ \varphi : C(X) \to \boldsymbol{R}$ *is continuous with respect to the uniform convergence topology on* $C(X)$. *Then, every subset of* X *satisfying the condition* (2) *satisfies* (1).

PROOF. Let S be a subset of X satisfying (2). Suppose that $f \in C(X)$ and $f|_S = 0$. For each $n \in \boldsymbol{N}$, define $f_n(x) = \max\{f(x) - n^{-1}, 0\} + \min\{f(x) + n^{-1}, 0\}$ for $x \in X$. Then, $f_n \in C(X)$ and $S \subseteq \{x : |f(x)| < 1/n\} \subseteq Z(f_n)$. Since S satisfies (2), $(\pi_y \circ \varphi)(f_n) = \varphi(f_n)(y) = 0$ for each $n \in \boldsymbol{N}$. Since $\{f_n\}$ converges to f with respect to the uniform convergence topology, it follows from our assumption that $\varphi(f)(y) = (\pi_y \circ \varphi)(y) = \lim_{n\to\infty}(\pi_y \circ \varphi)(f_n) = 0$. Hence, S satisfies (1). □

LEMMA 6. *Assume that $\pi_y \circ \varphi : C_0(X) \to \boldsymbol{R}$ is continuous. Then,* supp(y) *is compact and satisfies* (1), *and moreover, if there is $f_0 \in C(X)$ such that $\varphi(f_0)(y) \neq 0$, then* supp$(y) \neq \varnothing$.

PROOF. If $\varphi(f)(y) = 0$ for each $f \in C(X)$, then supp$(y) = \varnothing$ and it obviously satisfies (1). Now, assume that $\varphi(f)(y) \neq 0$ for some $f \in C(X)$. By our assumption, $\pi_y \circ \varphi$ is continuous with respect to the uniform convergence topology. By Lemmas 4 and 5, it suffices to show that $\mathscr{S}(y)$ contains a compact set. Since φ is continuous, there is $K \in \mathscr{K}(X)$ such that $\varphi[\langle 0, K, \varepsilon\rangle] \subseteq \langle 0, \{y\}, 1\rangle$. If $g \in C(X)$ and $g|_K = 0$, then by the linearity of φ, $n|\varphi(g)(y)| = |\varphi(ng)(y)| < 1$ for each $n \in \boldsymbol{N}$, which implies that $\varphi(g)(y) = 0$. Hence, $K \in \mathscr{S}(y)$. □

In the preceding corollary, that supp(y) is compact and satisfies (2) was proved in [3], but it was not stated that supp(y) satisfies (1). Lemma 6 and the following lemmas are used in the next section. For $B \subseteq Y$, the *support* of B with respect to φ is the set supp$B = \bigcup\{\text{supp}(y) : y \in B\}$. When φ is a bijection, the support of $A \subseteq X$ with respect to φ^{-1} is also denoted by the same symbol supp$A$. The next lemma was proved in [3].

LEMMA 7 ([3, Lemma 1.5.6]). *If $\varphi : C_0(X) \to C_0(Y)$ is continuous and B is a compact set in Y, then* $\text{cl}_X(\text{supp}\, B)$ *is compact.*

LEMMA 8. *If $\varphi : C_0(X) \to C_0(Y)$ is a homeomorphism, then* $x \in \text{cl}_X(\text{suppsupp}(x))$ *for each* $x \in X$.

PROOF. Suppose that $x \notin \text{cl}_X(\text{suppsupp}(x))$ for some $x \in X$. Then, there is $f \in C(X)$ such that $f(x) = 1$ and $f[\text{suppsupp}(x)] = \{0\}$. By Lemma 6, $\varphi(f)|_{\text{supp}(x)} = 0$ and hence $f(x) = 0$, which is a contradiction. □

3. Proof of Theorem 1

We need some more lemmas to prove Theorem 1. The following one was proved by Baars and de Groot [3].

LEMMA 9 ([3, Lemma 1.2.10]). *Let X and Y be normal spaces, K a non-empty compact set in Y, $\{U_n : n \in N\}$ a decreasing neighborhood base of K in Y, and $\{A_s : s \in S\}$ a locally finite family of subsets of X. Suppose that there is a linear continuous map $\varphi : C_0(X) \to C_0(Y)$. Then, there are $m \in N$ and $s_1, \ldots, s_m \in S$ such that* $(\mathrm{supp}\, U_m) \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} A_s = \varnothing$.

The following Lemmas 10 and 12 sharpen Baars and de Groot's idea frequently used in [3]. Lemma 11 is well known.

LEMMA 10. *Let X and Y be metric spaces and $\varphi : C_0(X) \to C_0(Y)$ a linear homeomorphism. Let A be a closed set in Y and $B = \mathrm{cl}_X(\mathrm{supp}\ A)$. Let U be an open set in X such that $A \cap \mathrm{cl}_Y(\mathrm{supp}\, U) = \varnothing$. Then, $C_0(A)$ is linearly homeomorphic to a subspace of $C_0(B \backslash U)$.*

PROOF. Let $S = B \cup \mathrm{cl}_X U$ and $T = \{f \in C_0(S) : f|_{\mathrm{cl}\, U} = 0\}$. Then, the subspace T of $C_0(S)$ is linearly homeomorphic to the subspace $\{f \in C_0(B \backslash U) : f|_{B \cap (\mathrm{cl}\, U \backslash U)} = 0\}$ of $C_0(B \backslash U)$. Thus, it suffices to show that there is a linear embedding $\lambda : C_0(A) \to T$. Define $r_S(f) = f|_S$ for each $f \in C_0(X)$ and $r_A(f) = f|_A$ for each $f \in C_0(Y)$. By the Dugundji extension theorem (cf. [3, Theorem 2.3.1]), there is a linear continuous map $e_S : C_0(S) \to C_0(X)$ such that $r_S \circ e_S = \mathrm{id}_{C(S)}$. Since $A \cap \mathrm{cl}_Y(\mathrm{supp}\, U) = \varnothing$, using the Dugundji theorem again, we can define a linear continuous map $e_A : C_0(A) \to C_0(Y)$ such that $r_A \circ e_A = \mathrm{id}_{C(A)}$ and $e_A(f)|_{\mathrm{supp}\, U} = 0$ for each $f \in C_0(A)$ (cf. [3, Lemma 4.1.11]). Define $\lambda = r_S \circ \varphi^{-1} \circ e_A$ and $\mu = r_A \circ \varphi \circ e_S$. Then, $\lambda : C_0(A) \to C_0(S)$ and $\mu : C_0(S) \to C_0(A)$ are linear continuous maps. For each $f \in C_0(A)$, since $e_A(f)|_{\mathrm{supp}\, U} = 0$, it follows from Lemma 6 that $\varphi^{-1}(e_A(f))|_U = 0$, which implies that $\lambda(f) \in T$. Hence, $\lambda[C_0(A)] \subseteq T$. It remains to show that $\mu \circ \lambda = \mathrm{id}_{C(A)}$. Let $g \in C_0(A)$. Since $r_S \circ e_S = \mathrm{id}_{C(S)}$ and $\lambda = r_S \circ \varphi^{-1} \circ e_A$,

$$e_S(\lambda(g))|_S = \lambda(g) = \varphi^{-1}(e_A(g))|_S. \tag{6}$$

Since $\mathrm{supp}\, A \subseteq S$, it follows from Lemma 6 that $\varphi(e_S(\lambda(g))|_A = e_A(g)|_A$. Since $\mu = e_A \circ \varphi \circ e_S$ and $r_A \circ e_A = \mathrm{id}_{C(A)}$, $(\mu \circ \lambda)(g) = g$. Hence, $\mu \circ \lambda = \mathrm{id}_{C(A)}$. □

LEMMA 11 (cf. [3, Proposition 2.2.4]). *Let A be a subspace of a space X and α an ordinal. Then, $A^{(\alpha)} \subseteq A \cap X^{(\alpha)}$, and if A is an open set, then $A^{(\alpha)} = A \cap X^{(\alpha)}$.*

For a scattered space X, let $\kappa(X)$ denote the smallest ordinal α such that $X^{(\alpha)} = \varnothing$. For a non-scattered space X, we write $\kappa(X) > \alpha$ for each ordinal α. For spaces X and Y, $X \approx Y$ means that X is homeomorphic to Y.

Lemma 12. *Under the same assumption as in Lemma* 8, *assume further that* $\kappa(A) > \alpha$ *for a prime ordinal* $\alpha \leq \omega_1$. *Then,* $\kappa(B\backslash U) > \alpha$.

Proof. If $B\backslash U$ is not scattered, then there is nothing to prove. So, we assume that $B\backslash U$ is scattered. We distinguish three cases:

Case 1. $\alpha = 0$. Since $\kappa(A) > 0$, $A \neq \varnothing$. Then, $B\backslash U \neq \varnothing$ by Lemma 10, and hence, $\kappa(B\backslash U) > 0$.

Case 2. $0 < \alpha < \omega_1$. Since $\kappa(A) > \alpha$, $A^{(\alpha)} \neq \varnothing$. By [3, Lemma 4.1.8], there is a compact set $K \subseteq A$ such that $K \approx \omega^\alpha + 1$. Put $L = \mathrm{cl}_X(\operatorname{supp} K)$; then $L \subseteq B$. By Lemma 10, $C_0(K)$ is linearly homeomorphic to a subspace of $C_0(L\backslash U)$. Thus, $L\backslash U \neq \varnothing$, and it is compact by Lemma 7. Moreover, since $B\backslash U$ is scattered, so is $L\backslash U$. Hence, $\kappa(L\backslash U) = \beta + 1$ for some $\beta < \omega_1$ and $(L\backslash U)^{(\beta)}$ consists of finitely many points, say $x_1, \dots, x_k$. By Sierpiński-Mazurkiewicz's theorem [3, Theorem 2.2.8], $L\backslash U \approx (\omega^\beta \cdot k) + 1$. Hence, $C_0(\omega^\alpha + 1)$ is linearly embedded in $C_0((\omega^\beta \cdot k) + 1)$. If $\alpha = 1$, then $\beta \geq 1$, because $C(\omega + 1)$ cannot be linearly embedded in a finitely dimensional space. Hence, $\kappa(B\backslash U) \geq \kappa(L\backslash U) = \beta + 1 > 1$. If $\alpha > 1$, since α is prime, it follows from [3, Lemma 2.6.7 (a)(ii)] that $\alpha \leq \beta + 1$. Since α is a limit, $\alpha < \beta + 1 = \kappa(L\backslash U) \leq \kappa(B\backslash U)$.

Case 3. $\alpha = \omega_1$. Suppose on the contrary that $\kappa(B\backslash U) \leq \omega_1$. Then, since $(B\backslash U)^{(\omega_1)} = \varnothing$, there is a locally finite cover $\{C_\gamma : \gamma < \omega_1\}$ of X by closed sets such that $C_\gamma \cap (B\backslash U)^{(\gamma)} = \varnothing$ for each $\gamma < \omega_1$. On the other hand, since $\kappa(A) > \omega_1$, there is $y \in A^{(\omega_1)}$. Let $\{V_n : n \in \omega\}$ be a decreasing neighborhood base of y in Y. By Lemma 9, there are $m < \omega$ and a finite set $F \subseteq \omega_1$ such that $\operatorname{supp} V_m \subseteq \bigcup_{\gamma \in F} C_\gamma$. Put $\delta = \max F$. Then

$$\mathrm{cl}_X \operatorname{supp} V_m \cap (B\backslash U)^{(\delta)} = \varnothing. \tag{7}$$

Choose a prime ordinal ρ with $\delta \leq \rho < \omega_1$. Since V_m is open, it follows from Lemma 11 that $(V_m \cap A)^{(\rho)} = V_m \cap A^{(\rho)} \supseteq V_m \cap A^{(\omega_1)} \neq \varnothing$. Hence, there is $K' \subseteq V_m \cap A$ with $K' \approx \omega^\rho + 1$ by [3, Lemma 4.1.8]. Put $L' = \mathrm{cl}_X(\operatorname{supp} K')$. Then, $L' \subseteq \mathrm{cl}_X \operatorname{supp} V_m$. By (7) this combined with Lemma 11 implies that $(L'\backslash U)^{(\delta)} \subseteq L' \cap (B\backslash U)^{(\delta)} = \varnothing$. Hence, $\kappa(L'\backslash U) \leq \delta < \rho$. Since $\kappa(K') > \rho$, this contradicts Case 2 we have proved above. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Since X and Y are l_0-equivalent, there is a linear homeomorphism $\varphi : C_0(X) \to C_0(Y)$.

(a) Suppose that $X^{(\alpha)} = \varnothing \neq Y^{(\alpha)}$ for a prime ordinal $\alpha \leq \omega_1$. Then, $\kappa(Y) > \alpha$. Since $X^{(\alpha)} = \varnothing$, $\kappa(\mathrm{cl}_X(\mathrm{supp}\, Y)) \leq \kappa(X) \leq \alpha$. This contradicts Lemma 12.

(b) Suppose that there is a prime ordinal $\alpha \leq \omega_1$ such that $X^{(\alpha)}$ is compact but $Y^{(\alpha)}$ is not. Then, there is a decreasing neighborhood base $\{U_n : n < \omega\}$ of $X^{(\alpha)}$ in X and a discrete family $\{V_n : n < \omega\}$ of open sets in Y such that $V_n \cap Y^{(\alpha)} \neq \varnothing$ for each $n < \omega$. By Lemma 9, there is $m < \omega$ such that $(\mathrm{supp}\, U_m) \cap V_m = \varnothing$. Let A be a closed set in Y such that $A \subseteq V_m$ and $\mathrm{int}_Y A \cap Y^{(\alpha)} \neq \varnothing$. Then, $\kappa(A) > \alpha$ by Lemma 11. Put $B = \mathrm{cl}_X(\mathrm{supp}\, A)$. Then, by Lemma 11, $(B \backslash U_m)^{(\alpha)} \subseteq (B \backslash U_m) \cap X^{(\alpha)} = \varnothing$. Hence, $\kappa(B \backslash U_m) \leq \alpha$, which contradicts Lemma 12.

(c) Suppose that $X^{(\alpha)}$ is locally compact for a prime ordinal $\alpha \leq \omega_1$. Then, there is a locally finite cover $\{C_s : s \in S\}$ of X by closed sets such that $C_s \cap X^{(\alpha)}$ is compact for each $s \in S$. Let $y \in Y^{(\alpha)}$ and $\{U_n : n < \omega\}$ be a decreasing neighborhood base of y in Y. Then, by Lemma 9, there is $k < \omega$ and a finite set $F \subseteq S$ such that $\mathrm{supp}\, U_k \subseteq \bigcup_{s \in F} C_s$. It suffices to show that $\mathrm{cl}_Y U_k \cap Y^{(\alpha)}$ is compact. Suppose not; then there is a discrete family $\{V_n : n < \omega\}$ of open sets in Y such that $V_n \subseteq U_k$ and $U_n \cap Y^{(\alpha)} \neq \varnothing$ for each $n < \omega$. Put $C = \bigcup_{s \in F} C_s$. Since $C^{(\alpha)} \subseteq C \cap X^{(\alpha)}$ by Lemma 11, $C^{(\alpha)}$ is compact. Hence, there is a decreasing neighborhood base $\{W_n : n < \omega\}$ of $C^{(\alpha)}$ in X. By Lemma 9 again, $(\mathrm{supp}\, W_m) \cap V_m = \varnothing$ for some $m < \omega$. Let A be a closed set in Y such that $A \subseteq V_m$ and $\mathrm{int}_Y V_m \cap Y^{(\alpha)} \neq \varnothing$. Then, $\kappa(A) > \alpha$ by Lemma 11. Put $B = \mathrm{cl}_X(\mathrm{supp}\, A)$. Since $B \subseteq \mathrm{cl}_X(\mathrm{supp}\, U_k) \subseteq C$,

$$(B \backslash W_m)^{(\alpha)} \subseteq (B \backslash W_m) \cap C^{(\alpha)} \tag{8}$$

by Lemma 11. Since $C^{(\alpha)} \subseteq W_m$, (8) implies that $(B \backslash W_m)^{(\alpha)} = \varnothing$, and hence, $\kappa(B \backslash W_m) \leq \alpha$. Since $\mathrm{cl}_Y(\mathrm{supp}\, W_m) \cap A = \varnothing$, this contradicts Lemma 12. □

REMARK 3. For each ordinal $\alpha < \omega_1$ which is not prime, there are l_0-equivalent spaces X and Y such that $X^{(\alpha)}$ is compact but $Y(\alpha)$ is not locally compact. To show this, let $\alpha < \omega_1$ be an ordinal which is not prime. Then, by [3, Corollary 2.1.18], there is the largest prime ordinal β less than α. Let $S = \omega^\beta + 1$ and $T = \omega^\alpha + 1$. Since $\beta\omega$ is prime, $\beta < \alpha < \beta\omega$. Hence, it follows from Bessaga-Pelczyński's theorem [3, Theorem 2.4.1] that S and T are l_0-equivalent. Observe that $S^{(\alpha)} = \varnothing$ and $T^{(\alpha)} = \{\omega^\alpha\}$ (cf. [3, Proposition 2.2.5]). Define $X = (S \times (\omega \times \omega)) \cup \{\infty\}$ and $Y = (T \times (\omega \times \omega)) \cup \{\infty\}$, where the subspace $S \times (\omega \times \omega)$ of X has the usual product topology, a basic neighborhood of $\infty \in X$ is a set of the form $(S \times ((\omega \backslash n) \times \omega)) \cup \{\infty\}$ for $n < \omega$, and the topology of Y is analogously defined. Then, it is easily checked that X and Y are l_0-equivalent and

$Y^{(\alpha)}$ is not locally compact. If $\beta + 1 < \alpha$, $X^{(\alpha)} = \emptyset$ and if $\beta + 1 = \alpha$, then $X^{(\alpha)} = \{\infty\}$. In each case, $X^{(\alpha)}$ is compact. The authors do not know if the statements (a), (b) and (c) in Theorem 1 hold for a prime ordinal greater than ω_1 (cf. [3, Question, p. 149]).

Gul'ko-Okunev [5] and McCoy-Ntantu [6] independently proved that for a first countable, paracompact space X, $C_0(X)$ is a Baire space if and only if X is locally compact. Since l_p-equivalent paracompact spaces are l_0-equivalent by [1, Corollary 5], we have: *For l_p-equivalent, first countable, paracompact spaces X and Y, if X is locally compact, then so is Y* (cf. also [3, Theorem 1.5.10]). In [3, Question 3, p. 37], Baars and de Groot asked if the paracompactness is essential in this statement. The following example answers their question positively.

Example. *There exist first countable, l_p- and l_0-equivalent spaces X and Y such that X is locally compact, but Y is not.*

Proof. Let $X = \omega_1 \times (\omega + 1)$, $A = \omega_1 \times \{\omega\} \subseteq X$, $Y = (X/A) \oplus A$, and $p : X \to X/A$ the quotient map. Since A is a retract of X, it is routinely proved that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ (cf. [2, Proposition 1]). Moreover, since $\mathrm{cl}_X p^{-1}[K \setminus p[A]]$ is compact for every compact set $K \subseteq Y$, it is also proved that $C_0(X)$ is linearly homeomorphic to $C_0(Y)$. Thus, X and Y are l_p- and l_0-equivalent. The space X is first countable and locally compact, but Y is not locally compact. Since every open set in X including A includes a set of the form $\omega_1 \times ((\omega + 1) \setminus n)$, Y is also first countable. □

Acknowledgment. In the first version of the paper, the authors proved: *For l_0-equivalent, first countable, paracompact spaces X and Y, if X is locally compact, then so is Y.* A. V. Arhangel'skiĭ kindly informed them that this result immediately follows from the theorem, by Gul'ko-Okunev [5] and McCoy-Ntantu [6], quoted before Example. M. Sakai and the referee also pointed out that the result follows from [6, Corollary 5.3.4]. The authors would like to thank them for their helpful comments.

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