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C^*-EMBEDDING AND C-EMBEDDING ON PRODUCT SPACES

By

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1. Introduction

Throughout this paper by a space we mean a topological space. Let X be a space and A its subspace. Then A is said to be *C^*-embedded* (resp. *C-embedded*) in X if every bounded real-valued (resp. real-valued) continuous function on A can be extended to a continuous function over X. For an infinite cardinal number γ, A is said to be *P^γ-embedded* in X if for every locally finite cozero-set cover $\mathscr{U}$ of A with Card $\mathscr{U} \leq \gamma$ there exists a locally finite cozero-set cover $\mathscr{V}$ of X such that $\mathscr{V} \cap A$ $(= \{V \cap A | V \in \mathscr{V}\}) < (=\text{refines})\mathscr{U}$; A is *P-embedded* in X if A is P^γ-embedded in X for every γ. P^γ- and P-embeddings were originally introduced by Shapiro [16]. For the case $\gamma = \aleph_0$ it is known that $P^{\aleph_0}$-embedding coincides with C-embedding. And a well-known fact is that collectionwise normal spaces are those spaces in which every closed subset is P-embedded. For basic facts of these embeddings the reader is referred to Alò and Shapiro [1] and Hoshina [3].

As for normality of product spaces we have known the following results due to Morita [4] and Rudin and Starbird [15], respectively, that a Hausdorff space X is γ-paracompact normal iff $X \times Y$ is normal for any compact Hausdorff space Y of weight $w(Y) \leq \gamma$, and that for a normal space X and a non-discrete metric space $Y, X \times Y$ is normal iff $X \times Y$ is countably paracompact. Being motivated by the first result Morita and Hoshina [7] and Przymusiński [12] independently proved that for a compact Hausdorff space Y with $w(Y) = \gamma$, A is P^γ-embedded in X iff $A \times Y$ is C^*-embedded in $X \times Y$. On the other hand, corresponding to the second result above, the following problem was posed in Przymusiński [13] (see Hoshina [3]) but still remains open: for a non-discrete metric space Y is it true that $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is C-embedded in $X \times Y$? Recently Ohta [10] proved this equivalence when $Y = \kappa^\omega$,

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the product of countably many copies of the discrete space with Card $\kappa \geq \aleph_0$ under an additional assumption on A; in particular, in case $\kappa = \aleph_0$, that is, Y is space of irrationals, he showed the problems is affirmative.

In this paper we study to obtain further such equivalences when X and Y belong to other classes of spaces. Indeed, it is known so far that in case either Y is non-discrete compact Hausdorff, or A is C-embedded in X and Y is locally compact paracompact Hausdorff, then in $X \times Y$ C^*-embedding of $A \times Y$ implies its C-embedding (see [3]). But, any other case of X and Y for which similar results hold seems to be unknown. In this paper first we prove the following theorem. As a corollary to this result, in case Y is non-discrete σ-locally compact metrizable we have a partial answer to the Przymusiński's problem, which seems to be interesting when compared with the Ohta's result above. Here, Y is *σ-locally compact* if Y is a union of countably many locally compact closed subspaces.

THEOREM 1.1. *Let A be a C-embedded subspace of a space X and Y a σ-locally compact paracompact Hausdorff space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is C-embedded in $X \times Y$.*

Using Theorem 1.1, in Theorem 2.4 we show further the corresponding result for the case of P-embedding. In the next two theorems we discuss for the case X is a P-space and Y is a Σ-space or a σ-space. It may be emphasized that these results seem to give a new possibility to discuss various embeddings such as C^*-, C- or P-embedding on products for known classes of generalized metric spaces.

THEOREM 1.2. *Let X be a normal P-space and A be C-embedded in X. Let Y be a paracompact Hausdorff Σ-space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is C-embedded in $X \times Y$.*

In case Y is a σ-space, Theorem 1.2 enables us futher to prove the following theorem which shows the equivalence of C^*-embedding and P-embedding of $A \times Y$ in $X \times Y$.

THEOREM 1.3. *Let X be a normal P-space and A be P-embedded in X. Let Y be a paracompact Hausdorff σ-space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is P-embedded in $X \times Y$.*

P-spaces, Σ-spaces and σ-spaces are due to Morita [5]. Nagami [8] and Okuyama [11], respectively. Our results are motivated by the results obtained in Nagami [8], [9] and Chiba [2] which show equivalences between normality and either countable paracompactness or collectionwise normality on product spaces.

2. Proof of Theorem 1.1

Throughout this paper N denotes the set of positive integers.

Let A be a subspace of a space X. Then it is well-known that A is C-embedded in X iff A is C^*-embedded in X and is completely separated from any zero-set Z of X which is disjoint from A. This fact will be frequently used in this paper. Moreover, for later use let us recall two lemmas below.

LEMMA 2.1 (see [3]). *Let B be a compact subset of a Tychonoff space Y. Then for any space X, $X \times B$ is P-embedded in $X \times Y$.*

(1) $\leftrightarrow$ (4) of the following lemma was mentioned in the introduction.

LEMMA 2.2 (Morita and Hoshina [7], Przymusiński [12]). *For a subspace A of a space X the following statements are equivalent.*

(1) *A is P^γ-embedded in X*

(2) *$A \times Y$ is P^γ-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$*

(3) *$A \times Y$ is C-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$*

(4) *$A \times Y$ is C^*-embedded in $X \times Y$ for some compact Hausdorff space Y with $w(Y) = \gamma$*

Let us now prove Theorem 1.1.

PROOF OF THEOREM 1.1. We only prove the "only if" part since the "if" part is clear. Suppose A is C-embedded in X and $A \times Y$ is C^*-embedded in $X \times Y$. To prove C-embedding of $A \times Y$ in $X \times Y$, let Z be zero-set of $X \times Y$ disjoint from $A \times Y$. We shall show that $A \times Y$ and Z are completely separated in $X \times Y$.

Since Y is σ-locally compact and paracompact, Y admits a σ-locally finite cover $\mathscr{C} = \bigcup_{i \in N} \mathscr{C}_i$ consists of compact subsets, where $\mathscr{C}_i$ is locally finite. Let

$\mathscr{C}_i = \{C_{i\lambda} | \lambda \in \Lambda_i\}$. For each $C_{i\lambda} \in \mathscr{C}_i$, put

$$G_{i\lambda} = (X \times C_{i\lambda}) \cap (X \times Y - Z).$$

Then $G_{i\lambda}$ is a cozero-set of $X \times C_{i\lambda}$ containing $A \times C_{i\lambda}$. Hence, since $C_{i\lambda}$ is compact, it is easy to see that we can take a cozero-set $H_{i\lambda}$ of X so that

$$A \times C_{i\lambda} \subset H_{i\lambda} \times C_{i\lambda} \subset G_{i\lambda}.$$

Since A is C-embedded in X, there exists a cozero-set $L_{i\lambda}$ of X such that

$$A \cap L_{i\lambda} = \varnothing \quad \text{and} \quad X - H_{i\lambda} \subset L_{i\lambda}.$$

On the other hand, since Y is paracompact and $\mathscr{C}_i$ is locally finite, there exists a locally finite collection $\{U_{i\lambda} | \lambda \in \Lambda_i\}$ of cozero-sets of Y such that $C_{i\lambda} \subset U_{i\lambda}$ for each $\lambda \in \Lambda_i$. Hence it follows that $\{L_{i\lambda} \times U_{i\lambda} | \lambda \in \Lambda_i\}$ is a locally finite cozero-set collection of $X \times Y$ which satisfies for each $\lambda \in \Lambda_i$

$$(A \times Y) \cap (L_{i\lambda} \times U_{i\lambda}) = \varnothing \quad \text{and} \quad L_{i\lambda} \times U_{i\lambda} \supset (X \times C_{i\lambda}) \cap Z$$

Let us now put

$$K = \bigcup \{L_{i\lambda} \times U_{i\lambda} | \lambda \in \Lambda; i \in N\}.$$

Since $\{L_{i\lambda} \times U_{i\lambda} | \lambda \in \Lambda; i \in N\}$ is σ-locally finite, K is a cozero-set of $X \times Y$, and we have

$$(A \times Y) \cap K = \varnothing \quad \text{and} \quad K \supset Z.$$

Hence $X \times Y - K$ is a zero-set of $X \times Y$ containing $A \times Y$ and disjoint from Z. Thus $A \times Y$ and Z are completely separated in $X \times Y$. This completes the proof of the theorem. □

COROLLARY 2.3. *Let A be a subspace of a space X and Y a non-discrete σ-locally compact metrizable space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is C-embedded in $X \times Y$.*

PROOF. Suppose that $A \times Y$ is C^*-embedded in $X \times Y$. Y being non-discrete metrizable, it is essentially proved in [3] that A is C-embedded in X. For completeness we give its proof. Y contains a convergent sequence $\{y_n | n \in N\}$ of distinct points having y_0 as its limit. Let $C = \{y_0\} \cup \{y_n | n \in N\}$. Then C is compact, and by assumption and Lemma 2.1 we see that $A \times C$ is C^*-embedded in $X \times Y$, especially in $X \times C$. Hence by Lemma 2.2 A is C-embedded in X. The corollary now follows from Theorem 1.1. □

REMARK. Let A be a C-embedded subset of a space X. It is known that if Y is locally compact metrizable, then $A \times Y$ is C-embedded in $X \times Y$ (see [3]). In case Y is σ-locally compact metrizable, then $A \times Y$ need not to be C^*-embedded in $X \times Y$. Indeed, Przymusiński [14] pointed out that there exists a normal space X with the property that for a non-locally compact metric space M (in particular, $Q =$ the space of rational numbers) X contains a closed subset A such that $A \times M$ is not C^*-embedded in $X \times M$.

For the case of P-embedding corresponding to Theorem 1.1 we have the following theorem. The result is motivated by a theorem of Chiba [2] that for a collectionwise normal space X and a σ-locally compact paracompact Hausdorff space Y, $X \times Y$ is normal iff $X \times Y$ is collectionwise normal.

A collection $\{C_\lambda | \lambda \in \Lambda\}$ of subsets of a space X is *uniformly locally finite* if there exist a locally finite cozero-set collection $\{G_\lambda | \lambda \in \Lambda\}$ and a zero-set collection $\{Z_\lambda | \lambda \in \Lambda\}$ such that $C_\lambda \subset Z_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$.

THEOREM 2.4. *Let A be a P-embedded subset of a space X and Y a σ-locally compact paracompact Hausdorff space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is P-embedded in $X \times Y$.*

PROOF. It is sufficient to show the "only if" part. Let $\mathscr{C} = \bigcup_{i \in N} \mathscr{C}_i$ be as in the proof of Theorem 1.1. Let $\mathscr{U}$ be a locally finite cozero-set cover of $A \times Y$. Put

$$F_i = \bigcup \{C_{i\lambda} | \lambda \in \Lambda_i\}.$$

Since A is P-embedded in X, by Lemmas 2.1 and 2.2 $A \times C_{i\lambda}$ is P-embedded in $X \times Y$. Since Y is paracompact Hausdorff, $\{C_{i\lambda} | \lambda \in \Lambda_i\}$ is uniformly locally finite in Y and so is also $\{A \times C_{i\lambda} | \lambda \in \Lambda_i\}$ in $X \times Y$. For $\lambda, \mu \in \Lambda_i$, $(A \times C_{i\lambda}) \cup (A \times C_{i\mu}) = A \times (C_{i\lambda} \cup C_{i\mu})$ is P-embedded in $X \times Y$. Hence by Morita [6] (see [3, Theorem 3.12]), $A \times F_i$ is P-embedded in $X \times Y$. Consequently there exists a locally finite cozero-set cover $\mathscr{V}_i$ of $X \times Y$ such that

$$\mathscr{V}_i \cap (A \times F_i) < \mathscr{U} \cap (A \times F_i) < \mathscr{U}.$$

For any $V \in \mathscr{V}_i$, select $U_V \in \mathscr{U}$ so that

$$V \cap (A \times F_i) \subset U_V.$$

Since $A \times Y$ is C^*-embedded in $X \times Y$, there exists a cozero-set U'_V of $X \times Y$

such that $U'_V \cap (A \times Y) = U_V$. Let us put

$$W_V = V \cap U'_V \quad \text{and} \quad \mathscr{W} = \{W_V | V \in \mathscr{V}_i, i \in N\}.$$

Then $\mathscr{W}$ is a σ-locally finite collection of cozero-sets of $X \times Y$ such that

$$\mathscr{W} \cap (A \times Y) < \mathscr{U} \quad \text{and} \quad A \times Y \subset \bigcup \mathscr{W}.$$

Since $\bigcup \mathscr{W}$ is a cozero-set of $X \times Y$ containing $A \times Y$, by Theorem 1.1 there exists a cozero-set H of $X \times Y$ such that

$$(A \times Y) \cap H = \varnothing \quad \text{and} \quad \bigcup \mathscr{W} \cup H = X \times Y.$$

Let

$$\mathscr{W}' = \mathscr{W} \cup \{H\},$$

then $\mathscr{W}'$ is a σ-locally finite cozero-sets cover of $X \times Y$ and

$$\mathscr{W}' \cap (A \times Y) < \mathscr{U}.$$

Thus, $A \times Y$ is P-embedded in $X \times Y$, which completes the proof. $\square$

3. Proofs of Theorems 1.2 and 1.3

Before proving these theorems, let us recall the definition of P-spaces and basic facts of Σ-spaces and σ-spaces. In the following we assume all spaces are Hausdorff.

A space X is a *P-space* [5] if for any index set Ω and for any collection $\{G(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega; n \in N\}$ of open subsets of X such that

$$G(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \text{ for } \alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega,$$

there exists a collection $\{F(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega; n \in N\}$ of closed subsets of X such that the conditions (i), (ii) below are satisfied:

(i) $F(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n)$ for $\alpha_1, \dots, \alpha_n \in \Omega$,

(ii) $X = \bigcup \{G(\alpha_1, \dots, \alpha_n) | n \in N\} \Longrightarrow X = \bigcup \{F(\alpha_1, \dots, \alpha_n) | n \in N\}$.

Let Y be a Σ-space. Then by [8, Lemmal. 4], Y has a sequence, called a *Σ-net*, $\{\mathscr{E}_n | n \in N\}$ of locally finite closed covers of Y which satisfies the following conditions:

(iii) $\mathscr{E}_n$ is written as $\{E(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega\}$ with an index set Ω,

(iv) $E(\alpha_1, \dots, \alpha_n) = \bigcup \{E(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) | \alpha_{n+1} \in \Omega\}$ for $\alpha_1, \dots, \alpha_n \in \Omega$,

(v) For every $y \in Y, C(y)$ is countably compact, and there exists a sequence

$\alpha_1, \alpha_2, \ldots, \in \Omega$ such that $C(y) \subset V$ with V open implies $C(y) \subset E(\alpha_1, \ldots, \alpha_n) \subset V$ for some n, where $C(y) = \bigcap\{E | y \in E \in \mathscr{E}_n, n \in N\}$.

If a regular space Y is a σ-space, then Y has a sequence, called a *σ-net*, $\{\mathscr{E}_n | n \in N\}$ of locally finite closed covers of Y which satisfies (iii), (iv) above and (v′) below ([9, Theorem 1]):

(v′) For each $y \in Y$ there exists a sequence $\alpha_1, \alpha_2, \ldots \in \Omega$ such that $y \in E(\alpha_1, \ldots, \alpha_n)$ for each $n \in N$ and $y \in V$ with V open implies $y \in E(\alpha_1, \ldots, \alpha_n) \subset V$ for some n.

A space Y is a *strong Σ-space* if Y has Σ-net $\{\mathscr{E}_n\}$ such that $C(y)$ is compact for each $y \in Y$. Nagami proved in [8, Theorem 4.10] that for a P-space X and a strong Σ-space Y if $X \times Y$ is normal, then $X \times Y$ is countably paracompact, and in [9, Theorem 5] that for a collectionwise normal P-space X and a paracompact σ-space Y if $X \times Y$ is normal, then $X \times Y$ is collectionwise normal. Our Theorems 1.2 and 1.3 are motivated by these results.

PROOF OF THEOREM 1.2. It is sufficient to show the "only if" part. Assume $A \times Y$ is C^*-embedded in $X \times Y$. Let Z be a zero-set of $X \times Y$ disjoint from $A \times Y$. First we observe that

$$(\bar{A} \times Y) \cap Z = \varnothing. \tag{1}$$

To see this, let y be any point of Y. Since A is C-embedded in X, $A \times \{y\}$ is C-embedded in $X \times Y$, and $(A \times \{y\}) \cap Z = \varnothing$. Hence there exists a zero-set Z' of $X \times Y$ such that $A \times \{y\} \subset Z'$ and $Z' \cap Z = \varnothing$, which implies $(\bar{A} \times \{y\}) \cap Z = \varnothing$. Hence we have (1).

Let $\{\mathscr{E}_n | n \in N\}$ be a Σ-net. Note that $C(y)$ is compact for any $y \in Y$ since Y is paracompact. Define

$$H(\alpha_1, \ldots, \alpha_n) = \bigcup\{P | P \text{ is open in } X; (P \times E(\alpha_1, \ldots, \alpha_n)) \cap Z = \varnothing\}$$

for $\alpha_1, \ldots, \alpha_n \in \Omega$. Put

$$G(\alpha_1, \ldots, \alpha_n) = H(\alpha_1, \ldots, \alpha_n) \cup (X - \bar{A}),$$

clearly $G(\alpha_1, \ldots, \alpha_n)$ is open in X. Since

$$H(\alpha_1, \ldots, \alpha_n) \subset H(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}),$$

we have

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for any $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$. Since X is a P-space, there exists a collection $\{F(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega; n \in N\}$ of closed subsets such that

$$F(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n)$$

and

$$X = \bigcup \{G(\alpha_1, \dots, \alpha_n) | n \in N\} \Longrightarrow X = \bigcup \{F(\alpha_1, \dots, \alpha_n) | n \in N\}. \tag{2}$$

Then $F(\alpha_1, \dots, \alpha_n) - H(\alpha_1, \dots, \alpha_n)$ is closed and contained in $X - \bar{A}$. Since X is normal, there exists a cozero-set $U(\alpha_1, \dots, \alpha_n)$ of X such that

$$F(\alpha_1, \dots, \alpha_n) - H(\alpha_1, \dots, \alpha_n) \subset U(\alpha_1, \dots, \alpha_n) \subset X - \bar{A}.$$

On the other hand, since Y is paracompact, for each $\mathscr{E}_n$, there exits a locally finite cozero-set cover $\mathscr{L}_n = \{L(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega\}$ of Y such that

$$E(\alpha_1, \dots, \alpha_n) \subset L(\alpha_1, \dots, \alpha_n).$$

Let us put

$$W_n = \bigcup \{U(\alpha_1, \dots, \alpha_n) \times L(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Omega\}.$$

Since $\mathscr{L}_n$ is locally finite, W_n is a cozero-set of $X \times Y$. Hence $W = \bigcup_{n \in N} W_n$ is a cozero-set of $X \times Y$, and we have $(A \times Y) \cap W = \varnothing$ because $\bar{A} \cap U(\alpha_1, \dots, \alpha_n) = \varnothing$.

Finally we shall show $Z \subset W$. To see this, pick $(x, y) \in Z$. For y, there exists a sequence $\alpha_1, \alpha_2, \dots \in \Omega$ satisfying (v) above. First we prove that $X = \bigcup_{n \in N} G(\alpha_1, \dots, \alpha_n)$. Pick $z \in X$. We may assume $z \in \bar{A}$. Then by (1) we have $(\{z\} \times C(y)) \cap Z = \varnothing$. Since $C(y)$ is compact, there exist open sets O in X and O' in Y such that

$$\{z\} \times C(y) \subset O \times O' \subset X \times Y - Z.$$

By (v), there exists $m \in N$ such that $C(y) \subset E(\alpha_1, \dots, \alpha_m) \subset O'$. Hence it follows that we have $z \in H(\alpha_1, \dots, \alpha_m)$. Therefore $z \in G(\alpha_1, \dots, \alpha_m)$, which shows that $X = \bigcup \{G(\alpha_1, \dots, \alpha_n) | n \in N\}$. Consequently by (2) $X = \bigcup \{F(\alpha_1, \dots, \alpha_n) | n \in N\}$. Select $k \in N$ so that $x \in F(\alpha_1, \dots, \alpha_k)$. Then $x \in X - H(\alpha_1, \dots, \alpha_k)$ because $(x, y) \in Z$ and $y \in E(\alpha_1, \dots, \alpha_k)$. So

$$x \in F(\alpha_1, \dots, \alpha_k) - H(\alpha_1, \dots, \alpha_k) \subset U(\alpha_1, \dots, \alpha_k),$$

therefore $(x, y) \in W_k \subset W$. Thus, we have $Z \subset W$. Now $X \times Y - W$ is a zero-set of $X \times Y$ containing $A \times Y$ and is disjoint from Z. Hence $A \times Y$ and Z are completely separated in $X \times Y$, which completes the proof. $\square$

PROOF OF THEOREM 1.3. Let $\mathscr{G}$ be a locally finite cozero-set cover of $A \times Y$. To prove our theorem it suffices to show that there exists a σ-locally finite cozero-set cover $\mathscr{W}$ of $X \times Y$ such that $\mathscr{W} \cap (A \times Y) < \mathscr{G}$. Since $\mathscr{G}$ is refined by a σ-discrete cozero-set cover of $A \times Y$, we may assume $\mathscr{G}$ itself is σ-discrete. Hence let $\mathscr{G} = \bigcup_{i \in N} \mathscr{G}_i$, where $\mathscr{G}_i = \{G_{i\lambda} | \lambda \in \Lambda_i\}$ is discrete. By assumption, for any $\lambda \in \Lambda_i$, $i \in N$, there exists a cozero-set $H_{i\lambda}$ of $X \times Y$ such that $H_{i\lambda} \cap (A \times Y) = G_{i\lambda}$.

First we shall show the following fact:

$$\bar{A} \times Y \subset \bigcup \{H_{i\lambda} | \lambda \in \Lambda_i, i \in N\}. \tag{3}$$

Pick $y \in Y$. Let $U_{i\lambda} = \{x \in A | (x, y) \in H_{i\lambda}\}$. Then $\{U_{i\lambda} | \lambda \in \Lambda_i; i \in N\}$ is a σ-discrete cozero-set cover of A. Since A is P-embedded in X, there exists a locally finite cozero-set cover $\{V_{i\lambda} | \lambda \in \Lambda_i; i \in N\}$ of X such that $V_{i\lambda} \cap A \subset U_{i\lambda}$ for each $\lambda \in \Lambda_i$ and $i \in N$. Since $\{(V_{i\lambda} \times Y) \cap H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}$ is a locally finite cozero-set collection of $X \times Y$, $\cup \{(V_{i\lambda} \times Y) \cap H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}$ is a cozero-set of $X \times Y$ and we have

$$A \times \{y\} \subset \bigcup \{(V_{i\lambda} \times Y) \cap H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}.$$

Since $A \times \{y\}$ is C-embedded in $X \times Y$, there exists a zero-set Z of $X \times Y$ such that

$$A \times \{y\} \subset Z \subset \bigcup \{(V_{i\lambda} \times Y) \cap H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}.$$

Hence

$$\bar{A} \times \{y\} \subset \bigcup \{(V_{i\lambda} \times Y) \cap H_{i\lambda} | \lambda \in \Lambda_i; i \in N\} \subset \bigcup \{H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}.$$

Therefore $\bar{A} \times Y \subset \bigcup \{H_{i\lambda} | \lambda \in \Lambda_i; i \in N\}$.

Let $\mathscr{E} = \{\mathscr{E}_n\}$ be a σ-net for Y. By the paracompactness of Y, there exists a locally finite cozero-set cover

$$\{K(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Omega\}$$

of Y such that

$$E(\alpha_1, \ldots, \alpha_n) \subset K(\alpha_1, \ldots, \alpha_n)$$

for each $\alpha_1, \ldots, \alpha_n \in \Omega$. Let us put

$$U_{i\lambda}(\alpha_1, \ldots, \alpha_n) = \bigcup \{U | U \text{ is open in } X, U \times E(\alpha_1, \ldots, \alpha_n) \subset H_{i\lambda}\}$$

for $\alpha_1, \ldots, \alpha_n \in \Omega$. Then $U_{i\lambda}(\alpha_1, \ldots, \alpha_n)$ is open in X. Define

$$U_i(\alpha_1, \ldots, \alpha_n) = \bigcup\{U_{i\lambda}(\alpha_1, \ldots, \alpha_n) | \lambda \in \Lambda_i\}$$

and

$$U(\alpha_1, \ldots, \alpha_n) = \bigcup\{U_i(\alpha_1, \ldots, \alpha_n) | i \in N\}.$$

For any $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$, clearly

$$U(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}). \tag{4}$$

Let λ and λ' be distinct elements of Λ_i. For $\alpha_1, \ldots, \alpha_n \in \Omega$, we have

$$\begin{aligned} &((U_{i\lambda}(\alpha_1, \ldots, \alpha_n) \cap A) \cap (U_{i\lambda'}(\alpha_1, \ldots, \alpha_n) \cap A)) \times E(\alpha_1, \ldots, \alpha_n) \\ &\subset H_{i\lambda} \cap (A \times Y) \cap H_{i\lambda'} \cap (A \times Y) = G_{i\lambda} \cap G_{i\lambda'} = \varnothing. \end{aligned}$$

Hence

$$U_{i\lambda}(\alpha_1, \ldots, \alpha_n) \cap U_{i\lambda'}(\alpha_1, \ldots, \alpha_n) \cap A = \varnothing.$$

Therefore

$$U_{i\lambda}(\alpha_1, \ldots, \alpha_n) \cap U_{i\lambda'}(\alpha_1, \ldots, \alpha_n) \cap \bar{A} = \varnothing. \tag{5}$$

Define

$$V(\alpha_1, \ldots, \alpha_n) = U(\alpha_1, \ldots, \alpha_n) \cup (X - \bar{A}).$$

Then $V(\alpha_1, \ldots, \alpha_n)$ is open in X and $V(\alpha_1, \ldots, \alpha_n) \subset V(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ by (4). Since X is a P-space and normal, there exist a zero-set $D(\alpha_1, \ldots, \alpha_n)$ and a cozero-set $L(\alpha_1, \ldots, \alpha_n)$ of X for $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in N$ such that

$$D(\alpha_1, \ldots, \alpha_n) \subset L(\alpha_1, \ldots, \alpha_n) \subset \overline{L(\alpha_1, \ldots, \alpha_n)} \subset V(\alpha_1, \ldots, \alpha_n), \tag{6}$$

$$X = \bigcup\{V(\alpha_1, \ldots, \alpha_n) | n \in N\} \Longrightarrow X = \cup\{D(\alpha_1, \ldots, \alpha_n) | n \in N\}. \tag{7}$$

Put

$$C(\alpha_1, \ldots, \alpha_n) = \overline{A \cap L(\alpha_1, \ldots, \alpha_n)}.$$

Then $C(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n) = \bigcup_{i \in N} U_i(\alpha_1, \ldots, \alpha_n)$. Since X is countably paracompact and normal, there exists a locally finite cozero-set collection $\{N_i(\alpha_1, \ldots, \alpha_n) | i \in N\}$ of X such that

$$\overline{N_i(\alpha_1, \ldots, \alpha_n)} \subset U_i(\alpha_1, \ldots, \alpha_n) = \bigcup_{\lambda \in \Lambda_i} U_{i\lambda}(\alpha_1, \ldots, \alpha_n) \tag{8}$$

and

$$C(\alpha_1, \dots, \alpha_n) \subset \bigcup_{i \in N} N_i(\alpha_1, \dots, \alpha_n).$$

Consequently by (5) and (8) we have that

$$\{\overline{N_i(\alpha_1, \dots, \alpha_n) \cap A} \cap U_{i\lambda}(\alpha_1, \dots, \alpha_n) | \lambda \in \Lambda_i\}$$

is a discrete collection of X and that $N_i(\alpha_1, \dots, \alpha_n) \cap U_{i\lambda}(\alpha_1, \dots, \alpha_n) \cap A$ is a cozero-set of A. Therefore

$$\{(X - D(\alpha_1, \dots, \alpha_n)) \cap A, N_i(\alpha_1, \dots, \alpha_n) \cap U_{i\lambda}(\alpha_1, \dots, \alpha_n) \cap A | \lambda \in \Lambda_i, i \in N\}$$

is a locally finite cozero-set cover of A. Since A is P-embedded in X, there exists a locally finite cozero-set cover

$$\mathscr{W}'(\alpha_1, \dots, \alpha_n) = \{W_0'(\alpha_1, \dots, \alpha_n), W_{i\lambda}'(\alpha_1, \dots, \alpha_n) | \lambda \in \Lambda_i, i \in N\}$$

of X such that

$$W_0'(\alpha_1, \dots, \alpha_n) \cap A \subset (X - D(\alpha_1, \dots, \alpha_n)) \cap A$$

and

$$W_{i\lambda}'(\alpha_1, \dots, \alpha_n) \cap A \subset N_i(\alpha_1, \dots, \alpha_n) \cap U_{i\lambda}(\alpha_1, \dots, \alpha_n) \cap A.$$

Let us put

$$W_{i\lambda}(\alpha_1, \dots, \alpha_n) = W_{i\lambda}'(\alpha_1, \dots, \alpha_n) \cap N_i(\alpha_1, \dots, \alpha_n),$$

and

$$\mathscr{W} = \{(W_{i\lambda}(\alpha_1, \dots, \alpha_n) \times K(\alpha_1, \dots, \alpha_n)) \cap H_{i\lambda} | \lambda \in \Lambda_i; \alpha_1, \dots, \alpha_n \in \Omega; i, n \in N\}.$$

Then $\mathscr{W}$ is a σ-locally finite cozero-set collection of $X \times Y$, and $\mathscr{W} \cap (A \times Y) < \mathscr{G}$.

Next we shall show that $A \times Y \subset \bigcup \mathscr{W}$. Pick $(x, y) \in A \times Y$. For this y, there exists a sequence $\alpha_1, \alpha_2, \dots \in \Omega$ which has the property (v′). Then we have $X = \bigcup \{V(\alpha_1, \dots, \alpha_n) | n \in N\}$. To see this, pick $z \in X$. If $z \in X - \bar{A}$, it is clear. Let $z \in \bar{A}$. Then by (3) shown above, there exists $i \in N$ and $\lambda \in \Lambda_i$ such that $(z, y) \in H_{i\lambda}$. So there exists open sets O in X and O' in Y such that

$$(z, y) \in O \times O' \subset H_{i\lambda}.$$

By the property (v′), there exists $m \in N$ such that

$$y \in E(\alpha_1, \ldots, \alpha_m) \subset O'.$$

Thus

$$z \in O \subset U_{i\lambda}(\alpha_1, \ldots, \alpha_m) \subset U(\alpha_1, \ldots, \alpha_m) \subset V(\alpha_1, \ldots, \alpha_m).$$

Hence it follows that $X = \bigcup\{V(\alpha_1, \ldots, \alpha_n) | n \in N\}$. Therefore $X = \bigcup\{D(\alpha_1, \ldots, \alpha_n) | n \in N\}$ by (7), so there exists $k \in N$ such that $x \in D(\alpha_1, \ldots, \alpha_k)$. Since $\mathscr{W}'(\alpha_1, \ldots, \alpha_k)$ covers X, there exist $i \in N$ and $\lambda \in \Lambda_i$ such that $x \in W'_{i\lambda}(\alpha_1, \ldots, \alpha_k)$. That is,

$$x \in N_i(\alpha_1, \ldots, \alpha_k) \cap U_{i\lambda}(\alpha_1, \ldots, \alpha_k) \cap A.$$

Hence $x \in W_{i\lambda}(\alpha_1, \ldots, \alpha_k)$ and $y \in E(\alpha_1, \ldots, \alpha_k) \subset K(\alpha_1, \ldots, \alpha_k)$. Since $x \in U_{i\lambda}(\alpha_1, \ldots, \alpha_k)$, we have $(x, y) \in H_{i\lambda}$. Consequently

$$(x, y) \in (W_{i\lambda}(\alpha_1, \ldots, \alpha_k) \times K(\alpha_1, \ldots, \alpha_k)) \cap H_{i\lambda}.$$

Thus we have shown $A \times Y \subset \bigcup \mathscr{W}$.
Since $\mathscr{W}$ is σ-locally finite, $\bigcup \mathscr{W}$ is a cozero-set of $X \times Y$, and

$$(X \times Y - \bigcup \mathscr{W}) \cap (A \times Y) = \varnothing.$$

By Theorem 1.2, there exists a cozero-set W in $X \times Y$ such that

$$(A \times Y) \cap W = \varnothing \quad \text{and} \quad X \times Y - \bigcup \mathscr{W} \subset W.$$

Define newly $\mathscr{W}$ by $\mathscr{W} \cup \{W\}$. Then the above shows that $\mathscr{W}$ is the required σ-locally finite cozero-set cover of $X \times Y$. This completes the proof. □

The following results, which are corollaries to Theorems 1.2 and 1.3, together with Theorems 1.1 and 2.4 are proved in the author's master thesis at Univ. Tsukuba (1995) (in Japanese).

COROLLARY 3.1. *Let X be a normal P-space and A closed in X. Let Y be a paracompact Σ-space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is C-embedded in $X \times Y$.*

COROLLARY 3.2. *Let X be a collectionwise normal P-space and A closed in X. Let Y be a paracompact σ-space. Then $A \times Y$ is C^*-embedded in $X \times Y$ iff $A \times Y$ is P-embedded in $X \times Y$.*

REMARK. A subset A of a space X is said to be *z-embedded* in X if every zero-set in A is the intersection of A with a zero-set in X. Clearly C^*-embedding implies z-embedding. We note that in all of our results except Corollary 2.3 "C^*-embedded" in the assumption can be weakened to "z-embedded".

For the normality of products, Yang [17] posed a problem whether it is true that for a collectionwise normal P-space X and a paracompact Σ-space Y normality of $X \times Y$ implies collectionwise normality of $X \times Y$. Likewise, in our case the following question remains open.

QUESTION. In Theorem 1.3 or Corollary 3.2 can "σ-space" be weakened to "Σ-space"?

Added in proof. Recently the author showed that "σ-space" in Corollary 3.2 can be weakened to "Σ-space", and solved the Yang's problem above affirmatively.

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