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A theorem in the geometry of numbers

By

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Let M be the exterior of a knot in the 3-sphere S^3 (or more generally a compact 3-manifold with a torus as boundary) and let $M(p,r)$ be the closed 3-manifold obtained from M by (p,r)-Dehn surgery. (p, r are co-prime integers.) Roughly speaking, the number of non-trivial representations of the fundamental group of $M(p,r)$ to PSL $(2,\boldsymbol{C})$ is given by the formula

$$\sum_{i=1}^{n}|\alpha_i p-\beta_i r|-e$$

So, if this number is positive, then $M(p,r)$ is not simply-connected. So, the calculation of this number is useful for studying Poincaré conjecture.

In this paper we shall prove a theorem about the functions of the above form, purely in the geometry of numbers, independent of the topology of 3-manifolds. We use Minkowski's theorem in proving this theorem. Moreover we introduce the notion of C-system as a tool for proving the theorem. In future we wish to apply this theorem to the study of Poincaré conjecture.

Let $L'=\boldsymbol{Z}\times\boldsymbol{Z}-\{(0,0)\}$

THEOREM 1. *Let α_i,β_i $(i=1,\dots,m)$, γ_j,δ_j $(j=1,\dots,n)$, e, f be real numbers such that $\alpha_i\beta_k-\beta_i\alpha_k\neq 0$ $(i\neq k)$, $\gamma_j\delta_\ell-\delta_j\gamma_\ell\neq 0$ $(j\neq l)$, $e>0$, $f>0$. Suppose that, for all $(x,y)\in L'$,*

$$\sum_{i=1}^{m}|\alpha_i x-\beta_i y|\geq e \tag{1}$$

and

$$\sum_{j=1}^{n}|\gamma_j x-\delta_j y|\geq f. \tag{2}$$

Then,

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$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\delta_j-\beta_i\gamma_j|\geq ef.$$

Moreover, the equlity in the last inequality holds only when $m=n=2$ *and* $|\alpha_1\beta_2-\beta_1\alpha_2|=e^2/2,\ |\gamma_1\delta_2-\delta_1\gamma_2|=f^2/2.$

Corollary 2. *Let* α_i,β_i $(i=1,\dots,m)$, γ_j,δ_j $(j=1,\dots,n)$, e,f *be as in the Theorem* 1. *Then, for all* $(u,v)\in L'$,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_ju-(\alpha_i\delta_j+\beta_i\gamma_j)v|\geq ef.$$

Moreover, the equality in the last inequality holds only when either
(i) $u=\pm 1,\ v=0$, *or* (ii) $m=n=2$.

We first prove that Theorem 1 implies Corollary 2. Let $(u,v)\in L'$.

Case 1. $v=0$.
Then, $u\neq 0$ and, since u is an integer, $|u|\geq 1$. According to the hypothesis (1). (2) of the Corollary 2 with $(x,y)=(1,0)$, we have

$$\sum_{i=1}^{m}|\alpha_i|\geq e \quad\text{and}\quad \sum_{j=1}^{n}|\gamma_j|\geq f.$$

Therefore,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_j|\geq ef.$$

So,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_ju-(\alpha_i\delta_j+\beta_i\gamma_j)v|=\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_j||u|\geq\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_j|\geq ef.$$

Moreover, if $|u|>1$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\gamma_ju-(\alpha_i\delta_j+\beta_i\gamma_j)v|>ef.$$

Case 2. $v\neq 0$.
Let $(x,y)\in L'$. Then, $(uy-vx,vy)\in L'$. Then, by the hypothesis (2), we have

$$\sum_{j=1}^{n}|\gamma_j(uy-vx)-\delta_j(vy)|\geq f,$$

that is,

$$\sum_{j=1}^{n} |(\gamma_j v)x - (\gamma_j u - \delta_j v)y| \geq f, \tag{3}$$

for all $(x, y) \in L'$. Moreover, if $j \neq \ell$,

$$(\gamma_j v)(\gamma_\ell u - \delta_\ell v) - (\gamma_j u - \delta_j v)(\gamma_\ell v) = -(\gamma_j \delta_\ell - \delta_j \gamma_\ell)v^2 \neq 0.$$

So, we can use Theorem 1 for (1) and (3) (that is, we take $\gamma_j v$ and $\gamma_j u - \delta_j v$ instead of γ_j and δ_j, respectively).

Then, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_i(\gamma_j u - \delta_j v) - \beta_i(\gamma_j v)| \geq ef,$$

that is,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_i \gamma_j u - (\alpha_i \delta_j + \beta_i \gamma_j)v| \geq ef,$$

as was to be proved. Moreover, if the equality holds in the last inequality, then by Theorem 1, we have $m = n = 2$ (Q.E.D.)

In order to prove Theorem 1, we can assume that $e = f = 1$ without loss of generality. First prove the following lemma, which is a special case of Theorem 1.

LEMMA 3. *Let α_i, β_i $(i = 1, \ldots, m)$ be real numbers such that $\alpha_i \beta_k - \beta_i \alpha_k \neq 0$ $(i \neq k)$. Suppose that, for all $(x, y) \in L'$,*

$$\sum_{i=1}^{m} |\alpha_i x - \beta_i y| \geq 1.$$

Then,

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i \beta_j - \beta_i \alpha_j| \left(= 2 \sum_{i<j} |\alpha_i \beta_j - \beta_i \alpha_j| \right) \geq 1.$$

Moreover, the equality holds in the last inequality only when $m = 2$ and $|\alpha_1 \beta_2 - \beta_1 \alpha_2| = \frac{1}{2}$.

PROOF OF LEMMA 3. Case 1. $m = 1$.

Clearly, the hypothesis of the lemma never holds.

Case 2. $m = 2$.

Then, the hypotheses are:

$$\alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$$

and

$$|\alpha_1 x - \beta_1 y| + |\alpha_2 x - \beta_2 y| \geq 1,$$

for all $(x, y) \in L'$. The conclusion of the theorem becomes

$$2|\alpha_1\beta_2 - \beta_1\alpha_2| \geq 1.$$

Now, the domain

$$\Delta = \{(x, y) \in \boldsymbol{R}^2 | . |\alpha_1 x - \beta_1 y| + |\alpha_2 x - \beta_2 y| < 1\}$$

is the interior of a parallelogram which is symmetric with respect to the origin, and the vertices of which are

$$\pm(\beta_1/|\alpha_1\beta_2 - \beta_1\alpha_2|, \alpha_1/|\alpha_1\beta_2 - \beta_1\alpha_2|)$$

and

$$\pm(\beta_2/|\alpha_1\beta_2 - \beta_1\alpha_2|, \alpha_2/|\alpha_1\beta_2 - \beta_1\alpha_2|).$$

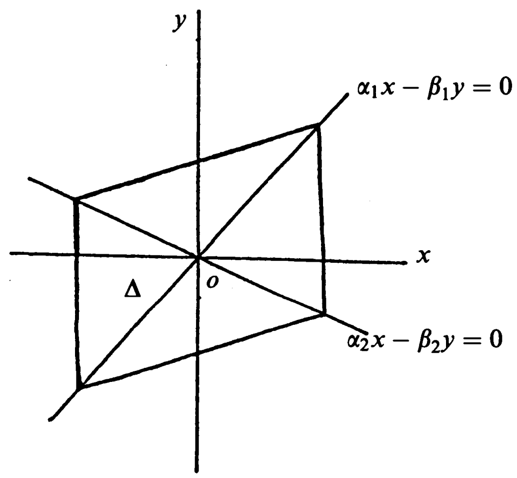

Figure 1

So, the area of Δ is $2/\alpha_1\beta_2 - \beta_1\alpha_2|$. By the hypothesis of the lemma, Δ does not contain elements of L'. So, by Minkowski's theorem, the area of Δ must be less than or equal to 4. Thus, $2/|\alpha_1\beta_2 - \beta_1\alpha_2| \leq 4$, that is, $2|\alpha_1\beta_2 - \beta_1\alpha_2| \geq 1$, as was to be proved. If the equality holds, then $|\alpha_1\beta_2 - \beta_1\alpha_2| = 1/2$.

Case 3. $m > 2$.

We shall prove that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i \beta_j - \beta_i \alpha_j| > 1.$$

Consider the domain

$$\Delta = \left\{ (x, y) \in \boldsymbol{R}^2 \middle| \sum_{i=1}^{m} |\alpha_i x - \beta_i y| < 1 \right\}.$$

Δ is the interior of a convex $2m$-gon D which is symmetric with respect to the origin. Let $P_1, P_2, \ldots, P_{2m}$ be the vertices of D (we assume that these are ordered counterclockwise). By changing the subscripts of α_i, β_i, if necessary, we can assume without loss of generality that P_i and P_{i+m} are on the line $\alpha_i x - \beta_i y = 0$ $(i = 1, \ldots, m)$. Let (a_i, b_i) be the coordinate of P_i. It holds that $a_{i+m} = -a_i$, $b_{i+m} = -b_i$.

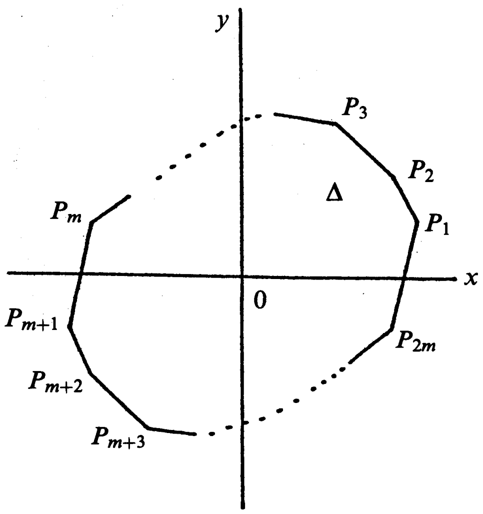

Figure 2

Since P_i is on the line $\alpha_i x - \beta_i y = 0$, there is a $\lambda_i \neq 0$ such that $\alpha_i = \lambda_i b_i$ and $\beta_i = \lambda_i a_i$. It holds that $\lambda_{i+m} = -\lambda_i$ $(i = 1, \ldots, m)$. Let $u_i = |\lambda_i| = |\lambda_{i+m}|$. Since $P_i = (a_i, b_i)$ is on D, we have

$$\sum_{k=1}^{m} |\alpha_k a_i - \beta_k b_i| = 1,$$

that is,

$$\sum_{k=1}^{m} \mu_k |b_k a_i - a_k b_i| = 1.$$

So, if we put $c(k,i) = |b_k a_i - a_k b_i|$, we have

$$\sum_{k=1}^{m} \mu_k c(k,i) = 1 \quad (i = 1, \ldots, m) \tag{4}$$

$\{c(k,i) : 1 \le k, i \le m\}$ satisfies the following.

C1. (i) $c(k,k) = 0$,
(ii) $c(k,i) > 0 \quad (k \neq i)$,
C2. $c(k,i) = c(i,k)$,
C3. for $1 \le i < j < k < \ell \le m$,

$$c(i,k)c(j,\ell) = c(i,j)c(k,\ell) + c(i,\ell)c(j,k).$$

C4. (i) $c(m,1) + c(1,2) > c(m,2)$,
(ii) for $2 \le i \le m-1$,

$$c(i-1,i) + c(i,i+1) > c(i-1,i+1),$$

(iii) $c(m-1,m) + c(m,1) > c(m-1,1)$.

C1 and C2 are obvious. We prove C3. From the Figure 2, we have $a_i b_j - b_i a_j > 0$, for $1 \le i < j \le m$. Hence, $c(i,j) = a_i b_j - b_i a_j$. Similarly, if $1 \le i < j < k < \ell \le m$, then

$$c(i,k) = a_i b_k - b_i a_k, \quad c(j,\ell) = a_j b_\ell - b_j a_\ell, \quad c(k,\ell) = a_k b_\ell - b_k a_\ell,$$

$$c(i,\ell) = a_i b_\ell - b_i a_\ell, \quad c(j,k) = a_j b_k - b_j a_k.$$

Thus, C3 holds.

Next we prove C4 (ii). C4 (i) and C4 (iii) can be proved similarly. Now, by (4) above,

$$1 = \sum_{\ell=1}^{m} \mu_\ell c(\ell, i+1) = \sum_{\ell=1}^{m} \mu_\ell c(\ell, i-1) = \sum_{\ell=1}^{m} \mu_\ell c(\ell, i).$$

So,

$$
\begin{aligned}
&c(i-1,i)+c(i,i+1)-c(i-1,i+1)\\
&\quad=\left\{\sum_{\ell=1}^{m}\mu_\ell c(\ell,i+1)\right\}c(i-1,i)+\left\{\sum_{\ell=1}^{m}\mu_\ell c(\ell,i-1)\right\}c(i,i+1)\\
&\qquad-\left\{\sum_{\ell=1}^{m}\mu_\ell c(\ell,i)\right\}c(i-1,i+1)\\
&\quad=\sum_{\ell=1}^{m}\mu_\ell\{c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)\}.
\end{aligned}
$$

If $\ell < i-1$ or $i+1<\ell$, then by C3,

$$c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)=0.$$

Also, if $\ell = i-1$, then by C1 and C2,

$$
\begin{aligned}
&c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)\\
&\quad=c(i-1,i+1)c(i-1,i)+c(i-1,i-1)c(i,i+1)-c(i-1,i)c(i-1,i+1)=0.
\end{aligned}
$$

Similarly, if $\ell = i+1$, then

$$
\begin{aligned}
&c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)\\
&\quad=c(i+1,i+1)c(i-1,i)+c(i+1,i-1)c(i,i+1)-c(i+1,i)c(i-1,i+1)=0.
\end{aligned}
$$

If $\ell = i$, then

$$
\begin{aligned}
&c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)\\
&\quad=2c(i-1,i)c(i,i+1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&c(i-1,i)+c(i,i+1)-c(i-1,i+1)\\
&\quad=\sum_{\ell=1}^{m}\mu_\ell\{c(\ell,i+1)c(i-1,i)+c(\ell,i-1)c(i,i+1)-c(\ell,i)c(i-1,i+1)\}\\
&\quad=\mu_i\{2c(i-1,i)c(i,i+1)\}>0. \qquad (*)
\end{aligned}
$$

Hence

$$c(i-1,i)+c(i,i+1)>c(i-1,i+1).$$

From $(*)$ follows that

M1. $\mu_1 = \dfrac{1}{2c(m,1)} + \dfrac{1}{2c(1,2)} - \dfrac{c(m,2)}{2c(m,1)c(1,2)}$,

M2. For $2 \le i \le m-1$,

$$\mu_i = \frac{1}{2c(i-1,i)} + \frac{1}{2c(i,i+1)} - \frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)},$$

M3. $\mu_m = \dfrac{1}{2c(m-1,m)} + \dfrac{1}{2c(m,1)} - \dfrac{c(m-1,1)}{2c(m-1,m)c(m,1)}$.

We must show

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i\beta_j - \beta_i\alpha_j| > 1.$$

Now,

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i\beta_j - \beta_i\alpha_j| &= \sum_{i=1}^{m} \sum_{j=1}^{m} \mu_i\mu_j c(i,j) \\
&= \sum_{i=1}^{m} \mu_i \left\{ \sum_{j=1}^{m} \mu_j c(i,j) \right\} = \sum_{i=1}^{m} \mu_i \\
&= \left\{ \frac{1}{2c(m,1)} + \frac{1}{2c(1,2)} - \frac{c(m,2)}{2c(m,1)c(1,2)} \right\} \\
&\quad + \sum_{i=2}^{m-1} \left\{ \frac{1}{2c(i-1,i)} + \frac{1}{2c(i,i+1)} - \frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)} \right\} \\
&\quad + \left\{ \frac{1}{2c(m-1,m)} + \frac{1}{2c(m,1)} - \frac{c(m-1,1)}{2c(m-1,m)c(m,1)} \right\} \\
&= \left[\left\{ \sum_{i=1}^{m-1} \frac{1}{c(i,i+1)} + \frac{1}{c(m,1)} \right\} - \left\{ \frac{c(m,2)}{2c(m,1)c(1,2)} \right.\right. \\
&\quad \left.\left. + \sum_{i=2}^{m-1} \frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)} + \frac{c(m-1,1)}{2c(m-1,m)c(m,1)} \right\} \right].
\end{aligned}$$

Here we have used (4) and M1, M2, M3. So, we have to show

$$\left\{\sum_{i=1}^{m-1}\frac{1}{c(i,i+1)}+\frac{1}{c(m,1)}\right\}-\left\{\frac{c(m,2)}{2c(m,1)c(1,2)}\right.$$

$$\left.+\sum_{i=2}^{m-1}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right\}>1. \tag{5}$$

Now, the area of the domain Δ is

$$\begin{aligned}\sum_{i=1}^{m}|a_ib_{i+1}-b_ia_{i+1}| &= \sum_{i=1}^{m-1}|a_ib_{i+1}-b_ia_{i+1}|+|a_mb_1-b_ma_1|\\ &= \sum_{i=1}^{m-1}c(i,i+1)+c(m,1).\end{aligned}$$

By the assumption of the lemma, Δ does not contain elements of L'. So by Minkowski's theorem, the area of Δ is less than or equal to 4. That is,

$$\sum_{i=1}^{m-1}c(i,i+1)+c(m,1)\leq 4.$$

So, in order to prove (5), it suffices to show that

$$\left\{\sum_{i=1}^{m-1}c(i,i+1)+c(m,1)\right\}\left[\left\{\sum_{i=1}^{m-1}\frac{1}{c(i,i+1)}+\frac{1}{c(m,1)}\right\}\right.$$

$$\left.-\left\{\frac{c(m,2)}{2c(m,1)c(1,2)}+\sum_{i=2}^{m-1}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right\}\right]>4. \tag{6}$$

Definition. A system $C=\langle c(i,j):i,j=1,\dots,m\rangle$ of real numbers ($m\geq 3$) is called a C-system if it satisfies the above C1–C4. We set $\mu(C)=m$.

If C satisfies C1–C3 and

C4′. (i) $c(m,1)+c(1,2)\geq c(m,2)$,

(ii) for $2\leq i\leq m-1$,

$$c(i-1,i)+c(i,i+1)\geq c(i-1,i+1),$$

(iii) $c(m-1,m)+c(m,1)\geq c(m-1,1)$,

instead of C4, then C is called a semi-C-system. Also, we set $\mu(C)=m$.

In order to prove Lemma 3, it suffices to show the following.

LEMMA 4. (i) *For every C-system C,* (6) *holds, i.e.*

$$\left\{\sum_{i=1}^{m-1} c(i,i+1)+c(m,1)\right\}\left\{\sum_{i=1}^{m-1}\frac{1}{c(i,i+1)}+\frac{1}{c(m,1)}-\frac{c(m,2)}{2c(m,1)c(1,2)}\right.$$
$$\left.-\sum_{i=2}^{m-1}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}-\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right\}>4. \tag{6}$$

(ii) *For every semi-C-system C, the following* (7) *holds.*

$$\left\{\sum_{i=1}^{m-1} c(i,i+1)+c(m,1)\right\}\left\{\sum_{i=1}^{m-1}\frac{1}{c(i,i+1)}+\frac{1}{c(m,1)}-\frac{c(m,2)}{2c(m,1)c(1,2)}\right.$$
$$\left.-\sum_{i=2}^{m-1}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}-\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right\}\geq 4. \tag{7}$$

PROOF. We prove Lemma 4 by the induction on $m=\mu(C)$.

Case 1. $m=3$.

Then,

$$L=\{c(1,2)+c(2,3)+c(3,1)\}\left\{\frac{1}{c(1,2)}+\frac{1}{c(2,3)}+\frac{1}{c(3,1)}-\frac{c(3,2)}{2c(3,1)c(1,2)}\right.$$
$$\left.-\frac{c(1,3)}{2c(1,2)c(2,3)}-\frac{c(2,1)}{2c(2,3)c(3,1)}\right\}-4$$
$$=\frac{\{c(1,2)+c(2,3)-c(3,1)\}\{c(2,3)+c(3,1)-c(1,2)\}\{c(3,1)+c(1,2)-c(2,3)\}}{2c(1,2)c(2,3)c(3,1)}$$

Hence if C is a C-system, then $L>0$, and if C is a semi-C-system, then $L\geq 0$.

Case 2. $m\geq 4$.

We assume that (6) holds for every C-system C with $\mu(C)<m$ and that (7) holds for every semi-C-system C with $\mu(C)<m$. We shall show that (6) holds for every C-system C with $\mu(C)=m$ and that (7) holds for every semi-C-system C with $\mu(C)=m$. (From now on to the end of the proof of Lemma 4, we fix m but C varies.)

Under the above hypothesis, we first prove the following.

PROPOSITION 5. *Let C be a semi-C-system with $\mu(C)=m$. Suppose that one of the following holds.*

(i) $c(m,1)+c(1,2)=c(m,2)$,
(ii) $c(k-1,k)+c(k,k+1)=c(k-1,k+1)$, *for some* k $(2\le k\le m-1)$,
(iii) $c(m-1,m)+c(m,1)=c(m-1,1)$.

Then, (7) *holds for* C.

Proof of Proposition 5. By a cyclic change of subscripts we can assume that

(iii) $c(m-1,m)+c(m,1)=c(m-1,1)$.

Let $C'=\langle c(i,j):i,j=1,\dots,m-1\rangle$. C' satisfies C1–C3. Also, it satisfies

$$c(i-1,1)+c(i,i+1)\ge c(i-1,i+1)$$

for $2\le i\le m-2$. Also,

$$\begin{aligned}&\{c(m-1,1)+c(1,2)-c(m-1,2)\}c(1,m)\\ &\quad=c(m-1,1)c(1,m)+c(1,2)c(1,m)-c(m-1,2)c(1,m)\\ &\quad=c(m-1,1)c(1,m)+c(1,2)c(1,m)-c(m-1,1)c(2,m)+c(1,2)c(m-1,m)\\ &\quad=c(m-1,1)c(1,m)-c(m-1,1)c(2,m)+c(1,2)\{c(1,m)+c(m-1,m)\}\\ &\quad=c(m-1,1)c(1,m)-c(m-1,1)c(2,m)+c(1,2)c(m-1,1)\\ &\quad=c(m-1,1)\{c(1,m)+c(1,2)-c(2,m)\}\ge 0.\end{aligned}$$

Hence

$$c(m-1,1)+c(1,2)\ge c(m-1,2).$$

Also,

$$\begin{aligned}&\{c(m-2,m-1)+c(m-1,1)-c(m-2,1)\}c(m-1,m)\\ &\quad=c(m-2,m-1)c(m-1,m)+c(m-1,1)c(m-1,m)-c(m-2,1)c(m-1,m)\\ &\quad=c(m-2,m-1)c(m-1,m)+c(m-1,1)c(m-1,m)-c(m-1,1)c(m-2,m)\\ &\qquad+c(1,m)c(m-2,m-1)\\ &\quad=c(m-2,m-1)\{c(m-1,m)+c(m,1)\}+c(m-1,1)c(m-1,m)\\ &\qquad-c(m-1,1)c(m-2,m)\end{aligned}$$

$$= c(m-2,m-1)c(m-1,1) + c(m-1,1)c(m-1,m) - c(m-1,1)c(m-2,m)$$

$$= c(m-1,1)\{c(m-2,m-1) + c(m-1,m) - c(m-2,m)\} \geq 0.$$

Hence

$$c(m-2,m-1) + c(m-1,1) \geq c(m-2,1).$$

Thus, C' satisfies C4′. Hence C' is a semi-C-system with $\mu(C') = m-1$. Hence (7) holds for C' by the assumption. So,

$$J = \left\{\sum_{i=1}^{m-2} c(i,i+1) + c(m-1,1)\right\}\left\{\sum_{i=1}^{m-2} \frac{1}{c(i,i+1)} + \frac{1}{c(m-1,1)}\right.$$

$$- \frac{c(m-1,2)}{2c(m-1,1)c(1,2)} - \sum_{i=2}^{m-2} \frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}$$

$$\left. - \frac{c(m-2,1)}{2c(m-2,m-1)c(m-1,1)}\right\} \geq 4.$$

We must show that

$$K = \left\{\sum_{i=1}^{m-1} c(i,i+1) + c(m,1)\right\}\left\{\sum_{i=1}^{m-1} \frac{1}{c(i,i+1)} + \frac{1}{c(m,1)} - \frac{c(m,2)}{2c(m,1)c(1,2)}\right.$$

$$\left. - \sum_{i=2}^{m-1} \frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)} - \frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right\} \geq 4.$$

It suffices to show that $J = K$. Now,

$$\sum_{i=1}^{m-2} c(i,i+1) + c(m-1,1) = \sum_{i=1}^{m-2} c(i,i+1) + c(m-1,m) + c(m,1)$$

$$= \sum_{i=1}^{m-1} c(i,i+1) + c(m,1).$$

Also,

$$\left[\sum_{i=1}^{m-2}\frac{1}{c(i,i+1)}+\frac{1}{c(m-1,1)}-\frac{c(m-1,2)}{2c(m-1,1)c(1,2)}-\sum_{i=2}^{m-2}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}\right.$$

$$\left.-\frac{c(m-2,1)}{2c(m-2,m-1)c(m-1,1)}\right]-\left[\sum_{i=1}^{m-1}\frac{1}{c(i,i+1)}+\frac{1}{c(m,1)}-\frac{c(m,2)}{2c(m,1)c(1,2)}\right.$$

$$\left.-\sum_{i=2}^{m-1}\frac{c(i-1,i+1)}{2c(i-1,i)c(i,i+1)}-\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}\right]$$

$$=-\frac{1}{c(m-1,m)}+\frac{1}{c(m-1,1)}-\frac{1}{c(m,1)}-\frac{c(m-1,2)}{2c(m-1,1)c(1,2)}+\frac{c(m,2)}{2c(m,1)c(1,2)}$$

$$+\frac{c(m-2,m)}{2c(m-2,m-1)c(m-1,m)}-\frac{c(m-2,1)}{2c(m-2,m-1)c(m-1,1)}$$

$$+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}$$

$$=-\frac{1}{c(m-1,m)}+\frac{1}{c(m-1,1)}-\frac{1}{c(m,1)}+\frac{-c(m-1,2)c(m,1)+c(m,2)c(m-1,1)}{2c(m-1,1)c(m,1)c(1,2)}$$

$$+\frac{c(m-2,m)c(m-1,1)-c(m-2,1)c(m-1,m)}{2c(m-2,m-1)c(m-1,m)c(m-1,1)}+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}$$

$$=-\frac{1}{c(m-1,m)}+\frac{1}{c(m-1,1)}-\frac{1}{c(m,1)}+\frac{c(1,2)c(m-1,m)}{2c(m-1,1)c(m,1)c(1,2)}$$

$$+\frac{c(1,m)c(m-2,m-1)}{2c(m-2,m-1)c(m-1,m)c(m-1,1)}+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}$$

$$=-\frac{1}{c(m-1,m)}+\frac{1}{c(m-1,1)}-\frac{1}{c(m,1)}+\frac{c(m-1,m)}{2c(m-1,1)c(m,1)}$$

$$+\frac{c(1,m)}{2c(m-1,m)c(m-1,1)}+\frac{c(m-1,1)}{2c(m-1,m)c(m,1)}$$

$$=\frac{1}{2c(m-1,m)c(m-1,1)c(m,1)}\{-2c(m-1,1)c(m,1)+2c(m-1,m)c(m,1)$$

$$-2c(m-1,m)c(m-1,1)+c(m-1,m)^2+c(m,1)^2+c(m-1,1)^2\}$$

$$=\frac{(c(m-1,m)+c(m,1)-c(m-1,1))^2}{2c(m-1,m)c(m-1,1)c(m,1)}=0.$$

Thus, $J=K$. Proposition 5 is proved. (Q.E.D.)

Let C be a semi-C-system with $\mu(C) = m$. We shall show that (7) holds for C and (6) holds when C is a C-system. For this purpose we define a one-parameter family of semi-C-systems $C(t)$ with $\mu(C(t)) = m$.

DEFINITION. Let t be a real number. We define

$$\tilde{C}(t) = \langle \tilde{c}(i,j;t) : 1 \leq i,j \leq m \rangle$$

as follows.

(i) $\tilde{c}(i,j;t) = c(i,j)$, for $1 \leq i,j \leq m-2$.
(ii) $\tilde{c}(m-1,m-1;t) = \tilde{c}(m,m;t) = 0$.
(iii) $\tilde{c}(m-1,m;t) = \tilde{c}(m,m-1;t) = c(m-1,m)$.
(iv) For $1 \leq i \leq m-2$,

$$\tilde{c}(i,m-1;t) = \tilde{c}(m-1,i;t) = \frac{c(i,m-2)t + c(1,i)c(m-2,m-1)}{c(1,m-2)}.$$

(v) For $1 \leq i \leq m-2$,

$$\tilde{c}(i,m;t) = \tilde{c}(m,i;t) = \frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)}.$$

Let

$$r_1 = \frac{c(1,m-1)c(m-2,m)}{c(m-2,m-1)+c(m-1,m)},$$

$$r_2 = \frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)\{c(m,1)+c(1,2)-c(m,2)\}+c(1,2)c(m-2,m)},$$

$$s_1 = c(m-1,m) + c(m,1),$$

$$s_2 = \frac{c(1,m-2)\{c(m-3,m-2)+c(m-2,m-1)-c(m-3,m-1)\}+c(1,m-1)c(m-3,m-2)}{c(m-3,m-2)}$$

$$a = \max(r_1,r_2), \quad b = \min(s_1,s_2).$$

PROPOSITION 6.

(i) $0 < a \leq b$.
(ii) *For* $t \in [a,b]$, $\tilde{C}(t)$ *is a semi-C-system.*
(iii) *If* $t = c(1,m-1)$, *then* $t \in [a,b]$ *and* $\tilde{C}(t) = C$.

If C is a C-system, then

(iv) $0 < a < b$.

(v) *For $t \in (a,b)$, $\tilde{C}(t)$ is a C-system.*

(vi) *If $t = c(1,m-1)$, then $t \in (a,b)$ and $\tilde{C}(t) = C$.*

PROOF. $0 < a$ is obvious. We show that $\tilde{C}(t)$ satisfies the conditions C1, C2 C3. C1 and C2 are obvious from the definition.

PROOF OF C3: Let $1 \le i < j < k < \ell \le m$. We show that

$$\tilde{c}(i,k;t)c(j,\ell;t) = \tilde{c}(i,j;t)\tilde{c}(k,\ell;t) + \tilde{c}(i,\ell;t)\tilde{c}(j,k;t).$$

If $\ell \le m-2$, this is obvious, since then

$$\tilde{c}(i,k;t) = c(i,k), \quad \tilde{c}(j,\ell;t) = c(j,\ell), \quad \tilde{c}(i,j;t) = c(i,j),$$
$$\tilde{c}(k,\ell;t) = c(k,\ell), \quad \tilde{c}(i,\ell;t) = c(i,\ell), \quad \tilde{c}(j,k;t) = c(j,k).$$

Next suppose that $\ell = m-1$. Then, $1 \le i < j < k \le m-2$. So,

$$\tilde{c}(i,k;t) = c(i,k), \quad \tilde{c}(i,j;t) = c(i,j), \quad \tilde{c}(j,k;t) = c(j,k),$$

$$\tilde{c}(j,\ell;t) = \frac{c(j,m-2)t + c(1,j)c(m-2,m-1)}{c(1,m-2)},$$

$$\tilde{c}(k,\ell;t) = \frac{c(k,m-2)t + c(1,k)c(m-2,m-1)}{c(1,m-2)},$$

$$\tilde{c}(i,\ell;t) = \frac{c(i,m-2)t + c(1,i)c(m-2,m-1)}{c(1,m-2)}.$$

Hence,

$$\tilde{c}(i,j;t)\tilde{c}(k,\ell;t) + \tilde{c}(i,\ell;t)\tilde{c}(j,k;t) - \tilde{c}(i,k;t)\tilde{c}(j,\ell;t)$$

$$= c(i,j)\frac{c(k,m-2)t + c(1,k)c(m-2,m-1)}{c(1,m-2)}$$
$$+ c(j,k)\frac{c(i,m-2)t + c(1,i)c(m-2,m-1)}{c(1,m-2)}$$
$$+ c(i,k)\frac{c(j,m-2)t + c(1,j)c(m-2,m-1)}{c(1,m-2)}$$

$$= \frac{\{c(i,j)c(k,m-2) + c(j,k)c(i,m-2) - c(i,k)c(j,m-2)\}t}{c(1,m-2)}$$

$$+ \frac{\{c(i,j)c(1,k) + c(j,k)c(1,i) - c(i,k)c(1,j)\}}{c(1,m-2)} = 0,$$

since

$$c(i,j)c(k,m-2) + c(j,k)c(i,m-2) - c(i,k)c(j,m-2) = 0,$$

$$c(i,j)c(1,k) + c(j,k)c(1,i) - c(i,k)c(1,j) = 0.$$

Note that the last equalities hold also when $k = m-2$ or $i = 1$.

Next suppose that $\ell = m$ and $k \leq m-2$. Then

$$\tilde{c}(i,k;t) = c(i,k), \quad \tilde{c}(i,j,t) = c(i,j), \quad \tilde{c}(j,k;t) = c(j,k),$$

$$\tilde{c}(j,\ell;t) = \frac{c(1,j)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(j,m-2)}{c(1,m-2)},$$

$$\tilde{c}(k,\ell;t) = \frac{c(1,k)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(k,m-2)}{c(1,m-2)},$$

$$\tilde{c}(i,\ell;t) = \frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)}.$$

Hence

$$\tilde{c}(i,j;t)\tilde{c}(k,\ell;t) + \tilde{c}(i,\ell;t)\tilde{c}(j,k;t) - \tilde{c}(i,k;t)\tilde{c}(j,\ell;t)$$

$$= c(i,j)\left\{\frac{c(1,k)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(k,m-2)}{c(1,m-2)}\right\}$$

$$+ c(j,k)\left\{\frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)}\right\}$$

$$- c(i,k)\left\{\frac{c(1,j)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(j,m-2)}{c(1,m-2)}\right\}$$

$$= \frac{c(1,m-1)c(m-2,m)}{c(1,m-2)t}\{c(i,j)c(1,k) + c(j,k)c(1,i) - c(1,k)c(1,j)\}$$

$$+ \frac{c(1,m)}{c(1,m-2)}\{c(i,j)c(k,m-2) + c(j,k)c(i,m-2) - c(i,k)c(j,m-2)\}$$

$$= 0.$$

Next suppose that $\ell = m$ and $k = m-1$. Then

$$\tilde{c}(i,j;t) = c(i,j), \quad \tilde{c}(k,\ell;t) = c(m-1,m),$$

$$\tilde{c}(i,k,t) = \frac{c(i,m-2)t + c(1,i)c(m-2,m-1)}{c(1,m-2)},$$

$$\tilde{c}(i,\ell;t) = \frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)},$$

$$\tilde{c}(j,k,t) = \frac{c(j,m-2)t + c(1,j)c(m-2,m-1)}{c(1,m-2)},$$

$$\tilde{c}(j,\ell;t) = \frac{c(1,j)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(j,m-2)}{c(1,m-2)}.$$

Hence

$$\begin{aligned}
&\tilde{c}(i,j;t)\tilde{c}(k,\ell;t) + \tilde{c}(i,\ell;t)\tilde{c}(j,k;t) - \tilde{c}(i,k;t)\tilde{c}(j,\ell;t)\\
&= c(i,j)c(m-1,m) + \left\{\frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)}\right\}\\
&\quad \times \left\{\frac{c(j,m-2)t + c(1,j)c(m-2,m-1)}{c(1,m-2)}\right\}\\
&\quad - \left\{\frac{c(i,m-2)t + c(1,i)c(m-2,m-1)}{c(1,m-2)}\right\}\\
&\quad \times \left\{\frac{c(1,j)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(j,m-2)}{c(1,m-2)}\right\}\\
&= \frac{1}{c(1,m-2)^2 t}[c(i,j)c(m-1,m)c(1,m-2)^2 t\\
&\quad + \{c(1,m)c(i,m-2)t + c(1,i)c(1,m-1)c(m-2,m)\}\\
&\quad \times \{c(j,m-2)t + c(1,j)c(m-2,m-1)\}\\
&\quad - \{c(1,m)c(j,m-2)t + c(1,j)c(1,m-1)c(m-2,m)\}\\
&\quad \times \{c(i,m-2)t + c(1,i)c(m-2,m-1)\}]\\
&= \frac{1}{c(1,m-2)^2 t}(Pt^2 + Qt + R),
\end{aligned}$$

where

$$P = c(1,m)c(i,m-2)c(j,m-2) - c(1,m)c(j,m-2)c(i,m-2) = 0,$$

$$\begin{aligned}
Q &= c(i,j)c(m-1,m)c(1,m-2)^2 + c(1,m)c(i,m-2)c(1,j)c(m-2,m-1) \\
&\quad + c(1,i)c(1,m-1)c(m-2,m)c(j,m-2) \\
&\quad - c(1,m)c(j,m-2)c(1,i)c(m-2,m-1) \\
&\quad - c(1,j)c(1,m-1)c(m-2,m)c(i,m-2) \\
&= c(i,j)c(m-1,m)c(1,m-2)^2 + c(1,m)c(m-2,m-1)\{c(i,m-2)c(1,j) \\
&\quad - c(j,m-2)c(1,i)\} - c(1,m-1)c(m-2,m)\{c(1,j)c(i,m-2) \\
&\quad - c(1,i)c(j,m-2)\} \\
&= c(i,j)c(m-1,m)c(1,m-2)^2 + c(1,m)c(m-2,m-1)c(1,m-2)c(i,j) \\
&\quad - c(1,m-1)c(m-2,m)c(1,m-2)c(i,j) \\
&= c(i,j)c(1,m-2)\{c(m-1,m)c(1,m-2) + c(1,m)c(m-2,m-1) \\
&\quad - c(1,m-1)c(m-2,m)\} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
R &= c(1,i)c(1,m-1)c(m-2,m)c(1,j)c(m-2,m-1) \\
&\quad - c(1,j)c(1,m-1)c(m-2,m)c(1,i)c(m-2,m-1) = 0.
\end{aligned}$$

Thus, C3 is proved.

Next suppose that $t = c(1,m-1)$. Then, for $1 \leq i \leq m-2$,

$$\begin{aligned}
\tilde{c}(i,m-1;t) = \tilde{c}(m-1,i;t) &= \frac{c(i,m-2)c(1,m-1) + c(1,i)c(m-2,m-1)}{c(1,m-2)} \\
&= \frac{c(1,m-2)c(i,m-1)}{c(1,m-2)} = c(i,m-1),
\end{aligned}$$

$$\begin{aligned}\tilde{c}(i,m;t) &= \tilde{c}(m,i;t)\\ &= \frac{c(1,i)c(1,m-1)c(m-2,m)}{c(1,m-2)c(1,m-1)} + \frac{c(1,m)c(i,m-2)}{c(1,m-2)}\\ &= \frac{c(1,i)c(m-2,m)+c(1,m)c(i,m-2)}{c(1,m-2)}\\ &= \frac{c(1,m-2)c(i,m)}{c(1,m-2)} = c(i,m).\end{aligned}$$

Thus, $\tilde{C}(t) = C$, when $t = c(1,m-1)$.

Next by C4′,

$$c(m-2,m) \le c(m-2,m-1) + c(m-1,m).$$

Hence,

$$c(1,m-1)c(m-2,m) \le c(1,m-1)\{c(m-2,m-1)+c(m-1,m)\}.$$

Hence,

$$\frac{c(1,m-1)c(m-2,m)}{c(m-2,m-1)+c(m-1,m)} \le c(1,m-1),$$

that is, $r_1 \le c(1,m-1)$.

Also by C4′,

$$c(m,1) + c(1,2) \ge c(m,2).$$

Hence,

$$\begin{aligned}c(1,2)c(1,m-1)c(m-2,m) \le{}& c(1,m-2)\{c(m,1)+c(1,2)-c(m,2)\}\\ &+ c(1,2)c(1,m-1)c(m-2,m).\end{aligned}$$

Hence,

$$\frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)\{c(m,1)+c(1,2)-c(m,2)\}+c(1,2)c(m-2,m)} \le c(1,m-1),$$

that is, $r_2 \le c(1,m-1)$.

Also by C4′,

$$c(1,m-1) \le c(m-1,m) + c(m,1).$$

Hence, $c(1,m-1) \le s_1$.

Also by C4′,

$$c(m-3,m-2)+c(m-2,m-1)\geq c(m-3,m-1).$$

Hence,

$$\begin{aligned}&c(1,m-1)c(m-3,m-2)\\&\quad\leq c(1,m-2)\{c(m-3,m-2)+c(m-2,m-1)-c(m-3,m-1)\}\\&\qquad+c(1,m-1)c(m-3,m-2).\end{aligned}$$

Hence,

$$\begin{aligned}&c(1,m-1)\\&\leq\frac{c(1,m-2)\{c(m-3,m-2)+c(m-2,m-1)-c(m-3,m-1)\}+c(1,m-1)c(m-3,m-2)}{c(m-3,m-2)},\end{aligned}$$

that is, $c(1,m-1)\leq s_2$.

Therefore,

$$a=\max(r_1,r_2)\leq c(1,m-1)\leq\min(s_1,s_2)=b.$$

Hence, $a\leq b$ and $c(1,m-1)\in[a,b]$.

Thus, (i) and (iii) of Proposition 6 are proved.

Next we shall show that C4′ holds for $C(t)$ with $t\in[a,b]$, that is,

$$\tilde{c}(m,1;t)+\tilde{c}(1,2;t)\geq\tilde{c}(m,2;t),\tag{8}$$

$$\tilde{c}(i-1,i;t)+\tilde{c}(i,i+1;t)\geq\tilde{c}(i-1,i+1;t),\tag{9}$$

for $2\leq i\leq m-1$, and

$$\tilde{c}(m-1,m;t)+\tilde{c}(m,1;t)\geq\tilde{c}(m-1,1;t)\tag{10}$$

First we prove (8). Now,

$$\tilde{c}(m,1;t)=c(m,1),\quad\tilde{c}(1,2;t)=c(1,2),$$

$$\tilde{c}(m,2;t)=\frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)t}+\frac{c(1,m)c(2,m-2)}{c(1,m-2)}.$$

Hence,

$$\tilde{c}(m,1;t)+\tilde{c}(1,2;t)-\tilde{c}(m,2;t)$$

$$=c(m,1)+c(1,2)-\frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)t}-\frac{c(1,m)c(2,m-2)}{c(1,m-2)}$$

$$=\frac{\{(c(m,1)+c(1,2))c(1,m-2)-c(1,m)c(2,m-2)\}t-c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)t}$$

$$=\frac{1}{c(1,m-2)t}[\{(c(m,1)+c(1,2))c(1,m-2)-c(1,m-2)c(2,m)$$

$$+c(1,2)c(m-2,m)\}t-c(1,2)c(1,m-1)c(m-2,m)]$$

$$=\frac{1}{c(1,m-2)t}[\{(c(m,1)+c(1,2)-c)(2,m))c(1,m-2)+c(1,2)c(m-2,m)\}t$$

$$-c(1,2)c(1,m-1)c(m-2,m)]$$

$$=\frac{\{c(m,1)+c(1,2)-c(2,m))c(1,m-2)+c(1,2)c(m-2,m)\}}{c(1,m-2)t}$$

$$\times\left\{t-\frac{c(1,2)c(1,m-1)c(m-2,m)}{(c(m,1)+c(1,2)-c(2,m))c(1,m-2)+c(1,2)c(m-2,m)}\right\}$$

$$=\frac{\{(c(m,1)+c(1,2)-c(2,m))c(1,m-2)+c(1,2)c(m-2,m)\}}{c(1,m-2)t}(t-r_2)\geq 0.$$

Next we prove (9). (9) for $i\leq m-3$ is obvious, since then

$$\tilde{c}(i-1,i;t)=c(i-1,i),\quad \tilde{c}(i,i+1;t)=c(i,i+1),$$
$$\tilde{c}(i-1,i+1;t)=c(i-1,i+1).$$

When $i=m-2$, (9) becomes

$$\tilde{c}(m-3,m-2;t)+\tilde{c}(m-2,m-1;t)\geq c(m-3,m-1;t).$$

Now,

$$\tilde{c}(m-3,m-2;t)=c(m-3,m-2),\quad \tilde{c}(m-2,m-1;t)=c(m-2,m-1),$$

$$\tilde{c}(m-3,m-1;t)=\frac{c(m-3,m-2)t+c(1,m-3)c(m-2,m-1)}{c(1,m-2)}.$$

Hence,

$$\tilde{c}(m-3,m-2;t)+\tilde{c}(m-2,m-1;t)-\tilde{c}(m-3,m-1;t)$$

$$=c(m-3,m-2)+c(m-2,m-1)-\frac{c(m-3,m-2)t+c(1,m-3)c(m-2,m-1)}{c(1,m-2)}$$

$$=\frac{\{c(m-3,m-2)+c(m-2,m-1)\}c(1,m-2)-c(1,m-3)c(m-2,m-1)-c(m-3,m-2)t}{c(1,m-2)}$$

$$=\frac{1}{c(1,m-2)}[\{c(m-3,m-2)+c(m-2,m-1)\}c(1,m-2)-c(1,m-2)c(m-1,m-3)$$

$$+c(1,m-1)c(m-3,m-2)-c(m-3,m-2)t]$$

$$=\frac{1}{c(1,m-2)}[c(1,m-2)\{c(m-3,m-2)+c(m-2,m-1)-c(m-3,m-1)\}$$

$$+c(1,m-1)c(m-3,m-2)-c(m-3,m-2)t]$$

$$=\frac{c(m-3,m-2)}{c(1,m-2)}$$

$$\times\left[\frac{c(1,m-2)\{c(m-3,m-2)+c(m-2,m-1)-c(m-3,m-1)\}+c(1,m-1)c(m-3,m-2)}{c(m-3,m-2)}-t\right]$$

$$=\frac{c(m-3,m-2)}{c(1,m-2)}(s_2-t)\geq 0.$$

When $i=m-1$, (9) becomes

$$\tilde{c}(m-2,m-1;t)+\tilde{c}(m-1,m;t)\geq\tilde{c}(m-2,m;t).$$

Now,

$$\tilde{c}(m-2,m-1;t)=c(m-2,m-1),\quad \tilde{c}(m-1,m;t)=c(m-1,m),$$

$$\tilde{c}(m-2,m;t)=\frac{c(1,m-2)c(1,m-1)c(m-2,m)}{c(1,m-2)t}+\frac{c(1,m)c(m-2,m-2)}{c(1,m-2)}$$

$$=\frac{c(1,m-1)c(m-2,m)}{t}.$$

Hence,

$$\tilde{c}(m-2,m-1;t)+\tilde{c}(m-1,m;t)-\tilde{c}(m-2,m;t)$$

$$=c(m-2,m-1)+c(m-1,m)-\frac{c(1,m-1)c(m-2,m)}{t}$$

$$=\frac{\{c(m-2,m-1)+c(m-1,m)\}}{t}\left\{t-\frac{c(1,m-1)c(m-2,m)}{c(m-2,m-1)+c(m-1,m)}\right\}$$

$$=\frac{\{c(m-2,m-1)+c(m-1,m)\}}{t}(t-r_1)\geq 0.$$

Next we prove (10). Now,

$$\tilde{c}(m-1,m;t) = c(m-1,m), \quad \tilde{c}(m,1;t) = c(m,1), \quad c(m-1,1;t) = t.$$

Hence,

$$\begin{aligned}\tilde{c}(m-1,m;t) + \tilde{c}(m,1;t) - \tilde{c}(m-1,1;t) &= c(m-1,m) + c(m,1) - t \\ &= s_1 - t \geq 0.\end{aligned}$$

Next suppose that C is a C-system. Then by the above proof we have

$$r_1 < c(1,m-1), \quad r_2 < c(1,m-1), \quad c(1,m-1) < s_1, \quad c(1,m-1) < s_2.$$

Hence,

$$a = \max(r_1,r_2) < c(1,m-1) < \min(s_1,s_2) = b.$$

Thus, (iv) and (vi) of Proposition 6 is proved.

Moreover, similarly to the proof of (8), (9), (10) above we can show that, for $t \in (a,b)$,

$$\tilde{c}(m,1;t) + \tilde{c}(1,2;t) > \tilde{c}(m,2;t),$$

$$\tilde{c}(i-1,i;t) + \tilde{c}(i,i+1;t) > \tilde{c}(i-1,i+1;t) \quad (2 \leq i \leq m-1),$$

and

$$\tilde{c}(m-1,m;t) + \tilde{c}(m,1;t) > \tilde{c}(m-1,1;t).$$

Thus, (v) of Proposition 6 is proved. (Q.E.D.)

Let

$$\begin{aligned}f(t) = &\left\{\sum_{i=1}^{m-1} \tilde{c}(i,i+1;t) + \tilde{c}(m,1;t)\right\}\left\{\sum_{i=1}^{m-1} \frac{1}{\tilde{c}(i,i+1;t)} + \frac{1}{\tilde{c}(m,1;t)}\right. \\ &- \frac{\tilde{c}(m,2;t)}{2\tilde{c}(m,1;t)\tilde{c}(1,2;t)} - \sum_{i=2}^{m-1} \frac{\tilde{c}(i-1,i+1;t)}{2\tilde{c}(i-1,i;t)\tilde{c}(i,i+1;t)} \\ &\left. - \frac{\tilde{c}(m-1,1;t)}{2\tilde{c}(m-1,m;t)\tilde{c}(m,1;t)}\right\} - 4.\end{aligned}$$

Then,

Proposition 7. *Let C be a semi-C-system. If $t \in [a,b]$, then, $f''(t) < 0$.*

PROOF. $\tilde{c}(i,i+1;t) = c(i,i+1)$ $(i = 1,\dots,m-1)$ and $\tilde{c}(m,1;t) = c(m,1)$ are constants. Also, $\tilde{c}(i-1,i+1;t) = c(i-1,i+1)$ $(2 \le i \le m-3)$ are constants. Moreover,

$$\tilde{c}(m-3,m-1;t) = \frac{c(m-3,m-2)t + c(1,m-3)c(m-2,m-1)}{c(1,m-2)},$$

$$\tilde{c}(m-2,m;t) = \frac{c(1,m-1)c(m-2,m)}{t},$$

$$\tilde{c}(m-1,1;t) = t,$$

$$\tilde{c}(m,2;t) = \frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)t} + \frac{c(1,m)c(2,m-2)}{c(1,m-2)}.$$

Hence,

$$\begin{aligned} f''(t) = {} & \left\{\sum_{i=1}^{m-1} c(i,i+1) + c(m,1)\right\}\left\{-\frac{\tilde{c}(m,2;t)''}{2c(m,1)c(1,2)}\right. \\ & - \frac{\tilde{c}(m-3,m-1;t)''}{2c(m-3,m-2)c(m-2,m-1)} - \frac{\hat{c}(m-2,m;t)''}{2c(m-2,m-1)c(m-1,m)} \\ & \left. - \frac{\tilde{c}(m-1,1;t)''}{2c(m-1,m)c(m,1)}\right\} \\ = {} & \left\{\sum_{i=1}^{m-1} c(i,i+1) + c(m,1)\right\}\left\{-\frac{c(1,2)c(1,m-1)c(m-2,m)}{2c(m,1)c(1,2)c(1,m-1)}\frac{2}{t^3}\right. \\ & \left. - \frac{c(1,m-1)c(m-2,m)}{2c(m-2,m-1)c(m-1,m)}\frac{2}{t^3}\right\} < 0. \end{aligned}$$

(Q.E.D.)

PROPOSITION 8. *If $f''(t) < 0$ for each $t \in [a,b]$, then $f(t) > \min\{f(a), f(b)\}$, for each $t \in (a,b)$.*

PROOF. Suppose that $f''(t) < 0$ for each $t \in [a,b]$. Then, $f'(t)$ is strictly decreasing in $[a,b]$.

Case 1. $f'(b) < f'(a) \le 0$.

Then, $f'(t) < 0$ for each $t \in (a,b)$. Hence, $f(t)$ is strictly decreasing in $[a,b]$. Hence,

$$f(t) > f(b) \ge \min\{f(a), f(b)\},$$

for each $t \in (a,b)$.

Case 2. $0 \leq f'(b) < f'(a)$.

Then, $f'(t) > 0$ for each $t \in (a,b)$. Hence, $f(t)$ is strictly increasing in $[a,b]$. Hence,

$$f(t) > f(a) \geq \min\{f(a), f(b)\},$$

for each $t \in (a,b)$.

Case 3. $f'(b) < 0 < f'(a)$.

Then, there is a unique $d \in (a,b)$ such that $f'(d) = 0$. $f(t)$ is strictly increasing in $[a,d]$ and strictly decreasing in $[d,b]$. So, if $t \in (a,d]$, then

$$f(t) > f(a) \geq \min\{f(a), f(b)\},$$

and if $t \in [d,b)$, then

$$f(t) > f(b) \geq \min\{f(a), f(b)\}.$$

This completes the proof of Proposition 8. (Q.E.D.)

Now we can show that (7) holds for the semi-C-system C. Since $\tilde{C}(c(1,m-1)) = C$, it suffices to show that $f(t) \geq 0$, for $t \in [a,b]$. By Proposition 7 and 8, it suffices to show that $f(a) \geq 0$ and $f(b) \geq 0$.

$a = \max\{r_1, r_2\}$. So, $a = r_1$ or $a = r_2$.

Case 1. $a = r_1$.

Then, $\tilde{C}(r_1)$ is a semi-C-system such that $\mu(\tilde{C}(r_1)) = m$ and

$$\begin{aligned}\tilde{c}(m-2,m;r_1) &= \frac{c(1,m-2)c(1,m-1)c(m-2,m)}{c(1,m-2)r_1} + \frac{c(1,m)c(m-2,m-2)}{c(1,m-2)} \\ &= \frac{c(1,m-1)c(m-2,m)}{r_1} = c(m-2,m-1) + c(m-1,m) \\ &= \tilde{c}(m-2,m-1;r_1) + \tilde{c}(m-1,m;r_1).\end{aligned}$$

Hence, by Proposition 5, (7) holds for $\tilde{C}(r_1)$, that *is*, $f(r_1) \geq 0$.

Cases 2. $a = r_2$.

Then, $\tilde{C}(r_2)$ is a semi-C-system such that $\mu(\tilde{C}(r_2)) = m$. Now,

$$\begin{aligned}\tilde{c}(m,2;r_2) &= \frac{c(1,2)c(1,m-1)c(m-2,m)}{c(1,m-2)r_2} + \frac{c(1,m)c(2,m-2)}{c(1,m-2)} \\ &= \frac{c(1,m-2)\{c(m,1) + c(1,2) - c(m,2)\} + c(1,2)c(m-1,m)}{c(1,m-2)} \\ &\quad + \frac{c(1,m)c(2,m-2)}{c(1,m-2)}\end{aligned}$$

$$= c(m,1) + c(1,2) - c(m,2) + \frac{c(1,2)c(m-2,m) + c(1,m)c(2,m-2)}{c(1,m-2)}$$

$$= c(m,1) + c(1,2) - c(m,2) + \frac{c(1,m-2)c(2,m)}{c(1,m-2)}$$

$$= c(m,1) + c(1,2)$$

$$= \tilde{c}(m,1;r_2) + \tilde{c}(1,2;r_2).$$

Hence, by Proposition 5, (7) holds for $\tilde{C}(r_2)$, that is $f(r_2) \geq 0$.

Thus, we have proved that $f(a) \geq 0$. Next we prove that $f(b) \geq 0$. $b = \min\{s_1, s_2\}$. So, $b = s_1$ or $b = s_2$.

Case 1. $b = s_1$.

Then, $\tilde{C}(s_1)$ is a semi-C-system such that $\mu(\tilde{C}(s_1)) = m$ and

$$\tilde{c}(1,m-1;s_1) = s_1 = c(m-1,m) + c(m,1)$$

$$= \tilde{c}(m-1,m;s_1) + \tilde{c}(m,1;s_1).$$

Hence, by Proposition 5, (7) holds for $\tilde{C}(s_1)$, that is $f(s_1) \geq 0$.

Case 2. $b = s_2$.

Then, $\tilde{C}(s_2)$ is a semi-C-system such that $\mu(\tilde{C}(s_2)) = m$ and

$$\tilde{c}(m-3,m-1;s_2) = \{c(m-3,m-2)s_2 + c(1,m-3)c(m-2,m-1)\}/c(1,m-2)$$

$$= [c(1,m-2)\{c(m-3,m-2) + c(m-2,m-1) - c(m-3,m-1)\}$$

$$+ c(1,m-1)c(m-3,m-2)$$

$$+ c(1,m-3)c(m-2,m-1)]/c(1,m-2)$$

$$= c(m-3,m-2) + c(m-2,m-1) - c(m-3,m-1)$$

$$+ \frac{c(1,m-2)c(m-3,m-1)}{c(1,m-2)}$$

$$= c(m-3,m-2) + c(m-2,m-1)$$

$$= \tilde{c}(m-3,m-2;s_2) + \tilde{c}(m-2,m-1;s_2).$$

Hence by Proposition 5, (7) holds for $\tilde{C}(s_2)$, that is, $f(s_2) \geq 0$. This completes the proof of (7) for the semi-C-system C.

Next suppose that C is a C-system. Then, by Proposition 6, $\tilde{C}(c(1,m-1)) = C$ and $c(1,m-1) \in (a,b)$. Hence by Proposition 7 and 8,

$$f(c(1,m-1)) > \min\{f(a), f(b)\} \geq 0.$$

Hence $f(c(1,m-1)) > 0$, that is, (6) holds for C. This completes the proof of Lemma 4 and hence of Lemma 3.

PROOF OF THEOREM 1.

Suppose that $\alpha_i\beta_k - \beta_i\alpha_k \neq 0$ $(i \neq k)$, $\gamma_j\delta_\ell - \delta_j\gamma_\ell \neq 0 (j \neq \ell)$, $e > 0, f > 0$ and that for all $(x,y) \in L$,

$$\sum_{i=1}^{m} |\alpha_i x - \beta_i y| \geq 1 \quad \text{and} \quad \sum_{i=1}^{n} |\gamma_j x - \delta_j y| \geq 1.$$

We shall prove that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_i\delta_j - \beta_i\gamma_j| \geq 1. \tag{11}$$

Now, by Lemma 3, we have

$$\sum_{i=1}^{m} \sum_{k=1}^{m} |\alpha_i\beta_k - \beta_i\alpha_k| \geq 1 \quad \text{and} \quad \sum_{j=1}^{n} \sum_{\ell=1}^{n} |\gamma_j\delta_\ell - \delta_j\gamma_\ell| \geq 1.$$

So, it suffices to prove that

$$\left\{\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_i\delta_j - \beta_i\gamma_j|\right\}^2 \geq \left\{\sum_{i=1}^{m} \sum_{k=1}^{m} |\alpha_i\beta_k - \beta_i\alpha_k|\right\}\left\{\sum_{j=1}^{n} \sum_{\ell=1}^{n} |\gamma_j\delta_\ell - \delta_j\gamma_\ell|\right\}. \tag{12}$$

Moreover, if (12) is proved and if the equality in (11) holds, then by (12) we have

$$\sum_{i=1}^{m} \sum_{k=1}^{m} |\alpha_i\beta_k - \beta_i\alpha_k| = 1 \quad \text{and} \quad \sum_{j=1}^{n} \sum_{\ell=1}^{n} |\gamma_j\delta_\ell - \delta_j\gamma_\ell| = 1.$$

By Lemma 3, we must have $m = 2$, $n = 2$ and $|\alpha_1\beta_2 - \beta_1\alpha_2| = 1/2$, $|\gamma_1\delta_2 - \delta_1\gamma_2| = 1/2$. Thus we have only to prove (12).

In order to prove (12), it suffices to prove that the quadratic form

$$F(x_1,\dots,x_m)=\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha_i\delta_j-\beta_i\gamma_j|x_i\right\}^2$$
$$-\left\{\sum_{i=1}^{m}\sum_{k=1}^{m}|\alpha_i\beta_k-\beta_i\alpha_k|x_ix_k\right\}\left\{\sum_{j=1}^{n}\sum_{\ell=1}^{n}|\gamma_j\delta_\ell-\delta_j\gamma_\ell|\right\}$$

is positive semi-definite, i.e. $F(x_1,\dots,x_m)\geq 0$ for every real numbers $x_1,\dots,x_m$.

Now, let

$$\xi_{ik}=\left\{\sum_{j=1}^{n}|\alpha_i\delta_j-\beta_i\gamma_j|\right\}\left\{\sum_{\ell=1}^{n}|\alpha_k\delta_\ell-\beta_k\gamma_\ell|\right\}$$
$$-|\alpha_i\beta_k-\beta_i\alpha_k|\left\{\sum_{j=1}^{n}\sum_{\ell=1}^{n}|\gamma_j\delta_\ell-\delta_j\gamma_\ell|\right\}.$$

Then, $\xi_{ik}=\xi_{ki}$ and

$$F(x_1,\dots,x_m)=\sum_{i=1}^{m}\sum_{k=1}^{m}\xi_{ik}x_ix_k.$$

PROPOSITION 9. *A quadratic form*

$$G(x_1,\dots,x_m)=\sum_{i=1}^{m}\sum_{k=1}^{m}p_{ik}x_ix_k$$

(*with* $p_{ik}=p_{ki}$) *is positive semi-definite if and only if*

($*$) *for each sequence* $i_1,\dots,i_r$ $(r\geq 1)$ *of natural numbers such that* $1\leq i_1<i_2<\cdots<i_r\leq m$,

$$\begin{vmatrix} p_{i_1i_1} & p_{i_1i_2} & \cdots & p_{i_1i_r}\\ p_{i_2i_1} & p_{i_2i_2} & \cdots & p_{i_2i_r}\\ \cdots & \cdots & \cdots & \cdots\\ p_{i_ri_1} & p_{i_ri_2} & \cdots & p_{i_ri_r}\end{vmatrix}\geq 0.$$

The proof of Proposition 9 on matrix theory is omitted.

By Proposition 9, it suffices to prove that

$$\begin{vmatrix} \xi_{i_1i_1} & \xi_{i_1i_2} & \cdots & \xi_{i_1i_r}\\ \xi_{i_2i_1} & \xi_{i_2i_2} & \cdots & \xi_{i_2i_r}\\ \cdots & \cdots & \cdots & \cdots\\ \xi_{i_ri_1} & \xi_{i_ri_2} & \cdots & \xi_{i_ri_r}\end{vmatrix}\geq 0,$$

for every sequence $i_1, \dots i_r$ such that $1 \le i_1 < \dots < i_r \le m$. By the change of subscript i_k to k, it suffices to prove that

$$\begin{vmatrix} \xi_{11} & \cdots\cdots & \xi_{1m} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} \\ \xi_{m1} & \cdots\cdots & \xi_{mm} \end{vmatrix} \ge 0,$$

for every α_i, β_i, γ_j, δ_j.

Now, let $a_i = \sum_{j=1}^{n} |\alpha_i \delta_j - \beta_i \gamma_j|$, $d_{ik} = |\alpha_i \beta_k - \beta_i \alpha_k|$, $t = \sum_{j=1}^{n} \sum_{\ell=1}^{n} |\gamma_j \delta_\ell - \delta_j \gamma_\ell|$. Then, $\xi_{ik} = a_i a_k - d_{ik} t$. Now,

$$\begin{vmatrix} \xi_{11} & \cdots\cdots & \xi_{1m} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} \\ \xi_{m1} & \cdots\cdots & \xi_{mm} \end{vmatrix} = \begin{vmatrix} a_1^2 - d_{11}t & a_1 a_2 - d_{12}t & \cdots\cdots & a_1 a_m - d_{1m}t \\ a_2 a_1 - d_{21}t & a_2^2 - d_{22}t & \cdots\cdots & a_2 a_m - d_{2m}t \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_m a_1 - d_{m1}t & a_m a_2 - d_{m2}t & \cdots\cdots & a_m^2 - d_{mm}t \end{vmatrix}$$

$$= a_1 \begin{vmatrix} a_1 & -d_{12}t & -d_{13}t & \cdots\cdots & -d_{1m}t \\ a_2 & -d_{22}t & -d_{23}t & \cdots\cdots & -d_{2m}t \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_m & -d_{m2}t & -d_{m3}t & \cdots\cdots & -d_{mm}t \end{vmatrix}$$

$$+ a_2 \begin{vmatrix} -d_{11}t & a_1 & -d_{13}t & \cdots\cdots & -d_{1m}t \\ -d_{21}t & a_2 & -d_{23}t & \cdots\cdots & -d_{2m}t \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ -d_{m1}t & a_m & -d_{m3}t & \cdots\cdots & -d_{mm}t \end{vmatrix}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ a_m \begin{vmatrix} -d_{11}t & -d_{12}t & \cdots\cdots & -d_{1m-1}t & a_1 \\ -d_{21}t & -d_{22}t & \cdots\cdots & -d_{2m-1}t & a_2 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ -d_{m1}t & -d_{m2}t & \cdots\cdots & -d_{mm-1}t & a_m \end{vmatrix}$$

$$+ \begin{vmatrix} -d_{11}t & -d_{12}t & \cdots\cdots & -d_{1m-1}t & -d_{1m}t \\ -d_{21}t & -d_{22}t & \cdots\cdots & -d_{2m-1}t & -d_{2m}t \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ -d_{m1}t & -d_{m2}t & \cdots\cdots & -d_{mm-1}t & -d_{mm}t \end{vmatrix}$$

$$= (-1)^{m-1} t^{m-1} a_1 \begin{vmatrix} a_1 & d_{12} & d_{13} & \cdots\cdots & d_{1m} \\ a_2 & d_{22} & d_{23} & \cdots\cdots & d_{2m} \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_m & d_{m2} & d_{m3} & \cdots\cdots & d_{mm} \end{vmatrix}$$

$$+(-1)^{m-1}t^{m-1}a_2\begin{vmatrix} d_{11} & a_1 & d_{13} & \cdots\cdots & d_{1m} \\ d_{21} & a_2 & d_{23} & \cdots\cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{m1} & a_m & d_{m3} & \cdots\cdots & d_{mm} \end{vmatrix}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(-1)^{m-1}t^{m-1}a_m\begin{vmatrix} d_{11} & d_{12} & \cdots\cdots & d_{1m-1} & a_1 \\ d_{21} & d_{22} & \cdots\cdots & d_{2m-1} & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{m1} & d_{m2} & \cdots\cdots & d_{mm-1} & a_m \end{vmatrix}$$

$$+(-1)^{m}t^{m}\begin{vmatrix} d_{11} & d_{12} & \cdots\cdots & d_{1m-1} & d_{1m} \\ d_{21} & d_{22} & \cdots\cdots & d_{2m-1} & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{m1} & d_{m2} & \cdots\cdots & d_{mm-1} & d_{mm} \end{vmatrix}$$

$$= t^{m-1}(-1)^{m}\begin{vmatrix} t & a_1 & a_2 & \cdots\cdots & a_m \\ a_1 & d_{11} & d_{12} & \cdots\cdots & d_{1m} \\ a_2 & d_{21} & d_{22} & \cdots\cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_m & d_{m1} & d_{m2} & \cdots\cdots & d_{mm} \end{vmatrix}.$$

Since $t > 0$, it suffices to prove that

$$K = (-1)^{m}\begin{vmatrix} t & a_1 & a_2 & \cdots\cdots & a_m \\ a_1 & d_{11} & d_{12} & \cdots\cdots & d_{1m} \\ a_2 & d_{21} & d_{22} & \cdots\cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_m & d_{m1} & d_{m2} & \cdots\cdots & d_{mm} \end{vmatrix} \geq 0.$$

Since

$$t = \sum_{j=1}^{n}\left(\sum_{\ell=1}^{n}|\gamma_j\delta_\ell - \delta_j\gamma_\ell|\right)$$

and

$$a_i = \sum_{j=1}^{n}|\alpha_i\delta_j - \beta_i\gamma_j| = \sum_{\ell=1}^{n}|\alpha_i\delta_\ell - \beta_i\gamma_l|,$$

we have

$$K=\sum_{j=1}^{n}(-1)^{n}\begin{vmatrix} \sum\limits_{\ell=1}^{n}|\gamma_j\delta_\ell-\delta_j\gamma_\ell| & |\alpha_1\delta_j-\beta_1\gamma_j| & |\alpha_2\delta_j-\beta_2\gamma_j| & \cdots & |\alpha_m\delta_j-\beta_m\gamma_j| \\ \sum\limits_{\ell=1}^{n}|\alpha_1\delta_\ell-\beta_1\gamma_\ell| & d_{11} & d_{12} & \cdots & d_{1m} \\ \sum\limits_{\ell=1}^{n}|\alpha_2\delta_\ell-\beta_2\gamma_\ell| & d_{21} & d_{22} & \cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{\ell=1}^{n}|\alpha_m\delta_\ell-\beta_m\gamma_\ell| & d_{m1} & d_{m2} & \cdots & d_{mm} \end{vmatrix}$$

$$=\sum_{j=1}^{n}\sum_{\ell=1}^{n}(-1)^{m}\begin{vmatrix} \gamma_j\delta_\ell-\delta_j\gamma_\ell| & |\alpha_1\delta_j-\beta_1\gamma_j| & |\alpha_2\delta_j-\beta_2\gamma_j| & \cdots & |\alpha_m\delta_j-\beta_m\gamma_j| \\ |\alpha_1\delta_\ell-\beta_1\gamma_\ell| & d_{11} & d_{12} & \cdots & d_{1m} \\ |\alpha_2\delta_\ell-\beta_2\gamma_\ell| & d_{21} & d_{22} & \cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ |\alpha_m\delta_\ell-\beta_m\gamma_\ell| & d_{m1} & d_{m2} & \cdots & d_{mm} \end{vmatrix}.$$

So, it suffices to prove that

$$L=(-1)^{m}\begin{vmatrix} |\alpha_0v-\beta_0u| & |\alpha_1v-\beta_1u| & |\alpha_2v-\beta_2u| & \cdots\cdots & |\alpha_mv-\beta_mu| \\ |\alpha_1\beta_0-\beta_1\alpha_0| & |\alpha_1\beta_1-\beta_1\alpha_1| & |\alpha_1\beta_2-\beta_1\alpha_2| & \cdots & |\alpha_1\beta_m-\beta_1\alpha_m| \\ |\alpha_2\beta_0-\beta_2\alpha_0| & |\alpha_2\beta_1-\beta_2\alpha_1| & |\alpha_2\beta_2-\beta_2\alpha_2| & \cdots & |\alpha_2\beta_m-\beta_2\alpha_m| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ |\alpha_m\beta_0-\beta_m\alpha_0| & |\alpha_m\beta_1-\beta_m\alpha_1| & |\alpha_m\beta_2-\beta_m\alpha_2| & \cdots & |\alpha_m\beta_m-\beta_m\alpha_m| \end{vmatrix}$$

$$\geq 0,$$

for every real numbers u, v, α_i, β_i $(i=0,1,\ldots,m)$.

The following Proposition is rather obvious. So the detailed proof of it is omitted.

PROPOSITION 10. *Let*

$$\varphi(u,v)=\sum_{i=1}^{n}\theta_i|\zeta_iu-\rho_iv|,$$

where θ_i, ζ_i, ρ_i *are real number. If* $\varphi(\rho_i,\zeta_i)\geq 0$ *for all* $i=1,2,\ldots,n$, *then* $\varphi(u,v)\geq 0$ *for all* u, v.

Now, the above L is of the form

$$L = L(u, v) = \sum_{i=0}^{m} |\alpha_i v - \beta_i u|.$$

By Proposition 10, $L(u, v) \geq 0$ for all u, v, if we prove that $L(\alpha_i, \beta_i) \geq 0$, $(i = 0, 1, \dots, m)$.

Now, $L(\alpha_i, \beta_i) = 0$ $(i = 1, 2, \dots, n)$ is obvious. So, it remains to prove that $L(\alpha_0, \beta_0) \geq 0$, i.e.

$$S = (-1)^m \begin{vmatrix} |\alpha_0\beta_0 - \beta_0\alpha_0| & |\alpha_0\beta_1 - \beta_0\alpha_1| & |\alpha_0\beta_2 - \beta_0\alpha_2| & \cdots & |\alpha_0\beta_m - \beta_0\alpha_m| \\ |\alpha_1\beta_0 - \beta_1\alpha_0| & |\alpha_1\beta_1 - \beta_1\alpha_1| & |\alpha_1\beta_2 - \beta_1\alpha_2| & \cdots & |\alpha_1\beta_m - \beta_1\alpha_m| \\ |\alpha_2\beta_0 - \beta_2\alpha_0| & |\alpha_2\beta_1 - \beta_2\alpha_1| & |\alpha_2\beta_2 - \beta_2\alpha_2| & \cdots & |\alpha_2\beta_m - \beta_2\alpha_m| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ |\alpha_m\beta_0 - \beta_m\alpha_0| & |\alpha_m\beta_1 - \beta_m\alpha_1| & |\alpha_m\beta_2 - \beta_m\alpha_2| & \cdots & |\alpha_m\beta_m - \beta_m\alpha_m| \end{vmatrix}$$
$$\geq 0.$$

Because of symmetry, we can assume that

$$\frac{\alpha_0}{\beta_0} \geq \frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \geq \cdots \geq \frac{\alpha_m}{\beta_m},$$

where $\beta_i \geq 0$ $(i = 0, 1, \dots, m)$ and if $\beta_i = 0$ then $\alpha_i > 0$ and α_i/β_i is regarded as $+\infty$. (If necessary, (α_i, β_i) is replaced by $(-\alpha_i, -\beta_i)$). Then, $|\alpha_i\beta_j - \beta_i\alpha_j| = \alpha_i\beta_j - \beta_i\alpha_j$, if $i \leq j$. Hence

$$S = (-1)^m \begin{vmatrix} 0 & \alpha_0\beta_1 - \beta_0\alpha_1 & \alpha_0\beta_2 - \beta_0\alpha_2 & \cdots & \alpha_0\beta_m - \beta_0\alpha_m \\ \alpha_0\beta_1 - \beta_0\alpha_1 & 0 & \alpha_1\beta_2 - \beta_1\alpha_2 & \cdots & \alpha_1\beta_m - \beta_1\alpha_m \\ \alpha_0\beta_2 - \beta_0\alpha_2 & \alpha_1\beta_2 - \beta_1\alpha_2 & 0 & \cdots & \alpha_2\beta_m - \beta_2\alpha_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_0\beta_m - \beta_0\alpha_m & \alpha_1\beta_m - \beta_1\alpha_m & \alpha_2\beta_m - \beta_2\alpha_m & \cdots & 0 \end{vmatrix}$$

So, $S \geq 0$ follows from the following.

PROPOSITION 11.

$$S = 2^{m-1}(\alpha_0\beta_1 - \beta_0\alpha_1)(\alpha_1\beta_2 - \beta_1\alpha_2)\cdots(\alpha_{m-1}\beta_m - \beta_{m-1}\alpha_m)(\alpha_0\beta_m - \beta_0\alpha_m).$$

PROOF. S is a homogeneous polynomial in (α_i, β_i) $(i = 0, 1, \dots, m)$ and vanishes when $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$. Also, it vanishes when $\alpha_0 = \alpha_m$, $\beta_0 = \beta_m$. Hence S can be divided by

$$(\alpha_0\beta_1 - \beta_0\alpha_1)(\alpha_1\beta_2 - \beta_1\alpha_2)\cdots(\alpha_{m-1}\beta_m - \beta_{m-1}\alpha_m)(\alpha_0\beta_m - \beta_0\alpha_m).$$

So,

$$S = k(\alpha_0\beta_1 - \beta_0\alpha_1)(\alpha_1\beta_2 - \beta_1\alpha_2)\cdots(\alpha_{m-1}\beta_m - \beta_{m-1}\alpha_m)(\alpha_0\beta_m - \beta_0\alpha_m),$$

for some constant k. In order to determine k, let $\alpha_i = m - i$, $\beta_i = 1$ $(i = 0, 1, \ldots, m)$. Then, $\alpha_i\beta_j - \beta_i\alpha_j = j - i$. So,

$$S = (-1)^m \begin{vmatrix} 0 & 1 & 2 & \cdots\cdots & m \\ 1 & 0 & 1 & \cdots\cdots & m-1 \\ 2 & 1 & 0 & \cdots\cdots & m-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m & m-1 & m-2 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} -1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & 0 & 1 & \cdots\cdots & m-1 \\ 2 & 1 & 0 & \cdots\cdots & m-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m & m-1 & m-2 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} -1 & 1 & 1 & \cdots\cdots & 1 \\ -1 & -1 & 1 & \cdots\cdots & 1 \\ 2 & 1 & 0 & \cdots\cdots & m-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m & m-1 & m-2 & \cdots & 0 \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= (-1)^m \begin{vmatrix} -1 & 1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ -1 & -1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ -1 & -1 & -1 & 1 & \cdots\cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots\cdots & -1 & 1 \\ m & m-1 & m-2 & m-3 & \cdots\cdots & 1 & 0 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} 0 & 2 & 0 & 0 & \cdots\cdots & 0 & 0 \\ -1 & -1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ -1 & -1 & -1 & 1 & \cdots\cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots\cdots & -1 & 1 \\ m & m-1 & m-2 & m-3 & \cdots\cdots & 1 & 0 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} 0 & 2 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots\cdots & 0 & 0 \\ -1 & -1 & -1 & 1 & \cdots\cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots\cdots & -1 & 1 \\ m & m-1 & m-2 & m-3 & \cdots\cdots & 1 & 0 \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= (-1)^m \begin{vmatrix} 0 & 2 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 2 & 0 \\ -1 & -1 & -1 & -1 & \cdots\cdots & -1 & 1 \\ m & m-1 & m-2 & m-3 & \cdots\cdots & 1 & 0 \end{vmatrix}$$

$$= 2^{m-1}m.$$

Hence,

$$2^{m-1}m = k \cdot 1 \cdot 1 \cdot \cdots \cdot 1 \cdot m.$$

Hence $k = 2^{m-1}$, as was to be proved.

This completes the proof of Theorem 1.