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THE SELF-EQUIVALENCE GROUPS IN CERTAIN COHERENT HOMOTOPY CATEGORIES

By

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Abstract. We study the self-equivalence groups associated with objects in (i) the track homotopy category over a fixed space B, (ii) the track homotopy category under a fixed space A and (iii) the category of homotopy pairs. In each case a short exact sequence decomposition of the self-equivalence group is available. In the case of (i) the group is isomorphic to the group of fibre-homotopy self-equivalences of an associated fibration, the decomposition (in other form) is known and has been used as the basis of computations. We make sample computations in the simplest situations for (i), (ii), and (iii), in each case solving the extension problem that arises by considering secondary operations and determining the Toda-Hopf invariant of relevant tracks. We indicate that in certain cases such computations can be used to determine the self-equivalence group of a mapping cone.

0. Introduction

Let $p : E \rightarrow B$ be a pointed Hurewicz fibration. The problem of determining the group $\mathscr{E}_F(E)$ of self fibre-homotopy equivalences of p seems to have been considered first by Nomura [9]. $\mathscr{E}_F(E)$ is formally the group of automorphisms of p in the classical homotopy category of (pointed) spaces over B. As discussed by the authors of [7] this category has certain inconvenient features and can (for many purposes with advantage) be replaced by $\mathscr{H}_B$, the corresponding *track homotopy category over B*, which has the same objects but whose arrows are

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equivalence classes $\{v, v_t\}$ of diagrams

(0.1)
$$\begin{array}{ccc} W & \xrightarrow{v} & X \\ {\scriptstyle w}\downarrow & \{v_t\}\nearrow & \downarrow{\scriptstyle f} \\ B & = & B \end{array}$$

with commuting *track* (i.e. relative homotopy class of homotopies). Moreover it was shown [6; Theorem 4.3] that the set $\pi(w, f/B)$ of morphisms from w to f lies in an exact sequence

(0.2) $$\pi_1^W(X; u) \xrightarrow{f.} \pi_1^W(B; fu) \xrightarrow{m_v} \pi(w, f/B)_v \xrightarrow{d} \pi(W, X)_u \xrightarrow{f.} \pi(W, B)_w$$

yielding a bijection

(0.3) $$\pi(w, f/B) \leftrightarrow \bigcup_{\{u\} \in f.^{-1}\{w\}} K(u, w),$$

where $K(u, w)$ denotes the set of left cosets of $f.\pi_1^W(X; u)$ in $\pi_1^W(B; fu)$. Here $\pi_1^W(X; u)$ denotes the u-based track group in the sense of Rutter [10].

If we are interested in the automorphism group of an object f, we may enquire what happens to the bijection 0.3 when we take $W = X$, $w = f$, $u = 1 = 1_X$. Then it turns out that the operator m_v, (which in general depends on the choice of preferred element v in $\pi(w, f/B)$) becomes a homomorphism if we take $v = 1_f$ in $\pi(f, f/B)$, and $f.\pi_1^X(X; 1)$ becomes a normal subgroup of $\pi_1^X(B; f)$. Moreover the image of m_v is contained in $\mathscr{E}(f/B)$, the self-equivalence group of f in $\mathscr{H}_B$ and, if we further restrict d to this subgroup, its image lies in

(0.4) $$\bar{\mathscr{E}}_f(X) = \{\{v\} | v \text{ is a self-equivalence of } X \text{ with } fv \simeq f\}.$$

Accordingly the bijection 0.3 takes the form of a short exact sequence of (not necessarily abelian) groups:

(0.5) $$coker(f.) \rightarrowtail \mathscr{E}(f/B) \twoheadrightarrow \bar{\mathscr{E}}_w(X).$$

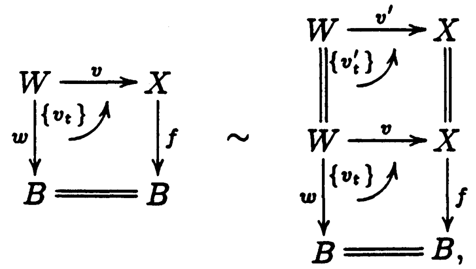
Details for the decomposition 0.5 are given in section 1, together with a technique for settling the extension problem, which we successfully use in a sample computation. Corresponding discussions of self-equivalences under a fixed A and of homotopy pair self-equivalences are given in sections 2 and 3. In a final section we show that the homotopy pair self-equivalence group of a map f may coincide with the self-equivalence group of its associated mapping cone.

Although we began the introduction by considering a pointed fibration it is worth noting that the argument leading up to the sequence 0.5 applies equally well to the basepoint free case and, indeed, remains valid if the category of topological spaces is replaced by an arbitrary 2-category with invertible 2-morphisms.

1. Track self-equivalences over B

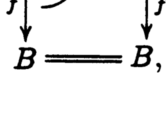
We recall [6] that the morphism set $\pi(w, f/B)$ from w to f in $\mathscr{H}_B$ is obtained from the set of diagrams of form 0.1 by factoring out by the equivalence relation

(1.1)

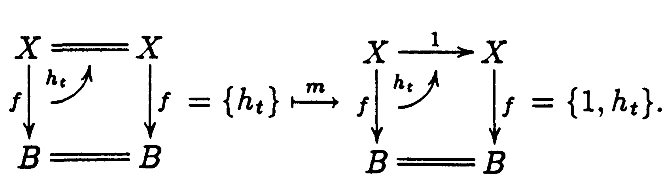


where the diagram on the right is the composite in the obvious sense of the two rectangles. We note that each isomorphism in $\mathscr{H}_B$ is the equivalence class of a diagram of form 0.1 in which the map v is a homotopy equivalence [7; Theorem 1.3].

For the general case of the sequence 0.2, apart from the first homomorphism, the operators are functions between pointed sets in which there is some freedom in the choice of preferred elements. Since we are interested in the case $W = X$, $w = f$ we are free to select preferred elements $\{f\} \in \pi(X, B)$, $\{u\} = \{1\} \in \pi(X, X)$. Moreover, choosing v to be the class of the diagram



m_v becomes the function



With d the operator that selects the class of the top arrow of diagram 0.1 we have the following.

1.2. PROPOSITION. *The following sequence of groups and homomorphisms is exact.*

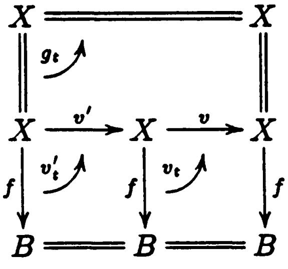
$$\pi_1^X(X;1) \xrightarrow{f.} \pi_1^X(B;f) \xrightarrow{m} \mathscr{E}(f/B) \overset{d}{\twoheadrightarrow} \mathscr{E}_f(X)$$

PROOF. Although the result can be obtained by specialisation from [6; Theorem 4.3] it will be convenient to argue directly. Clearly $dm = 0$. Suppose that $d\{v, v_t\} = 0$. Then there exists a homotopy $g_t : v \simeq 1$. Then $m\{v_t + fg_t\} = \{1, v_t + fg_t\} \sim \{v, v_t\}$. Clearly $mf. = 0$. Suppose $m\{h_t\} = 0$. Then $\{1, h_t\} \sim \{1, 1\}$, hence there exists $g_t : 1 \to 1$ such that h_t is relatively homotopic to fg_t. Then $f.\{g_t\} = \{h_t\}$. The exactness of 0.5 is an immediate consequence.

1.3. REMARK. If $\mathscr{E}(f/B)$ and $\bar{\mathscr{E}}_f(X)$ are replaced by $M(f/B)$ (the endomorphism monoid of f in the track homotopy category over B) and $\bar{M}_f(X) = \{\{u\} | u : X \to X, fu \simeq f\}$ respectively, the sequence remains exact.

The sequence 0.5 is, of course, a formulation in another setting of a basic fact that, in the context of fibre homotopy self-equivalences, has been exploited by several authors (cf. [9], [3], [13], [11], [2]). The simplicity of the situation, however, facilitates the enunciation of a principle that can sometimes be used to resolve the problems of group extension that arise when 0.5 is used to compute $\mathscr{E}(f/B)$. Following the pioneering work of Toda [12], secondary operations have been used systematically to resolve extension problems. Suppose that α, α' in $\bar{\mathscr{E}}_f(X)$ are elements with composition $\alpha.\alpha' = 1$ in $\bar{\mathscr{E}}_f(X)$. In view of the surjectivity of d there exist elements β, β' in $\mathscr{E}(f/B)$ such that $d\beta = \alpha$, $d\beta' = \alpha'$. For the extension problem it becomes important to know whether $\beta.\beta'$ is the image under m of a non-zero element of $coker(f.)$. Let $\beta = \{v, v_t\}$, $\beta' = \{v', v'_t\}$ and consider the element

(1.4) $\gamma = \gamma(g_t) =$



$\in \pi_1^X(B; f)$,

where $g_t : vv' \simeq 1$, and let

$$(1.5) \qquad \{\beta, \beta'\}_B = \{\gamma | g_t : vv' \simeq 1\} \subseteq \pi_1^X(B; f).$$

Then we have

1.6. Proposition. *The subset* $\{\beta, \beta'\}_B$ *is a coset in* $\pi_1^X(B; f)$ *of the image of the homomorphism* $f. : \pi_1^X(X; 1) \to \pi_1^X(B; f)$ *and is independent of the choice of representatives* (v, v_t), (v', v_t') *of* β, β' *respectively. Moreover* $m\{\beta, \beta'\}_B = \beta.\beta'$.

Proof. Composing the homotopies in diagram 1.4 we see that $\gamma(g_t) = \{v_t' + v_t v' + f g_t\}$. Hence $\gamma(g_t)^{-1}\gamma(g_t') = \{f g_{1-t} + v_{1-t} v' + v_{1-t}' + v_t' + v_t v' + f g_t'\} = f.\{g_{1-t} + g_t'\}$. The independence of $\{\beta, \beta'\}_B$ on the choice of representatives is an easy consequence of the relation 1.1 and, to verify the equation $m\{\beta, \beta'\} = \beta\beta'$, we need only apply the definition of m to the composite square 1.4, bearing in mind 1.1.

1.7. Example. Let $f : S^5 \to S^3$ be a representative of the generator η_3^2 of $\pi_5(S^3)$ (using Toda's notation [12]). The homomorphism $f.$ is equivalent to Rutter's homomorphism $\Delta(f, 1_X) : [X, \Omega X] \to [X, \Omega B]$, [10; §1.2]. In this case, since $X = S^5$ is a suspension, it follows from [10; Corollary 1.4.4] that $f.$ is equivalent to $\eta_{3*}^2 : \pi_6(S^5) \to \pi_6(S^3)$. Since $\pi_6(S^3) \approx \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_3$, with $2\nu' = \eta_3^3$ [12], we have *coker*$(f.) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$. Moreover, $\bar{\mathscr{E}}_f(S^5) \approx \boldsymbol{Z}_2$ generated by the degree minus one class $-\iota$. Then we have:

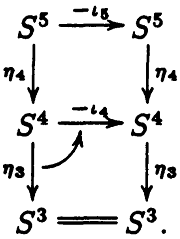
1.7.1. Proposition. $\mathscr{E}(f/S^3) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$.

Proof. Before indicating certain generators of $\mathscr{E}(f/S^3)$ we develop an appropriate notation. Since the homotopy set $\pi(w, f/B)$ depends (up to bijection) only on the homotopy classes of f and w [7; Corollary 1.4], we may replace maps by homotopy classes when using diagrams to indicate elements. We obtain generators as follows. Since $2\eta_3 = 0$ there is an element

$$(1.7.2) \qquad \beta = \begin{array}{ccc} S^5 & \xrightarrow{-\iota_5} & S^5 \\ {\scriptstyle \eta_4}\downarrow & \nearrow & \downarrow{\scriptstyle \eta_4} \\ S^4 & \xrightarrow{\iota_4} & S^4 \\ {\scriptstyle \eta_3}\downarrow & & \downarrow{\scriptstyle \eta_3} \\ S^3 & = & S^3. \end{array}$$

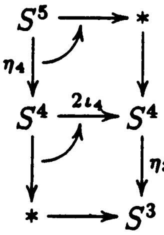
Cells of the diagram that do not contain a curving arrow are understood to carry the identity track. Strictly speaking diagram 1.7.2 does not define a unique element but rather a coset determined by the possible tracks that could occupy the position of the curving arrow. We nevertheless regard 1.7.2 as representing a typical element of the coset. We have another generator

(1.7.3) $\beta' =$



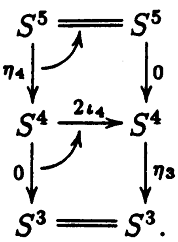
Although it may not be obvious a priori that $\beta \neq \beta'$, this will be verified. Now recall that there is an element α of $\pi_6(S^3)$ of order three. Since the f-based track group $\pi_1^{S^5}(S^3; f)$ is isomorphic with $\pi_6(S^3)$ there is a generator of $\pi_1^{S^5}(S^3; f)$ corresponding to α. We claim that $\beta.\beta' \neq 1$, $\beta.\beta = 1$, $\beta'.\beta' = 1$ and $\beta m(\alpha)\beta^{-1} = \beta m(\alpha)\beta = m(\alpha)$. In view of the nature of the possible extensions these are sufficient to prove 1.7.1. In order to establish 1.7.1 we utilise Proposition 1.6 and the discussion preceding the proposition. First note that the composite track

(1.7.4)

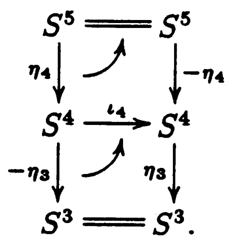


represents the Toda bracket $\{\eta_3, 2\iota_4, \eta_4\}$ [12; Proposition 1.3] (see also [4]), and hence the two elements ν' and $-\nu'$, [12; (5.4)]. Note that, regarded as a coset in $\pi_1^{S^5}(S^3; *)$, we can re-express 1.7.4 as

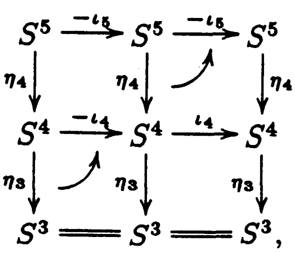
(1.7.5)



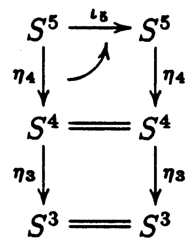
The diagram 1.7.5 can be modified (while still representing the same coset) by subtracting the same element from each route from source to sink. Applying this principle to 1.7.5 we see that 1.7.4 can also represented by the diagram



(The general principle can be justified by considering Rutter's isomorphism $f_\flat$, see [10; p381].) Moreover, arranging the position of the tracks a little differently, we see that 1.7.4 also represents the element



and hence coincides with $\beta.\beta'$. As for $\beta.\beta$, we note that it is represented by a diagram of form



and hence lies in the image of the homomorphism $\eta_{3_*} : \mathscr{E}(f'/S^4) \to \mathscr{E}(f/S^3)$, where f' is a representative of η_4. Considering the sequence 0.5 associated with $\mathscr{E}(f'/S^4)$, it is clear that $\beta.\beta = 1$. Similarly for $\beta'.\beta'$ which belongs to the image of the homomorphism $\eta_4^* : \mathscr{E}(f''/S^3) \to \mathscr{E}(f/S^3)$, where f'' is a representative of η_3. Since $\beta.\beta \neq \beta.\beta'$, it follows that $\beta \neq \beta'$ as claimed earlier.

It remains to verify that $\beta m(\alpha)\beta^{-1} = \beta m(\alpha)\beta = m(\alpha)$. To achieve this we need to recall that α is detected by the Toda-Hopf invariant. There is a homotopy equivalence $S^2_\infty \cong \Omega S^3$ between the James space and the space of loops on S^3. Via the adjoint isomorphism α is equivalent to a class in $\pi_5(S^2_\infty)$ and it is known that $\alpha = i_*\bar{\alpha}$, where $\bar{\alpha} \in \pi_5(S^2_2)$ and $\bar{\alpha}$ is of infinite order, with $3\bar{\alpha} = [\iota_2]^3$, the attaching class of the 6-cell of S^2_∞. (We have used i to represent the class of the inclusion map $S^2 \to S^2_\infty$.) As described in [8] the Toda-Hopf invariant is a homomorphism $\bar{H}_p$ of homotopy groups associated with an odd prime p (in our case $p = 3$ and $n = 1$), induced by a map

$$\bar{h} : \Omega S^{2n}_{p-1} \to \Omega S^{2np-1}$$

which, composed with the inclusion map $S^{2n-1} \to \Omega S^{2n} \to \Omega S^{2n}_{p-1}$ is trivial. Then (with $p = 3$ and $n = 1$) $\bar{H}_3 : \pi_r(S^2_2) \to \pi_r(S^5)$ detects α in the sense that $\bar{H}_3(\bar{\alpha}) = \iota_5$ in $\pi_5(S^5)$. Let $\bar{\eta}$ in $\pi_2(\Omega S^2_2)$ denote the adjoint of $i_2 \circ \eta_2$ in $\pi_3(S^2_2)$. (Here i_2 refers to the class of the inclusion $S^2 \to S^2_2$.) It follows that the adjoint of $\bar{\alpha}$ is represented by a diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\bar{\eta}\circ\eta_2} & \Omega S^2_2 \\ {\scriptstyle \iota}\uparrow & \nearrow & \| \\ S^3 & \xrightarrow{\bar{\eta}\circ\eta_2} & \Omega S^2_2, \end{array}$$

regarded as an element of $\pi_1^{S^3}(\Omega S^2_2; f''')$, where f''' is a representative of $\bar{\eta} \circ \eta_2$. Moreover, denoting by $\bar{\beta}$ the corestriction to ΩS^2_2 of the adjoint of β, the following diagram represents $\bar{H}_3\bar{\beta}\bar{\alpha}\bar{\beta}$.

$$\begin{array}{ccccccc} S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5 \\ {\scriptstyle -\iota}\uparrow & \nearrow & \uparrow{\scriptstyle \iota} & & \| & & \| \\ S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5 \\ \| & \nearrow & & & \| & & \| \\ S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5 \\ {\scriptstyle -\iota}\uparrow & \nearrow & \uparrow{\scriptstyle \iota} & & \| & & \| \\ S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5 \end{array} \qquad (1.7.6)$$

Note that the top three rectangles of 1.7.6 can be rewritten

$$\begin{array}{ccccccc} S^3 & \xrightarrow{-\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5 \\ {\scriptstyle \iota}\uparrow & \nearrow & \uparrow{\scriptstyle \iota} & & \| & & \| \\ S^3 & \xrightarrow{\eta_2} & S^2 & \xrightarrow{\bar{\eta}} & \Omega S^2_2 & \xrightarrow{\{\bar{h}\}} & \Omega S^5. \end{array} \qquad (1.7.7)$$

Now certainly both $\{\bar{h}\} \circ \bar{\eta} \circ \eta_2 = 0$ and $\{\bar{h}\} \circ \bar{\eta} \circ (-\eta_2) = 0$ in $\pi_3(\Omega S^5)$. It follows that 1.7.7 represents a class in $\pi_5(S^5)$ that one might describe as the Toda-Hopf invariant of the track $\bar{\beta}$. Moreover it is clear that the class represented by 1.7.7 is zero since it belongs to $\{\bar{h}\} \circ \bar{\eta} \circ \pi_4(S^2)$. In fact each row of rectangles in 1.7.6 represents an element of $\pi_5(S^5)$, with 1.7.6 itself the sum of these, which is ι_5. Therefore $\beta\alpha\beta = \alpha$. A similar argument shows that $\beta'\alpha\beta' = \alpha$, completing the proof of Proposition 1.7.1.

2. Track self-equivalences under A

Let $v : A \to W$, $g : A \to X$ be (pointed or unpointed) continuous maps. Recall [6], [7] that the morphism set $\pi(A/v, g)$ from v to g in the *track homotopy category* $\mathscr{H}^A$ *under* A is obtained from the set of track commutative diagrams under A by factoring out by the equivalence relation

(2.1)
$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow & \{w_t\}\nearrow & \downarrow{\scriptstyle g} \\ W & \xrightarrow[w]{} & X \end{array} \quad \sim \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow & \{w_t\}\nearrow & \downarrow{\scriptstyle g} \\ W & \xrightarrow{w} & X \\ \| & \{w'_t\}\nearrow & \| \\ W & \xrightarrow[w']{} & X. \end{array}$$

We may express the relation 2.1 in the non-diagramatic form $(\{w_t\}, w) \sim (\{w'_t v + w_t\}, w')$, and use $\{w_t, w\}$ to denote the equivalence class concerned.

Let $\mathscr{E}(A/g)$ be the self-equivalence group of g under A and consider the operators $n : \pi_1^A(X; g) \to \mathscr{E}(A/g)$ and $c : \mathscr{E}(A/g) \to \underline{\mathscr{E}}^g(X)$ given by the rules

(2.2)
$$\begin{array}{ccc} A & = & A \\ {\scriptstyle g}\downarrow & \{h_t\}\nearrow & \downarrow{\scriptstyle g} \\ X & = & X \end{array} \quad \overset{n}{\longmapsto} \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle g}\downarrow & \{h_t\}\nearrow & \downarrow{\scriptstyle g} \\ X & \xrightarrow{1} & X \end{array}$$

(2.3)
$$c\{w_t, w\} = \{w\},$$

where $\underline{\mathscr{E}}^g(X)$ is the subgroup of homotopy classes $\{w\}$ of self-equivalences $w : X \to X$ satisfying $\{wg\} = \{g\}$. Then we have

2.4. Proposition. *The following sequence of groups and homomorphisms is exact.*

$$\pi_1^X(X; 1) \xrightarrow{.g} \pi_1^A(X; g) \xrightarrow{n} \mathscr{E}(A/g) \overset{c}{\twoheadrightarrow} \underline{\mathscr{E}}^g(X)$$

The proof is dual to that of Proposition 1.2. As a corollary we obtain a short exact sequence dual to 0.5

(2.5)
$$coker(.g) \rightarrowtail \mathscr{E}(A/g) \twoheadrightarrow \mathscr{E}^g(X)$$

and a remark dual to 1.3 can be made.

To formulate a dual to Proposition 1.6, let $\gamma = \{w_t, w\}$ and $\gamma' = \{w'_t, w'\}$ be elements of $\mathscr{E}(A/g)$ such that $c\gamma.c\gamma' = 1$ in $\underline{\mathscr{E}}^g(X)$ and consider the element

(2.6)
$$\delta = \delta(k_t) = \begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle g}\downarrow & \{w'_t\}\nearrow & {\scriptstyle g}\downarrow & \{w_t\}\nearrow & \downarrow{\scriptstyle g} \\ X & \xrightarrow{w'} & X & \xrightarrow{w} & X \\ \| & \{k_t\}\nearrow & & & \| \\ X & = & = & = & X \end{array} \in \pi_1^A(X; g),$$

where $k_t : 1 \simeq ww'$. Let

(2.7)
$$\{\gamma, \gamma'\}^A = \{\delta | k_t : 1 \simeq ww'\} \subseteq \pi_1^A(X; g).$$

Then we have

2.8. PROPOSITION. *The subset $\{\gamma, \gamma'\}^A$ is a coset in $\pi_1^A(X; g)$ of the image of the homomorphism $.g$ and is independent of the choice of representatives $(\{w_t\}, w)$ and $(\{w'_t\}, w')$. Moreover $n\{\gamma, \gamma'\}^A = \gamma.\gamma'$.*

The proof is dual to that of Proposition 1.6.

2.9 EXAMPLE. Take $A = S^5$, $X = S^3$ and let $g : S^5 \to S^3$ be a representative of η_3^2 in $\pi_5(S^3)$. The homomorphism $.g : \pi_1^X(X; 1) \to \pi_1^A(X; g)$ is equivalent to Rutter's homomorphism $\Gamma(1_{S^3}, g) : \pi_4(S^3) \to \pi_6(S^3)$, [10]. Since S^3 is an H-space it follows from [10; Corollary 3.3.4] that $\Gamma(1_{S^3}, g) = (Sg)^* = (\eta_4^2)^*$, where S denotes suspension, so that we have $coker(.g) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$. Moreover since $(-\iota_3) \circ \eta_3 = -\eta_3 = \eta_3$, we have $\underline{\mathscr{E}}^g(S^3) \approx \boldsymbol{Z}_2$ and we have an extension

(2.9.1)
$$\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3 \rightarrowtail \mathscr{E}(S^5/g) \twoheadrightarrow \boldsymbol{Z}_2.$$

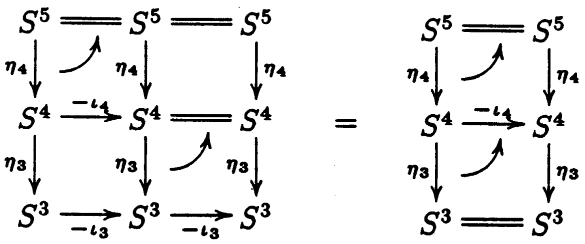
As with Proposition 1.7.1, we again find it to be split.

2.9.2. PROPOSITION. $\mathscr{E}(S^5/g) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$.

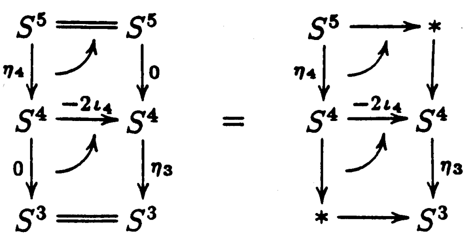
PROOF. The following generators of $\mathscr{E}(S^5/g)$ can be identified.

(2.9.3)
$$\gamma = \begin{array}{ccc} S^5 & = & S^5 \\ {\scriptstyle \eta_4}\downarrow & & \downarrow{\scriptstyle \eta_4} \\ S^4 & = & S^4 \\ {\scriptstyle \eta_3}\downarrow & \nearrow & \downarrow{\scriptstyle \eta_3} \\ S^3 & \xrightarrow[-\iota_3]{} & S^3 \end{array}, \qquad \gamma' = \begin{array}{ccc} S^5 & = & S^5 \\ {\scriptstyle \eta_4}\downarrow & \nearrow & \downarrow{\scriptstyle \eta_4} \\ S^4 & \xrightarrow[-\iota_4]{} & S^4 \\ {\scriptstyle \eta_3}\downarrow & & \downarrow{\scriptstyle \eta_3} \\ S^3 & \xrightarrow[-\iota_3]{} & S^3 \end{array}.$$

We claim that $\{\gamma, \gamma'\}^{S^5}$ in $\pi_6(S^3)$ is the coset containing the elements ν' and $-\nu'$. To see this, note that $\{\gamma, \gamma'\}^{S^5}$ contains the element



which we may interpret as an element of $\pi_1^{S^5}(S^3; g)$. Subtracting η_3^2 from each route, according to the principle stated earlier, we obtain the element

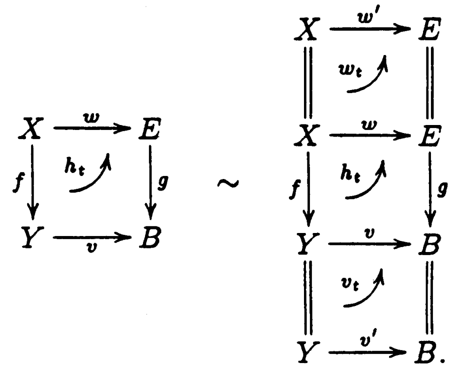


which belongs to the Toda bracket $\{\eta_3, 2\iota_4, \eta_4\}$. The remainder of the proof is similar to that for Proposition 1.7.1.

3. Homotopy pair self-equivalences

The objects of the *category of homotopy pairs* [5] are usually taken to be pointed continuous maps but in this section (except for Example 3.8) all considerations will apply equally to the basepoint free case. The morphisms from f to g are equivalence classes of track commutative squares, where

(3.1)



In non-diagramatic form the relation can be expressed:

(3.2) $$(v, \{h_t\}, w) \sim (v', \{v_t f + h_t + g w_t\}, w')$$

and $\{v, h_t, w\} \in \pi(f, g)$ used to denote the corresponding equivalence class.

Let $\mathscr{E}(f)$ be the homotopy pair self-equivalence group of $f : X \to Y$ and consider the operators $d : \pi(f,g) \to \pi(X,X)$, $c : \pi(f,g) \to \pi(Y,Y)$ such that $d\{v,h_t,w\} = \{w\}$, $c\{v,h_t,w\} = \{v\}$. Consider also the pull-back diagram (in the category of pointed sets)

$$\begin{array}{ccc} \mathscr{E}(X)\sqcap\mathscr{E}(Y) & \longrightarrow & \mathscr{E}(Y) \\ \big\downarrow & & \big\downarrow{\scriptstyle \cdot f} \\ \mathscr{E}(X) & \xrightarrow{\ f\cdot\ } & \pi(X,Y), \end{array}$$

where $\mathscr{E}(X)$ and $\mathscr{E}(Y)$ denote the self-equivalence groups of X and Y respectively. By [5; Theorem 1.3], for every $\{v,h_t,w\}$ in $\mathscr{E}(f)$ we have $\{v\} \in \mathscr{E}(Y)$ and $\{w\} \in \mathscr{E}(X)$ so that the operators d and c define a function

$$(d,c) : \mathscr{E}(f) \to \mathscr{E}(X) \sqcap \mathscr{E}(Y).$$

Moreover it is easy to check that the operations in $\mathscr{E}(X)$ and $\mathscr{E}(Y)$ induce a group structure in $\mathscr{E}(X) \sqcap \mathscr{E}(Y)$ and that (d,c) is a homomorphism. Let $\nabla : \pi_1^X(Y;f) \to \mathscr{E}(f)$ be the operator such that $\nabla\{h_t\} = \{1_Y, h_t, 1_X\}$ and let $(.f,f.) : \pi_1^Y(Y;1) \oplus \pi_1^X(X;1) \to \pi_1^X(Y;f)$ be the function (in general *not* a homomorphism) such that $(.f,f.)(\{v_t\},\{w_t\}) = \{v_tf + fw_t\}$.

3.3 REMARK. $\pi_1^Y(Y;1)$ and $\pi_1^X(X;1)$ are abelian and, if Y^X is an H_0-space [10; p381], then $\pi_1^X(Y;f)$ is abelian and $(.f,f.)$ is equivalent to the homomorphism

$$(\Gamma(1_Y,f), \Delta(f,1_X)) : \pi(Y,\Omega Y) \oplus \pi(\textstyle\sum X, X) \to \pi(\textstyle\sum X, Y),$$

where Γ and Δ are the homomorphisms described in [10; 3.2, 1.2].

3.4. PROPOSITION. *The following sequence in which* ∇ *and* (d,c) *are (group)* homomorphisms *is exact in the category of pointed sets. If* Y^X *is an* H_0*-space then it is exact in the category of groups.*

$$\pi_1^Y(Y;1) \oplus \pi_1^X(X;1) \xrightarrow{(.f,f.)} \pi_1^X(Y;f) \xrightarrow{\nabla} \mathscr{E}(f) \overset{(d,c)}{\twoheadrightarrow} \mathscr{E}(X) \sqcap \mathscr{E}(Y)$$

PROOF. Clearly $\nabla(.f,f.) = 0$. Suppose that $\nabla\{h_t\} = \{1,f,1\}$. Then in view of 3.2 there exist homotopies $v_t : 1_Y \simeq 1_Y$ and $w_t : 1_X \simeq 1_X$ such that $\{1,f,1\} = \{1,h_t,1\} = \{1, v_tf + fw_t, 1\}$ and $\{h_t\} = \{v_tf + fw_t\} = (.f,f.)(\{w_t\},\{v_t\})$. Clearly $(d,c)\nabla = 0$. Suppose that $(d,c)\{v,h_t,w\} = (\{1_X\},\{1_Y\})$. Then there exist homotopies $w_t : w \simeq 1_X$ and $v_t : 1_Y \simeq v$ so that $\{v, h_t, w\} = \nabla\{v_t f + h_t + f w_t\}$, completing the proof.

As a corollary of Proposition 3.4, we obtain the short exact sequence

$$(3.5) \qquad \pi_1^X(Y; f)/ker(\nabla) \rightarrowtail \mathscr{E}(f) \twoheadrightarrow \mathscr{E}(X) \sqcap \mathscr{E}(Y).$$

To obtain an analogue of Proposition 1.6, let $\delta = \{v, h_t, w\}$ and $\delta' = \{v', h'_t, w'\}$ be elements of $\mathscr{E}(f)$ such that $c\delta.c\delta' = \{1_Y\}$ and $d\delta.d\delta' = \{1_X\}$. Then there exist homotopies $g_t : 1_Y \simeq vv'$ and $k_t : ww' \simeq 1_X$ giving rise to an element

$$(3.6) \qquad \mu = \mu(g_t, k_t) = \begin{array}{ccccc} X & = & = & = & X \\ \| & & k_t \nearrow & & \| \\ X & \xrightarrow{w'} & X & \xrightarrow{w} & X \\ f\downarrow & h'_t \nearrow & f\downarrow & h_t \nearrow & \downarrow f \\ Y & \xrightarrow{v'} & Y & \xrightarrow{v} & Y \\ \| & & g_t \nearrow & & \| \\ Y & = & = & = & Y \end{array} \in \pi_1^X(Y; f).$$

Let

$$(3.7) \qquad \{\delta, \delta'\} = \{\mu | g_t : 1_Y \simeq vv', k_t : ww' \simeq 1_X\} \subseteq \pi_1^X(Y; f).$$

Then we have

3.8. Proposition. *The subset* $\{\delta, \delta'\}$ *is a coset in* $\pi_1^X(Y; f)$ *of* $ker(\nabla)$. *Moreover* $\nabla\{\delta, \delta'\} = \delta.\delta'$.

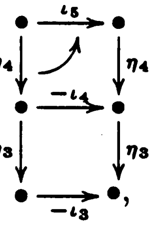
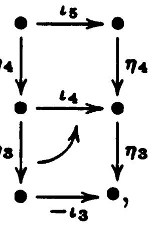
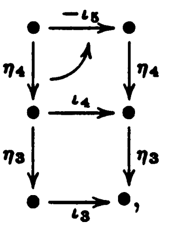
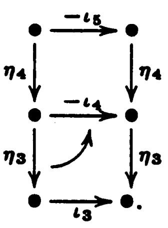
Proof. The second assertion is a consequence of the homotopy pair relation 3.2. For the first assertion, it is sufficient to observe that $\{\delta, \delta'\} = \nabla^{-1}(\delta.\delta')$.

3.9 Example. Take $X = S^5$, $Y = S^3$ and again let f be a representative of η_3^2. As indicated in Example 1.7 (respectively Example 2.9), the homomorphism f (respectively the homomorphism $f.$) is equivalent to $\eta_4^{2^*}$ (respectively to $\eta^2_{3_*}$). Since $coker(\eta_4^{2^*}, \eta^2_{3_*}) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_3$ and since $-\iota_3 \circ \eta_3^2 = \eta_3^2 \circ (-\iota_5) = \eta_3^2$, we have $\mathscr{E}(X) \sqcap \mathscr{E}(Y) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the sequence 3.5 becomes

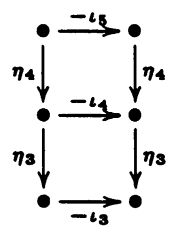
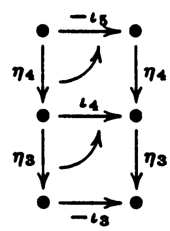
$$\mathbf{Z}_2 \oplus \mathbf{Z}_3 \rightarrowtail \mathscr{E}(f) \twoheadrightarrow \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

3.10. Proposition. $\mathscr{E}(f) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$.

The style of proof is similar to that for Propositions 1.7.1 and 2.9.2. The following elements can be identified.

$\gamma =$  $\gamma' =$  $\beta =$  $\beta' =$ 

Then we find $\beta\beta' = \beta'\beta = \gamma\gamma' = \gamma'\gamma = \nabla(v')$,

$\beta\gamma = \gamma\beta = \beta'\gamma' = \gamma'\beta' =$  , $\beta\gamma' = \gamma'\beta = \beta'\gamma = \gamma\beta' =$ 

with each element above of order 2 and the elements of order 3 commuting with them. One possible choice of representatives of generators of the $\mathbf{Z}_2$ summands would be γ, β and $\nabla(v')$.

4. The self-equivalence group of a mapping cone

In this final section we consider the relation between the homotopy pair self-equivalence group $\mathscr{E}(f)$ of a map f and the self-equivalence group $\mathscr{E}(C_f)$ of the associated mapping cone. Given a pointed map $f : X \to Y$, let $Pf : Y \to C_f$ denote the inclusion of Y into the mapping cone of f. It is well known that a homotopy commutative diagram, such as the left hand diagram of 3.1 induces a map $\chi : C_f \to C_g$ given by equations

$$(4.1) \qquad \chi(x,t) = (wx,t) + h_{1-t}x, (x \in X, t \in I), \quad \chi y = vy (y \in Y).$$

In particular the assignment $\Theta\{v, h_t, w\} = \{\chi\} \in \pi(C_f, C_g)$ is functorial and restricts further to a homomorphism

$$\Theta : \mathscr{E}(f) \to \mathscr{E}(C_f).$$

We shall be interested in conditions under which Θ becomes an isomorphism.

4.2. THEOREM. *Suppose that X and Y are CW-complexes whose homotopy groups (based at $*$) $\pi_i(X,*)$, $\pi_i(Y,*)$ vanish for all $i \leq a-1$ and for all $i \leq b-1$, respectively. If $dim(X) \leq Min(a+b-2, 2a-1)$ and $dim(Y) \leq a$ then the* *homomorphism* $\Theta : \mathscr{E}(f) \to \mathscr{E}(C_f)$ *is a surjection. If, further,* $dim(X) < 2a - 1$, $dim(Y) < a$ *and* $f = \sum f' : \sum X' \to \sum Y'$ *then* Θ *is an isomorphism.*

PROOF. The homomorphism is surjective if and only if every map $C_f \to C_f$ is *principal*, see [1; Chapter V.2], and according to [1; V, 7.7, 7.8, and 7.9], this is the case under the conditions given. Furthermore if $f = \sum f' : \sum X' \to \sum Y'$ then injectivity is a consequence of the cofibre sequence (Puppe sequence) since Θ is compatible with the extension in (3.5), compare the extension [1; V.7.19].

4.3. REMARK. In the case of Example 3.9 it is easy to check that the conditions of Theorem 4.2 are satisfied. It follows that Proposition 3.10 also computes the self-equivalence group $\mathscr{E}(C_f)$, for f a representative of the class η_3^2.

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