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THE CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATION WITH DEGENERACY

By

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§1. Introduction and Results

In this paper we consider the Cauchy problem for Schrödinger type equation

$$(1.1)\qquad \begin{cases} \left(\partial_t + i\frac{1}{2}\sum_{j,k=1}^{n} D_j(a_{jk}(x)D_k) + \sum_{j=1}^{n} b_j(t,x)D_j + c(t,x)\right)u(t,x) \\ \quad = f(t,x) \text{ in } \mathscr{D}'((0,T)\times \boldsymbol{R}_x^n) \\ u(0,x) = u_0(x) \end{cases}$$

where $i=\sqrt{-1}$, $D_j = -i\partial_j = -i\partial/\partial x_j$, $T>0$. We assume

$$(A1)\qquad \begin{cases} a_{jk}\in \mathscr{B}^\infty(\boldsymbol{R}^n) \text{ real valued}, a_{jk}(x) = a_{kj}(x), 1\le j,k\le n, \\ b_j, c\in C([0,T];\mathscr{B}^\infty(\boldsymbol{R}^n)), 1\le j\le n, \\ \text{and there exists } \delta\ge 0 \text{ such that} \\ \frac{1}{2}\sum_{j,k=1}^{n} a_{jk(x)}\xi_j\xi_k \ge \delta|\xi|^2 \quad \text{for } x,\xi\in \boldsymbol{R}^n. \end{cases}$$

Here $\mathscr{B}^\infty(\boldsymbol{R}^n) = \{f\in C^\infty(\boldsymbol{R}^n); \partial^\alpha f\in L^\infty \text{ for all } \alpha\in N^n\}$.

Put $a_2(x,\xi) = \frac{1}{2}\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k, a_1(t,x,\xi) = \sum_{j=1}^n b_j(t,x)\xi_j$.

The purpose of this paper is to give a sufficient condition for the Cauchy problem (1.1) to be well-posed in the framework of the Sobolev spaces H^s. First we recall the related results. About the necessity, the following theorem has been shown by Ichinose [Ic.2] (resp. Hara [Ha]) in case of L^2 (resp. H^∞) well-posedness.

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THEOREM ([Ic.2], [Ha]). *Suppose* (A1) *with* $b_j(t,x) = b_j(x), 1 \le j \le n$, *and with* $\delta > 0$. *If* (1.1) *is* L^2 *well-posed, then there exists* $C > 0$ *such that for any* $t \ge 0$

$$\sup_{y,\eta \in \boldsymbol{R}^n, |\eta|=1} \left| \int_0^t Re \sum_{j=1}^n b_j(X(\tau,y,\eta)) \Xi_j(\tau,y,\eta)\, d\tau \right| \le C.$$

In the H^∞ *case, the above condition should be replaced by*

$$\sup_{y,\eta \in \boldsymbol{R}^n, |\eta|=1} \left| \int_0^t Re \sum_{j=1}^n b_j(X(\tau,y,\eta)) \Xi_j(\tau,y,\eta)\, d\tau \right| \le C \log(1+t) + C'$$

for any $t \ge 0$ *with* C, $C' > 0$. *Here* $(X(t,y,\eta), \Xi(t,y,\eta))$ *is the integral curve of the Hamilton vector field* H_{a_2} *through* (y,η) *at* $t = 0$. (*See* [Ic.2] *and* [Ha] *for the precise statements.*)

About the sufficiency, [Ta], [Mi], [Ic.1], [Do.1], etc in the case of $a_2(x,\xi) = \frac{1}{2}|\xi|^2$ and [Ka], [Do.2] in the case of $a_2(x,\xi)$ with variable coefficients are known.

In those results, the case where $\delta > 0$ (i.e. $a_2(x,\xi)$ is uniformly elliptic) is treated. So, in the present paper, we shall treat the case where $\delta = 0$. In such case, of course, it is not obvious how condition for the lower order term $a_1(t,x,\xi)$ is sufficient. So our aim is to give a sufficient condition for the lower order term $a_1(t,x,\xi)$.

Since it seems not easy to treat a general case for principle part $a_2(x,\xi)$, here we will consider a special case. More precisely, we consider the following Cauchy problem

$$(1.2)\quad \begin{cases} \left(\partial_t + i\frac{1}{2}(D_1^2 + \psi(x_1)\sum_{j,k=2}^n D_j(a_{jk}(x')D_k)) + \sum_{j=1}^n b_j(t,x)D_j + c(t,x)\right)u(t,x) \\ \quad = f(t,x) \text{ in } \mathscr{D}'((0,T)\times \boldsymbol{R}_x^n) \\ u(0,x) = u_0(x) \end{cases}$$

where $x' = (x_2, \dots, x_n), \xi' = (\xi_2, \dots, \xi_n), n \ge 2$. Here we assume (A2) which is composed of the following conditions (1.3) ~ (1.6):

$$(1.3)\quad \begin{cases} \psi \in \mathscr{B}^\infty(\boldsymbol{R}), \psi(0) = 0, \sup_t |t\psi'(t)| < \infty, \text{and there exist } \mu, \nu > 0 \\ \text{such that } t\psi'(t) \ge 0 \text{ for } |t| \le \mu \text{ and } \psi(t) \ge \nu \text{ for } |t| \ge \mu. \end{cases}$$

$$(1.4)\quad \begin{cases} a_{jk} \in \mathscr{B}^\infty(\boldsymbol{R}^{n-1}) \text{ real valued}, a_{jk}(x') = a_{kj}(x'), 2 \le j,k \le n, \\ b_j, c \in C([0,T]; \mathscr{B}^\infty(\boldsymbol{R}^n)), 1 \le j \le n. \end{cases}$$

$$(1.5)\quad \begin{cases} \text{There exists } C_1 > 0 \text{ such that} \\ a'_2(x',\xi') \ge C_1|\xi'|^2 \quad \text{for } x',\xi' \in \boldsymbol{R}^{n-1}, \text{where } a'_2(x',\xi') = \sum_{j,k=2}^n a_{jk}(x')\xi_j\xi_k. \end{cases}$$

$$(1.6)\quad \begin{cases} \text{There exist } \theta_j \in C^\infty(\boldsymbol{R}^{n-1})(2 \le j \le n) \text{ and } C_2, C_\alpha > 0 \text{ such that} \\ |\partial_{x'}^\alpha \theta_j(x')| \le C_\alpha(1+|x'|) \quad \text{for } x' \in \boldsymbol{R}^{n-1}, 2 \le j \le n, \alpha \in \mathbf{N}^{n-1}, \\ H_{a'_2}\theta(x',\xi') \ge C_2|\xi'|^2 \quad \text{for } x',\xi' \in \boldsymbol{R}^{n-1}, \text{where } \theta(x',\xi') = \sum_{j=2}^n \theta_j(x')\xi_j. \end{cases}$$

Remark. The condition (1.6) was introduced by [Do.2] (see Theorem 1.2 of [Do.2]). If the following Kajitani type condition

$$2a'_2(x',\xi') - \sum_{j=2}^n x_j\partial_j a'_2(x',\xi') \ge \delta|\xi'|^2 \quad \text{for } x',\xi' \in \boldsymbol{R}^{n-1} \quad \text{with } \delta > 0$$

are fulfilled, then (1.6) is satisfied with $\theta_j(x') = x_j, 2 \le j \le n$.

Before stating our main results, we prepare notation. For usual notation,

$$\langle\xi\rangle = (10+|\xi|^2)^{1/2}(\xi \in \boldsymbol{R}^n), L^2 = L^2(\boldsymbol{R}^n), (\cdot,\cdot) = (\cdot,\cdot)_{L^2}, \|\cdot\| = \|\cdot\|_{L^2}.$$

$$H^s = H^s(\boldsymbol{R}^n) = \{u \in S'(\boldsymbol{R}^n); \langle\xi\rangle^s\hat{u}(\xi) \in L^2\}, \|u\|_s = \|\langle\xi\rangle^s\hat{u}(\xi)\|, (s \in \boldsymbol{R}).$$

$$H^\infty = \bigcap_{s\in R} H^s, H^{-\infty} = \bigcup_{s\in R} H^s, H_p = \sum_{j=1}^n(\partial_{\xi_j}p\partial_{x_j} - \partial_{x_j}p\partial_{\xi_j}).$$

$$C^k([0,T];X) = \{f; f(t,\cdot) \in C^k([0,T]) \text{ in the topology of } X\},$$

$$L^p([0,T];X) = \{f; f(t,\cdot) \in L^p([0,T]) \text{ in the topology of } X\},$$

where X is a Fréchet space, $k = 0,1,2,\cdots, 1 \le p \le \infty$.

$$C([0,T];H^{-\infty}) = \bigcup_{s\in R} C([0,T];H^s), L^p([0,T];H^\infty) = \bigcap_{s\in R} L^p([0,T];H^s)\ (1 \le p \le \infty).$$

For $S^m_{\rho,\delta}$ and $S(m,g)$, see Chapter 18 in [Hö].

Now we state our main results.

THEOREM 1.1. *Assume* (A2). *In addition, we assume*

$$\text{(A3)} \qquad Re\, b_1 \in C^1([0,T]; \mathscr{B}^\infty(\mathbf{R}^n)).$$

(1) *If there exists a positive nonincreasing function* $\lambda(t) \in C([0,\infty)) \cap L^1([0,\infty))$ *satisfying*

$$(1.7) \quad \begin{cases} |Re\, b_1(t,x)| \le \lambda(|x|) \quad \text{for } x \in \mathbf{R}^n, \quad 0 \le t \le T, \\ |Re\, \partial_j b_1(t,x)| \le \lambda(|x'|) \quad \text{for } |x_1| \le 2\mu, x' \in \mathbf{R}^{n-1}, 0 \le t \le T, 2 \le j \le n, \\ |Re\, b_j(t,x)| \le \psi(x_1)\lambda(|x|) \quad \text{for } x \in \mathbf{R}^n, 0 \le t \le T, 2 \le j \le n, \end{cases}$$

then for any $u_0 \in H^s$ *and* $f \in L^1([0,T]; H^s)$ *there exists a solution* $u \in C([0,T]; H^s)$ *of* (1.2), *and it is unique in* $C([0,T]; H^{-\infty})$.

(2) *If there exists a positive nonincreasing function* $\lambda(t) \in C([0,\infty))$ *satisfying*

$$\int_0^t \lambda(\tau)\, d\tau \le C \log(1+t) + C' \text{ for } t \ge 0, \text{with } C, C' > 0 \text{ and } (1.7),$$

then for any $u_0 \in H^s$ *and* $f \in L^1([0,T]; H^s)$ *there exists a solution* $u \in C([0,T]; H^{s-\gamma})$ *of* (1.2), *and it is unique in* $C([0,T]; H^{-\infty})$. *Here* $\gamma = \gamma(T) > 0$ *is independent of s. Especially for any* $u_0 \in H^\infty$ *and* $f \in L^1([0,T]; H^\infty)$ *there exists a unique solution* $u \in C([0,T]; H^\infty)$ *of* (1.2).

To prove Theorem 1.1 we shall use the following theorem, which is a *degenerate* version of Theorem 1.4 in [Do.2].

THEOREM 1.2. *Assume* (A1). *In addition, we assume the following conditions* (A4) *and* (A5):

(A4) *There exists* $e \in S^1_{1,0}$ *such that* $e(x,\xi) \ge \delta\langle\xi\rangle$ *with some* $\delta > 0$ *and that* $\{e, a_2\} \in S^1_{1,0}$.

(A5) *There exists a real valued function* $q \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ *such that with* $C_{\alpha\beta}, C_1, C_2 > 0$

$$(1.8) \qquad |\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \le C_{\alpha\beta} m(x) \langle\xi\rangle^{-|\alpha|} \quad \text{for } x, \xi \in \mathbf{R}^n, \alpha, \beta \in \mathbf{N}^n,$$

$$(1.9) \qquad H_{a_2} q \ge C_1 \rho(x)|\xi| - C_2 \quad \text{for } x, \xi \in \mathbf{R}^n,$$

where $m \in C^\infty(\mathbf{R}^n)$ *and* $\rho(x) \in C(\mathbf{R}^n)$ *satisfy that with* $C, C', C'', C_\alpha > 0$

$$(1.10) \quad \begin{cases} \sqrt{10} \le m(x) \le C\langle x\rangle, |\partial^\alpha m(x)| \le C_\alpha \quad \text{for } x \in \mathbf{R}^n, \alpha \in N^n, |\alpha| \ge 1, \\ 0 \le \rho(x) \le C' \quad \text{for } x \in \mathbf{R}^n, \end{cases}$$

and

$$(1.11)\qquad |(\nabla_\xi a_2 \cdot \nabla_x m)(x,\xi)| \le C''\rho(x)|\xi| \quad \text{for } x,\xi \in \boldsymbol{R}^n.$$

(1) *If there exists a positive nonincreasing function* $\lambda(t) \in C([0,\infty)) \cap L^1([0,\infty))$ *satisfying*

$$(1.12)\qquad |Re\, b_j(t,x)| \le \rho(x)\lambda(|x|) \quad \text{for } x \in \boldsymbol{R}^n, 0 \le t \le T, 1 \le j \le n,$$

then for any $u_0 \in H^s$ *and* $f \in L^1([0,T];H^s)$ *there exists a solution* $u \in C([0,T];H^s)$ *of* (1.1), *and it is unique in* $C([0,T];H^{-\infty})$. *Moreover the unique solution* $u \in C([0,T];H^s)$ *of* (1.1) *corresponding to* $u_0 \in H^s$ *and* $f \in L^2([0,T];H^s)$ *belongs to* $L^2([0,T];X^s)$. *Here* X^s *is the Hilbert space with the following norm*

$$\|u\|_{X^s}^2 = (\rho(x)\lambda(|x|)\langle D\rangle^{s+1/2}u, \langle D\rangle^{s+1/2}u) + \|u\|_s^2.$$

(2) *If there exists a positive nonincreasing function* $\lambda(t) \in C([0,\infty))$ *satisfying* $\int_0^t \lambda(\tau)\,d\tau \le C\log(1+t) + C'$ *for* $t \ge 0$, *with* $C, C' > 0$ *and* (1.12), *then for any* $u_0 \in H^s$ *and* $f \in L^1([0,T];H^s)$ *there exists a solution* $u \in C([0,T];H^{s-\gamma})$ *of* (1.1), *and it is unique in* $C([0,T];H^{-\infty})$. *Here* $\gamma = \gamma(T) > 0$ *is independent of* s. *Especially for any* $u_0 \in H^\infty$ *and* $f \in L^1([0,T];H^\infty)$ *there exists a unique solution* $u \in C([0,T];H^\infty)$ *of* (1.1).

§ 2. Preliminaries

In this section we consider the following Cauchy problem

$$(2.1)\qquad \begin{cases} (\partial_t + a(t,x,D))u = f \text{ in } \mathscr{D}'((0,T)\times \boldsymbol{R}^n) \\ u(0,x) = u_0(x) \end{cases}$$

Here we assume (B1) and (B2);

$$(\mathrm{B1})\qquad a(t,x,\xi) = ia_2(x,\xi) + a_1(t,x,\xi) + a_0(t,x,\xi),$$

where $a_2 \in S^2_{1,0}, a_2(x,D)^* = a_2(x,D), a_j \in C([0,T];S^j_{1,0}), j = 0,1.$

(B2) There exists $e \in S^1_{1,0}$ such that $e(x,\xi) \ge \delta\langle\xi\rangle$ with some $\delta > 0$ and that $\{e, a_2\} \in S^1_{1,0}$.

Now we shall recall the results in [Do.1], [Do.2].

LEMMA 2.1 (see Lemma 2.1 and Lemma 2.2 in [Do.2]). *Assume* (B1), (B2). *If there exist* $p \in S^0_{1,0}$ *of real value and* $C > 0$ *satisfying*

$$H_{a_2}p + Re\, a_1 \ge -C \quad \text{for } x,\xi \in \boldsymbol{R}^n, 0 \le t \le T.$$

Then for any $u_0 \in H^s$ *and* $f \in L^1([0,T]; H^s)$ *there exists a solution* $u \in C([0,T]; H^s)$ *of* (2.1), *and it is unique in* $C([0,T]; H^{-\infty})$. *Moreover if* $f \in L^2([0,T]; H^s)$, *then* $u \in L^2([0,T]; X^s)$ *and satisfies*

$$\int_0^t \|u(\tau)\|_{X^s}^2 \, d\tau \le C_1(\|u(0)\|_s^2 + \int_0^t \|f(\tau)\|_s^2 \, d\tau) \quad \text{for } 0 \le t \le T, \text{with } C_1 > 0.$$

Here

$$\|u(t)\|_{X^s}^2 = ((H_{a_2}p + Re\, a_1)(t,x,D)\langle D\rangle^s u(t), \langle D\rangle^s u(t)) + C_{s,T}\|u(t)\|_s^2,$$

with a large constant $C_{s,T} > 0$.

If there exist $p \in S_{1,0}(\log\langle\xi\rangle) = S(\log\langle\xi\rangle, |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$ *of real value and* $C, C' > 0$ *satisfying*

$$H_{a_2}p + Re\, a_1 \ge -C\log\langle\xi\rangle - C' \quad \text{for } x, \xi \in \boldsymbol{R}^n, 0 \le t \le T,$$

then for any $u_0 \in H^s$ *and* $f \in L^1([0,T]; H^s)$ *there exists a solution* $u \in C([0,T]; H^{s-\gamma})$ *of* (2.1), *and it is unique in* $C([0,T]; H^{-\infty})$. *Here* $\gamma = \gamma(T) > 0$ *is independent of* s. *Especially for any* $u_0 \in H^\infty$ *and* $f \in L^1([0,T]; H^\infty)$ *there exists a unique solution* $u \in C([0,T]; H^\infty)$ *of* (2.1).

We need the following *degenerate* version of Lemma 2.3 in [Do.2]:

Lemma 2.2. *Assume* (A1), (A4) *and* (A5). *Let* $\lambda(t)$ *be a positive non-increasing function in* $C([0,\infty))$.

(1) *If* $\lambda \in L^1([0,\infty))$, *then there exist* $p \in S_{1,0}^0$ *of real value and* $C > 0$ *such that*

$$H_{a_2}p \ge \rho(x)\lambda(|x|)|\xi| - C \quad \text{for } x, \xi \in \boldsymbol{R}^n.$$

(2) *If* $\int_0^t \lambda(\tau)\, d\tau \le C\log(1+t) + C'$ *for* $t \ge 0$, *with* $C, C' > 0$, *then there exist* $p \in S_{1,0}(\log\langle\xi\rangle)$ *of real value and* $C_1, C_2 > 0$ *such that*

$$H_{a_2}p \ge \rho(x)\lambda(|x|)|\xi| - C_1\log\langle\xi\rangle - C_2 \quad \text{for } x, \xi \in \boldsymbol{R}^n.$$

§3. Proofs

Proof of Lemma 2.2 (cf. Proof of Lemma 2.3 in [Do.2]). Take $K, L \ge 1$ such that $|q(x,\xi)| \le Km(x), m(x) \le L\langle x\rangle$ for $x, \xi \in \boldsymbol{R}^n$. Extend $\lambda(t)$ by $\lambda(t) = \lambda(0)$ for $t \le 0$. By Lemma 3.1 in [Do.2], there exists a nonnegative function

$f \in C^\infty([0,\infty))$ such that $\lambda(K^{-1}L^{-1}t - 10) \le f'(t), t \ge 0$, and that for all $m \in \boldsymbol{Z}_+$

$$|f^{(m)}(t)| \le C_m\left(\lambda(0) + \int_0^t \lambda(s)\,ds\right)(1+t)^{-m} \quad \text{for } t \ge 0,$$

with $C_m > 0$. Then

$$f'(|q|) \ge \lambda(K^{-1}L^{-1}|q| - 10) \ge \lambda(\langle x\rangle - 10) \ge \lambda(|x|) \quad \text{for } x, \xi \in \boldsymbol{R}^n.$$

Let $0 < \varepsilon \ll 1$ be a parameter fixed later. Take $\phi(t) \in C^\infty(\boldsymbol{R})$ such that $\phi(t) = 0$ if $t \le 1$, $\phi(t) = 1$ if $t \ge 2$ and $\phi'(t) \ge 0$ on $\boldsymbol{R}$. Set $\phi_+(t) = \phi(t/\varepsilon)$, $\phi_-(t) = \phi_+(-t)$ and $\phi_0 = 1 - \phi_+ - \phi_-$. Define $\psi_0, \psi_\pm$ by $\psi_0 = \phi_0(q/m(x))$, $\psi_\pm = \phi_\pm(q/m(x))$. By (1.8) and (1.10), we have ψ_0, $\psi_\pm \in S^0_{1,0}$. Noting (1.9) and (1.11), we have for small $\varepsilon > 0$ that

$$H_{a_2}\left(\frac{q}{m}\right) = \frac{H_{a_2}q}{m} - \frac{q}{m}\frac{\nabla_\xi a_2 \cdot \nabla_x m}{m} \ge C_1\frac{\rho(x)|\xi|}{m} - C_2 \quad \text{on } \operatorname{supp}\psi_0.$$

Replacing $\langle x\rangle$ by $m(x)$ and noting (A4), (A5), we can prove (1), (2) of the lemma similarly as in the proof of those of Lemma 2.3 in [Do.2].

Proof of Theorem 1.2. In view of (1.12), Theorem 1.2 is the direct consequence of Lemma 2.1 and Lemma 2.2 if we replace p by $p/2$. Indeed, the last statement of (1) in Theorem 1.2 follows from Lemma 2.1 and

$$H_{a_2}p + Re\,a_1 \ge \rho(x)\lambda(|x|)|\xi| - C \quad \text{for } x, \xi \in \boldsymbol{R}^n, 0 \le t \le T, \quad \text{with } C > 0,$$

which is obtained from (1.12) and

$$H_{a_2}p \ge 2\rho(x)\lambda(|x|)|\xi| - C' \quad \text{for } x, \xi \in \boldsymbol{R}^n \quad \text{with } C' > 0.$$

The proof of Theorem 1.2 is completed.

Before proceeding to the proof of Theorem 1.1, we shall prepare a little. Take $\chi_0(t), \chi_1(t) \in C^\infty(\boldsymbol{R})$ such that $\chi_0(t) = 1$ if $|t| \le \mu$, $\chi_0(t) = 0$ if $|t| \ge 2\mu$, $\chi_0(t) \ge 0$ on $\boldsymbol{R}$ and $\chi_1(t) = 1 - \chi_0(t)$. Set $\sigma(t) = \chi_0(t) + \chi_1(t)/\psi(t)$ and $\tilde\psi(t) = \sigma(t)\psi(t)$. $\sigma(t) \in \mathscr{B}^\infty(\boldsymbol{R})$, $\sigma(t) = 1$ if $|t| \le \mu$, $\sigma(t) = 1/\psi(t)$ if $|t| \ge 2\mu$ and $C^{-1} \le \sigma(t) \le C$ for all $t \in \boldsymbol{R}$ with $C > 0$, hence $\tilde\psi(t) \in \mathscr{B}^\infty(\mathbf{R})$, $\tilde\psi(t) = \psi(t)$ if $|t| \le \mu$, $\tilde\psi(t) = 1$ if $|t| \ge 2\mu$ and $C^{-1}\psi(t) \le \tilde\psi(t) \le C\psi(t)$ for all $t \in \boldsymbol{R}$ with $C > 0$.

Proof of Theorem 1.1. Set $\phi(t,x) = \exp\{(1 - \tilde\psi^2(x_1))\int_0^{x_1} Re\,b_1(t,s,x')\,ds\}$. Since $|x_1| \le 2\mu$ on $\operatorname{supp}(1 - \tilde\psi^2)$, we get $\phi \in C^1([0,T]; \mathscr{B}^\infty(\boldsymbol{R}^n))$ from (A.3).

Multiplying (1.2) by $\phi(t,x)$, we have

$$\left\{\partial_t + i\frac{1}{2}(D_1^2 + \psi(x_1)\sum_{j,k=2}^{n} D_j(a_{jk}(x')D_k))\right\}(\phi u)$$

$$+ \{b_1(t,x)\phi - \partial_1\phi\}D_1 u + \sum_{j=2}^{n}\left\{b_j(t,x)\phi - \psi(x_1)\sum_{k=2}^{n} a_{jk}(x')\partial_k\phi\right\}D_j u$$

$$+ \left\{c(t,x)\phi - \partial_t\phi - i\frac{1}{2}(D_1^2 + \psi(x_1)\sum_{j,k=2}^{n} D_j(a_{jk}(x')D_k))\phi\right\}u = \phi f.$$

Since it follows from (A3) that $\partial_t^j\partial_x^\alpha\phi = r_{j,\alpha}(t,x)\times\phi$, where $r_{j,\alpha}\in C^{1-j}([0,T];\mathscr{B}^\infty(\boldsymbol{R}^n))$ for $j=0,1$, $\alpha\in\boldsymbol{N}^n$, we have from (1.2)

(3.1)
$$\left\{\partial_t + i\frac{1}{2}(D_1^2 + \psi(x_1)\sum_{j,k=2}^{n} D_j(a_{jk}(x')D_k)) + \sum_{j=1}^{n}\tilde{b}_j(t,x)D_j + \tilde{c}(t,x)\right\}(\phi u) = \phi f,$$

where

$$\tilde{b}_1(t,x) = \tilde{\psi}^2(x_1)Re\, b_1(t,x) + 2\tilde{\psi}(x_1)\tilde{\psi}'(x_1)\int_0^{x_1} Re\, b_1(t,s,x')\,ds + iIm\, b_1(t,x),$$

$$\tilde{b}_j(t,x) = b_j(t,x) - \psi(x_1)(1-\tilde{\psi}^2(x_1))\sum_{k=2}^{n} a_{jk}(x')\int_0^{x_1} Re\,\partial_k b_1(t,s,x')\,ds$$

for $2\le j\le n$,

$$\tilde{b}_j, \tilde{c}\in C([0,T];\mathscr{B}^\infty(\boldsymbol{R}^n)) \quad \text{for } 1\le j\le n.$$

Since ϕ and $\phi^{-1}\in C^1([0,T];\mathscr{B}^\infty(\boldsymbol{R}^n))$, it suffices to solve (3.1) instead of (1.2). By (1.7),

$$|Re\ \tilde{b}_1(t,x)| \le \tilde{\psi}^2(x_1)\lambda(|x|) + 2\tilde{\psi}(x_1)|\tilde{\psi}'(x_1)|\,|x_1|\lambda(|x'|),$$

$$|Re\ \tilde{b}_j(t,x)| \le \psi(x_1)\lambda(|x|) + C\psi(x_1)\sum_{k=2}^{n}\left|(1-\tilde{\psi}^2(x_1))\int_0^{x_1} Re\,\partial_k b_1(t,s,x')\,ds\right|$$

for $x\in\boldsymbol{R}^n, 0\le t\le T, 2\le j\le n$.

Noting the property of $\tilde{\psi}$ and that $|x|\le 2\mu + |x'|$ on $\operatorname{supp}\tilde{\psi}'$, $\operatorname{supp}(1-\tilde{\psi}^2)$, and using (1.7), we obtain $|Re\ \tilde{b}_j(t,x)| \leq C'\psi(x_1)\lambda(|x|-2\mu)$, $1 \leq j \leq n$. Hence for (3.1) we have (1.12) of Theorem 1.2 with $\rho(x)$ and $\lambda(\cdot)$ replaced by $\psi(x_1)$ and $C'\lambda(\cdot-2\mu)$. The proof of Theorem 1.1 is reduced to check the assumptions (A4) and (A5) of Theorem 1.2. Put $e(x,\xi) = \sqrt{a_2(x,\xi) + a_2'(x',\xi') + 10}$ and $q(x,\xi) = \{\int_0^{x_1} \tilde{\psi}(t)\,dt \cdot \xi_1 + M\theta(x',\xi')\} \cdot e(x,\xi)^{-1}$, where $M \gg 1$ is a large constant fixed later. (A4) follows from (1.5). (A5) is satisfied with $m(x) = \sqrt{(\int_0^{x_1} \tilde{\psi}(t)\,dt)^2 + |x'|^2 + 10}$ and $\rho(x) = \psi(x_1)$. In fact,

$$H_{a_2}q = \left(\xi_1 \cdot \tilde{\psi}(x_1)\xi_1 + \frac{M}{2}\,\psi(x_1)H_{a_2'}\theta - \frac{1}{2}\psi'(x_1)a_2'(x',\xi') \cdot \int_0^{x_1} \tilde{\psi}(t)\,dt\right)e^{-1}.$$

By (1.3), $|\psi'(x_1)\int_0^{x_1} \tilde{\psi}(t)\,dt| = |\psi'(x_1)\int_0^{x_1} \psi(t)\,dt| \leq |\psi'(x_1)x_1|\psi(x_1) \leq C_1\psi(x_1)$ if $|x_1| \leq \mu$, $|\psi'(x_1)\int_0^{x_1} \tilde{\psi}(t)\,dt| \leq C_2|\psi'(x_1)x_1| \leq C_3 \leq C_4\psi(x_1)$ if $|x_1| \geq \mu$. By using (1.6), we have for large $M \gg 1$ that

$$\begin{aligned} H_{a_2}q &\geq (\tilde{\psi}(x_1)\xi_1^2 + MC_5\psi(x_1)|\xi'|^2 - C_6\psi(x_1)|\xi'|^2)e(x,\xi)^{-1} \\ &\geq C_7\psi(x_1)|\xi| - C_8, \end{aligned}$$

which shows (1.9). (1.10) is obvious from the definition. In view of (A4), it is easy to see (1.8) and (1.11) by the direct calculation. Hence the proof of Theorem 1.1 is completed.

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